Note Title — 18-10-2011

MIMO Observability + Controllability

MIMO challenges:

1) Given a transfer matrix how to create a S.S. realization of minimal order?
2) How is minimality, coprimeness and Contr + Obs. related to each other?
3) Given a realization $\{A, B, C, D\}$ which of the SISO S.S. results can be extended?

* We will address (2) first →

MIMO : $\{A, B, C\}$

$A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{p \times n}$

Observability: Try same method as SISO

$$\begin{array}{c} p \\ p \\ \\ p \end{array}\begin{bmatrix} y(t) \\ y^{(1)}(t) \\ \vdots \\ y^{(n-1)}(t) \end{bmatrix} = \mathcal{O}\, x(t) + T \begin{bmatrix} u(t) \\ \vdots \\ u^{(n-1)}(t) \end{bmatrix}\begin{array}{c} \}m \\ \\ \\ \}m \end{array}$$

$np \times 1$; $\mathcal{O}$: $np \times n$; $x(t)$: $n \times 1$; $nm \times 1$

$$T = \begin{bmatrix} h_0 & & & 0 \\ h_1 & \ddots & & \\ \vdots & & \ddots & \\ h_{n-1} & \cdots & h_1 & h_0 \end{bmatrix}_{np \times nm}$$

$h_0 = 0$

$h_i = CA^{i-1}B$ ($p \times m$)

$$\left(\mathcal{Y} - T\mathcal{U}\right) = \begin{bmatrix} \\ \mathcal{O} \\ \\ \end{bmatrix} \begin{bmatrix} x \end{bmatrix}$$

These eqns
less variables

Observability $\Rightarrow$ $\mathcal{O}$ has rank n

Proof: Otherwise $\mathcal{O}$ has non-zero kernel
and hence x cannot be
determined uniquely.

$\rho(\mathcal{O}) = n \quad \Rightarrow$ Observability

Proof: $\mathcal{O}^T\mathcal{Y} = \mathcal{O}^T\mathcal{O}x(t) + \mathcal{O}^T T\mathcal{U}$

Now, $\rho(\mathcal{O}) = n \Leftrightarrow \det(\mathcal{O}^T\mathcal{O}) \neq 0$
(Exercise)

Hence $x(t) = \left[\mathcal{O}^T\mathcal{O}\right]^{-1}\left\{\mathcal{O}^T\mathcal{Y} - \mathcal{O}^T T\mathcal{U}\right\}$

Q. Is this unique solⁿ. for (1)?

Let x_1 and x_2 be two different solⁿ.
then
$$\mathcal{O}(x_1 - x_2) = 0$$
But $\mathcal{O}$ is full rank $\Rightarrow x_1 = x_2$

FACT: A realization $\{A, B, C\}$ is observable iff the $np \times n$ observability matrix $\mathcal{O}(C, A)$ has full rank

$$\begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 2 & -2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \to \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \end{bmatrix} \; - - - -$$

Non-unique sol^{n} ; $\rho(A) = 1 < 2$

$$\begin{bmatrix} 2 \\ 5 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \end{bmatrix}$$

$\to$ Unique sol^{n}

$$\begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \to \text{No sol}^{n}$$

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \to \text{No sol}^{n}.$$

Arbitrary vector cannot be achieved.

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$v_1 = x_1$
$v_2 = x_2$ | Any $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ can be achieved.

Controllability: Let's try to extend the derivation with impulsive inputs

$$B = [b_1 \cdots b_m]$$

$$\dot{x}(t) = Ax(t) + b_1 u_1(t) + \cdots b_m u_m(t)$$

$$u(t) = g_1 \delta(t) + \cdots g_n \delta^{(n-1)}(t)$$

$$g_i = \begin{bmatrix} g_i^1 \\ g_i^2 \\ \vdots \\ g_i^m \end{bmatrix} \in \mathbb{R}^m$$

$$x(0^+) = x(0^-) + \int_0^t e^{-A\tau} \overbrace{B}^{n \times m} [\underbrace{g_1}_{m \times 1} \delta(t) + \cdots + g_n \delta^{(n-1)}(t)] dt$$

$$= x(0^-) + \underbrace{[B \;\; AB \; \cdots \; A^{n-1}B]}_{n \times nm} \underbrace{\begin{bmatrix} g_1 \\ g_2 \\ \vdots \\ g_n \end{bmatrix}}_{nm \times 1}$$

$$x(0^+) - x(0^-) = \mathcal{C} \begin{bmatrix} g_1 \\ g_2 \\ \vdots \\ g_n \end{bmatrix}$$

less eqns more variables

Any arbitrary LHS can be achieved iff $\mathcal{C}$ has full rank.
(Obviously the solⁿ is <u>non-unique</u>)

Similarity Transform : Identical to SISO

$\bar{A} = T^{-1}AT \qquad \bar{B} = T^{-1}B \qquad \bar{C} = CT$

$$x(t) = T\bar{x}(t)$$
$$H(s) = C(sI-A)^{-1}B = \bar{C}(sI-\bar{A})^{-1}\bar{B}$$

$$\bar{\mathcal{O}} = \mathcal{O}T \qquad \bar{\mathcal{C}} = T^{-1}\mathcal{C}$$

Decomposition of Uncontrollable Realizations :

