Matrix Fraction Description

Note Title 20-10-2010

Matrices over polynomials

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ a_{n1} & - & - & a_{nn} \end{pmatrix} \qquad a_{ij} \in R[x]$$

Ex: $D = \mathbb{R}[z]$, $n = 2$

$$A = \begin{bmatrix} z+1 & z^2+3z+2 \\ 15z^3 & 17z^5 \end{bmatrix}$$

Determinant: defined as for real matrices

adj (A) - : "

1) $A(\mathrm{adj}(A)) = (\det A) I = (\mathrm{adj}\, A) A$

2) $\det(AB) = \det(A)\det(B)$

3) $\det(A^T) = \det A$

Defn: Unimodular Matrices: A $n \times n$ matrix U over $R[x]$ is unimodular if there is an $n \times n$ matrix U' over D such that

$$UU' = U'U = I$$

FACT: A $n \times n$ matrix over $R[x]$ is unimodular if and only if $\det U$ is invertible in $R[x]$ i.e. $\det U \neq 0$.

Note Title 06-11-2008

Proof: "if" Assume U is unimodular
Then $\exists\, U'$ s.t. $UU' = I$
$\det(UU') = 1 \iff \det(U)\det(U') = 1$

"only if" $\det U$ is invertible
Let $U' = [\det(U)]^{-1} \operatorname{adj} U$
Then $UU' = [\det U]^{-1} U(\operatorname{adj} U)$
$= [\det U]^{-1}(\det U) I$
$= I$

Example: $D = \mathbb{R}[x]$: U is unimodular if $\det(U)$ is a non-zero real no.

$$U = \begin{bmatrix} 1 & x+2 \\ 0 & 5 \end{bmatrix} \qquad U' = \begin{bmatrix} 1 & -\frac{(x+2)}{5} \\ 0 & \frac{1}{5} \end{bmatrix}$$

$\det U = 5$

$$X = \begin{bmatrix} x & x+2 \\ 0 & 5 \end{bmatrix} \qquad \det(X) = 5x$$

not invertible

Matrices over polynomials

Let A, B be two $n \times n$ matrices over polynomials $F[x]$.

Left Associates: A and B are left associates if there is a $n \times n$ unimodular matrix U s.t.

$$A = UB$$

Relation to elementary row operation

FACT: Any elementary row operation can be represented by left multiplication by a unimodular matrix U.

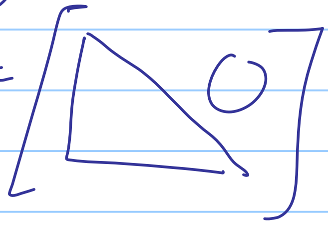
\# U can be obtained by performing the corresponding operation on the identity matrix.

Hermite Form:

1) Every $n \times n$ matrix over the polynomials $R[x]$ is a left associate of a lower triangular matrix H (the Hermite form of A)

2) The Hermite form of A can be obtained from A by a finite no of elementary row operations.

$$H = MA$$

$H =$ 

M is unimodular.

How to find H

$$A = \begin{bmatrix} - & - & - & a_{1n} \\ - & - & - & a_{2n} \\ & & & \vdots \\ - & - & - & a_{nn} \end{bmatrix}$$

look at last col. of A. If there is a non-zero entry, switch rows to put the lowest degree non-zero entry at the bottom

After row switch:

$$A' = \begin{bmatrix} - & - - & a_{1n}' \\ - & - - & a_{2n}' \\ & & \vdots \\ - & \cdot & a_{nn}' \end{bmatrix}$$

Use the polynomial division algo to write $a_{in}' = q_{in} a_{nn}' + r_{in}$ ($\deg r_{in} < \deg a_{nn}'$)

Perform the following elementary row operations: Subtract $q_{in} \times$ (last row) from row i ($i = 1, 2, \ldots, n-1$) to get

$$\begin{bmatrix} - - - & r_{1n} \\ & r_{2n} \\ & \vdots \\ & r_{(n-1)n} \\ & a_{nn}' \end{bmatrix}$$

Repeat from beginning.
After a finite no. of cycles

$$\left[\begin{array}{c|c} A'_{(n-1)\times(n-1)} & \begin{matrix} 0 \\ 0 \\ \vdots \\ 0 \end{matrix} \\ \hline & b_{nn} \end{array}\right]_{n\times n}$$

Repeat for matrix A'. To find M, perform same operation on I.

