Realizations for MIMO tr. matrices

Note Title 11-06-2008

Q. How to get (smallest ?) realizations from given tr. matrix

<u>Example 1</u>. $H(s) = \begin{bmatrix} \dfrac{1}{(s-1)^2} & \dfrac{1}{(s-1)(s+3)} \\ \dfrac{-6}{(s-1)(s+3)^2} & \dfrac{s-2}{(s+3)^2} \end{bmatrix}$

<u>Obvious method</u>: Controller form real. for each subsystem and connect.

$$A = \left[\begin{array}{cc|cc|ccc|cc} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & -2 & 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & -5 & -3 & 9 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & -6 & -9 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

$$B = \left[\begin{array}{cc} 1 & 0 \\ 0 & 0 \\ \hline 0 & 1 \\ 0 & 0 \\ \hline 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ \hline 0 & 1 \\ 0 & 0 \end{array}\right]; \quad C = \left[\begin{array}{cc|cc|ccc|cc} 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -6 & 1 & -2 \end{array}\right]$$

$n = 9 = \Sigma$ degrees of denominators of each subsystem

* This is not the smallest

<u>Ex 2</u> : Rewrite $H(s) = \dfrac{N(s)}{d(s)}$

$d(s) = $ lcm of denominators

$$= s^r + d_1 s^{r-1} + \cdots + d_r$$

$$\left(= (s-1)^2 (s+3)^2 = s^4 + 4s^3 - 2s^2 - 12s + 1 \right.$$

$$N(s) = N_1 s^3 + N_2 s^2 + N_3 s + N_4$$

$$N_1 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \quad N_2 = \begin{bmatrix} 1 & 1 \\ 0 & -4 \end{bmatrix} \quad N_3 = \begin{bmatrix} 6 & 2 \\ -6 & 5 \end{bmatrix}$$

$$N_4 = \begin{bmatrix} 9 & -3 \\ 6 & -2 \end{bmatrix}$$

$$A = \begin{bmatrix} -d_1 I_m & -d_2 I_m & \cdots & -d_r I_m \\ I_m & 0 & - & 0 \\ & \ddots & & \\ 0 & & \ddots \; I_m & 0 \end{bmatrix} ; B = \begin{bmatrix} I_m \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$C = \left[N_1 \; \cdots \; N_r \right]$$

$$\dot{x}_1 = -d_1 x_1 - d_2 x_2 \cdots - d_r x_r + u$$

$$\dot{x}_2 = x_1$$

$$\vdots$$

$$\dot{x}_r = x_{r-1}$$

$$y = N_1 x_1 + \cdot + N_r x_r$$

$$sX_1 = -d_1 X_1 - \cdots - d_r X_r + U$$

$$sX_2 = X_1$$

$$\vdots$$

$$sX_{r-1} = X_{r-2}$$

$$sX_r = X_{r-1}$$

$$X_r = \frac{1}{s^{r-1}} X_1$$

$$X_{r-1} = \frac{1}{s^{r-2}} X_1$$

$$X_2 = \frac{1}{s} X_1$$

size of $A = r \times m = 4 \times 2 = 8$

Q. Can we make size $A = \deg d(s)$?

Ans: No in general.

[Put this page after coprime Fraction rep. section

$$H(s) = \begin{bmatrix} \frac{s}{(s+1)^2(s+2)^2} & \frac{s}{(s+2)^2} \\ \frac{-s}{(s+2)^2} & \frac{-s}{(s+2)^2} \end{bmatrix}$$

