Sampled - data K.F.

\# Real processes are continuous time, whereas the KF implementation is usually discrete time.

$$\dot{x} = F(t)x(t) + B(t)u(t) + G(t)w(t)$$

$$E(w(t)) = 0$$

$$E\left[w(t)w(t')^T\right] = Q(t)\,\delta(t-t')$$

\# Observations are available at discrete intervals:

$$(t_1, t_2, \ldots\ldots t_i, \ldots\ldots)$$

$$y(t_i) = H(t_i)\,x(t_i) + v(t_i)$$

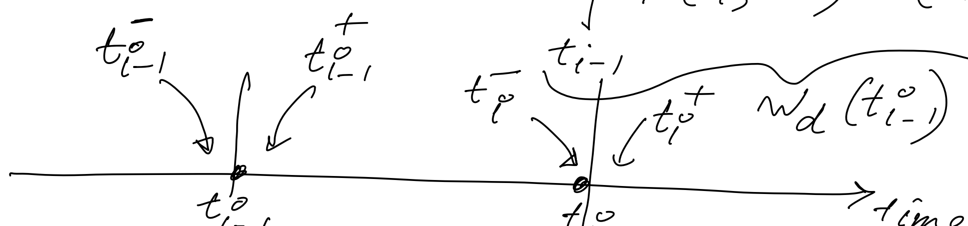
$$E(v(t_i)) = 0$$

$$E\left[v(t_i)v^T(t_j)\right] = R(t_i)\,\delta_{ij}$$

The discretized state eqn:

$$x(t_i) = \underbrace{\Phi(t_i, t_{i-1})}_{\text{state transition Matrix}} x(t_{i-1}) + \int_{t_{i-1}}^{t_i} \Phi(t_i,\tau)B(\tau)u(\tau)\,d\tau$$

$$+ \underbrace{\int_{t_{i-1}}^{t_i} \Phi(t_i, \tau)\,G(\tau)w(\tau)\,d\tau}_{w_d(t_{i-1})}$$



The K.F. Eqns in T.U + M.U. form:

$\boxed{\text{T.U.:}}$

$$\hat{x}(t_i^-) = \Phi(t_i, t_{i-1})\,\hat{x}(t_{i-1}^+) + \int_{t_{i-1}}^{t_i} \Phi(t_i,\tau)B(\tau)u(\tau)\,d\tau$$

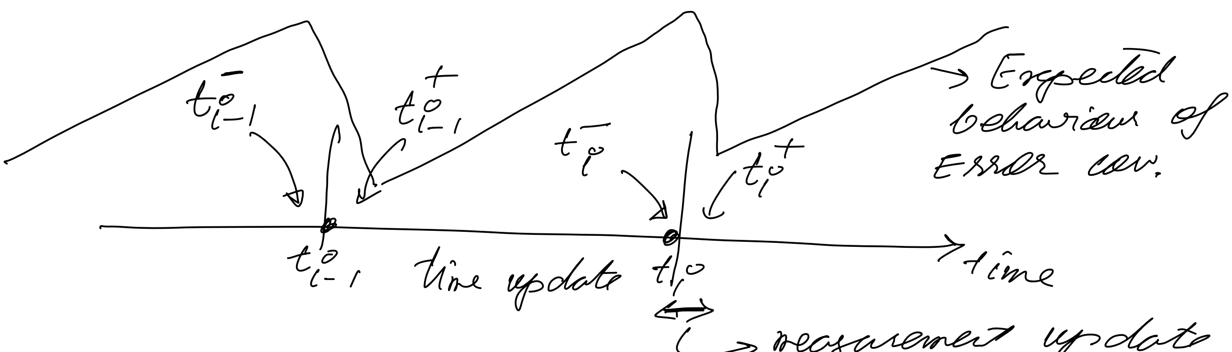
$$P(t_i^-) = \Phi(t_i, t_{i-1}) P(t_{i-1}^+) \Phi^T(t_i, t_{i-1}) +$$

$$+ \int_{t_{i-1}}^{t_i} \Phi(t_i, \tau) G(\tau) Q(\tau) G^T(\tau) \Phi^T(t_i, \tau) d\tau$$

$\boxed{\text{M.U}}$ $$K(t_i) = P(t_i^-) H^T(t_i) \left[H(t_i) P(t_i^-) H^T(t_i) + R(t_i) \right]^{-1}$$

$$\hat{x}(t_i^+) = \hat{x}(t_i^-) + K(t_i) \left[y_i - H(t_i) \hat{x}(t_i^-) \right]$$

$$P(t_i^+) = P(t_i^-) - K(t_i) H(t_i) P(t_i^-)$$



Kalman Filter Examples

Ex: $x(t) = u + w(t)$; u is constant

$E(w(t), w(t+\tau)) = \sigma_w^2 \delta(\tau)$ $w(t)$ is zero mean white Gaussian

$y(t_i) = x(t_i) + v(t_i)$ $\longrightarrow E(v(t_i), v(t_j)) = \sigma_v^2 \delta_{ij}$

Hence: $F = 0 \Rightarrow \phi = 1$, $B = 1$, $G = 1$, $Q = \sigma_w^2$

$H = 1$, $R = \sigma_v^2$,

$\boxed{\text{T.U. Eqns}}$

$$\hat{x}(t_i^-) = 1 . \hat{x}(t_{i-1}^+) + \int_{t_{i-1}}^{t_i} \Phi(t_i, \tau) B(\tau) u(\tau) d\tau$$

$$= \hat{x}(t_{i-1}^+) + \int_{t_{i-1}}^{t_i} 1 . 1 . u(\tau) d\tau$$

$$= \hat{x}(t_{i-1}^{o+}) + (t_i^o - t_{i-1}^o)\,u \quad \left[\text{since } u \text{ is constant}\right]$$

$$P(t_i^{o-}) = 1 \,.\, P(t_{i-1}^{o+}) \,.\, 1 + \int_{t_{i-1}}^{t_i^o} 1 \,.\, 1 \,.\, \sigma_\omega^2 \,.\, 1 \,.\, 1 \,.\, d\tau$$