Example: Find the Hermite form of

$$\overbrace{\begin{bmatrix} z^2 & z^3 & z^5+z^4+z^2 \\ 0 & 0 & z^3 \\ z & 0 & z^4 \end{bmatrix}}^{A} \quad \overbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}^{I}$$

Switch rows to
Bring the lowest degree poly. of the last column to the last row.

$$\begin{bmatrix} z^2 & z^3 & z^5+z^4+z^2 \\ z & 0 & z^4 \\ 0 & 0 & z^3 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Divide 1) $z^5+z^4+z^2 \div z^3 = z^2+z$

$$\begin{array}{l} z^5+z^4+z^2 \\ \underline{z^5} \\ \quad z^4+z^2 \\ \quad \underline{z^4} \\ \qquad z^2 \end{array}$$

2) $z^4 \div z^3 = z$

Multiply row 3 by (z^2+z) and subtract from row 1

$$\begin{bmatrix} z^2 & z^3 & z^2 \\ z & 0 & z^4 \\ 0 & 0 & z^3 \end{bmatrix} \quad \begin{bmatrix} 1 & -(z^2+z) & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Switch rows 1, 3

$$\begin{bmatrix} 0 & 0 & z^3 \\ z & 0 & z^4 \\ z^2 & z^3 & z^2 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -(z^2+z) & 0 \end{bmatrix}$$

Multiply row 3 by z, sub from row 1

$$\begin{bmatrix} -z^3 & -z^4 & 0 \\ z & 0 & z^4 \\ z^2 & z^3 & z^2 \end{bmatrix} \quad \begin{bmatrix} -z & 1+z^3+z^2 & 0 \\ 0 & 0 & 1 \\ 1 & (-z^2-z) & 0 \end{bmatrix}$$

Multiply row 3 by z^2, sub from row 2

$$\left[\begin{array}{cc|c} -z^3 & -z^4 & 0 \\ z-z^4 & -z^5 & 0 \\ \hline z^2 & z^3 & z^2 \end{array}\right] \quad \begin{bmatrix} -z & 1+z^3+z^2 & 0 \\ -z^2 & z^4+z^3 & 1 \\ 1 & -(z^2+z) & 0 \end{bmatrix}$$

Switch row 1, 2

$$\left[\begin{array}{cc|c} z-z^4 & -z^5 & 0 \\ -z^3 & -z^4 & 0 \\ \hline z^2 & z^3 & z^2 \end{array}\right] \quad \begin{bmatrix} -z^2 & z^4+z^3 & 1 \\ -z & 1+z^2+z^3 & 0 \\ 1 & -(z^2+z) & 0 \end{bmatrix}$$

Multiply row 2 by z, sub from row 1

$$\underbrace{\begin{bmatrix} z & 0 & 0 \\ -z^3 & -z^4 & 0 \\ z^2 & z^3 & z^2 \end{bmatrix}}_{= H} \quad \underbrace{\begin{bmatrix} 0 & -z & 1 \\ -z & 1+z^2+z^3 & 0 \\ 1 & -(z^2+z) & 0 \end{bmatrix}}_{= M}$$

$$\underline{\underline{MA = H}}$$

<u>Note</u>: The non-zero entry of the last column of H is the gcd of all element of the last column of A!

Hint: Find $\gcd(a_1, .., a_n)$.
Calculate $b_1 = \gcd(a_1, a_2)$; $b_2 = \gcd(b_1, a_3)$ --- $b_{n-1} = \gcd(b_{n-2}, a_n)$. Then $b_{n-1} = \gcd(a_1, \ldots, a_n)$

Divisibility of Matrices

A, B, C are matrices with $A = BC$

B = left divisor of A
C = right divisor of A
A = left multiple of C
= right " " B

A matrix D is a common right divisor of A & B if it is right divisor of A and B.