<u>MFD - 1</u>

$$H(s) = \begin{bmatrix} s & s(s+1)^2 \\ -s(s+1)^2 & -s(s+1)^2 \end{bmatrix} \begin{bmatrix} (s+1)^2(s+2)^2 & 0 \\ 0 & (s+1)^2(s+2)^2 \end{bmatrix}^{-1}$$

$$\qquad\qquad N_1(s) \qquad\qquad\qquad D_1(s)^{-1}$$

$$\deg_s \det D_1(s) = 8$$

<u>MFD - 2</u>

$$H(s) = \begin{bmatrix} s & s \\ -s(s+1)^2 & -s \end{bmatrix} \begin{bmatrix} (s+1)^2(s+2)^2 & 0 \\ 0 & (s+2)^2 \end{bmatrix}^{-1}$$

$$= N_2(s) \quad D_2(s)^{-1}$$

$$\deg \det D_2(s) = 6$$

MFD - 3

$$H(s) = \begin{bmatrix} s & 0 \\ s(s+1)^2 & s^2 \end{bmatrix} \begin{bmatrix} (s+1)^2(s+2)^2 & -(s+1)^2(s+2) \\ 0 & (s+2) \end{bmatrix}^{-1}$$

$$= N_3(s)\, D_3^{-1}(s)$$

$$\deg \det D_3(s) = 5$$

Exercise: Calculate the GCRD of $N_1(s)$ and $D_1(s)$

<u>Problem</u>: Given a strictly proper right MFD:

$$H(s) = N(s)D^{-1}(s)$$

construct a controllable realization of order equal to deg det

$$H(s) = N(s)D^{-1}(s)$$

$$D(s)\xi(s) = u(s) \quad ; \quad y(s) = N(s)\xi(s)$$

$$\begin{bmatrix} d_{11} & \cdots & d_{1m} \\ \vdots & & \vdots \\ d_{m1} & & d_{mm} \end{bmatrix} \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_m \end{bmatrix} = \begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix}$$

Q. What is the highest derivative of ξ_i?

$$D(s) = D_{hc} S(s) + D_{lc} \Psi(s)$$

$$S(s) = \begin{bmatrix} s^{k_1} & & 0 \\ & \ddots & \\ 0 & & s^{k_m} \end{bmatrix}$$

k_i = col. degrees of $D(s)$

D_{hc} = leading coeff matrix

$$\Psi^T(s) = \left[\begin{array}{c|c|c|c} s^{k_1-1} \cdots s \; 1 & 0 & \cdots & 0 \\ \hline 0 & s^{k_2-1} \cdots 1 & \cdots & 0 \\ \hline & & \ddots & \\ \hline & & \cdots & s^{k_m-1} \cdots s \; 1 \end{array}\right]$$

D_{lc} = a matrix of coefficients

So: $\left[D_{hc} S(s) + D_{lc} \Psi(s) \right] \xi(s) = u(s)$

Assuming D_{hc} is invertible

[i.e. assuming $D(s)$ is column reduced.

This is OK since if $D(s)$ is not col. reduced to begin with we can make it col. red. without affecting the deg det $D(s)$

Q. Why deg det $D(s)$ remains the same?

→ Exercise (Hint: mult. by unimodular matrix)

Assume $k_1 \leq k_2 \leq \cdots \leq k_m$.

Then $S(s)\xi = -D_{hc}^{-1} D_{lc} \Psi(s) \xi + D_{hc}^{-1} u$

$y(s) = N(s)\xi(s) = N_{lc} \Psi(s) \xi(s)$

Example: $N(s) = \begin{bmatrix} s & 0 \\ -s & s^2 \end{bmatrix}$

$$D(s) = \begin{bmatrix} 0 & -(s^3+4s^2+5s+2) \\ (s+2)^2 & (s+2) \end{bmatrix}$$

$k_1 = 2 \qquad k_2 = 3$

$k_1 + k_2 = 5 = \deg\det D(s)$ (col red)

$$\begin{bmatrix} 0 & -(s^3+4s^2+5s+2) \\ (s+2)^2 & (s+2) \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