$$= P(t_{i-1}^{o+}) + \sigma_\omega^2 (t_i^o - t_{i-1}^o)$$

$\boxed{\text{M.U.}}$

$$K(t_i^o) = P(t_i^{o-}) \Big/ \left[P(t_i^{o-}) + \sigma_v^2\right]$$

$$\hat{x}(t_i^{o+}) = \hat{x}(t_i^{o-}) + K(t_i^o)\left[y(t_i^o) - \hat{x}(t_i^{o-})\right]$$

$$P(t_i^{o+}) = P(t_i^{o-}) - K(t_i^o)\,P(t_i^{o-})$$

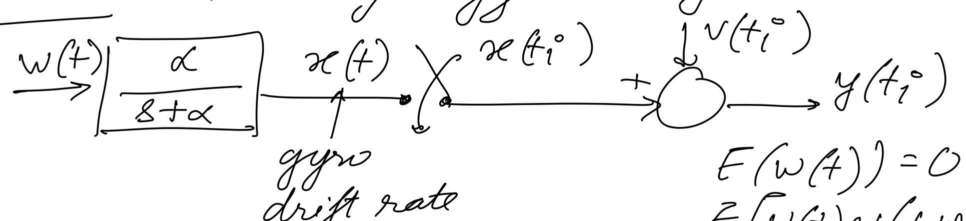
Note: ~~# In~~ T.U step, error cov increases proportional to process noise var × (time gap)

\# In M.U., error cov:

$$P(t_i^{o+}) = P(t_i^{o-})\left[1 - \frac{P(t_i^{o-})}{P(t_i^{o-}) + \sigma_v^2}\right]$$

Error cov would be reduced to zero if observation was noise free. ($\sigma_v = 0$)

Example 2 : Modeling gyro drift rate



$E(w(t)) = 0$

$E[w(t)\,w(t+\tau)] = Q\,\delta(\tau)$

$\alpha = 1$ rad/hr

$W \rightarrow$ in deg/hr

$Q = 2$ deg^2/hr (strange unit since $\delta(\tau)$ is in /hr)

$E[v(t_i^o)\,v(t_j)] = R\,\delta_{ij}$

$R = 0.5$ deg^2/hr^2

Clearly, $\dot{x} = -\alpha x(t) + \alpha w(t)$

$$F = -\alpha = -1 ; \quad G = \alpha = 1$$

Sampling time $= 0.25$ hr

$$\Phi(t_i^o, t_{i-1}^o) = \exp\left[-\alpha(t_i^o - t_{i-1}^o)\right]$$
$$= \exp\left[-1(0.25)\right]$$
$$\simeq 0.78$$

T.U: $\hat{x}(t_i^-) = 0.78\, \hat{x}(t_{i-1}^+)$

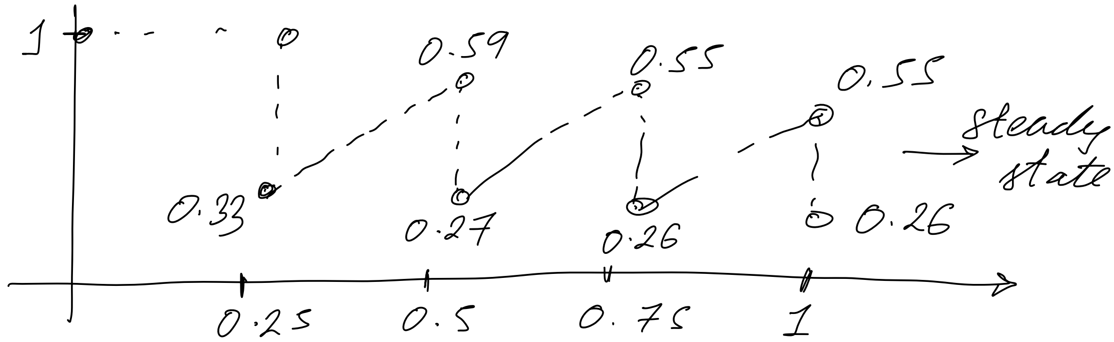
$$P(t_i^-) = \Phi^2(t_i, t_{i-1}) P(t_{i-1}^+) + \int_{t_{i-1}}^{t_i} \Phi^2(t_i^o, \tau) G^2 Q\, d\tau$$
$$= (0.78)^2 P(t_{i-1}^+) + 2 \int \exp\left[-2(t_i^o - \tau)\right] d\tau$$
$$= 0.61\, P(t_{i-1}^{o+}) + 0.39$$

M.U.: $K(t_i^o) = \dfrac{P(t_i^{o-})}{P(t_i^{o-}) + 0.5}$

$$\hat{x}(t_i^{o+}) = \hat{x}(t_i^{o-}) + K(t_i^o)\left[y_i^o - \hat{x}(t_i^{o-})\right]$$

$$P(t_i^{o+}) = \frac{0.5\, P(t_i^{o-})}{P(t_i^{o-}) + 0.5}$$

Assume: $\hat{x}(t_0) = 0$; $P(t_0) = P_0 = 1\ \text{deg}^2/\text{hr}^2$



Steady state Calculation:

$$P(t_i^{+}) = \frac{0.5\,P(t_i^{-})}{P(t_i^{-}) + 0.5} = \frac{0.5\left[0.61P(t_{i-1}^{+}) + 0.39\right]}{\left[0.61P(t_{i-1}^{+}) + 0.39\right] + 0.5}$$

$$= \frac{0.30\,P(t_{i-1}^{+}) + 0.19}{0.61\,P(t_{i-1}^{+}) + 0.89}$$

At S.S. $P(t_i^{+}) = P(t_{i-1}^{+}) = P^{+}$ (say)

$$0.61\,P^{+2} + 0.59P^{+} - 0.19 = 0$$

The +ve soln is $P^{+} = 0.255$

The s.s. for $P^{-} = 0.61P^{+} + 0.39 = 0.546$

Effect of Q & R:

		0.25	0.5	0.75	1.00
$Q = 2$	$P(t_i^{-})$	1	0.59	0.55	0.55
$R = 0.5$	$P(t_i^{+})$	0.33	0.27	0.26	0.26
	$K(t_i)$	0.67	0.54	0.52	0.52
$Q = 4$	$P(t_i^{-})$	2	1.02	0.99	0.98
$R = 0.5$	$P(t_i^{+})$	0.4	0.34	0.33	0.33
	$K(t_i)$	0.8	0.67	0.66	0.66
$Q = 2$	$P(t_i^{-})$	1	0.69	0.64	0.63
$R = 1$	$P(t_i^{+})$	0.5	0.41	0.39	0.39
	$K(t_i)$	0.5	0.41	0.39	0.39
$Q = 4$	$P(t_i^{-})$	2	1.19	1.11	1.09
$R = 1$	$P(t_i^{+})$	0.67	0.54	0.52	0.52
	$K(t_i)$	"	"	"	"