$A = C_1 D$, $B = C_2 D$

Common left divisor: $A = DC_1$, $B = DC_2$

D is greatest common right divisor (GCRD) of A, B if:
1) it is a common right divisor
2) any common right divisor of A, B is a right divisor of D.

Similarly, GCLD.

D is a common left multiple of A, B if left multiple of A & B.

$D = C_1 A = C_2 B$

D is least common left multiple (LCLM) of A, B if

1) D is a common left multiple of A, B
2) D is a RIGHT divisor of every common left multiple of A, B.

Similarly LCRM

The Bezout Identity over matrices over $R[x]$

FACT: Let A, B be two matrices over $R[z]$ having the same number of columns

Then (1) A, B have a gcrd D

(2) There are matrices P, Q over $F[z]$ satisfying $PA + QB = D$

$$\begin{pmatrix} A = C_1 D \\ B = C_2 D \end{pmatrix}$$

→ Required since if A, B have CRD then they must have same no. of columns.

$A = C_1 D$
$B = C_2 D$

Proof: Let A be $p \times n$, B is $n \times n$

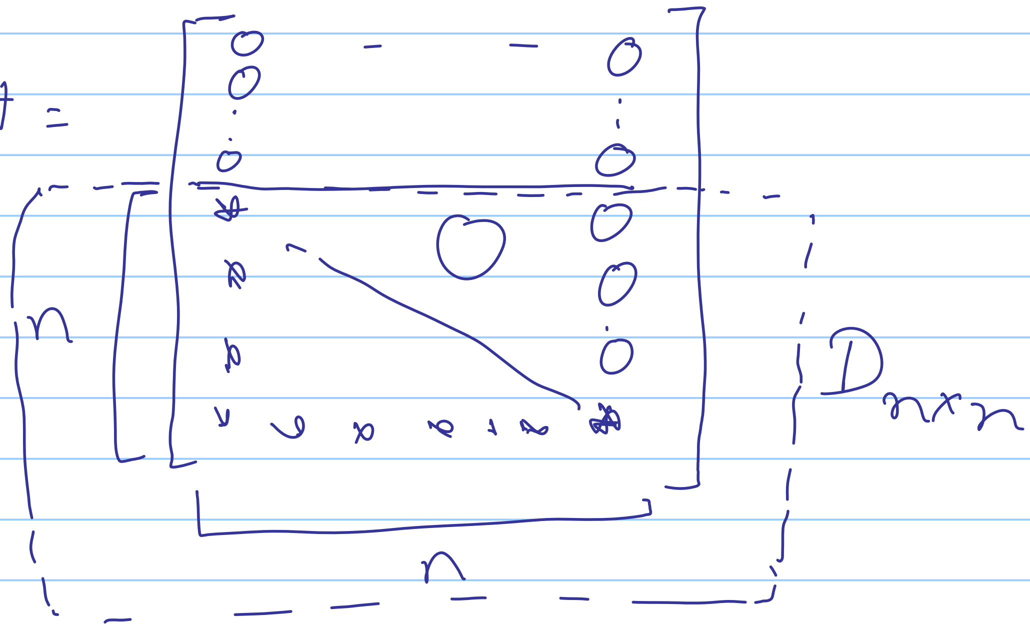
Build the matrix $\begin{bmatrix} A \\ B \end{bmatrix} \Rightarrow (p+n) \times n$

Bring $\begin{bmatrix} A \\ B \end{bmatrix}$ into Hermite form:

$$\underbrace{X}_{(n+p)\times(n+p)}\underbrace{\begin{bmatrix}A\\B\end{bmatrix}}_{(n+p)\times n} = \underbrace{H}_{(n+p)\times n}$$

H → Hermite form

$H =$



So $H = \begin{array}{c} \begin{bmatrix} O \\ \hline D \end{bmatrix} \begin{array}{l} p \\ n \end{array} \\ n \end{array}$

Let's partition $X = \begin{array}{cc} \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} & \begin{array}{l} p \\ n \end{array} \\ \quad p \quad\quad n & \end{array}$

$$\begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} O \\ D \end{bmatrix}$$

So $X_{21}A + X_{22}B = D$ — $(*)$

<u>Claim:</u> D is the GCRD of A, B.

