Errors

Recall $\begin{bmatrix} \varepsilon & 1 \\ 1 & \pi \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$. Using $\begin{bmatrix} \varepsilon & 1 \\ 1 & \pi \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 \\ \frac{1}{\varepsilon} & 1 \end{bmatrix}}_{L}\underbrace{\begin{bmatrix} \varepsilon & 1 \\ 0 & \pi - \frac{1}{\varepsilon} \end{bmatrix}}_{U}$

suppose the solution is $\hat{x}$, while the "true" solution is (say) x^*.

Q. How do we guarantee $\|\hat{x} - x^*\|$ to be small?

Subsidiary questions:

1) Why is $\hat{x} \neq x^*$?
2) How to calculate $\|\hat{x} - x^*\|$?
3) Does $\|\hat{x} - x^*\|$ depend on data (i.e. A, b) or on the algo (LU, LU with pivot, Rook-LU etc) or on both?

[Darve 3.2]

Floating Point Arithmetic:

(we will use Julia commands — same for many other lang.)

E.g.s Int8 $\boxed{\text{Integers}}$, Float 32, Float64 (Reals)

Int8 $\longrightarrow$ $\underbrace{d_7 d_6 \cdots d_0}_{\text{binary}}$ with $x = \sum_{i=0}^{6} d_i \cdot 2^i - d_7 2^7$

bitstring(Int8(1)) = "00000001" | $x = 1 \cdot 2^0 = 1$

" (Int8(-1)) = "11111111" | $x = 2^0 + \cdots + 2^6 - 2^7 = 127 - 128 = -1$

" (Int8(-128)) = "10000000"

Float32: $\pm \underbrace{\left(1 + \sum_{i=1}^{p-1} d_i 2^{-i}\right)}_{\substack{\text{Significand} \\ \text{/ mantissa}}} \underbrace{2}_{\text{Base } 2}^{\,e}$

$p-1 \rightarrow$ precision

$e \rightarrow$ exponent

E.g: $3.140625 = (\overbrace{1}^{\text{Not stored} \to \text{always 1 by convention}}.\underbrace{5703125}_{\text{significand}})\, 2^1$ | $e = 1$, sign $= +$

$$0.5703125 = \underset{0.5}{\frac{1}{2}} + \underset{0.0625}{\frac{1}{16}} + \underset{0.0078125}{\frac{1}{128}} = 2^{-1} + 2^{-4} + 2^{-7}$$

bitstring(Float32(3.140625))

$= \underbrace{0}_{\text{Sign}} \quad \underbrace{\overset{d_7}{1}000000\overset{d_0}{0}}_{e = \sum_{i=0}^{7} d_i 2^i - (2^7 - 1)} \quad \overbrace{1001\,00100\cdots\cdots 0}^{23 \text{ bits}}$

Sign: $0 \to +$, $1 \to -$

Here $e = 2^7 - 2^7 + 1 = 1$

$d_1 = d_4 = d_7 = 1$

Ex: bitstring(Float32(1.5703125)) $\to e = 01111111$ (Why?)

Ex: What is bitstring(Float32(0.5703125)) ? Why?

Ex: Show that Float32 can be used to represent nos between 10^{-38} to 10^{38}.

Float64 : Same setup : Sign $\to$ 1 bit
$e \to$ 11 bits

Errors in FPA

Basic Errors:
1> Overflow
2> Underflow
} when results go out of range for a particular data type
3> Roundoff

Roundoff in base 10 (easier than base 2 for explanation)

[Golub 2.7]

Let $x = \pm \underbrace{d_0 . d_1 d_2}_{\text{significand}} \times 10^e$ where $1 \leq d_0 \leq 9$
$0 \leq d_1 \leq 9$
$0 \leq d_2 \leq 9$
$-9 \leq e \leq 9$

Precision is defined by length of significand
e.g. $\pi = 3.14 \times 10^0$ (Error $\approx 10^{-3}$)

Not enough room:
$(1.23 \times 10^6) + (4.56 \times 10^4) = 1275600$
$(1.23 \times 10^1) * (4.56 \times 10^2) = 5608.8$
Results must be rounded
round$(1275600) = 1.28 \times 10^6$
round $(5608.8) = 5.61 \times 10^3$

There is a lowest & highest no (leading to underflow & overflow)

Set of nos is finite (Here $2 \times 9 \times 10 \times 10 \times 19 + 1 = 34201$)

Spacing between consecutive nos. vary
e.g. diff between 1×10^2 & 1.01×10^2 is 1.
but diff between 1×10^3 & 1.01×10^3 is 10.