Sensor Fusion using KF:

Example 1: INS aided by position data

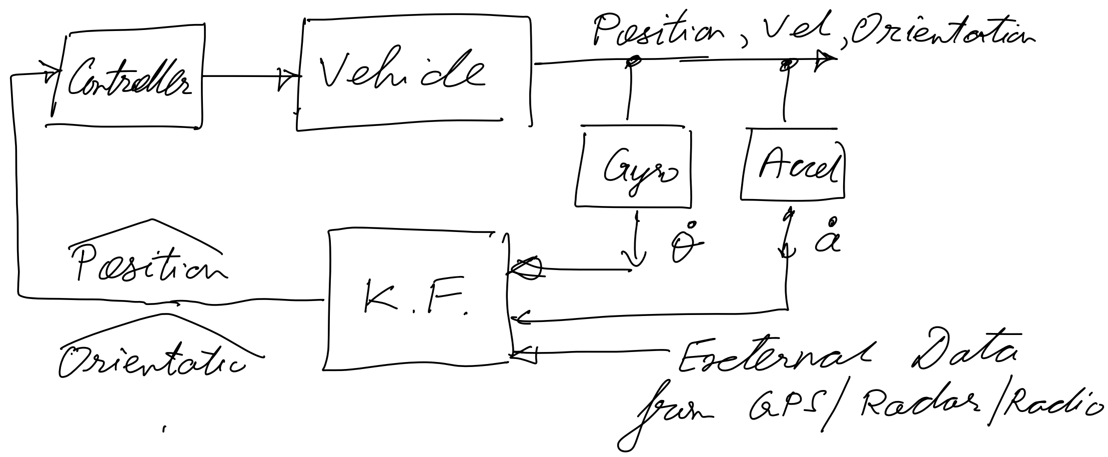
Position data: from (Radar / Radio / GPS)

INS: Gyro + Accelerometers

Various Configurations possible:

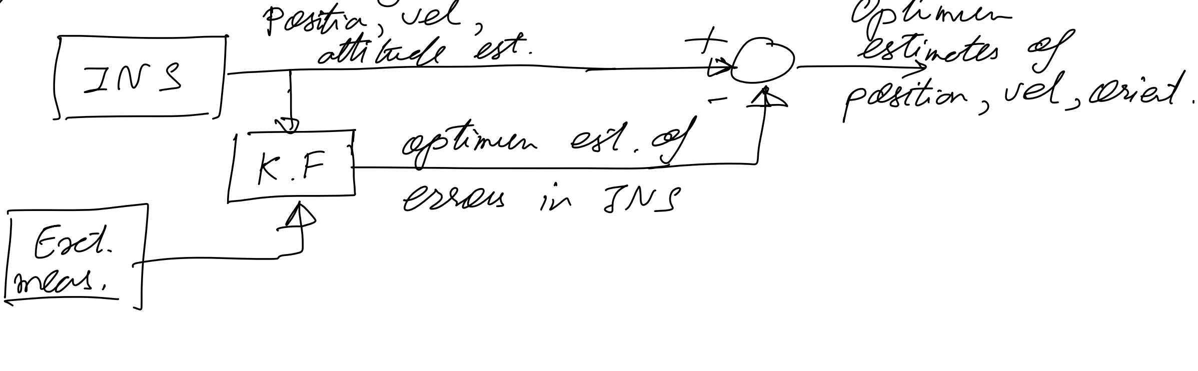
1) Direct / Total state KF
2) Indirect / Error State → a) Indirect Feed forward
   b) Indirect Feedback

In 1)



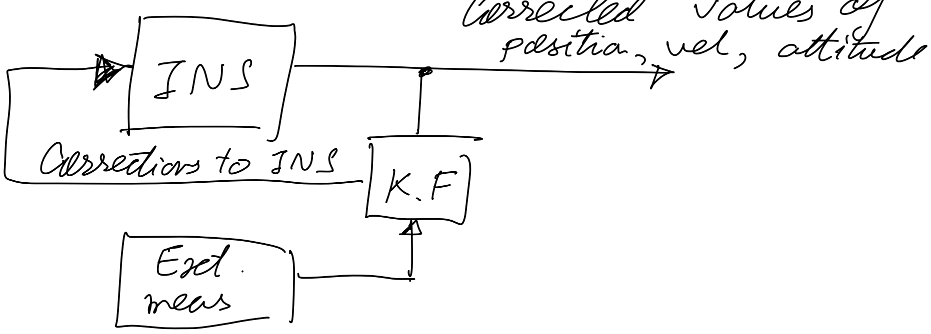
Problems: 1) K.F. requires vehicle model
2) K.F. requires to be as fast as the Control loop freq.
3) Non-linear model