Let C be any CRD of A, B

$$\left.\begin{aligned} A &= A_1 C \\ B &= B_1 C \end{aligned}\right] \quad \text{Substituting in } (*),$$

$$X_{21} A_1 C + X_{22} B_1 C = D$$

$$(X_{21} A_1 + X_{22} B_1) C = D$$

i.e. C is a CRD of D.

So we need to show the D is a CRD of A, B.

Since X is unimodular, $Y = X^{-1}$ is also a polynomial matrix.

$$\begin{bmatrix} A \\ B \end{bmatrix} = Y \begin{bmatrix} 0 \\ D \end{bmatrix}$$

$$\begin{bmatrix} A \\ B \end{bmatrix} = \begin{array}{cc} \left[\begin{array}{c|c} Y_{11} & Y_{12} \\ \hline Y_{21} & Y_{22} \end{array}\right] & \begin{array}{c} p \\ n \end{array} \\ \begin{array}{cc} \quad p \quad & \quad n \quad \end{array} & \end{array} \begin{array}{cc} \left[\begin{array}{c} 0 \\ \hline D \end{array}\right] & \begin{array}{c} p \\ n \end{array} \\ n & \end{array}$$

$$\left.\begin{aligned} A &= Y_{12} D \\ B &= Y_{22} D \end{aligned}\right] \Rightarrow D \text{ is a CRD}$$

Example: $\mathbb{R}[z]$, $n = p = 2$

$$A = \begin{bmatrix} z & z+1 \\ z+2 & z+3 \end{bmatrix} \qquad B = \begin{bmatrix} z+4 & z+5 \\ z+6 & z+7 \end{bmatrix}$$

Find GCRD and solve the Bezout Identity.

$$\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} z & z+1 \\ z+2 & z+3 \\ z+4 & z+5 \\ z+6 & z+7 \end{bmatrix}$$ ← Step 1: get this into Hermite form.

$$\left[\begin{array}{cc|cccc} z & z+1 & 1 & 0 & 0 & 0 \\ z+2 & z+3 & 0 & 1 & 0 & 0 \\ z+4 & z+5 & 0 & 0 & 1 & 0 \\ z+6 & z+7 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{cc|cccc} 0 & 0 & \frac{1}{3} & 0 & -1 & \frac{2}{3} \\ 0 & 0 & -\frac{1}{3} & 1 & -1 & \frac{1}{3} \\ \hline -1 & 0 & \frac{1}{6}(z+5) & 0 & 1 & -\frac{1}{6}(z+5) \\ -6 & -6 & 1 & 0 & 0 & -1 \end{array}\right]$$

$$\underbrace{\underbrace{\begin{bmatrix} \frac{1}{6}(z+5) & 0 \\ 1 & 0 \end{bmatrix}}_{X_{21}} \quad \underbrace{\begin{bmatrix} 1 & -\frac{1}{6}(z+5) \\ 0 & -1 \end{bmatrix}}_{X_{22}}}_{X}$$

$$\overset{X_{21}}{\begin{bmatrix} \frac{1}{6}(z+5) & 0 \\ 1 & 0 \end{bmatrix}} \overset{A}{\begin{bmatrix} z & z+1 \\ z+2 & z+3 \end{bmatrix}} + \overset{X_{22}}{\begin{bmatrix} 1 & -\frac{1}{6}(z+5) \\ 0 & -1 \end{bmatrix}} \overset{B}{\begin{bmatrix} z+4 & z+5 \\ z+6 & z+7 \end{bmatrix}} = \overset{D}{\begin{bmatrix} -1 & 0 \\ -6 & -6 \end{bmatrix}}$$

<u>Note</u> 1) When matrix is square, the Hermite form is unique i.e P and UP (U unimodular) have the same Hermite form

2) GCRD's are not unique

But for any two gcrd's R_1 and R_2

$$R_1 = W_2 R_2 \qquad R_2 = W_1 R_1 \qquad \left\{ \begin{array}{l} W_1, W_2 \\ \text{are poly. matrices} \end{array} \right\}$$

$$\Rightarrow \quad R_1 = W_2 W_1 R_1$$

If R_1 is non-singular $\Rightarrow W_i$ unimodular
$\Rightarrow R_2$ non-singular

FACT: If one GCRD is non-singular then all GCRD's are non-singular (differing only by a uni-modular left factor)

FACT: If a GCRD is unimodular, then all GCRD's are unimodular.