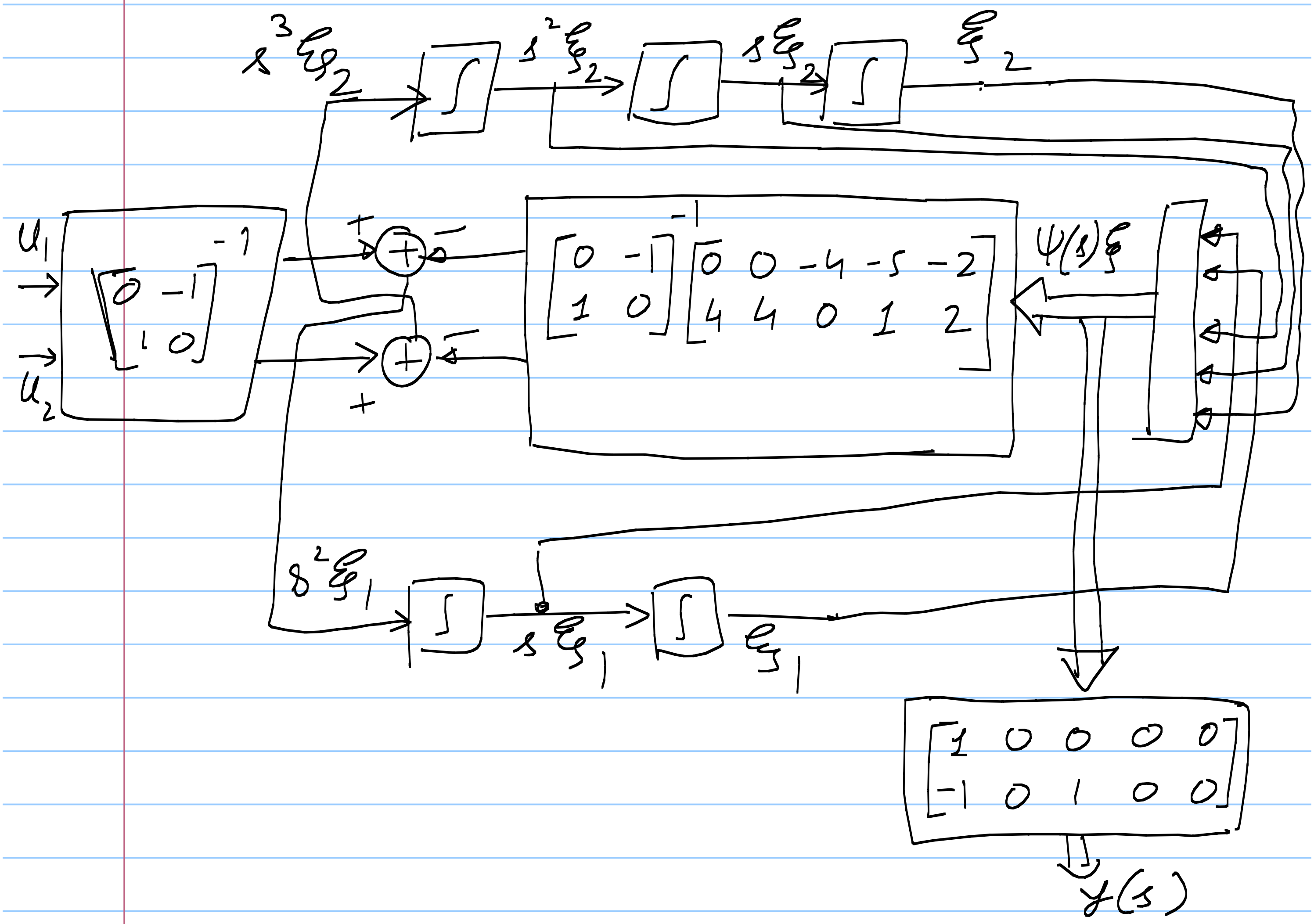
$$\Rightarrow \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} s^2 & 0 \\ 0 & s^3 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & 0 & -4 & -5 & -2 \\ 4 & 4 & 0 & 1 & 2 \end{bmatrix} \overset{\Psi}{\begin{bmatrix} s & 0 \\ 1 & 0 \\ 0 & s^2 \\ 0 & s \\ 0 & 1 \end{bmatrix}} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

