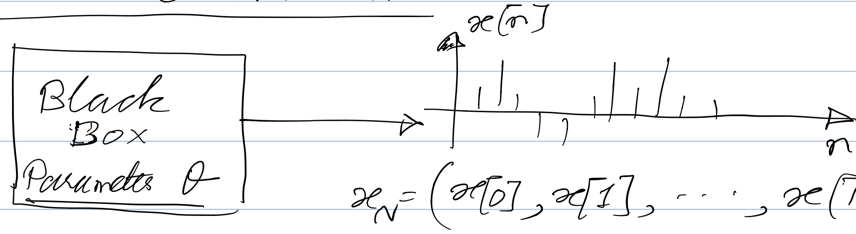
EE638 - Introduction

Estimation → Parameter Estimation ⇔ Identification

Specialized for various model structures.

Estimation → Signal Estimation ⇔ Filtering / Smoothing / Prediction

Parameter Estimation



$$x_N = (x[0], x[1], \cdots, x[N-1])$$

Assumptions: 1) x_N is somehow dependent on θ

2) θ is unknown but deterministic

(No Bayesian estimation in this course)

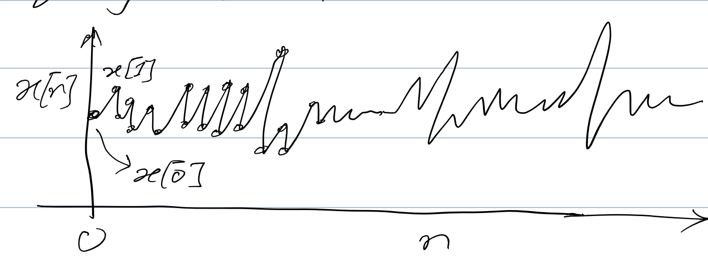
We try to "infer" θ from x_N using

$$\hat{\theta} = g(x[0], x[1], \cdots, x[N-1])$$

$\hat{\theta}$: estimate of θ ; g: estimator of θ

Examples: Bio-medicine, Military, Space, Comm, Control & everything else.

Priliminary Ideas:



Q. Where is θ here?

Clearly we need to guess 1) some model for the data OR

2) OR probability distribution of $x[1], \cdots, x[N-1]$

3) or both

Example

Let 1) $x[n] = A + w[n]$

2) $w[n]$ is zero mean, uncorrelated with equal variance σ^2. (full pdf would be better)

Let $\theta = A$. Can we estimate A?

Guess 1: $\hat{A} = \frac{1}{N}\sum_{n=0}^{N-1} x[n]$

Guess 2: $\bar{A} = x[0]$

1) Both are random variables.

2) Which one is better?

Expectation: $E(\hat{A}) = E\left(\frac{1}{N}\sum_{n=0}^{N-1} x[n]\right) = \frac{1}{N}\sum_{n=0}^{N-1} E(x[n])$
$= \frac{1}{N}\sum_{n=0}^{N-1}\left[E(A) + E(w[n])\right] = A$

$E(\bar{A}) = E(x[0]) = A \leftarrow$ Both are giving the true value.

"Intuitively" $\hat{A}$ is better. Check variance.

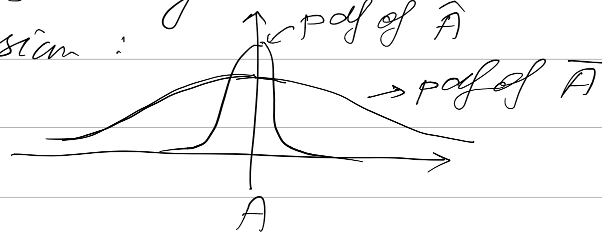
$Var(\hat{A}) = var\left(\frac{1}{N}\sum_{n=0}^{N-1} x[n]\right) = \frac{1}{N^2}\sum_{n=0}^{N-1} var(x[n])$
$= \frac{1}{N^2} N\sigma^2 = \frac{\sigma^2}{N}$

Exercise (since $w[n]$'s are uncorrelated)

$$\mathrm{Var}(\bar{A}) = \mathrm{var}(x[0]) = \sigma^2 > \mathrm{var}(\hat{A})$$

verifies our intuition

If $w[n]$ is further assumed to be Gaussian:



<u>Questions</u>: 1) What is the best estimator? → min variance / → other criteria | MLE, MAP, MMSE

2) Is it unbiased?

3) Is the best estimator linear?

4) Best linear estimator?

