(1)

• Change of Basis:

(1) $A = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$ What is the representation of A w.r.t. the following basis: $\left\{ \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{v_1}, \underbrace{\begin{bmatrix} 0.7 \\ 0.7 \end{bmatrix}}_{v_2} \right\}$

A is a linear operator i.e. $A : \mathbb{R}^2 \to \mathbb{R}^2$

(i) Act A on v_1 & v_2. So $Av_1 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

$$Av_2 = \begin{bmatrix} 2.1 \\ 2.1 \end{bmatrix}$$

(ii) Express Av_1 & Av_2 in terms of basis v_1 & v_2.

$$2v_1 + 0 \cdot v_2 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

& $0 \cdot v_1 + 3 \cdot v_2 = \begin{bmatrix} 2.1 \\ 2.1 \end{bmatrix}$

Hence we can write $\underbrace{\begin{bmatrix} v_1 & v_2 \end{bmatrix}}_{T} \underbrace{\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}}_{D} = \begin{bmatrix} 2 & 2.1 \\ 0 & 2.1 \end{bmatrix}$

So

$$AT = TD$$

Hence $T^{-1}AT = D$

This says that A can be represented as a diagonal matrix by choosing proper bases.

Infact in this example it can be varified that v_1 & v_2 are the eigenvectors of A & 2, 3 are the corresponding eigenvalues.

→ Now let us consider another example.

EX-2

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \quad \text{So } Ab = \begin{bmatrix} 5 \\ 9 \end{bmatrix}$$

Represent A w.r.t $\{\underset{\substack{\uparrow \\ v_1}}{b}, \underset{\substack{\uparrow \\ v_2}}{Ab}\}$ as basis.

So according to the procedure

$$Av_1 = \begin{bmatrix} 5 \\ 9 \end{bmatrix} \quad \& \quad Av_2 = \begin{bmatrix} 19 \\ 27 \end{bmatrix}$$

Now
$$0 \cdot v_1 + 1 \cdot v_2 = \begin{bmatrix} 5 \\ 9 \end{bmatrix}$$
$$-6 \cdot v_1 + 5 \cdot v_2 = \begin{bmatrix} 19 \\ 27 \end{bmatrix}$$

So we can write $[v_1 \ \ v_2] \begin{bmatrix} 0 & -6 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} 5 & 19 \\ 9 & 27 \end{bmatrix}$

i.e. $T \underbrace{\begin{bmatrix} 0 & -6 \\ 1 & 5 \end{bmatrix}}_{\hat{A}} = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} T \Rightarrow \hat{A} = T^{-1}AT$

Notice that the similar matrix $\hat{A}$ is formed by taking the coefficients associated with v_1 & v_2 to express the new vectors Av_1 & Av_2. Furthermore, the 1^{st} column of $\hat{A}$ is formed by taking the coefficients associated with v_1 & v_2 to form Av_1. Similarly for 2^{nd} column of $\hat{A}$, coefficients of associated with v_1 & v_2 to form Av_2.

$$\begin{array}{cc} \begin{bmatrix} 0 & -6 \\ 1 & 5 \end{bmatrix} \\ \uparrow \quad\quad \uparrow \end{array}$$

coefficients to express Av_2 coefficients to express Av_2.

Note that T is the controllability matrix associated with the pair (A, b).

In the previous example we observed that if we choose controllability matrix as similarity transformation matrix T then we will get an observer canonical form.

So the next question is how to get a controller canonical form with similarity transformation. Following are the steps to get a controller canonical form.

- Controller Canonical form for Single-input System:

The controllability matrix $\mathcal{C}$ associated with the pair $(A, b)_n$ is given by

$$\mathcal{C} := [b \quad Ab \quad A^2 b \quad \cdots \quad A^{n-1} b]$$

Assume that $|\mathcal{C}| \neq 0$ i.e. $\mathcal{C}$ is non-singular or the pair (A, b) is fully controllable. Then follow the following steps to get a controller canonical form.

Steps:

(i) Take the inverse of $\mathcal{C}$.

(ii) Take the last row of $\mathcal{C}^{-1}$ & denote it as q.