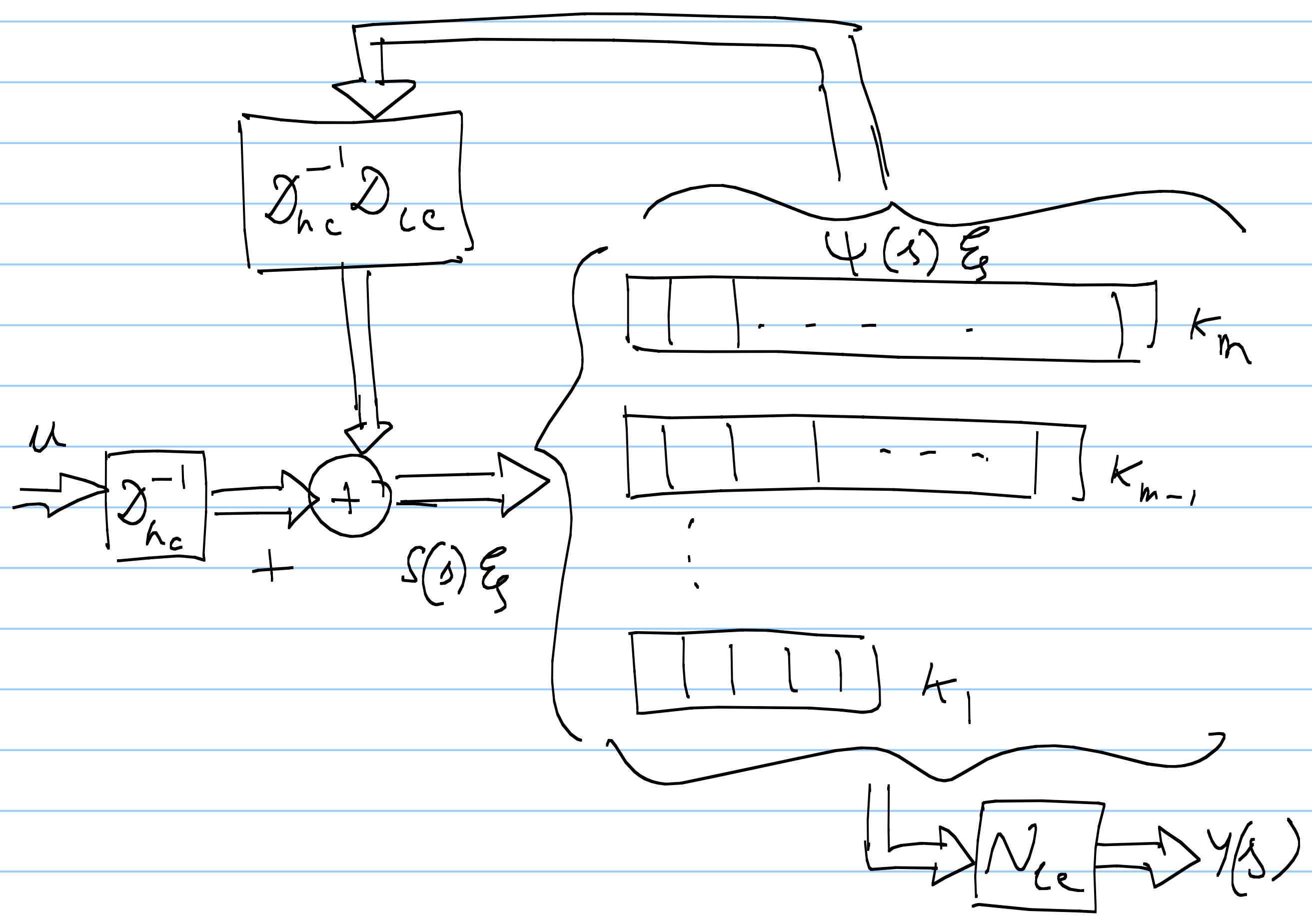
$$\Psi(s)\,\xi = \begin{bmatrix} s\xi_1 \\ \xi_1 \\ s^2\xi_2 \\ s\xi_2 \\ \xi_2 \end{bmatrix}$$

$$Y(s) = N(s)\,\xi = \begin{bmatrix} s & 0 \\ -s & s^2 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} s\xi_1 \\ \xi_1 \\ s^2\xi_2 \\ s\xi_2 \\ \xi_2 \end{bmatrix}$$