2) a) Indirect Feedforward



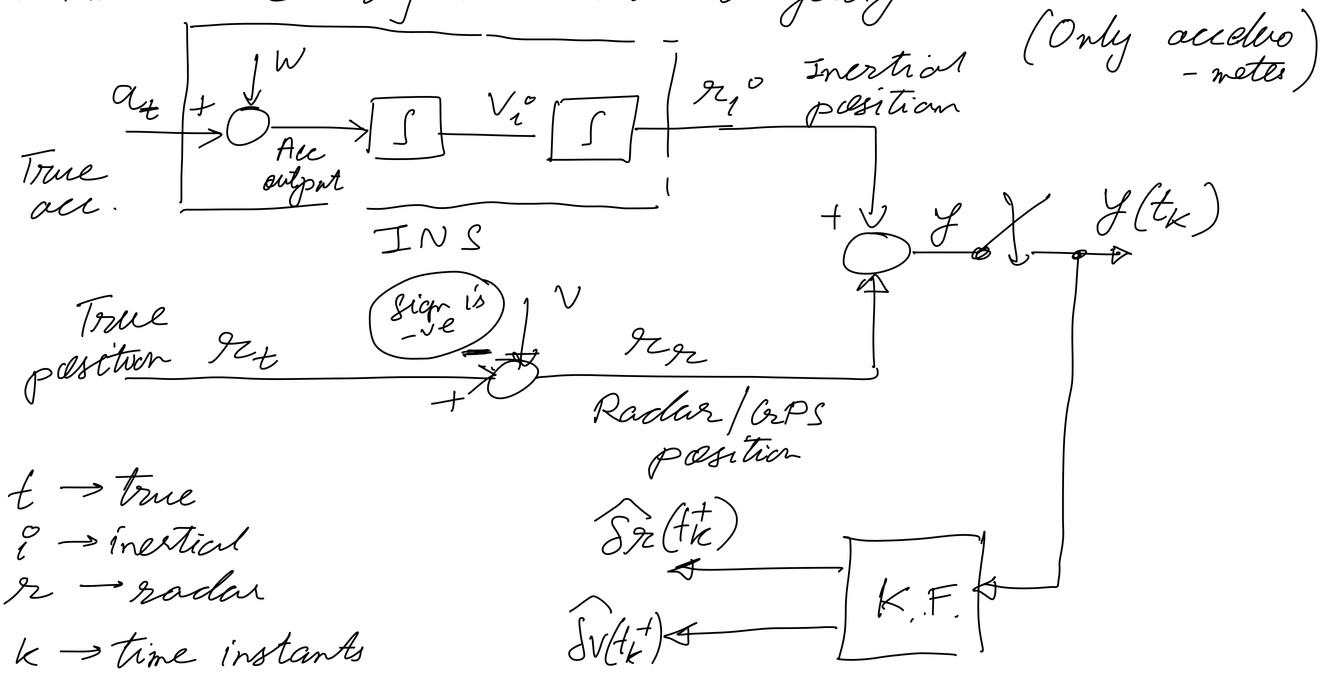
1) K.F. can be slow
2) Works even if K.F. fails temp.
3) Problem if INS deviates & the error grows too big, the K.F. (working with the linear model assumption) is no longer valid.

2b) Indirect Feedback



Combines all the advantages. & none of the disadv.

First we try the indirect feedforward case: (Only accelero-meter)



t → true
i → inertial
r → radar
k → time instants

$$E[w(t)\,w(t+\tau)] = Q\,\delta(\tau) \qquad E\,w(t) = 0$$
$$E[v(t)\,v(t+\tau)] = R_c\,\delta(\tau) \qquad E\,v(t) = 0$$

<u>Error States:</u> $\delta r(t) = r_i^\circ(t) - r_t(t)$

$\delta v(t) = v_i^\circ(t) - v_t(t)$

<u>Observation:</u> $y(t_k) = r_i^\circ(t_k) - r_r(t_k)$

$= r_t(t_k) + \delta r(t_k) - [r_t(t_k) - v(t_k)]$

$= \underbrace{\delta r(t_k) + v(t_k)}_{(3)}$

the initial sign was −ve to make this +ve.

<u>Now consider:</u>

<u>Inertial:</u>
$$\underbrace{\begin{bmatrix} \dot{r}_i^\circ(t) \\ \dot{v}_i^\circ(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} r_i^\circ(t) \\ v_i^\circ(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} [a_t(t) + w(t)]}_{(1)}$$

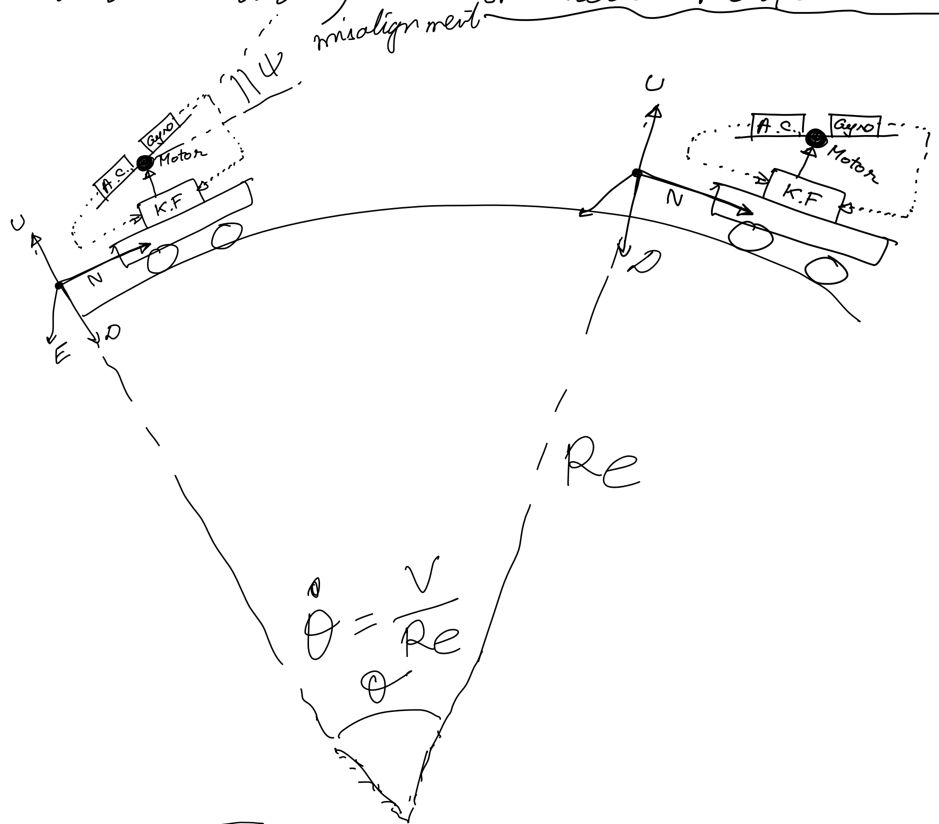
<u>True:</u>
$$\underbrace{\begin{bmatrix} \dot{r}_t(t) \\ \dot{v}_t(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} r_t(t) \\ v_t(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} a_t(t)}_{(2)}$$