Def$^{\underline{n}}$: A polynomial matrix has full column rank if no non-trivial linear combination of its columns, with either rational or polynomial coefficients is identically zero ($\forall s$)

$\Leftrightarrow$ Rank is full for almost all s

Def$^{\underline{n}}$: A polynomial matrix is irreducible if the rank is full for ALL s.

FACT: $\begin{bmatrix} A \\ B \end{bmatrix}$ has full column rank
(i.e. full col. rank for a.a.s)

$\Rightarrow$ All GCRD's of (A, B) are non-singular

Proof: Follows from Hermit form

$$\begin{bmatrix} U \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ D \end{bmatrix}$$ (U unimodular)

$\Rightarrow$ rank $\begin{bmatrix} A \\ B \end{bmatrix}$ = rank $\begin{bmatrix} 0 \\ D \end{bmatrix}$

$\Rightarrow$ D is invertible

* In MFDs e.g. PQ^{-1}, GCRD(P, Q) is always non-singular.

Proof: Later. (Exercise)

Fraction representation of linear system :

Consider a system with transfer matrix

$$F = \left(\frac{p_{ij}}{q_{ij}} \right)_{p \times m} \quad \text{where } p_{ij}, q_{ij} \in R[\bar{s}]$$

Consider, $q = \prod_{i,j} q_{ij}$

Define $Q := q I_m$

and $P := FQ$

So $F = PQ^{-1}$ =: expresses F as a fraction of two polynomial matrices = right fraction representation

Left Fraction Representation

Define $G := q I_p$ & $T := GF$

Then $F = G^{-1} T$ = left fraction representation

Coprime Fraction representations

Let P, Q be two matrices over $R[x]$ with same no. of columns.

P, Q are right coprime over $R[x]$ if gcrd of P & Q is a unimodular matrix (over $R[x]$)

(i.e. all their GCRD's are unimodular)

This is equivalent to the existence of matrices S, R over $\mathbb{R}[x]$ s.t.
$SP + RQ = I$

[Verify: $SP + RQ = M$ (unimodular)
$$\underbrace{(M^{-1}S)}_{\text{new } S} P + \underbrace{(M^{-1}R)}_{\text{new } R} Q = I$$
]

Right Coprime Fraction Representation

Let $F = P_1 Q_1^{-1}$ and let $D = \operatorname{gcrd}(P_1, Q_1)$

Then $\left.\begin{array}{l} P_1 = PD \\ Q_1 = QD \end{array}\right\}$ Since Q_1 is invertible
$\det(Q_1) = \det(QD)$
$= \det Q . \det D \neq 0$

$\Rightarrow \det D \neq 0 \Rightarrow D$ is invertible over rational functions

[For matrices over fields all the usual vector space concepts of rank etc can be used]

So $F = P_1 Q_1^{-1} = (PD)(QD)^{-1} = PDD^{-1}Q$
$= PQ^{-1}$

and $F = PQ^{-1}$ is a right coprime factor representation over $\mathbb{R}[x]$

Formulae for $\left.\begin{array}{l} P = P_1 D^{-1} \\ Q = Q_1 D^{-1} \end{array}\right\}$ over $\mathbb{R}[x]$

FACT: $P(s)$ and $Q(s)$ are right coprime iff $\begin{bmatrix} P(s) \\ Q(s) \end{bmatrix}$ has full rank for every s (i.e. iff $\begin{bmatrix} P(s) \\ Q(s) \end{bmatrix}$ is irreducible.