Def^n: Unit Roundoff Error:
$u = \frac{1}{2} \times$ [gap between 1 & next largest fl. point no]

For Float32 $u \simeq 10^{-7}$
Float64 $u \simeq 10^{-16}$

Can be checked with nextfloat(1.0) command! $\boxed{\square}$

<u>FACT</u>: For floating pt. nos. addition is not associative

If $x \in \mathbb{R}$, then, $fl(x) = x(1+\varepsilon)$ $|\varepsilon| < u$

the fl. pt. representation of x

Let $x = 1.24 \times 10^{0}$ (for our example 3 digit calc. above)
$y = -1.23 \times 10^{0}$
$z = 1.00 \times 10^{-3}$

$fl(x+y) = 1 \times 10^{-2}$
$fl[fl(x+y) + z] = 1.10 \times 10^{-2}$

But, $fl(y+z) = -1.23 \times 10^{0}$
$fl(x + fl(y+z)) = 1 \times 10^{-2}$

error $\not\approx u$!

Catastrophic Cancellation

Q) Why is this happening?

$$fl(x) + fl(y) = x(1+\varepsilon_1) + y(1+\varepsilon_2)$$

$|\varepsilon_1| \approx |u| \ll 1$
$|\varepsilon_2| = |u| \ll 1$

$$= x + y + x\varepsilon_1 + y\varepsilon_2$$
$$= (x+y)\left[1 + \underbrace{\left(\frac{x}{x+y}\right)\varepsilon_1 + \left(\frac{y}{x+y}\right)\varepsilon_2}\right]$$

[Watkins 2-5 pg 144]

If either $x \gg x+y$ or $y \gg x+y$ then error blows up (larger than u)

(get magnified)

Q. It seems that such errors can accumulate through the steps of any algo. Then how do we guarantee accuracy?

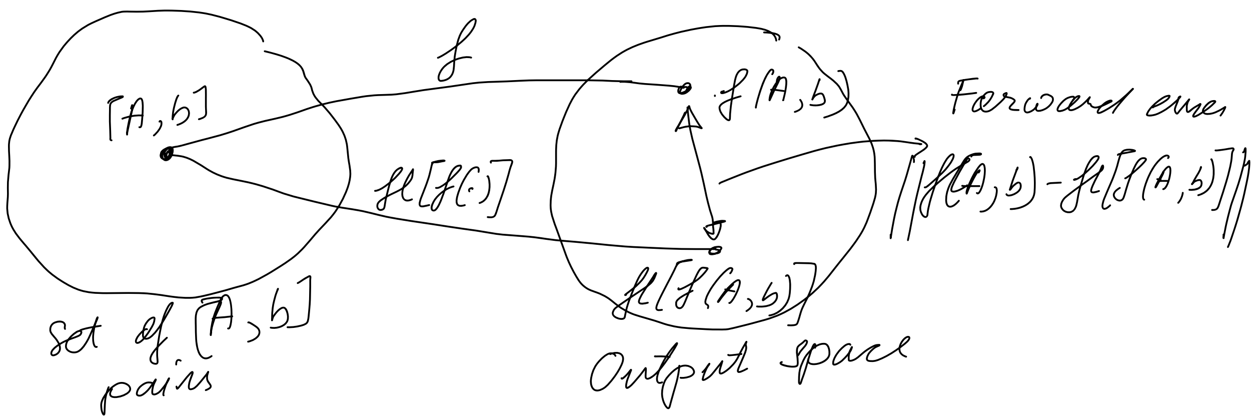
Show code for $\boxed{\text{Watkins } 2.6.1,\ 2.6.2,\ 2.6.9}$

<u>Some definitions</u>: Let any algo be defined by a map $f : [\text{data}] \longrightarrow [\text{Result}]$

e.g. in solving $Ax = b$ using LU

$$f(A, b) = A^{-1}b \qquad \text{Data} : [A, b]$$

Since this algo is implemented on a comp. the output is $fl[f(A,b)]$



Directly computing "Forward error" is difficult (catastropic cancellation can occur somewhere in the intermediate steps)