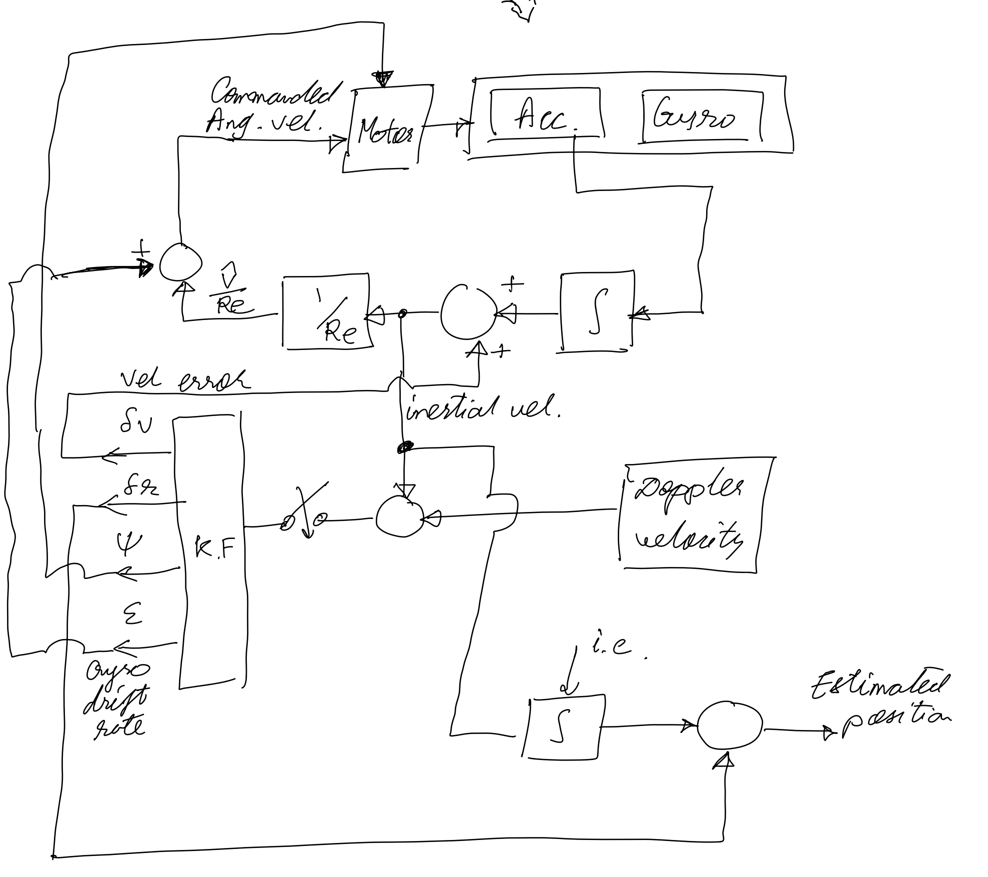
Subtracting (2) from (1)
$$\begin{bmatrix} \delta\dot{r}(t) \\ \delta\dot{v}(t) \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{F} \begin{bmatrix} \delta r(t) \\ \delta v(t) \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{G_2} w(t)$$

From (3) above,
$$y(t_k) = \underbrace{[1 \quad 0]}_{H} \begin{bmatrix} \delta r(t_k) \\ \delta v(t_k) \end{bmatrix} + v(t_k)$$

Q. How to use $\delta r(t_k^+) / \delta r(t_k^-)$ & $\delta v(t_k^+) / \delta v(t_k^-)$ to correct the INS?

Example 2: Acc. + Gyro + Doppler fusion (1-D motion with tilt) — Indirect Feedback Config

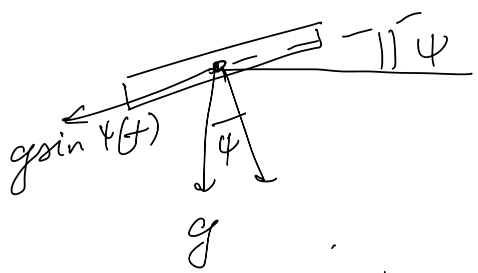




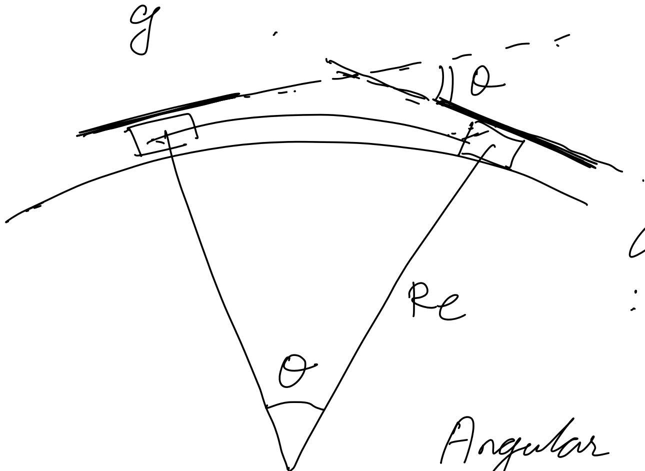
δr = error in INS-indicated position
δv = error in INS- " velocity
ψ = platform tilt
ε = gyro drift rate

(1) $\delta \dot{r}(t) = \delta v(t)$ | Recall: $\left.\begin{aligned}\delta r &= r_i(t) - r_t(t)\\ \delta v &= v_i(t) - v_t(t)\end{aligned}\right\}\ \delta \dot{r} = \delta v$

(2) $\delta \dot{v}(t) = g \sin \psi(t)$ Acceleration error (Otherwise accelerometer is assumed to be precise)



(3)



$$\dot{\psi}(t) = -\frac{\delta v(t)}{R_e} - \varepsilon(t)$$

Correction required : $\theta = \dfrac{\text{dist. travelled}}{R_e}$

Angular rate command required to keep the platform horizontal

$$= \dot{\theta} = -\frac{v(t)}{R_e} \Rightarrow \dot{\psi}_1(t) = -\frac{\delta v(t)}{R_e}$$

Also $\dot{\psi}_2 = -\varepsilon(t)$ → gyro drift rate.

$$\dot{\psi} = \dot{\psi}_1 + \dot{\psi}_2 = -\frac{\delta v(t)}{R_e} - \varepsilon(t)$$