Proof: $$U \begin{bmatrix} P(s) \\ Q(s) \end{bmatrix} = \begin{bmatrix} 0 \\ D(s) \end{bmatrix}$$

So $\begin{bmatrix} P(s) \\ Q(s) \end{bmatrix}$ has full rank for every s $\Longleftrightarrow$ $\det(D(s)) \neq 0 \ \forall s$

$\Longleftrightarrow$ $D(s)$ is unimodular

Corr: An irreducible square matrix is unimodular.

Column / Row Reduced Matrices

Proper Transfer Matrices: A rational transfer function matrix $H(s)$ is said to be *proper* if

$$\lim_{s \to \infty} H(s) < \infty \qquad \boxed{H(s) \text{ is } p \times m}$$

and strictly proper if

$$\lim_{s \to \infty} H(s) = 0$$

[Degree of $\begin{bmatrix} p_1(s) \\ p_2(s) \\ \vdots \\ p_m(s) \end{bmatrix}$ = highest degree of $p_i(s)$ $i = 1, \dots, m$

FACT: If $H(s)$ is strictly proper (proper) and $H(s) = N(s) D^{-1}(s)$ then every column of $N(s)$ has degree strictly less ($\leq$) that of corresponding column of $D(s)$.

Proof: $N(s) = H(s) D(s)$

For the j^{th} column:

$$n_{ij}(s) = \sum_{k=1}^{m} \underbrace{H_{ik}(s)}_{\text{strictly proper}} d_{kj}(s) \qquad i = 1, \dots, p$$

$$\begin{array}{c} j \\ \begin{bmatrix} i\,+ \end{bmatrix} \end{array} = \begin{bmatrix} i \text{———} \end{bmatrix} \begin{array}{c} j \\ \begin{bmatrix} \,|\, \end{bmatrix} \end{array}$$

Note: Converse not true

Example: $N(s) = \begin{bmatrix} 2s^2+1 & 2 \end{bmatrix}$

$$D(s) = \begin{bmatrix} s^3+s & s \\ s^2+s+1 & 1 \end{bmatrix}$$

$$N(s)D^{-1}(s) = \begin{bmatrix} \dfrac{-s^2+s}{s^2+s-1} , & \dfrac{s^3+s-1}{s^2+s-1} \end{bmatrix}$$

$\uparrow$ Not proper

Notation: $k_i :=$ degree of ith column of $D(s)$

Then $\deg[\det D(s)] \leq \sum_{i=1}^{m} k_i$

$$\det \begin{bmatrix} s^3+s & s+2 \\ s^2+s+1 & 1 \end{bmatrix} = (s^3+s) - (s+2)(s^2+s+1)$$
$$= -3s^2 - 2s - 2$$

$\rightarrow \deg = 2$

$\sum k_i = 3 + 1 = 4$

Def$\stackrel{n}{=}$: $D(s)$ is column reduced if

$$\deg[\det D(s)] = \sum_{i=1}^{m} k_i$$

$$D(s) = D_{hc}\, S(s) + L(s)$$

$$S(s) = \text{diag}\{s^{k_i}, i=1,\cdots,m\}$$

D_{hc} = leading coeff matrix

$$\begin{bmatrix} s^3+s & s+2 \\ s^2+s+1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} s^3 & 0 \\ 0 & s \end{bmatrix} + \begin{bmatrix} s & 2 \\ s^2+s+1 & 1 \end{bmatrix}$$

$$\det D(s) = \det[D_{hc}]\, s^{\Sigma k_i} + \text{lower degree terms}$$

<u>FACT</u>: A non-singular polynomial matrix is column reduced iff its leading coeff matrix is non-singular.

<u>Proof</u>: above

<u>FACT</u>: If $D(s)$ is column reduced, then $H(s) = N(s)D^{-1}(s)$ is strictly proper (proper) iff each column of $N(s)$ has degree less than ($\leq$) the degree of the corresponding column of $D(s)$.

$\longrightarrow$ No proof.

Reduction to column reduced form

Any polynomial matrix can be made column reduced by elementary column operations.