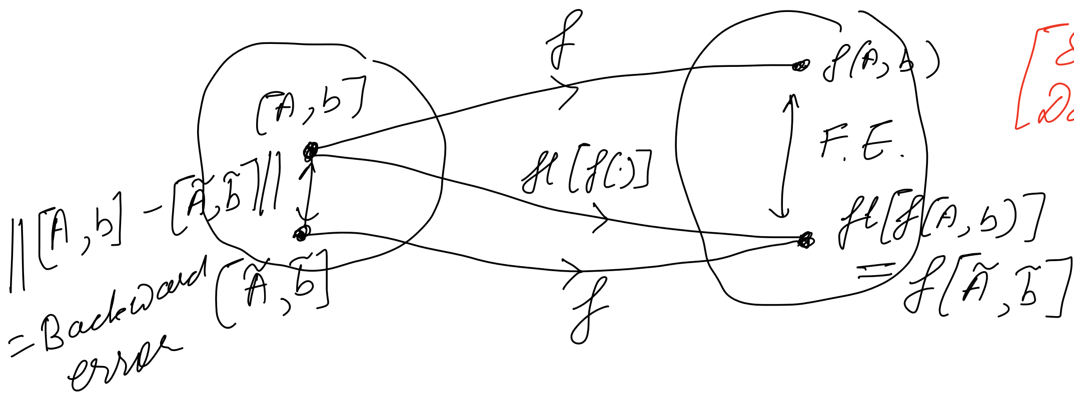
We ask a "strange" new question!

Q> Which $[\tilde{A}, \tilde{b}]$ gives

$$f[\tilde{A}, \tilde{b}] = fl[f(A, b)]$$

Then define Backward error $= \|(A, b) - [\tilde{A}, \tilde{b}]\|$

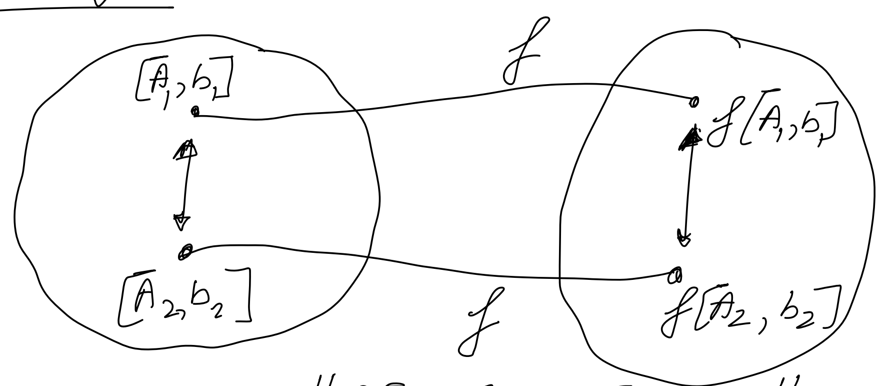
[Similar to Darve 3.3]



It is often easy to bound backward error for a <u>specific algorithm</u>.

Q. How does B.E (even if easy to compute) help in calculating F.E?

<u>Sensitivity</u>



$$\text{Sensitivity} = \frac{\| f[A_1, b_1] - f[A_2, b_2] \|}{\| [A_1, b_1] - [A_2, b_2] \|}$$

<u>Note</u>: 1) Sensitivity has nothing to do with floating point operation.

2) Sensitivity depends on data & f. → hence can be bdd. easily.

\# $F.E. = \| f(\hat{A},b) - fl(f(A,b)) \| = \| f(A,b) - f(\tilde{A},\tilde{b}) \|$

$$= \frac{\| f(A,b) - f(\tilde{A},\tilde{b}) \|}{\| (A,b) - (\tilde{A},\tilde{b}) \|} \cdot \| (A,b) - (\tilde{A},\tilde{b}) \|$$

$$= \text{Sensitivity} \times B.E$$

\# If we have bounds on Sensitivity & B.E separately, then we can bound F.E also.