4) $\dot{\varepsilon}(t) = w(t)$

$$
\begin{bmatrix} \delta\dot{r}(t) \\ \delta\dot{v}(t) \\ \dot{\psi}(t) \\ \dot{\varepsilon}(t) \end{bmatrix}
=
\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & g & 0 \\ 0 & -1/R_e & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}
\begin{bmatrix} \delta r(t) \\ \delta v(t) \\ \psi(t) \\ \varepsilon(t) \end{bmatrix}
+
\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} w(t)
$$

$$\dot{x}(t) = F\, x(t) + G\, w(t)$$

Assume
$$
P_0 = \begin{bmatrix} \sigma_{r0}^2 & & & 0 \\ & \sigma_{v_0}^2 & & \\ & & \sigma_{\psi 0}^2 & \\ 0 & & & \sigma_{\varepsilon 0}^2 \end{bmatrix}
\qquad E[x(t_0)] = 0_{4\times 1}
$$

$$v_{INS}(t_k) = v_{true}(t_k) + \delta v(t_k)$$

$$v_{doppler}(t_k) = v_{true}(t_k) - \underbrace{v(t_k)}_{\text{noise}}$$

$$y(t_k) = v_{INS}(t_k) - v_{doppler}(t_k) \qquad E(v(t_k)\, v(t_j)) = R\delta_{ij}$$

$$= \delta v(t_k) + v(t_k)$$

$$= \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \delta r(t_k) \\ \delta v(t_k) \\ \psi(t_k) \\ \varepsilon(t_k) \end{bmatrix} + v(t_k)$$

$$y(t_k) = H\, x(t_k) + v(t_k)$$

<u>K.F. Equations</u>: When the measurement is sampled, M.U. equations are computed and corrective signals are applied to the INS:

$$\hat{x}(t_{i-1}^{+c}) = \begin{bmatrix} \hat{\delta r}(t_{i-1}^{+c}) \\ \hat{\delta v}(t_{i-1}^{+c}) \\ \hat{\psi}(t_{i-1}^{+c}) \\ \hat{\varepsilon}(t_{i-1}^{+c}) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \hat{\psi}(t_{i-1}^{+}) \\ 0 \end{bmatrix}$$

$c \rightarrow$ denotes the time instant after the control is fed back to the INS.

\# Then $\psi(t)$ is zeroed out ASAP.

$$\hat{x}(t_i^-) = \begin{bmatrix} \hat{\delta r}(t_i^-) \\ \hat{\delta v}(t_i^-) \\ \hat{\psi}(t_i^-) \\ \hat{\varepsilon}(t_i^-) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Hence no need to compute $\hat{x}(t_i^-)$ onboard.

$$P(t_i^-) = \Phi(t_i, t_{i-1}) P(t_{i-1}^+) \Phi^T(t_i, t_{i-1}) + \int_{t_{i-1}}^{t_i} \Phi(t_i, \tau) G Q G^T \Phi^T(t_i, \tau) d\tau$$

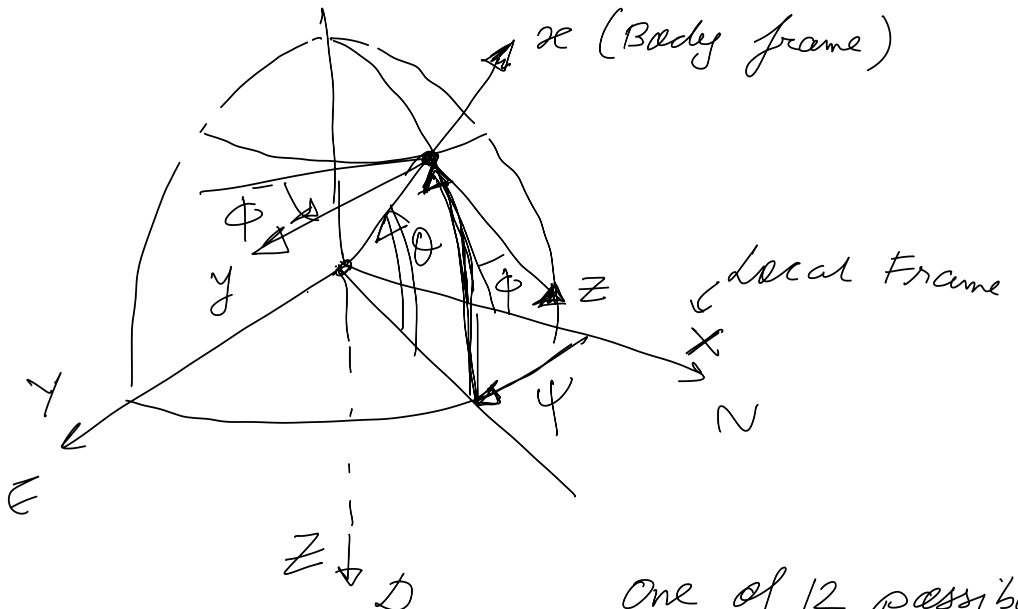
$$K(t_i) = P(t_i^-) H^T \left[H P(t_i^-) H^T + R \right]^{-1}$$

$$= \begin{bmatrix} k_1(t_i) \\ k_2(t_i) \\ k_3(t_i) \\ k_4(t_i) \end{bmatrix} = \frac{1}{P_{22}(t_i^-) + R} \begin{bmatrix} P_{12}(t_i^-) \\ P_{22}(t_i^-) \\ P_{32}(t_i^-) \\ P_{42}(t_i^-) \end{bmatrix}$$

$$\hat{x}(t_i^+) = \cancel{\hat{x}(t_i^-)}^{=0} + \begin{bmatrix} k_1(t_i) \\ k_2(t_i) \\ k_3(t_i) \\ k_4(t_i) \end{bmatrix} \left[v_{INS}(t_i) - v_{doppler}(t_i) \right]$$

$$P(t_i^{+}) = P(t_i^{-}) - K(t_i)HP(t_i^{-})$$

3D — Attitude Estimation : INS
(Acc + Gyro) using Error State K.F



One of 12 possible sequences

$$\text{Local Frame }(E) \longrightarrow \underset{\text{Yaw}}{\underset{z}{\psi}} \longrightarrow \underset{\text{Pitch}}{\underset{y}{\theta}} \longrightarrow \underset{\text{Roll}}{\underset{x}{\phi}} \longrightarrow \text{Body frame }(B)$$