(iii) Form the following Q matrix

$$Q = \begin{bmatrix} q \\ qA \\ qA^2 \\ \vdots \\ qA^{n-1} \end{bmatrix}$$

Note that from the relation $\mathcal{C}^{-1}\mathcal{C} = I$, it is clear that $qb = qAb = \cdots = qA^{n-2}b = 0$

& $qA^{n-1}b = 1$

$$\overset{\mathcal{C}^{-1}}{\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}} \begin{bmatrix} b & Ab & A^2 b \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Hence we can write

$$\hat{b} := Qb = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

Now we need to represent A interms of new basis $\mathcal{B} := \{q, qA \cdots qA^{n-1}\}$ where $q, qA \cdots$ are the row vectors.

According to the procedure, we have to operate A on each $q, qA \cdots$ & represent that in terms of basis $\mathcal{B}$.

- The similar matrix $\hat{A}$ can be formed by considering the coefficients required to express the new vector obtained by operating A on each basis. The i^{th} row of $\hat{A}$ would be the coefficients required to express qA^i vector. For instance, 1^{st} row of $\hat{A}$ would be $[0 \ 1 \ 0 \cdots 0]$. Hence we can write

$$QA = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & \ddots & \\ & & & & 1 \\ -a_0 & -a_1 & & \cdots & -a_{n-1} \end{bmatrix} Q$$

$$\Rightarrow QAQ^{-1} = \hat{A}$$

(iv) Do QAQ^{-1} to get $\hat{A}$ & Qb to get $\hat{b}$. So the pair $(\hat{A}, \hat{b})$ would be in controller canonical form.

So
$$\hat{A} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ & & & \ddots & 1 \\ -a_0 & -a_1 & \cdots & \cdots & -a_{n-1} \end{bmatrix}_{n\times n} ; \quad \hat{b} = Qb = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}_{n\times 1} \quad --- \; (*)$$

where $|sI - \hat{A}| = s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0$

Ex:
$$A = \begin{bmatrix} 1 & -2 & 1 \\ 2 & 1 & 1 \\ -1 & 2 & -3 \end{bmatrix} ; \quad eig(A) = -2.8897,\ 0.94 \pm 1.6i$$

$b = [1 \;\; 1 \;\; 1]^T$

$$\mathcal{C} = \begin{bmatrix} 1 & 0 & -10 \\ 1 & 4 & 2 \\ 1 & -2 & 14 \end{bmatrix} ; \quad \mathcal{C}^{-1} = \begin{bmatrix} 0.5 & 0.167 & 0.33 \\ -0.1 & 0.2 & -0.1 \\ -0.05 & 0.01 & 0.03 \end{bmatrix} \leftarrow q_1$$

So the generating vector q_1 is $[-0.05 \;\; 0.01 \;\; 0.03]$.

$$Q = \begin{bmatrix} -0.05 & 0.016 & 0.033 \\ -0.05 & 0.183 & -0.133 \\ 0.45 & 0.016 & 0.53 \end{bmatrix}$$

$$\hat{A} = QAQ^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -10 & 2 & -1 \end{bmatrix} \quad \& \quad \hat{b} = Qb = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$|sI - \hat{A}| = s^3 + s^2 - 2s + 10$$

Till now we have seen how to get a controller canonical form for single-input system. Now we will discuss a procedure to obtain controllable companion form for multi-input system.

→ Controllable Companion form for multi-input system:

Let us consider a system with dynamics

$$\dot{x} = Ax + Bu$$

where $A \in \mathbb{R}^{n\times n}$ $\quad B \in \mathbb{R}^{m}$ $\quad m \le n$.

Assume that B is of full rank i.e. all m vectors are linearly independent. Furthermore, assume that the controllability matrix

$$\mathcal{C} := [B \quad AB \quad \cdots \quad A^{n-1}B]$$

is of full rank i.e. $r(\mathcal{C}) = n$. In otherwords, the system is fully controllable.

Assume that the columns of B are named as follows:

$$B = [b_1 \; b_2 \cdots b_m]$$

Step-1

Form a matrix L in following ways

$$L = [\underbrace{b_1 \; Ab_1 \cdots A^{d_1-1} b_1}_{L_1} \quad \underbrace{b_2 \; Ab_2 \cdots A^{d_2-1} b_2}_{L_2} \cdots \underbrace{b_m \; Ab_m \cdots A^{d_m-1} b_m}_{L_m}]$$

• Take 1^{st} vector of B i.e. b_1 & find $Ab_1, A^2b_1 \cdots$ untill to get a linearly dependent vector. Consider only 1st

For instance, assume that $A^{d_1}b_1$ is linearly dependent on $b_1, Ab_1, \cdots A^{d_1-1}b_1$ i.e. $A^{d_1}b_1$ can be expressed as a linear combination of vectors $\{b_1, Ab_1, \cdots A^{d_1-1}b_1\}$. Form a matrix L_1 by taking the vectors $b_1, Ab_1, \cdots A^{d_1-1}b_1$.