Big - O Notation : A function $f(n)$ is $O(g(n))$ if $\exists\ n_0 > 0$ & $c > 0$ s.t. for all $n \geq n_0$, $f(n) \leq c\,g(n)$

1) Sensitivity of Linear Systems $(Ax = b)$

[Golub 2.6.2]

Let $(A + \varepsilon F)\,x(\varepsilon) = b + \varepsilon f$ $\quad x(0) = x$

$(A$ is non-singular$)$ $\quad F \in \mathbb{R}^{n\times n}, f \in \mathbb{R}^n$

\# Clearly, $x(\varepsilon)$ is diff. in the neighborhood of 0.

$$Ax(\varepsilon) + \varepsilon F x(\varepsilon) = b + \varepsilon f$$

Diff. w.r.t ε : $A\dot{x}(\varepsilon) + Fx(\varepsilon) + \varepsilon F\dot{x}(\varepsilon) = f$

$$(A + \varepsilon F)\,\dot{x}(\varepsilon) + Fx(\varepsilon) = f$$

$$\Rightarrow \quad A\dot{x}(0) + Fx = f \Rightarrow \dot{x}(0) = A^{-1}[f - Fx]$$

Taylor series of $x(\varepsilon)$ around "0":

$$x(\varepsilon) = x + \varepsilon\,\dot{x}(0) + O(\varepsilon^2)$$

Then $\|x(\varepsilon) - x\| = \|\varepsilon A^{-1}[f - Fx] + O(\varepsilon^2)\|$

or

$$\frac{\|x(\varepsilon) - x\|}{\|x\|} \leq |\varepsilon| \|A^{-1}\| \left[\frac{\|f\| + \|F\|\|x\|}{\|x\|}\right] + O(\varepsilon^2)$$

$$\leq |\varepsilon| \|A\| \|A^{-1}\| \left[\frac{\|F\|}{\|A\|} + \frac{\|f\|}{\|A\|\|x\|}\right] + O(\varepsilon^2)$$

$$\leq \underbrace{\|A\| \|A^{-1}\|} \left[\frac{|\varepsilon| \|F\|}{\|A\|} + \frac{|\varepsilon| \|f\|}{\|b\|}\right] + O(\varepsilon^2)$$

[since $\|A\|\|x\| \geq \|b\|$]

$$\leq \kappa(A) \left[\underbrace{\frac{|\varepsilon| \|F\|}{\|A\|}}_{\text{Relative error in } \|A\|} + \underbrace{\frac{|\varepsilon| \|f\|}{\|b\|}}_{\text{Relative error in}}\right] + O(\varepsilon^2)$$

$\therefore$

<u>Def</u>: Condition No: $\kappa(A) = \|A\| \|A^{-1}\|$
with the convention $\kappa(A) = \infty$ if A singular

\# Non-parametric bounds on relative errors are also easily derived:

<u>Thm:</u> If A is non-singular, $\overbrace{\frac{\|\delta A\|}{\|A\|} < \frac{1}{\kappa(A)}}^{\text{small perturbation assumption}}$, $b \neq 0$
$Ax = b$ and $(A + \delta A)(x + \delta x) = b + \delta b$, then

$$\frac{\|\delta x\|}{\|x\|} \leq \frac{\kappa(A)\left[\frac{\|\delta A\|}{\|A\|} + \frac{\|\delta b\|}{\|b\|}\right]}{1 - \kappa(A)\frac{\|\delta A\|}{\|A\|}}$$

[Watkins 2.3]

Proof: $\cancel{Ax} + \delta A x + A\delta x + \delta A . \delta x = \cancel{b} + \delta b$

$$\delta x = A^{-1}\left[-\delta A x - \delta A . \delta x + \delta b\right]$$

$$\|\delta x\| \leq \|A^{-1}\| \left[\|\delta A\| \|x\| + \|\delta A\| \|\delta x\| + \|\delta b\|\right]$$

$$\leq K(A)\left[\frac{\|\delta A\|}{\|A\|}\|x\| + \frac{\|\delta A\|}{\|A\|}\|\delta x\| + \frac{\|\delta b\|}{\|A\|}\right]$$

$$= \underbrace{K(A)\frac{\|\delta A\|}{\|A\|}}_{<1}\|\delta x\| + K(A)\left[\frac{\|\delta A\|}{\|A\|} + \frac{\|\delta b\|}{\|A\|\|x\|}\right]\|x\|$$

$\|b\| \leq \|A\|\|x\|$

or $$\underbrace{\left[1 - K(A)\frac{\|\delta A\|}{\|A\|}\right]}_{>0}\frac{\|\delta x\|}{\|x\|} \leq K(A)\left[\frac{\|\delta A\|}{\|A\|} + \frac{\|\delta b\|}{\|b\|}\right]$$

or $$\frac{\|\delta x\|}{\|x\|} \leq \frac{K(A)}{1 - K(A)\frac{\|\delta A\|}{\|A\|}}\left[\frac{\|\delta A\|}{\|A\|} + \frac{\|\delta b\|}{\|b\|}\right]$$