Example: $D(s) = \begin{bmatrix} (s+1)^2(s+2)^2 & -(s+1)^2(s+2) \\ 0 & s+2 \end{bmatrix}$

$$= \begin{bmatrix} s^4+6s^3+13s^2+12s+4 & -(s^3+4s^2+5s+2) \\ 0 & s+2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} s^4 & 0 \\ 0 & s^3 \end{bmatrix} + L(s)$$

$\hookrightarrow$ Not column reduced

Elementary col. operations

$$\begin{bmatrix} s^4 & -s^3 \\ 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\downarrow \qquad\qquad \downarrow$$

$$\begin{bmatrix} 0 & -s^3 \\ 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 \\ s & 1 \end{bmatrix}$$

$$D(s)\begin{bmatrix} 1 & 0 \\ s & 1 \end{bmatrix} = \begin{bmatrix} 2s^3+8s^2+10s+4 & -(s^3+4s^2+5s+2) \\ s^2+2s & (s+2) \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} s^3 & 0 \\ 0 & s^3 \end{bmatrix} + L_2(s)$$

↳ still not col. reduced

$$\begin{bmatrix} 2s^3 & -s^3 \\ 0 & 0 \end{bmatrix} \downarrow \begin{bmatrix} 0 & -s^3 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \downarrow \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$

$$D(s) \begin{bmatrix} 1 & 0 \\ s & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 0 & s^3 + 4s^2 + 5s + 2 \\ s^2 + 4s + 4 & s + 2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} s^2 & 0 \\ 0 & s^3 \end{bmatrix} + L_3(s)$$

↳ col. reduced.

Invariance of Column Degrees of Column Reduced Matrices

Let $D(s)$ and $\bar{D}(s)$ be col-red poly. matrices with col. degrees arranged in order (ascending)

FACT: If $D(s) = \bar{D}(s)\, U(s)$ [U unimodular] Then $D(s)$ and $\bar{D}(s)$ have same col. degrees.

Extracting the strictly proper part of a given MFD:

Thm: Let $D(s)$ be an $m \times m$ non-singular polynomial matrix. Then, for any $p \times m$ poly. matrix $N(s)$, there exist unique polynomial matrices $\{Q(s), R(s)\}$ s.t.
$$N(s) = Q(s)D(s) + R(s)$$
and $R(s)D^{-1}(s)$ is strictly proper

Proof:
$$H(s) = N(s)D^{-1}(s)$$
$$= \underbrace{H_{sp}(s)}_{\text{strictly proper}} + \underbrace{P(s)}_{\text{polynomial}}$$

Then,
$$N(s) = H(s)D(s)$$
$$= \underbrace{H_{sp}(s)D(s)}_{R(s)} + P(s)D(s)$$

Then
$$R(s) = N(s) - P(s)D(s)$$
$\Rightarrow R(s)$ is polynomial

Also, $R(s) = H_{sp}(s)D(s) \Rightarrow R(s)D^{-1}(s) = H_{sp}(s)$

Hence $R(s)D^{-1}(s)$ is S.P.

Uniqueness $\rightarrow$ skipped

Smith Form :

FACT : For any $p \times m$ poly. matrix $P(s)$, we can find elementary row and column operations OR corr. unimodular matrices $\{U(s), V(s)\}$ s.t.

$$U(s)\,P(s)\,V(s) = \Lambda(s)$$

where,

$$\Lambda(s) = \begin{array}{c} \left[\begin{array}{ccc|c} \lambda_1(s) & & & \\ & \ddots & & 0 \\ & & \lambda_r(s) & \\ \hline & 0 & & 0 \end{array}\right] \begin{array}{l} r \\ \\ \\ p-r \end{array} \\ \begin{array}{cc} \qquad r \qquad & \; m-r \end{array} \end{array}$$

r = rank of $P(s)$

and the $\lambda_i(s)$ are unique monic polynomials obeying a division property

$$\lambda_i(s) \mid \lambda_{i+1}(s) \qquad i = 1, \cdots, r-1$$

Moreover, if we define

$\Delta_i(s)$ = the gcd of all $i \times i$ minors det. of $P(s)$

then

$$\lambda_i(s) = \frac{\Delta_i(s)}{\Delta_{i-1}(s)}, \quad \Delta_0(s) = 1$$

$\Lambda(s) \rightarrow$ Smith form of $P(s)$

$\lambda_i(s) \rightarrow$ invariant polynomials of $P(s)$

$\det(P(s)) = \lambda_1(s)\,\lambda_2(s) \cdots \lambda_r(s)$

<u>Proof</u>: 1) Least degree element to $(1,1)$
2) Use elementary row + column ops to make

$$\left[\begin{array}{c|ccc} \lambda_1'(s) & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & P_1'(s) & \\ 0 & & & \end{array}\right]$$