- Once we get the linearly dependent vector i.e. $A^{d_1}b_1$, we have to consider the 2^{nd} column of B i.e. b_2 to form the matrix L_2 with similar procedure. However, the linear dependency should be checked with L_1 vectors also.
  This procedure has to be continued untill to get 'n' linearly independent vectors. This is guaranteed because we have assumed that the system is fully controllable. Finally the matrix L can be formed by stacking L_is as shown in ⊛.

We define the 'm' integers 'd_i' as the "controllability indices" of the system & 'μ' as the "controllability index" of the system where $\mu = \max_{d_i}(d_i)$ for $i = 1, 2 \cdots m$.

Step-2

Define $\sigma_k = \sum_1^k d_i \qquad k = 1, 2, \cdots m$

i.e. $\sigma_1 = d_1$

$\sigma_2 = d_1 + d_2$

$\vdots$

$\sigma_m = d_1 + d_2 + \cdots + d_m$

Note that $\sigma_m = n$

Step-3

- Take inverse of matrix L.
- Set $q_k = \sigma_k^{th}$ row of L^{-1} for $k = 1, 2 \cdots m$.
- Form the Q matrix by following way:

$$Q = \begin{bmatrix} q_1 \\ q_1 A \\ \vdots \\ q_1 A^{d_1-1} \\ q_2 \\ q_2 A \\ \vdots \\ q_2 A^{d_2-1} \\ \vdots \\ q_m \\ q_m A \\ \vdots \\ q_m A^{d_m-1} \end{bmatrix}$$

Step-4

- Find $\hat{A}$ by doing QAQ^{-1} & $\hat{B}$ by QB. Now $(\hat{A}, \hat{B})$ pair will be the "multivariable controllable companion form" of (A, B).

$$\hat{A} = QAQ^{-1} = \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} & \cdots & \hat{A}_{1m} \\ \hat{A}_{21} & \hat{A}_{22} & \cdots & \hat{A}_{2m} \\ \vdots & & & \\ \hat{A}_{m1} & \hat{A}_{m2} & \cdots & \hat{A}_{mm} \end{bmatrix}$$

$$
= \left[\begin{array}{ccccc:ccccc:c:ccccc}
0 & 1 & 0 & \ldots & 0 & 0 & 0 & \ldots & & 0 & & 0 & 0 & \ldots & & 0 \\
0 & 0 & 1 & 0 \ldots & 0 & 0 & 0 & \ldots & & 0 & & \vdots & & & & \vdots \\
& & \ddots & & & & & & & & & & & & & \\
& & & & 1 & & & & & & & & & & & \\
x & x & \ldots & \ldots & x & x & x & \ldots & \ldots & x & & x & x & \ldots & \ldots & x \\
\hdashline
0 & 0 & \ldots & \ldots & 0 & 0 & 0 & 1 & 0 \ldots & 0 & & 0 & 0 & \ldots & & 0 \\
\vdots & & & & \vdots & 0 & 0 & 0 & 1 & 0 \ldots 0 & & \vdots & & & & \vdots \\
& & & & & \vdots & & & \ddots & & & & & & & \\
& & & & & & & & & 1 & & & & & & \\
x & x & \ldots & \ldots & x & x & x & \ldots & \ldots & x & & x & x & \ldots & \ldots & x \\
\hdashline
& & \vdots & & & & \vdots & & & \ddots & & & & \vdots & & \\
\hdashline
0 & 0 & \ldots & & 0 & 0 & 0 & 0 & \ldots & 0 & & 0 & 1 & 0 & \ldots & 0 \\
\vdots & & & & \vdots & \vdots & & & & \vdots & & 0 & 0 & 1 & 0 \ldots & 0 \\
& & & & & & & & & & & & & \ldots & & \\
& & & & & & & & & & & & & & & 1 \\
x & x & \ldots & \ldots & x & x & x & \ldots & \ldots & x & & x & x & \ldots & \ldots & x
\end{array}\right]; \qquad 3.6.9a
$$