<u>Rotation Matrix</u> : $R_E^B = R_\phi^x R_\theta^y R_\psi^z \quad \| \quad x_B = R_E^B x_E$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\phi & s\phi \\ 0 & -s\phi & c\phi \end{bmatrix} \begin{bmatrix} c\theta & 0 & -s\theta \\ 0 & 1 & 0 \\ s\theta & 0 & c\theta \end{bmatrix} \begin{bmatrix} c\psi & s\psi & 0 \\ -s\psi & c\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \left[\begin{array}{l|l|l} c\theta c\psi & c\theta s\psi & -s\theta \\ s\phi s\theta c\psi - c\phi s\psi & - \; - \; - & - \\ - \; - \; - & - \; - \; - & - \; - \; - \end{array}\right]$$

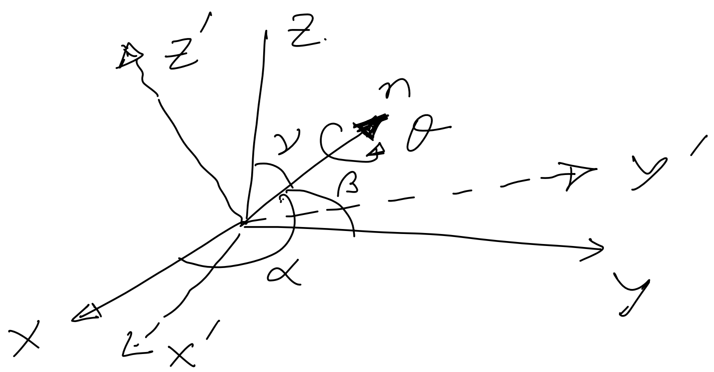
E.g. the gravity vector in Body frame

$$g_B = R_E^B \begin{bmatrix} 0 \\ 0 \\ g \end{bmatrix} = g \begin{bmatrix} -s\theta \\ s\phi\, c\theta \\ c\phi\, c\theta \end{bmatrix}$$

If $\omega_B = \begin{bmatrix} p \\ q \\ r \end{bmatrix}$ (angular vel. in body frame)

then
$$\begin{bmatrix} \dot\phi \\ \dot\theta \\ \dot\psi \end{bmatrix} = \begin{bmatrix} 1 & \frac{s\phi\, c\theta}{c\theta} & \frac{s\phi\, s\theta}{c\theta} \\ 0 & c\phi & -s\phi \\ 0 & \frac{s\phi}{c\theta}, & \frac{c\phi}{c\theta} \end{bmatrix} \begin{bmatrix} p \\ q \\ r \end{bmatrix}$$

<u>Quaternions:</u>



Recall Euler's thm of rotation.

$$q = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} = \begin{bmatrix} \cos\alpha \sin\theta/2 \\ \cos\beta \sin\theta/2 \\ \cos\gamma \sin\theta/2 \\ \cos\theta/2 \end{bmatrix}$$

clearly:

$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

$q_1^2 + q_2^2 + q_3^2 + q_4^2 = 1$

Sometimes q is represented as

$$q = \cos\theta/2 + \sin\theta/2 \left[\cos\alpha\, \hat{i} + \cos\beta\, \hat{j} + \cos\gamma\, \hat{k}\right]$$

quaternion $=: q_4 + \bar{q}$

Basic quaterion algebra:

$$q_4 + |\bar{q}|^2 = 1$$

$$q^a + q^b = (\bar{q}^a + \bar{q}^b)\, q_4^a + q_4^b$$

$$q^a \otimes q^b =$$

$$q^* =$$

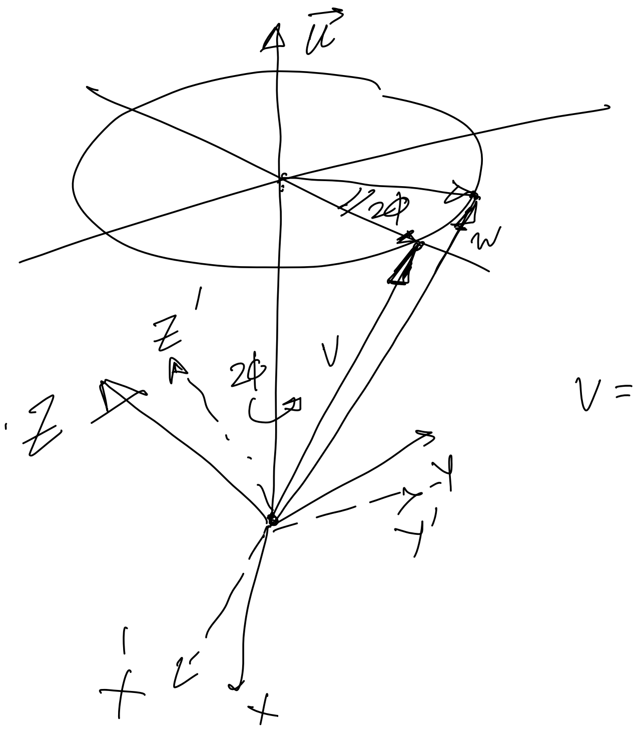
Let $\theta/2 =: \phi$

unit vector

FACT: For any unit quaternion $q = \cos\phi + \vec{u}\sin\phi$ and for any vector $v \in \mathbb{R}^3 \binom{0}{\vec{v}}$ the action of $L_q(v) = qvq^*$ is equivalent to a rotation of v through an angle 2ϕ about $\vec{u}$ as the axis of rotation

(ii) The action of $L_{q^*}(v) = q^* v q$ is equivalent to a rotation of the coordinate frame w.r.t the vector v through an angle 2ϕ about $\vec{u}$ as the axis

(iii) $L_{q^*}(v) = q^* v q$ is also equivalent to an <u>opposite</u> rotation of v w.r.t the coordinate frame through an angle 2ϕ about $\vec{u}$ as axis.