$s^3\xi_2$
$s^2\xi_2$
$s\xi_2$
ξ_2
u_1
u_2
$\begin{bmatrix}0 & -1\\1 & 0\end{bmatrix}^{-1}$
$\begin{bmatrix}0 & -1\\1 & 0\end{bmatrix}^{-1}\begin{bmatrix}0 & 0 & -4 & -5 & -2\\4 & 4 & 0 & 1 & 2\end{bmatrix}$
$\Psi(s)\xi$
$s^2\xi_1$
$s\xi_1$
ξ_1
$\begin{bmatrix}1 & 0 & 0 & 0 & 0\\-1 & 0 & 1 & 0 & 0\end{bmatrix}$
$y(s)$



$D_{hc}^{-1}D_{lc}$
$\Psi(s)\xi$
K_m
K_{m-1}
K_1
u
D_{hc}^{-1}
$S(s)\xi$
N_{lc}
$Y(s)$

State Space Eqns:

Case Realization

$$A_c^o = \text{block diag}\left\{\begin{bmatrix} 0 & \cdots & 0 \\ 1 & \ddots & \\ & \ddots & \vdots \\ 0 & \cdots 1 & 0 \end{bmatrix}_{k_i \times k_i} ; i = 1, \cdots, m\right\}$$

$$B_c^o = \text{block diag}\left\{\begin{bmatrix} 1 \\ \vdots \\ 0 \end{bmatrix}_{k_i \times 1}, i = 1, \cdots, m\right\}$$

$$C_0^o = I_n \qquad n = \det \deg D(s) = \sum k_i$$

Ex:

$$A_c^o = \left[\begin{array}{cc|ccc} 0 & 0 & & 0 & \\ 1 & 0 & & & \\ \hline & & 0 & 0 & 0 \\ & 0 & 1 & 0 & 0 \\ & & 0 & 1 & 0 \end{array}\right] \qquad B_c^o = \left[\begin{array}{c|c} 1 & 0 \\ 0 & 0 \\ \hline 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{array}\right]$$

$n = 5$

Now: closed loop ~~through~~ (state feedback $D_{hc}^{-1} D_{\ell c}$ and input $D_{hc}^{-1} u$)

Then
$$A_c = A_c^o - B_c^o (D_{hc}^{-1} D_{\ell c})$$
$$B_c = B_c^o D_{hc}^{-1}$$
$$C_c = N_{\ell c}$$

It is similar to give an feedback $u = G[v - kx]$ to $\{A, B, C\}$

$\Downarrow$

$\{A - BGK, BG, C\}$

Example: $D_{hc}^{-1} D_{lc} = \begin{bmatrix} 4 & 4 & 0 & 1 & 2 \\ 0 & 0 & 4 & 5 & 2 \end{bmatrix}$

$$D_{hc}^{-1} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$A_c = \left[\begin{array}{cc|ccc} -4 & -4 & 0 & -1 & -2 \\ 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & -4 & -5 & -2 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] \qquad B_c = \left[\begin{array}{cc} 0 & 1 \\ 0 & 0 \\ \hline -1 & 0 \\ 0 & 0 \\ 0 & 0 \end{array}\right]$$

$$C_c = N_{lc}$$

Properties

1) $\{A_c, B_c\}$ is controllable

Proof: $\{A_{c_0}, B_{c_0}\}$ is controllable & $\{A_c, B_c\}$ is obtained by state feedback around $\{A_c, B_c\}$

→ Observability is NOT guaranteed.

2) $N(s) D^{-1}(s) = C_c (sI - A_c)^{-1} B_c$

3) $\det(sI - A_c) = (\det D_{hc})^{-1} \det D(s)$

Proof: Exercise

Theorem: Any realization of an MFD $N(s)D^{-1}(s)$ with order $= \deg \det D(s)$ will be a minimal realization ($\equiv$ contr + obs) iff $N(s)$ and $D(s)$ are right coprime.