$\{A, B, C\} \qquad \rho[\mathcal{C}(A,B)] = r \le n$

$\exists\, T$ s.t. $\bar{A} = T^{-1}AT$, $\bar{B} = T^{-1}B$, $\bar{C} = CT$ are of the form :

$$\bar{A} = \begin{array}{c c} \left[\begin{array}{c|c} \bar{A}_c & \bar{A}_{12} \\ \hline 0 & \bar{A}_{\bar{c}} \end{array}\right] & \begin{array}{l} \}\, r \\ \}\, n-r \end{array} \\ \begin{array}{cc} \underbrace{\quad}_{r} & \underbrace{\quad}_{n-r} \end{array} & \end{array} \qquad \bar{B} = \left[\begin{array}{c} \bar{B}_c \\ \hline 0 \end{array}\right] \begin{array}{l} r \\ \}\, n-r \end{array}$$

$$\bar{C} = \begin{bmatrix} \bar{C}_c & \bar{C}_{\bar{c}} \end{bmatrix}$$

with
1) $\{\bar{A}_c, \bar{B}_c\}$ is controllable
2) $\bar{C}_c(sI - \bar{A}_c)^{-1}\bar{B}_c = C(sI-A)^{-1}B$

Proof: Identical to SISO

Similarly for Unobservable realizations

Thm: A realization $\{A, B, C\}$ is minimal iff it is controllable and observable.

Proof: i) Minimal $\Rightarrow$ Controllable + Observable

Let $\{A, B\}$ be un-controllable. Then by above FACT, $\exists$
$\{\bar{A}_c, \bar{B}_c, \bar{C}_c\}$ with same tr. fn.
but less no of states.
$\Rightarrow \{A, B, C\}$ not minimal.

ii) Contr + Obs $\Rightarrow$ Minimal

Let $\{A, B, C\}$ contr. + Obs. but not minimal.
Let $\{\bar{A}, \bar{B}, \bar{C}\}$ be another C+O real. with lower no. of states $\bar{n} < n$.
(Transfer functions are same)

$$CA^i B = \bar{C}\bar{A}^i\bar{B} \quad \forall i$$

$$\Rightarrow \quad \mathcal{O}\mathcal{C} = \bar{\mathcal{O}}_{n-1}\bar{\mathcal{C}}_{n-1} \qquad (1)$$

where $\bar{\mathcal{C}}_{n-1} := [\bar{B} \ \bar{A}\bar{B} \ \cdots \ \bar{A}^{n-1}\bar{B}]$

$$\bar{\mathcal{O}}_{n-1} := \begin{bmatrix} \bar{C} \\ \bar{C}\bar{A} \\ \vdots \\ \bar{C}\bar{A}^{n-1} \end{bmatrix}$$

Sylvester's Inequality :

If $A \in R^{m \times n}$, $B \in R^{n \times p}$

$$\rho(A) + \rho(B) - n \leq \rho(AB) \leq \min\{\rho(A), \rho(B)\}$$

Ex: $A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ $\quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$AB = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$

$\rho(AB) = 1$

$\rho(A) + \rho(B) - n$

$= 1 + 2 - 2 = 1$

$\min\{\rho(A), \rho(B)\} = 1$

Here,

$$\rho(\mathcal{O}) + \rho(\mathcal{C}) - n \leq \rho(\mathcal{O}\mathcal{C}) \leq \min\{\rho(\mathcal{O}), \rho(\mathcal{C})\}$$

Since $\mathcal{O}$, $\mathcal{C}$ has rank n (Cont + O as)

$\rho(\mathcal{O}\mathcal{C}) = n$ (by Sylvester's ineq. above)

But $\{\bar{A}, \bar{B}, \bar{C}\}$ is also C + O.

$\Rightarrow \rho(\bar{\mathcal{O}}_{n-1} \; \bar{\mathcal{C}}_{n-1}) = \bar{n}$

Now by ①,

$$\rho(\mathcal{O}\mathcal{C}) = \rho(\bar{\mathcal{O}}_{n-1} \bar{\mathcal{C}}_{n-1})$$

ie $\quad n = \bar{n} \quad$ (Contradiction)

FACT: If $\{A_i, B_i, C_i\}$ $i=1,2$ are two minimal realizations of a tr. fn. $\exists$ a unique invertible matrix T s.t.

$$A_2 = T^{-1} A_1 T \, , \quad B_2 = T^{-1} B_1 \, , \quad C_2 = C_1 T$$

$$T = \mathcal{C}_1 \mathcal{C}_2^T (\mathcal{C}_2 \mathcal{C}_2^T)^{-1}$$

$$T^{-1} = (\mathcal{O}_2^T \mathcal{O}_1)^{-1} \mathcal{O}_2^T \mathcal{O}_1$$

Proof: Exercise

PBH Eigenvector Tests

1) A pair $\{A, B\}$ will be controllable iff $\exists$ no left eigenvector of A that is orthogonal to all columns of B i.e. iff

$$p^T A = \lambda p^T \quad p^T B = 0 \Rightarrow p \equiv 0$$

2) Dual st. for observable

PBH rank test

1) A pair $\{A, B\}$ will be controllable iff the matrix

$$[sI - A \quad B] \text{ has rank } n \ \forall s$$

2) Dual st. for obs

$\begin{bmatrix} C \\ sI - A \end{bmatrix}$ has rank n $\forall s$