Many similar bdds are possible

Thm: Let A be non-singular & let $Ax = b$ and $\hat{A}\hat{x} = \hat{b}$ where $\hat{A} = A + \delta A$, $\hat{x} = x + \delta x \neq 0$, $\hat{b} = b + \delta b \neq 0$, then

$$\frac{\|\delta x\|}{\|\hat{x}\|} \leq K(A)\left[\frac{\|\delta A\|}{\|A\|} + \frac{\|\delta b\|}{\|\hat{b}\|} + \frac{\|\delta A\|}{\|A\|}\frac{\|\delta b\|}{\|\hat{b}\|}\right]$$

(Small perturb. not req)

Bottom line: $K(A)$ should be small.

[Well-conditioned]

Ill-conditional $\iff$ $K(A)$ large

Backward Error for Gaussian Elimination

[Watkins 2.7]

FACT: Suppose we compute $\sum_{j=1}^{n} w_j$ using floating point arithmetic with unit roundoff u.

Then: $$fl\left(\sum_{j=1}^{n} w_j\right) = \sum_{j=1}^{n} w_j(1+\gamma_j)$$

where $|\gamma_j| \le (n-1)u + O(u^2)$ $\longrightarrow (*)$

regardless of the order in which the terms are accumulated.

Proof: By induction, $\rightarrow$ true for $n=1$.

Let $(*)$ be true for all int. $m<n$.

$$\sum_{j=1}^{m} w_j = \sum_{j=1}^{k} w_j + \sum_{j=k+1}^{m} w_j$$

By induction hypothesis, $fl\left(\sum_{j=1}^{k} w_j\right) = \sum_{j=1}^{k} w_j(1+\alpha_j)$

& $fl\left(\sum_{j=k+1}^{m} w_j\right) = \sum_{j=k+1}^{m} w_j(1+\alpha_j)$

where $|\alpha_j| \le \begin{cases} (k-1)u + O(u^2) & j=1\ldots k \\ (m-k-1)u + O(u^2) & j=k+1,\ldots,m \end{cases}$ $\leftarrow (\partial)$

But $k-1 < m-2$ & $(m-k-1) \le m-2$

Hence for (∂) $|\alpha_j| \le (m-2)u + O(u^2)$ $j=1,\ldots,m$

Now, $$fl\left(\sum_{j=1}^{m} w_j\right) = fl\left[fl\left(\sum_{j=1}^{k} w_j\right) + fl\left(\sum_{j=k+1}^{m} w_j\right)\right]$$

$$= \left[\sum_{j=1}^{k} w_j(1+\alpha_j) + \sum_{j=k+1}^{m} w_j(1+\alpha_j)\right](1+\beta)$$

$$= \sum_{j=1}^{m} w_j(1+\alpha_j)(1+\beta) \qquad [|\beta| < u]$$

Let $\gamma_j^{\circ} = \alpha_j^{\circ} + \beta + \alpha_j \beta$

$$\Rightarrow fl\left[\sum_{j=1}^{m} w_j\right] = \sum_{j=1}^{m} w_j(1+\gamma_j)$$

Then
$$\begin{aligned}|\gamma_j| &\leq |\alpha_j| + |\beta| + O(u^2)\\ &\leq (m-2)u + O(u^2) + u + O(u^2)\\ &= (m-1)u + O(u^2)\end{aligned}$$

Notation: $C \in \mathbb{R}^{m\times n}$, then $|C| \in \mathbb{R}^{m\times n}$ with $[|C|]_{ij} = |c_{ij}|$
$C \leq F \Leftrightarrow c_{ij} \leq f_{ij} \quad \forall i,j$.