Lemma: If there exist one controllable + obs realization of $N(s)D^{-1}(s)$, with order $n = \deg \det D(s)$, then all realizations of order n will also be contr + obs

Proof: Recall minimality $\equiv$ contr + obs

Let $\{A_i, B_i, C_i\}$ $i = 1, 2$ be two realization of $N(s)D^{-1}(s)$ of the same order

Then $\quad \mathcal{O}_1 \mathcal{C}_1 = \mathcal{O}_2 \mathcal{C}_2 \quad$ —— (1)

If $\{A_1, B_1, C_1\}$ is minimal

$$\rho[\mathcal{O}_1 \mathcal{C}_1] = n$$

$$\Rightarrow \rho[\mathcal{O}_2 \mathcal{C}_2] = n \quad (\text{by } (1))$$

But $\rho[\mathcal{O}_2 \mathcal{C}_2] \leqslant \min[\rho(\mathcal{O}_2); \rho(\mathcal{C}_2)]$

$$\Rightarrow \rho(\mathcal{O}_2) = n = \rho(\mathcal{C}_2)$$

$\Rightarrow \{A_2, B_2, C_2\}$ is contr + obs

Lemma: A controller form realization of $N(s)D^{-1}(s)$ of order equal to deg det $D(s)$ will also be observable ($\equiv$ minimal) iff MFD is irreducible.

Proof of Necessity: A reducible MFD (of order equal = deg det $D(s)$) cannot be minimal.
Extract GCD and a lower det deg results $\Rightarrow$ realization of lower order

Sufficiency: skipped for now.

Thm: Suppose $\{N_i(s)D_i^{-1}(s), i=1,2\}$ are two irreducible MFDs. Then there exists a unimodular matrix $U(s)$ s.t.

$$D_1(s) = D_2(s)U(s)$$
$$N_1(s) = N_2(s)U(s)$$

Proof: $N_1 D_1^{-1} = N_2 D_2^{-1}$

$$\Rightarrow \quad N_1 = N_2 D_2^{-1} D_1$$

Let $U(s) = D_2^{-1} D_1$

Claim: $D_2^{-1}D_1$ and $D_1^{-1}D_2$ are both polynomial matrices.

Proof: Since N_1, D_1 are coprime

$\exists$ X and Y s.t. $XN_1 + YD_1 = I$

$\Rightarrow$ $X N_2 D_2^{-1} D_1 + Y D_2 D_2^{-1} D_1 = I$

$\Rightarrow$ $\underbrace{[X N_2 + Y D_2]}\, U = I$

$\Rightarrow$ U^{-1} is a polynomial

Similarly for U.

Examples:

1> A_c, B_c, C_c in the last example are contr + obs.

So $N(s) = \begin{bmatrix} s & 0 \\ -s & s^2 \end{bmatrix}$ and

$$D(s) = \begin{bmatrix} 0 & -(s^3+4s^2+5s+2) \\ (s+2)^2 & (s+2) \end{bmatrix}$$

are coprime

Exercise: Check coprimeness by bringing the Hermit form.

2> $H(s) = \begin{bmatrix} \dfrac{s+1}{s^2} , & \dfrac{s+2}{s^3} \end{bmatrix}$

$$= \underbrace{\begin{bmatrix} s+1 & s+2 \end{bmatrix}}_{N_R} \underbrace{\begin{bmatrix} s^2 & 0 \\ 0 & s^3 \end{bmatrix}}_{D_R}^{-1}$$

Following above procedure; controller form realization:

$$A_c = \left[\begin{array}{cc|ccc} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] \qquad B_c = \left[\begin{array}{cc} 1 & 0 \\ 0 & 0 \\ \hline 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{array}\right]$$

$$C_c = \left[\begin{array}{cc|ccc} 1 & 1 & 0 & 1 & 2 \end{array}\right]$$

Q. Is this observable ?

Check $\text{rank} \begin{bmatrix} c \\ sI - A \end{bmatrix}_{s=0} < 5$

$\Rightarrow$ unobservable.

→ Calculate GCRD

Let GCRD be $M(s)$

$$\begin{array}{l|l} N(s) = N_R(s) M(s) & N_R(s) = N(s) M^{-1}(s) \\ D(s) = D_R(s) M(s) & D_R(s) = D(s) M^{-1}(s) \end{array}$$

→ Find the controller form realization of $N_R(s) D_R^{-1}(s)$.