and

$$
\hat{B} = QB = \left[\begin{array}{cccc}
0 & 0 & \ldots & 0 \\
0 & 0 & \ldots & 0 \\
\vdots & \vdots & & \vdots \\
1 & x & \ldots & x \\
\hdashline
0 & 0 & \ldots & 0 \\
0 & 0 & \ldots & 0 \\
\vdots & \vdots & & \vdots \\
0 & 1 & x \ldots & x \\
\vdots & \vdots & & \vdots \\
\hdashline
\vdots & \vdots & & \vdots \\
0 & 0 & \ldots & 1
\end{array}\right], \qquad 3.6.9b
$$

→ MIMO - Poleplacement:

$$A = \begin{bmatrix} 2 & 1 & 0 & 1 \\ -1 & 0 & 2 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 3 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \quad \text{rank}(\text{ctrb}(A,B)) = 4$$

$$\qquad\qquad\qquad\qquad\qquad\qquad b_1 \quad b_2$$

eig(A) = 1, 1, −1, 1

It can be easily varified that with input b_1 the controllability matrix is not full rank 2 its rank is 2. So we form the L matrix

Step-1

$$L = [b_1 \quad Ab_1 \quad b_2 \quad Ab_2]$$

$$= \begin{bmatrix} 1 & 3 & 0 & 2 \\ 1 & -1 & 1 & 3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 4 \end{bmatrix} \qquad \left.\begin{matrix} d_1 = 2 \\ d_2 = 2 \end{matrix}\right\} \text{Controllability indices.}$$

2 is controllability index

Step-2

$$\sigma_1 = d_1 = 2$$
$$\sigma_2 = d_1 + d_2 = 4$$

Step-3

$$\text{Take } L^{-1} = \begin{bmatrix} 0.25 & 0.75 & -0.05 & -0.7 \\ 0.25 & -0.25 & 0.15 & 0.1 \\ 0 & 0 & 0.8 & 0.2 \\ 0 & 0 & -0.2 & 0.2 \end{bmatrix}$$

$q_1 = \sigma_1^{th}$ row of L^{-1} i.e. 2^{nd} row of L^{-1}

$q_2 = \sigma_2^{th}$ " L^{-1} i.e. 4^{th} row of L^{-1}

It can be noticed that the 'm' diagonal blocks $\hat{A}_{ii}$ are each an upper right identity companion matrix of dimension d_i, while the off diagonal blocks are each identically zero except for their last rows.

Hence it can be noted that all informations regarding the similar matrix $\hat{A}$ can be derived from knowledge of 'm' ordered controllability indices d_is. and 'm' ordered σ_k rows of $\hat{A}$. Similarly in $\hat{B}$, only these same order σ_k rows are nonzero.

→ Pole Placement via the Controllable Companion form:

Let us consider a LTI multi-input system represented by

$$(*) \quad \dot{x} = Ax + Bu \quad \text{where } A \in \mathbb{R}^{n\times n} \ \& \ B \in \mathbb{R}^{n\times m}$$

We are interested in finding a statefeedback of the form $u = Fx$; $F \in \mathbb{R}^{m\times n}$ to improve the performance of closed loop system.

So the closed loop system would be

$$(**) \quad \dot{x} = (A+BF)x$$

Assuming the system $(*)$ be fully controllable, we can transform it to a controllable companion form by introducing a new variable of the form

$$\hat{x} = Qx$$

$$\Rightarrow x = Q^{-1}\hat{x}$$

So we can write

$$\dot{\hat{x}} = Q\dot{x}$$

$$= Q(A+BF)x \qquad (\text{using } **)$$

$$= Q(A+BF)Q^{-1}\hat{x}$$

$$= (QAQ^{-1} + QBFQ^{-1})\hat{x}$$

$$= (\hat{A} + \hat{B}\hat{F})\hat{x}$$

where $\hat{A} = QAQ^{-1}$, $\hat{B} = QB$ & $\hat{F} = FQ^{-1}$ —— $***$

So, with proper transformation (as discussed in previous section) we can get $\hat{A}, \hat{B}$ in controllable companion form. (Refer equation 3·6·9a & 3·6·9b for the structure of ~~an~~ $\hat{A}$ & $\hat{B}$)