$w = q v q^*$ → vector rotation

$v_{X'Y'Z'} = q^* v_{XYZ}\, q$ ↳ frame rotation

$V = [v \;\; 0]$

$$\# \; R(q) = \begin{bmatrix} q_1^2 - q_2^2 - q_3^2 + q_4^2 & 2(q_1q_2 + q_3q_4) & 2(q_1q_3 - q_2q_4) \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \end{bmatrix}$$

Kinematics

$$\frac{d}{dt} R_E^B = \begin{bmatrix} 0 & \omega_z & -\omega_y \\ -\omega_z & 0 & \omega_x \\ \omega_y & -\omega_x & 0 \end{bmatrix} R_E^B$$

$$\dot{q} = \frac{1}{2} \underbrace{\begin{bmatrix} 0 & \omega_z & -\omega_y & \omega_x \\ -\omega_z & 0 & \omega_x & \omega_y \\ \omega_y & -\omega_x & 0 & \omega_z \\ -\omega_x & -\omega_y & -\omega_z & 0 \end{bmatrix}}_{\Omega(\omega_B)} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix}$$

$$q(t) = \exp\left[(t - t_0)\,\Omega(\omega_B)\right] q(t_0)$$

<u>Gyro model</u>: $\dot{\theta} = \omega_m + b + n_r$

$b \rightarrow$ drift rate bias

$n_r \rightarrow$ drift rate noise (Gaussian white with var N_r)

$\dot{b} = n_\omega \longrightarrow n_\omega$: drift rate ramp noise (Cov N_ω)

n_ω & n_r are uncorrelated

<u>Bias Error</u>: $\Delta\vec{b} = \vec{b}_{true} - \vec{b}_i^{\circ}$

<u>Quaternion error</u>: $\delta q = q_{true} \otimes q_i^{\circ\,-1}$

or $q_{true} = \delta q \otimes q_i^{\circ}$

Since small angles are involved, $\delta q \approx [\vec{\delta q} \;\; 1]^T$

From: $\dot{q}_{true} = \frac{1}{2}\Omega(\vec{\dot{\theta}}_{true})\, q_{true}$

& $\dot{q}_i^{\circ} = \frac{1}{2}\Omega(\vec{\dot{\theta}}_i^{\circ})\, q_i^{\circ}$

$\vec{\dot{\theta}}_{true} \rightarrow$ true rate of change of attitude

$\vec{\dot{\theta}}_i \rightarrow$ estimated rate of . . .

Steps

$$\frac{d}{dt}(\vec{\delta q}) = \underbrace{\begin{bmatrix} 0 & \omega_z & -\omega_y \\ -\omega_z & 0 & \omega_x \\ \omega_y & -\omega_x & 0 \end{bmatrix}}_{\text{gyro output}} \vec{\delta q} - \frac{1}{2}(\Delta\vec{b} + \vec{n}_r)$$

$$\frac{d}{dt}\delta q_4 = 0$$

Assuming small angles, $\vec{\delta q} = [s\frac{\theta}{2}\; s\frac{\theta}{2} \quad \frac{1}{2}\vec{\delta\theta}$, here

$$\frac{d}{dt}\vec{\delta\theta} = \begin{bmatrix} 0 & \omega_z & -\omega_y \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix} \vec{\delta\theta} - (\Delta\vec{b} + \vec{n}_r)$$

Also $\quad \Delta\vec{b} = \vec{b}_{true} - \vec{b}_{i}$

$$\dot{\Delta\vec{b}} = \dot{\vec{b}}_{true} - \dot{\vec{b}}_{i} \qquad \Big| \quad \text{Assume} \quad \dot{\vec{b}}_{i} = 0$$

$$\simeq \vec{n}_{\omega} - 0$$

Combining:

$$\frac{d}{dt}\begin{bmatrix} \delta\vec{\theta} \\ \Delta\vec{b} \end{bmatrix} = \left[\begin{array}{ccc|ccc} 0 & \omega_z & -\omega_y & -1 & 0 & 0 \\ -\omega_z & 0 & \omega_x & 0 & -1 & 0 \\ \omega_y & -\omega_x & 0 & 0 & 0 & -1 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{bmatrix} \delta\vec{\theta} \\ \hline \Delta\vec{b} \end{bmatrix}$$

$$+ \begin{bmatrix} -I_{3\times3} & 0_{3\times3} \\ 0_{3\times3} & I_{3\times3} \end{bmatrix} \begin{bmatrix} \vec{n}_{r} \\ \vec{n}_{\omega} \end{bmatrix}$$

Discretization:

$$\vec{\omega}_{k/k-1} = \vec{\omega}_m(t_k) - \vec{b}_{k/k-1}$$

$$\vec{\omega}_{k/k} = \vec{\omega}_m(t_k) - \vec{b}_{k/k}$$

$$q_{k/k-1} = \Phi(k, k-1)\, q_{k-1/k-1}$$

where $\Phi(k, k-1)$ is derived from the sol$^{\underline{n}}$ of the diff. eq$^{\underline{n}}$ (between $k-1 \to k$)

$$\dot{q}(t) = \frac{1}{2}\,\Omega(\vec{\omega}_{avg})\, q(t)$$

$$\vec{\omega}_{avg} = \frac{\vec{\omega}_{k/k-1} + \vec{\omega}_{k-1/k-1}}{2}$$

Assume $\bar{b}_{k/k-1} = \bar{b}_{k-1/k-1}$

Clearly, error cov. update:

$$P_{k/k-1} = \Phi(k, k-1) P_{k-1/k-1} \Phi^T(k, k-1) + Q_k$$

<u>Measurement Update:</u>

$$K_k = P_{k/k-1} H_k^T \left(H_k P_{k/k-1} H_k^T + R_k\right)^{-1}$$

$$P_{k/k} = P_{k/k-1} - K_k H_k P_{k/k-1}$$

Design based on available sensors → acc. or sun sensors

$$\Delta x_{k/k} = \Delta x_{k/k-1} + K_k \underline{\underline{\Delta z(t_k)}} \quad (*)$$

But $\Delta x_{k/k-1}$ can be made zero:

\# $q_{k/k} = \delta q_{k/k} \otimes q_{k/k-1} = [\delta \bar{q}_{k/k} \;\; 1]^T \otimes q_{k/k-1}$

where $\delta \bar{q}_{k/k} = \frac{1}{2} \delta \bar{\theta}_{k/k}$

\# $\vec{b}_{k/k} = \bar{b}_{k/k-1} + \Delta \bar{b}_{k/k}$

states after measurement update

Here (*) becomes:

$$\begin{bmatrix} \delta \theta_{k/k} \\ \Delta \vec{b}_{k/k} \end{bmatrix} = K_k \Delta z(t_k)$$