3) If $\lambda_1'(s)$ divides all elements in $P_1'(s)$ stop, otherwise ~~bring~~ go to step 1.
4) After finite no. of steps:

$$\left[\begin{array}{c|ccc} \lambda_1(s) & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & P_1(s) & \\ 0 & & & \end{array}\right]$$

$\lambda_1(s)$ divides each element of $P_1(s)$

5) Repeat steps 1 – 4 on $P_1(s)$

<u>FACT</u>: $\Delta_i(s)$ = gcd of all $i \times i$ minors of $P(s)$ depends only on $P(s)$ and is independent on row/col. operations on $P(s)$

Using fact: $\Delta_i(s)$ = gcd of all $i \times i$ minors of $\Lambda(s)$

Then,

$\Delta_1(s) = \lambda_1$

$\Delta_2(s) = \lambda_1 \lambda_2$

$\vdots$

$\Delta_i(s) = \lambda_1 \lambda_2 \cdots \lambda_i$

Assume $\Delta_0 = 1$

$\Rightarrow$

$$\lambda_1 = \Delta_1 \;,\; \lambda_2 = \frac{\Delta_2}{\Delta_1} \;,\; \ldots \;,\; \lambda_i = \frac{\Delta_i}{\Delta_{i-1}}$$

Moreover, uniqueness of Δ_i implies uniqueness of λ_i.

Properties / Uses of Smith Form

FACT: $A(s)$ and $B(s)$ are right coprime iff the Smith form of $\begin{bmatrix} A(s) \\ B(s) \end{bmatrix}$ is $\begin{bmatrix} 0 \\ I \end{bmatrix}$

Proof:
$$U(s) \begin{bmatrix} A(s) \\ B(s) \end{bmatrix} = \begin{bmatrix} 0 \\ D(s) \end{bmatrix} = \begin{bmatrix} 0 \\ I \end{bmatrix} D(s)$$

If A, B are coprime $D(s)$ unimodular

then
$$U(s) \begin{bmatrix} A(s) \\ B(s) \end{bmatrix} D^{-1}(s) = \begin{bmatrix} 0 \\ I \end{bmatrix}$$

polynomial unimodular

Converse : Exercise

FACT: If P is unimodular, Smith form is I

$$P = P . I . I$$

FACT: $sI - A$ and $sI - B$ have same Smith form iff A & B are similar.

Smith Form Example:

$$A = \begin{bmatrix} z & 0 \\ 0 & z+1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \begin{bmatrix} z & 0 \\ 0 & z+1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$c1 = c1 + c2$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \begin{bmatrix} z & 0 \\ z+1 & z+1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$r2 = r2 - r1$

$$\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \quad \begin{bmatrix} z & 0 \\ 1 & z+1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

switch $r1$ & $r2$

$$\begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & z+1 \\ z & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$r2 = r2 - z(r1)$

$$\begin{bmatrix} -1 & 1 \\ z+1 & -z \end{bmatrix} \quad \begin{bmatrix} 1 & z+1 \\ 0 & -z^2-z \end{bmatrix} \quad \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$c2 = c2 - (z+1)c1$$

$$\underset{U}{\begin{bmatrix} -1 & 1 \\ z+1 & -z \end{bmatrix}} \quad \underset{\Lambda}{\begin{bmatrix} 1 & 0 \\ 0 & -z^2-z \end{bmatrix}} \quad \underset{V}{\begin{bmatrix} 1 & -z-1 \\ 1 & -z \end{bmatrix}}$$

smith form of A

$$UAV = \Lambda$$