We now define $\hat{A}_m$ as the $(m \times n)$ matrix consisting of the 'm' ordered σ_k rows of $\hat{A}$, and $\hat{B}_m$ as the $(m \times m)$ matrix consisting of 'm' ordered σ_k rows of $\hat{B}$. From 3·6·9-b it is clear that $\hat{B}_m$ is an upper Δ^r matrix with 1s along the diagonal.

$$\hat{B}_m = \begin{bmatrix} 1 & * & * & \cdots & * \\ 0 & 1 & * & \cdots & * \\ \vdots & \vdots & & \ddots & \vdots \\ & & & & * \\ 0 & 0 & 0 & & 1 \end{bmatrix}_{m \times m} \text{——} \ \Delta$$

clearly $|\hat{B}_m| = 1$

If we now define $\bar{F} = \hat{B}_m \hat{F}$, it follows from the particular structure (3.6.9) for the pair $(\hat{A}, \hat{B})$, that each of the (mn) non-trivial entries of $\hat{A}_m$ will be replaced by some new members under the state feedback control law. This can be seen in following ways.

$$\underbrace{\begin{bmatrix} \hat{a}_{m11} & \hat{a}_{m12} & \cdots & \hat{a}_{m1n} \\ \vdots & & & \\ \hat{a}_{mm1} & \hat{a}_{mm2} & \cdots & \hat{a}_{mmn} \end{bmatrix}}_{\hat{A}_m} + \underbrace{\begin{bmatrix} \bar{f}_{11} & \bar{f}_{12} & \cdots & \bar{f}_{1n} \\ \vdots & & & \\ \bar{f}_{m1} & \bar{f}_{m2} & \cdots & \bar{f}_{mn} \end{bmatrix}}_{\bar{F}}$$

$$= \begin{bmatrix} \hat{a}_{m11}+\bar{f}_{11} & \hat{a}_{m12}+\bar{f}_{12} & \cdots & \hat{a}_{m1n}+\bar{f}_{1n} \\ \vdots & & & \\ \hat{a}_{mm1}+\bar{f}_{m1} & \hat{a}_{mm2}+\bar{f}_{m2} & \cdots & \hat{a}_{mmn}+\bar{f}_{mn} \end{bmatrix}_{m \times n}$$

Hence the closed loop system matrix

$$\hat{A} + \hat{B}\hat{F} = \hat{A} + \hat{B}\hat{B}_m^{-1}\bar{F}$$

with $\hat{A} + \hat{B}\hat{F}$ retaining the exact structure as of $\hat{A}$ (3.6.9-a).

- It can be observed that every one of the (mn) non-trivial elements of $\hat{A}$, denoted as $*$ in 3.6.9-a, can be arbitrarily altered via $\hat{F}$ $(\hat{B}_m^{-1}\bar{F})$.

Hence, with proper choice of $\hat{F}$ we can make

$$\hat{A} + \hat{B}\hat{F} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \ddots & \\ & & & & 1 \\ -\alpha_0 & -\alpha_1 & -\alpha_2 & \cdots & -\alpha_{n-1} \end{bmatrix}_{n \times n} \quad \text{—— (v)}$$

an 'n' dimensional companion matrix, where the scalars $\sigma_0, \sigma_1 \cdots \sigma_{n-1}$ represent the co-efficients of the desired closed loop characterstic polynomial i.e.

$$\sigma(s) = \left| sI - \hat{A}_0 - \hat{B}_0 \hat{F} \right| = s^n + \sigma_{n-1} s^{n-1} + \cdots + \sigma_1 s + \sigma_0$$

It is well known that the eigenvalues of a matrix do not change under similarity transformation. Hence we can write

$$\left| sI - A - BF \right| = \left| sI - \hat{A} - \hat{B}\hat{F} \right| = \sigma(s)$$

In order to explicitly determine an $\hat{F}$, let us denote $\hat{A}^*_m$ as the 'm' ordered σ_k rows of $(\hat{A} + \hat{B}\hat{F})$. Hence we can write

$$\hat{A}_m + \hat{B}_m \hat{F} = \hat{A}^*_m$$

$$\Rightarrow \hat{B}_m \hat{F} = \hat{A}^*_m - \hat{A}_m$$

$$\boxed{\Rightarrow \hat{F} = \hat{B}_m^{-1} \left[\hat{A}^*_m - \hat{A}_m \right]} \quad \text{——— (1)}$$

Note that $\hat{B}_m$ is a square non-singular upper matrix (see (A)) & hence it's ~~always~~ invertible.

