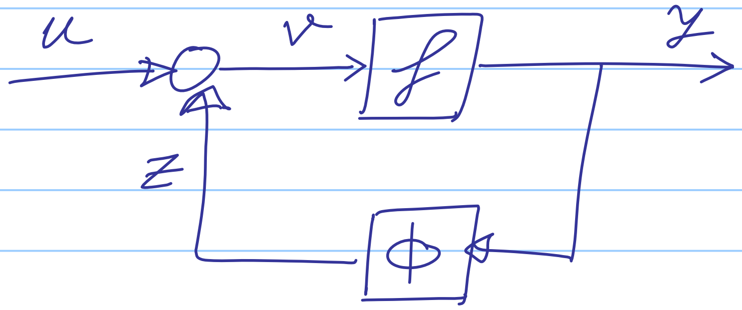
EE640-14 : Bezout Identity

Note Title 24-07-2008

SISO Systems:

$f = \frac{p}{q}$ where p, q are relatively prime polynomials



The closed loop t.f $f_\phi = \frac{y}{u} = \frac{f}{1+\phi f}$

Assume $\phi = \frac{s}{t}$ where s, t are two polynomials.

So $f_\phi = \dfrac{p/q}{1+\frac{s}{t}\cdot\frac{p}{q}} = \dfrac{pt}{tq+sp}$

p, q are known & are coprime.
So the stabilization problem becomes: Choose r which has desired roots
Find polynomials s, t s.t.

$$\boxed{tp + sp = r}$$ ← Bezout Identity

FACT:
If p, q are coprime, then s, t exist for every r. Then $\phi = \frac{s}{t}$

What we don't know:
* how to find s, t.

* Find all such s, t : this yields all feedback controllers for this pole assignment
* We have not addressed the causality of Φ.

Polynomials : $R[x]$ $R \rightarrow$ real nos.

$f = f_0 + f_1 x + f_2 x^2 + \cdots + f_m x^m \equiv$ polynomial

$f_0, f_1, \ldots f_m \in R$

f is a polynomial over R

Zero polynomial $\rightarrow$ All coeff $= 0 \in R$

"1" polynomial $\longrightarrow$ $f_0 = 1$, all other coff $= 0 \in R$

Addition : $(f+g)_i = f_i + g_i$

$\downarrow$ i^{th} coeff of $(f+g)$

Multiplication : $(fg)_i = f_0 g_i + f_1 g_{i-1} + f_2 g_{i-2} + \cdots + f_i g_0$

Properties of Divisibility

Let $a, b \in R[x]$

* a is a divisor of b (a/b) if $b = ca$ for some $c \in K[x]$

Ex: $D = R[x]$ $(x+1) \mid (x^2 + 2x + 1)$

$$\underbrace{x^2 + 2x + 1}_{b} = \underbrace{(x+1)}_{a} \underbrace{(x+1)}_{c}$$

Associates: Let $a, b \in R[x]$ a is an associate of b if $a|b$ and $b|a$

Ex: $-(x+1)$ is an associate $(x+1)$
$3x^2$ " " " x^2

Associate of '1': An element $u \in R[x]$ is invertible (in $R[x]$) if $\exists$ $c \in R[x]$ s.t $uc = 1$ $(c = u^{-1})$

Ex: The invertibles are the polynomials with deg 0.

So $p(x)$, $q(x)$ are associates iff $p = cq$ where c is a non-zero real number.

Consequence: Every non-zero poly. is an associate of a monic polynomial

Common Divisor: Let $a, b \in R[x]$. Then c is a common divisor of a & b if $c|a$ & $c|b$.

GCD: d is a greatest common divisor (gcd) of a, b if
(1) d is a common divisor of a, b
(2) any common divisor c of a, b is a divisor of d : $c|d$.

FACT: If d, d' are both gcds of a, b, then d, d' are associates.

FACT: Every pair of elements $a, b \in \mathbb{R}[x]$ has a greatest common divisor d. Moreover there are elements $s, t \in D$ s.t.

$$ta + sb = d$$

Special case: Two elements $a, b \in D$ are **coprime** if $\gcd(a, b) = 1$

Using the previous result: When a, b are coprime, then there are $s, t \in D$ such that $sa + tb = 1$

The converse also holds: If there are $s, t \in D$ such that $sa + tb = 1$ then $\gcd(a, b) = 1$

Example: $\mathbb{R}[x]$ $\quad a = x^2 + 2x + 1$
$\qquad b = x^2 + 4x + 4$

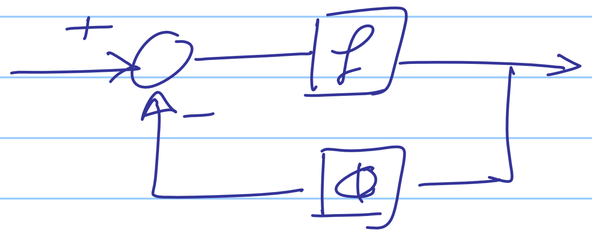
There are polynomials $s, t \in \mathbb{R}[x]$ such that
$s(x^2 + 2x + 1) + t(x^2 + 4x + 4) = 1$

NOTE: If a, b are coprime
$sa + tb = 1$

Let $r \in D$, Then $(rs)a + (rt)b = r$

So: If a, b are coprime, then for any $r \in D$, $\exists\ s, t \in D$, such that $sa + tb = r$.