B.E. for forward/backward substitution

Thm: Let $G \in \mathbb{R}^{n\times n}$ be non-singular, lower(upper) triangular matrix, and let $b \neq 0$. If $Gy = b$ is solved by any variant of forward (backward) substitution using fl. pt. arithmetic, then the answer $\hat{y}$ satisfies:
$$(G + \delta G)\hat{y} = b$$
where $|\delta G| \leq 2nu|G| + O(u^2)$ → $n\times m$ matrix each entry is of order u^2.
(i.e $|\delta g_{ij}| \leq 2nu|g_{ij}| + O(u^2) \quad \forall i,j$)

Proof: At the ith step: ideally. $y_i^{\circ} = \dfrac{b_i^{\circ} - \sum_{j=1}^{i-1} g_{ij} y_j^{\circ}}{g_{ii}}$

In reality: $\hat{y}_i^{\circ} = fl\left(\dfrac{b_i^{\circ} - \sum_{j=1}^{i-1} g_{ij}\hat{y}_j^{\circ}}{g_{ii}}\right)$

claim: $fl(g_{ij}\hat{y}_j) = g_{ij}\hat{y}_j(1+\alpha_{ij})$, $|\alpha_{ij}| \leq u$

Proof: Exercise.

From previous FACT:

$$fl\left(b_i - \sum_{j=1}^{i-1} g_{ij}\hat{y}_j(1+\alpha_{ij})\right)$$

$$= b_i(1+\gamma_{ii}) - \sum_{j=1}^{i-1} g_{ij}(1+\alpha_{ij})(1+\gamma_{ij})$$

where $|\gamma_{ij}| \leq (i-1)u + O(u^2)$ $j=1,\dots,i$

Next division:

$$\hat{y}_i = \left[\frac{b_i(1+\gamma_{ii}) - \sum g_{ij}\hat{y}_j(1+\alpha_{ij})(1+\gamma_{ij})}{g_{ii}}\right](1+\beta_i)$$

where $|\beta_i| < u$

$$= \frac{b_i - \sum_{j=1}^{i-1} g_{ij}(1+\varepsilon_{ij})\hat{y}_j}{g_{ii}(1+\varepsilon_{ii})}$$

for $1+\varepsilon_{ij} = \begin{cases} \dfrac{(1+\alpha_{ij})(1+\gamma_{ij})}{1+\gamma_{ii}}, & j<i \\ \dfrac{1}{(1+\gamma_{ii})(1+\beta_i)} & j=i \end{cases}$

or $\sum g_{ij}(1+\varepsilon_{ij})\hat{y}_j = b_i$

or $(G + \delta G)\hat{y} = b$

L.T with $\delta g_{ij} = \varepsilon_{ij} g_{ij}$ for $i>j$

Claim: $|\varepsilon_{ij}| \leq 2nu + O(u^2)$ $\forall i,j$

Proof: $\dfrac{1}{1+\gamma_{ii}} = 1 - \gamma_{ii} + \gamma_{ii}^2 - \gamma_{ii}^3, \dots$
$= 1 - \gamma_{ii} + O(u^2)$

since $\gamma_{ii} = O(u)$

Then for $j<i$, $1+\varepsilon_{ij} = (1+\alpha_{ij})(1+\gamma_{ij})(1-\gamma_{ii}+O(u^2))$
$= 1+\alpha_{ij}+\gamma_{ij}-\gamma_{ii}+O(u^2)$

$$\Rightarrow \quad |\varepsilon_{ij}| \leq |\alpha_{ij}| + |\gamma_{ij}| + |\gamma_{ii}| + O(u^2)$$
$$\leq u + (i-1)u + (i-1)u + O(u^2)$$
$$= (2i-1)u + O(u^2)$$
$$\leq 2nu + O(u^2)$$

Similarly $|\varepsilon_{ii}| \leq iu + O(u^2) \leq 2nu + O(u^2)$

B.E. in LU.

Thm 1: ~~Suppose~~ the LU decomposition of A is computed using fl. pt. arithmetic, & suppose no zero pivots are encountered. Let $\hat{L}$ & $\hat{U}$ are the computed factors. Then:

$$A + E = \hat{L}\hat{U}$$

where $|E| \leq 2nu|\hat{L}||\hat{U}| + O(u^2)$

& $\|E\|_\infty \leq 2nu\|\hat{L}\|_\infty\|\hat{U}\|_\infty + O(u^2)$

Thm 2: (Same assumptions as above thm), suppose we solve $Ax = b$ numerically using forward & backward substitution using $\hat{L}, \hat{U}$ resp. Then the solution $\hat{x}$ satisfies:

$$(A + \delta A)\hat{x} = b$$

where $|\delta A| \leq 6nu|\hat{L}||\hat{U}| + O(u^2)$

& $\|\delta A\|_\infty \leq 6nu\|\hat{L}\|_\infty\|\hat{U}\|_\infty + O(u^2)$

<u>Backward Stable</u> : (rather loosely used) → either 'B.E. is "small" or $\frac{B.E.}{\|Data\|}$ $\left(\text{e.g. } \frac{\|\delta A\|}{\|A\|}\right)$

From the above thm, B.E. is small only when $\|L\|$ & $\|U\|$ are small.