Finally F can be calculated from the relations given in (*)(**).

→ Procedure for Pole placement:

- Check whether the pair (A, B) is fully controllable or not.
- If it is fully controllable then do similarity transformation, as discussed in previous section, to get controller canonical form $(\hat{A}, \hat{B})$.
- Select 'n' eigenvalues $\lambda_1, \lambda_2 \cdots \lambda_n$. If they are complex then complex conjugate must be included.
- Compute the desired characteristic polynomial.

$$\sigma(s) = (s-\lambda_1)(s-\lambda_2)\cdots(s-\lambda_n)$$
$$= s^n + \sigma_{n-1}s^{n-1} + \cdots + \sigma_1 s + \sigma_0$$

- Form the matrix $\hat{A}_m^*$ by taking σ_k rows of $(\hat{A} + \hat{B}\hat{F})$ which is shown in ✓.
- Form matrices $\hat{A}_m$ & $\hat{B}_m$ as discussed in previous section.
- Calculate $\hat{F}$ from the relation given in 1.
- Calculate $F = \hat{F}Q$.

→ Partially Controllable System:

Let us consider the system (single-input)

$$\dot{x} = Ax + bu \qquad A \in \mathbb{R}^{n\times n} \; ; \; b \in \mathbb{R}^n$$

Assume that the pair (A, b) is not fully controllable. This means the controllability matrix associated with (A, b) is singular. Hence without loss of generality we can assume that the rank of controllability matrix is '$\bar{n}$' with $\bar{n} < n$. This means we can find $\bar{n}$ independent columns of $\mathcal{C}$.

i.e. $[b \; Ab \cdots A^{\bar{n}-1} b]_{n\times \bar{n}}$.

Now we can say that, the $\bar{n}$ linearly independent vectors of $\mathcal{C}$ form a basis for some subspace S of $\mathbb{R}^n$. Let us assume that there is an another subspace $\mathcal{S}$ of $\mathbb{R}^n$ such that

$$S \oplus \mathcal{S} = \mathbb{R}^n \qquad \oplus := \text{Direct sum}$$

This means the direct sum of S & $\mathcal{S}$ span the whole space $\mathbb{R}^n$. It follows that any vector $v \in \mathbb{R}^n$ can be expressed as a linear combination of some vector s in S & some vector r in $\mathcal{S}$. In particular

$$v = \alpha s + \beta r \qquad \forall\, v \in \mathbb{R}^n.$$

Now let $r_1, r_2 \cdots r_q$ form the basis for subspace $\mathcal{S}$.

Note that $q + \bar{n} = n$.

Consider the following extended state representation.

$$\dot{x} = Ax + B_e u_e \quad \text{——} \quad (2)$$

Where B_e is the $n \times (1+g)$ matrix obtained by appending 'g' number of basis vectors to the vector b i.e. $B_e = [b \; r_1 \cdots r_g]$.
Also u_e is an $(1+g)$ dimensional input obtained by appending to $u(t)$, 'g' additional input elements.

i.e. $u_e(t) = [u_1(t), u_2(t), \cdots, u_{(1+g)}(t)]$

Clearly, the extended system (2) is completely controllable systems hence possible to apply algorithm discussed in the previous section. However, following steps would be useful.

Steps :

(i) Form the extended controllability matrix

$$\mathcal{C}_e = [b \; Ab \cdots A^{\bar{n}-1}b \,|\, r_1 \; r_2 \cdots r_g]_{n \times n}$$

(ii) Take the inverse of $\mathcal{C}_e$.

(iii) Set $q = \bar{n}^{th}$ row of $\mathcal{C}_e^{-1}$.

(iv) Form the Q matrix as follows:

$$Q = \begin{bmatrix} q \\ qA \\ \vdots \\ qA^{\bar{n}-1} \\ r_1 \\ r_2 \\ \vdots \\ r_g \end{bmatrix}$$

Note that $qb = qAb = \cdots = qA^{\bar{n}-2}b = 0$ where as $qA^{\bar{n}-1}b = 1$.