* This proves the "FACT" stated in the control problem.



$f = \frac{p}{q}$; $\Phi = \frac{s}{t}$

p, q are coprime
$\gcd(p, q) = 1$

Stabilize $\equiv$ find s, t such that closed loop is stable

$$f_\Phi = \frac{f}{1 + \Phi f} = \frac{pt}{sp + tq} \quad ; \quad sp + tq = r$$
$= \text{desired poly.}$

Equivalent problem: Given p, q find s, t such that $sp + tq = \gcd(p, q)$

We need $\left.\begin{array}{c} \gcd(p,q) \\ s \\ t \end{array}\right\}$ Euclidean Algorithm

First we need:

The Division Algorithm (for polynomials)

Let K be a comm ring and let $g \in K[x]$ be a polynomial with an invertible (in K) leading coeff. Then, with every $f \in K[x]$, there is associated a pair $q, r \in K[x]$ such that $f = qg + r$ where $\deg r < \deg g$ ($r \equiv$ remainder)

If $r = 0$, then g is a divisor of f.

Ex: $K = \mathbb{R}$, $f = x^3 + 2x + 1$, $g = x^2 + 1$

Use long division:

$$\begin{array}{l} x^3 + 2x + 1 \div x^2 + 1 = x \\ \underline{x^3 + x \quad\quad} \\ \quad\quad x + 1 \end{array}$$

So $\underbrace{x^3 + 2x + 1}_{f} = \underbrace{(x)}_{q}\underbrace{(x^2+1)}_{g} + \underbrace{(x+1)}_{r}$

The Euclidean Algorithm

Given $b_0, b_1 \in F[x]$ find $\gcd(b_0, b_1)$

Use the division algorithm repeatedly.

$b_0 = q_1 b_1 + b_2 \qquad \deg b_2 < \deg b_1$

$b_1 = q_2 b_2 + b_3 \qquad \deg b_3 < \deg b_2$

$b_2 = q_3 b_3 + b_4 \qquad \deg b_4 < \deg b_3$

Since the degree of the remainder keeps dropping by 1 at every step, we must reach a step with zero remainder

$b_{n-2} = q_{n-1} b_{n-1} + b_n$

$b_{n-1} = q_n b_n$

The last non-zero remainder b_n is the $\gcd(b_0, b_1)$

Example: $b_0 = x^2 + 3x + 2$
$b_1 = x^2 + 4x + 3$

$$\overbrace{x^2+3x+2}^{b_0} \div \overbrace{x^2+4x+3}^{b_1} = \overbrace{1}^{q_1}$$

$$\begin{array}{l} x^2+3x+2 \\ x^2+4x+3 \\ \hline \underbrace{-x-1}_{b_2} \end{array}$$

$$\overbrace{x^2+3x+2}^{b_0} = \overbrace{(1)}^{q_1}\overbrace{(x^2+4x+3)}^{b_1} + \overbrace{(-x-1)}^{b_2}$$

$$\underbrace{x^2+4x+3}_{b_1} = \underbrace{(-x-3)}_{q_2}\underbrace{(-x-1)}_{b_2}$$

So $\gcd = b_2 = (-x-1)$

Example: Suppose:

(3) $b_0 = q_1 b_1 + b_2$
(2) $b_1 = q_2 b_2 + b_3$
(1) $b_2 = q_3 b_3 + b_4$

$b_3 = q_4 b_4$

So $b_4 = \gcd(b_0, b_1)$

From (1), $b_4 = b_2 - q_3 b_3$
" (2) $b_4 = b_2 - q_3[b_1 - q_2 b_2]$
$= -q_3 b_1 + (1 + q_3 q_2) b_2$
From (3) $b_4 = -q_3 b_1 + (1+q_3q_2)[b_0 - q_1 b_1]$

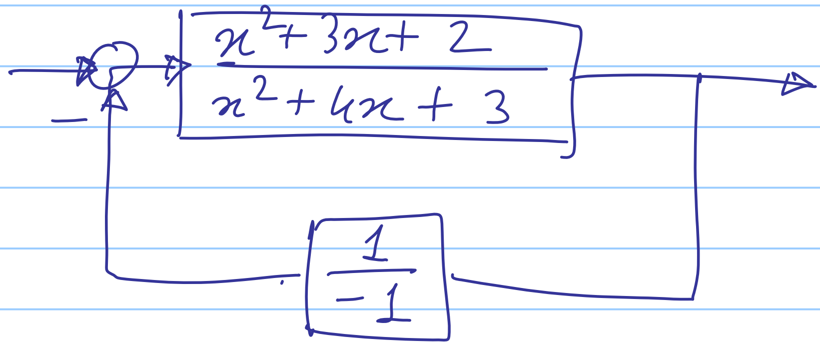
So $b_4 = \underbrace{(1+q_3q_2)}_{s} b_0 + [\underbrace{-q_3 - q_1(1+q_3q_2)}_{t}] b_1$

Exam: $b_0 = x^2 + 3x + 2$

$b_1 = x^2 + 4x + 3$

$$(-x-1) = (1)\left(x^2+3x+2\right) + (-1)\left(x^2+4x+3\right)$$

All feedback compensators with C.L. poles at -1 are



Exercise: What is the degree of the C.L. system? Explain.