# Lecture 7: LQR from E-L eqns

$$\dot{x} = A(t)x + B(t)u$$

$$x(t_0) \longrightarrow x(t_f) \cong 0$$

$t_0$ & $t_f$ fixed.

using acceptable levels of control + not exceeding acceptable levels of state on the way

$$J = \frac{1}{2} x^T(t_f) S_f x(t_f) + \frac{1}{2}\int_{t_0}^{t_f} \left[x^T Q x + u^T R u\right] dt$$

$$S_f, Q, R > 0$$

## E-L Necessary Conditions:

$$\dot{x} = Ax + Bu$$

$$\dot{\lambda}^T = -\frac{\partial H}{\partial x} \qquad \lambda(t_f) = S_f x(t_f)$$

$$0 = \frac{\partial H}{\partial u} \qquad H = \frac{1}{2}x^T Q x + \frac{1}{2}u^T R u + \lambda^T(Ax + Bu)$$

---

$$\dot{\lambda} = -Qx - A^T\lambda$$

$$0 = Ru + B^T\lambda \Rightarrow \boxed{u = -R^{-1}B^T\lambda} \hookrightarrow (1)$$

Using (1):

$$\begin{bmatrix} \dot{x} \\ \dot{\lambda} \end{bmatrix} = \begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix} \qquad \begin{array}{l} x(t_0) \text{ given} \\ \lambda(t_f) = S_f x(t_f) \end{array}$$

Now, if $x(t_f)$ is known, the solutions to these eqns can be written as:

$$x(t) = X(t, t_f)\,x(t_f)$$
$$\lambda(t) = \Lambda(t, t_f)\,x(t_f)$$

where $X(t_f, t_f) = I$
& $\Lambda(t_f, t_f) = S_f$

At $t = t_0$: $x(t_0) = X(t_0, t_f)\,x(t_f)$
or $x(t_f) = [X(t_0, t_f)]^{-1} x(t_0)$

$$x(t) = X(t, t_f)[X(t_0, t_f)]^{-1} x(t_0)$$
$$\lambda(t) = \Lambda(t, t_f)[X(t_0, t_f)]^{-1} x(t_0)$$

Substituting into $u(t) = -R^{-1}B^T\lambda$

$$u(t) = -R^{-1}B^T\Lambda(t, t_f)[X(t_0, t_f)]^{-1} x(t_0)$$

Clearly we can replace $t_0$ by any $t$, hence

$$u(t) = -R^{-1}B^T\Lambda(t, t_f)[X(t, t_f)]^{-1} x(t)$$

Similarly $\lambda(t) = \underbrace{\Lambda(t, t_f)[X(t, t_f)]^{-1}}_{S(t)} x(t)$

## Differential Riccatti Eqn

Use $\lambda(t) = S(t)\,x(t)$
in
$$\dot{\lambda} = -Qx - A^T\lambda \quad \text{to get}$$

$$\dot{S}x + S\dot{x} = -Qx - A^T S x$$

[Now substituting $\dot{x} = Ax - BR^{-1}B^T\lambda$
$= Ax - BR^{-1}B^T S x$ ]

$$\dot{S}x + S\left[Ax - BR^{-1}B^T S x\right] = -Qx - A^T S x$$

$$\left[\dot{S} + SA + A^T S - SBR^{-1}B^T S + Q\right]x = 0$$

Since $x(t) \not\equiv 0$ :

$$\boxed{\dot{S} + SA + A^T S - SBR^{-1}B^T S + Q = 0}$$

$$S(t_f) = S_f \qquad \hookrightarrow \underline{DRE}$$

* must be solved backwards in time. to find $S(t_0)$
* then $\lambda(t_0) = S(t_0)\,x(t_0)$
* then $\begin{bmatrix} \dot{x} \\ \dot{\lambda} \end{bmatrix} = \begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix}$

can be solved to compute optimal trajectories

* Otherwise $S(t)$ can be used to generate the feedback law

__Example__ : $\dot{x} = u \qquad x(t_0),\ t_0,\ t_f$ sp.
$x, u$ - scalar

$$\min J = \frac{1}{2}c\left[x(t_f)\right]^2 + \frac{1}{2}\int_{t_0}^{t_f} u^2\,dt$$

$$H = \frac{1}{2} u^2 + \lambda u \qquad (\lambda - \text{scalar})$$

$$\dot{\lambda} = -\frac{\partial H}{\partial x} = 0 \Rightarrow \lambda = \text{constant}$$

$$\frac{\partial H}{\partial u} = 0 \Rightarrow u + \lambda = 0 \Rightarrow u = -\lambda$$

$$\lambda(t_f) = c\,x(t_f) \qquad (\text{bdd condition})$$

$$x(t) = -[c\,x(t_f)](t - t_0) + x(t_0)$$

$$\Rightarrow x(t_f) = \frac{x(t_0)}{1 + c(t_f - t_0)}$$

$$u(t, t_0) = -\frac{1}{\frac{1}{c} + (t_f - t_0)} x(t_0)$$

$$u(t) = -\frac{1}{\frac{1}{c} + t_f - t} x(t)$$

Note: $x(t_f) \to 0$ as $c \to \infty$.