Further more $r_1 b, r_2 b \cdots$ all elements will be some arbitrary scalars. Hence

$$\hat{b} := Qb = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \hline * \\ * \\ \vdots \\ * \end{bmatrix} = \begin{bmatrix} \hat{b}_c \\ \hline \hat{b}_{\bar{c}} \end{bmatrix}$$

(iv) Do QAQ^{-1} to get $\hat{A}$ & Qb to get $\hat{b}$.

Notice that the structure of $\hat{A}$ would be

$$\hat{A} = \left[\begin{array}{c|c} \hat{A}_c & 0 \\ \hline \hat{A}_{c\bar{c}} & \hat{A}_{\bar{c}} \end{array}\right] \quad \& \quad \hat{b} = \begin{bmatrix} \hat{b}_c \\ \hline \hat{b}_{\bar{c}} \end{bmatrix} \qquad \text{——— (3)}$$

where the pair $(\hat{A}_c, \hat{b}_c)$ is in $\bar{n}$-dimensional controllable companion form. On closer inspection, it becomes clear that the controllable & uncontrollable ($\hat{A}_{\bar{c}}$) portions of the system have been seperated.

Furthermore it can be noticed that in view of (3) the characteristic polynomial of A can be written as the product of the characteristic polynomials of the controllable & completely uncontrollable portions of the system i.e.

$$|sI - A| = |sI - \hat{A}| = |sI - \hat{A}_c| \times |sI - \hat{A}_{\bar{c}}|$$

In view of the above, we call the roots of $|sI-\hat{A}_c|$ the "controllable modes" & roots of $|sI-\hat{A}_{\bar{c}}|$ the "uncontrollable modes" of the system. If the uncontrollable modes of the systems are stable i.e. are in left half of complex plane then we will say the system is "stabilizable".

→ Pole Placement:

As we have seen in the previous section that $(\hat{A}_c, \hat{b}_c)$ pair is controllable, we can place poles of $\hat{A}_c$ in arbitrary locations with state feedback of the form $u: \hat{f} = [\hat{f}_c \;\vdots\; 0]$. $\hat{f}_c^T \in \mathbb{R}^{\bar{n}}$

So

$$\hat{A} + \hat{b}\hat{f} = \left[\begin{array}{c|c} \overbrace{\begin{matrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ & & & \ddots & 1 \\ * & * & * & \cdots & * \end{matrix}}^{\hat{A}_c} & 0 \\ \hline \hat{A}_{c\bar{c}} & \hat{A}_{\bar{c}} \end{array}\right] + \left[\begin{array}{c} \hat{b}_c \\ \hline \hat{b}_{\bar{c}} \end{array}\right] [\hat{f}_c \;\vdots\; 0]$$

It can be observed from the above that with the state feedback $\hat{f}_c$, we can change the elements, marked $*$ in $\hat{A}_c$ block & hence the characteristics polynomial of $\hat{A}_c$. However, the characteristic polynomial of $\hat{A}_{\bar{c}}$ will be uneffected with the above state feedback.

EX:
$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \qquad b = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \qquad \text{rank}(\mathcal{C}) = 2$$

Hence not fully controllable

$$\mathcal{C} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & -1 & 1 \\ 1 & 2 & 4 \end{bmatrix}$$

Form $\mathcal{C}_e$ by considering $r_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$$\mathcal{C}_e = \begin{bmatrix} 0 & 0 & 1 \\ 1 & -1 & 1 \\ 1 & 2 & 1 \end{bmatrix} \qquad \mathcal{C}_e^{-1} = \begin{bmatrix} -1 & 0.667 & 0.33 \\ 0 & -0.33 & 0.33 \\ 1 & 0 & 0 \end{bmatrix} \begin{matrix} \\ \leftarrow q \\ \\ \end{matrix}$$

$$Q = \begin{bmatrix} 0 & -0.33 & 0.33 \\ 0 & 0.33 & 0.66 \\ 1 & 1 & 1 \end{bmatrix}$$

So
$$QA = \underbrace{\left[\begin{array}{cc|c} 0 & 1 & 0 \\ 2 & 1 & 0 \\ \hline 5 & -1 & 1 \end{array}\right]}_{\hat{A}} Q \qquad \hat{b} = Qb = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

So $\hat{A}_c = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$ $\quad \hat{b}_c = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $\quad \hat{A}_{\bar{c}} = 1$

~~Hence~~ The system is not stabilizable because the uncontrollable mode '1' is in RHP. However the eigenvalues of $\hat{A}_c$ can be altered.