Signal - (State) Estimation / Kalman filter (1960's)

(prediction, smoothing)

$$x_{i+1} = F x_i + G(r_i + u_i) \quad \text{process noise} \qquad i \geq 0$$

$$y_i = H x_i + v_i \quad \text{measurement noise}$$

$$E\begin{bmatrix} u_i \\ v_i \\ x_0 \end{bmatrix}\begin{bmatrix} u_j^* & v_j^* & x_0^* & 1 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix}\delta_{ij} & \begin{matrix} 0 \\ 0 \end{matrix} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & \Pi_0 & 0 \end{bmatrix}$$

Note: 1) u_i, v_i are "white noise" → zero mean; $E u_i u_j^* = Q\delta_{ij}$; $E v_i v_j^* = R\delta_{ij}$

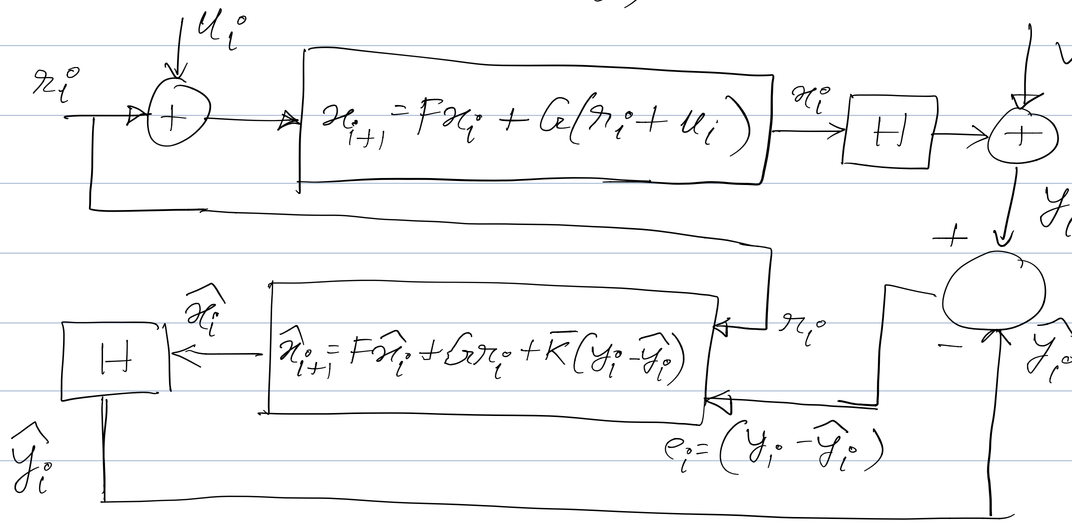
2) u_i, v_i are uncorrelated with x_0

3) x_0 is zero mean with variance Π_0

4) u_i, v_i are correlated. $E u_i v_j^* = S \delta_{ij}$

5) $Q = Q^* \geq 0$, $R = R^* \geq 0$, $\begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \geq 0$

(Q & R are not nec. diag)



We can make $E\hat{x}_0 = 0 = Ex_0$

Define $\tilde{x}_i = x_i - \hat{x}_i$. Then it is easy to show (look-up EE640 notes / Kailath etc):

$$E\tilde{x}_{i+1} = (F - \bar{K}H)\, E\tilde{x}_i \quad \text{with } E\tilde{x}_0 = 0.$$

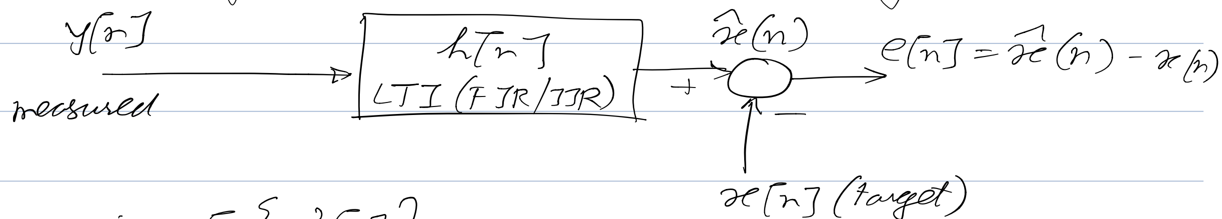
Q. Find $\bar{K}$ (or $\bar{K}_i$) to minimize $E\tilde{x}_i\tilde{x}_i^*$ for all i? → Solution is Kalman filter

Q. The observer structure seems ad hoc — is this the best estimator of x_i?

Q. What's the big deal → Efficient Recursion.

#Almost every signal estimation problem is Kalman filtering. → We go through innovations process, Wiener filtering to Kalman filter.

<u>Wiener Filtering</u> (1940's) → We will cover this in brief enroute to Kalman filter



$$\min_{h[\cdot]} E\{e^2[n]\}$$

↓

The minimizing filter is called Wiener filter

<u>Assumption</u>: Power spectra of $x[n]$ & $y[n]$ are known! → as opposed to (F, G, H) in Kalman filter.

<u>Important Limitation</u>: Does not work well with vector processes! → Kalman filter does.

All other filters (RLS, LMS etc) are derivatives of these two ideas.