$\Rightarrow$ LU without pivoting, LU with partial pivoting are both NOT backward stable.

$\Rightarrow$ LU with full pivoting is Backward Stable.

<u>Proof</u>: 1) If $A = LU$, $l_{ij} = \dfrac{a_{ij} - \sum_{k=1}^{j-1} l_{ik} u_{kj}}{u_{jj}}$ $i > j$

& $u_{ij} = a_{ij} - \sum_{k=1}^{i-1} l_{ik} u_{kj}$ $i < j$

In actual computation : $\hat{l}_{ij} = fl(\cdots \hat{l}_{ik} \hat{u}_{kj})$

$\hat{u}_{ij} = fl(\cdots\cdots)$

Then : $\hat{l}_{ij} = \dfrac{a_{ij}(1+\gamma_{ij}) - \sum_{k=1}^{j-1} \hat{l}_{ik} \hat{u}_{kj} (1+\alpha_k)(1+\gamma_{ik})}{\hat{u}_{jj}} (1+\beta)$

$|\gamma_{ij}| \leq (j-1)u + O(u^2)$ → from previous thm.

$|\alpha_k| \leq u$ ← roundoff for mult.

$|\beta| \leq u$ ← roundoff for div.

Proceeding as in last thm (Exercise),

$\hat{l}_{ij} = \dfrac{a_{ij} - \sum_{k=1}^{j-1} \hat{l}_{ik} \hat{u}_{kj} (1+\delta_{ik})}{\hat{u}_{jj}(1+\delta_{ij})}$; with $|\delta_{ik}| \leq 2nu + O(u^2)$ $k = 1, \cdots, j$

or $\quad a_{ij} + e_{ij} = \sum_{k=1}^{j} \hat{l}_{ik} \hat{u}_{kj}$ [Consolidating all errors terms in e_{ij}]

where $\quad e_{ij} = - \sum_{k=1}^{j} \hat{l}_{ik} \hat{u}_{kj} \delta_{ik} \quad [i > j]$

$$\Rightarrow \quad |e_{ij}| \leq |\delta_{ik}| \sum_{k=1}^{j} |\hat{l}_{ik}| |\hat{u}_{kj}| + O(u^2)$$

$$= 2nu \sum_{k=1}^{j} |\hat{l}_{ik}| |\hat{u}_{kj}| + O(u^2) \quad (*)$$

For $i < j$, same results can be obtained using the u_{ij} equation.

Hence $(*)$, is true for $\forall \, i, j$.

In matrix form:

$$A + E = \hat{L}\hat{U}$$

& $$|E| \leq 2nu |\hat{L}| |\hat{U}| + O(u^2).$$

Proof of 2:

$$(\hat{L} + \delta\hat{L}) \hat{y} = b \Rightarrow |\delta\hat{L}| \leq 2nu |\hat{L}|$$
$$(\hat{U} + \delta\hat{U}) \hat{x} = \hat{y} \Rightarrow |\delta\hat{U}| \leq 2nu |\hat{U}|$$

$$(\hat{L} + \delta\hat{L})(\hat{U} + \delta\hat{U}) \hat{x} = b$$

$$[\hat{L}\hat{U} + \delta\hat{L}\hat{U} + \hat{L}\delta\hat{U} + \delta\hat{L}\delta\hat{U}] \hat{x} = b$$

$$[A + \underbrace{E + \cdots\cdots}_{\delta A}] \hat{x} = b$$

Assume
$A + E = \hat{L}\hat{U}$
$|E| \leq 2nu |\hat{L}| |\hat{U}| + O(u^2)$

$$\delta A = E + \delta\hat{L}\hat{U} + \hat{L}\delta\hat{U} + \delta\hat{L}\delta\hat{U}$$

$$|\delta A| \leq (2nu + 2nu + 2nu) |\hat{L}| |\hat{U}| + O(u^2)$$

$$= 6nu |\hat{L}| |\hat{U}| + O(u^2)$$