→ Check whether it is observable.

→ What is deg det $D_R(s)$?

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3) $H(s) = \begin{bmatrix} \frac{1}{s} & \frac{1}{s^2} \end{bmatrix} = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} s & 0 \\ 0 & s^2 \end{bmatrix}^{-1}$

$$A_c = \left[\begin{array}{c|cc} 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right]; \quad B_c = \left[\begin{array}{cc} 1 & 0 \\ \hline 0 & 1 \\ 0 & 0 \end{array}\right]$$

$$C_c = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}$$

$$\mathcal{C} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

$$\mathcal{O} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Controllable but not observable.

$$\begin{bmatrix} s & 0 \\ 0 & s^2 \\ 1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} s & 0 \\ -s^2 & 0 \\ 1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} -s^2 & 0 \\ s & 0 \\ 1 & 1 \end{bmatrix} \rightarrow \left[\begin{array}{cc} 0 & 0 \\ \hline s & 0 \\ 1 & 1 \end{array}\right]$$

$$GCRD = \begin{bmatrix} s & 0 \\ 1 & 1 \end{bmatrix}$$

↳ invertible but not unimodular

$\Rightarrow N(s), D(s)$ are not coprime

$$N_R(s) = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} s & 0 \\ 1 & 1 \end{bmatrix}^{-1} \qquad \begin{bmatrix} \frac{1}{s} & 0 \\ -\frac{1}{s} & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{s} & 0 \\ -1/s & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \end{bmatrix}$$

$$D_R(s) = \begin{bmatrix} s & 0 \\ 0 & s^2 \end{bmatrix} \begin{bmatrix} \frac{1}{s} & 0 \\ -1/s & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -s & s^2 \end{bmatrix}$$

$$H_R(s) = N_R(s) D_R^{-1}(s) = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -s & s^2 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{s} & \frac{1}{s^2} \end{bmatrix}$$

But $D_R(s) = \begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} s & 0 \\ 0 & s^2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

↳ Not col. reduced.

$$\begin{bmatrix} 1 & 0 \\ -s & s^2 \end{bmatrix} \downarrow \begin{bmatrix} 1 & s \\ -s & 0 \end{bmatrix} \to D_R' \qquad \Bigg| \qquad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \downarrow \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix} \downarrow U$$

$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix}$ ↓ col. red.

$$D_R' = \begin{bmatrix} 1 & 0 \\ -s & s^2 \end{bmatrix} \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix}$$

↓ col red; D_R, U

$D_R = D_R' U^{-1}$ | $D_R' = D_R U$

$N_R = N_R' U^{-1}$ | $N_R' = N_R U$

$H(s) = N_R D_R^{-1} = (N_R' U^{-1})(U D_R'^{-1}) = N_R' D_R'^{-1}$

$$N_R' = N_R U = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \end{bmatrix}$$

$$H'(s) = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & s \\ -s & 0 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & \frac{-1}{s} \\ \frac{1}{s} & \frac{1}{s^2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{s} & \frac{1}{s^2} \end{bmatrix}$$

Realization for $N_R' D_R'^{-1}$

$A_c = ?$
$B_c = ?$
$C_c = ?$

Exercise

2) Calculate $\mathcal{C}$ and $\mathcal{O}$
→ both should be full rank.

3) Draw the interconnection diagram.

Smith-McMillan Form:

$$H(s) = \frac{N(s)}{d(s)}$$ [Assume the elements of $H(s)$ are in reduced form]

$d(s)$ = the monic lcm of the denominators of the entries of $H(s)$.

$$d(s)\,H(s) = N(s) = U_1(s)\,\Delta(s)\,U_2(s)$$

where $\{U_i(s)\}$ $i=1,2$ are unimodular

$\Delta(s)$ is in Smith form.

Then $$U_1^{-1}(s)\,H(s)\,U_2^{-1}(s) = \frac{\Delta(s)}{d(s)}$$

$$= \mathrm{diag}\left\{\frac{\lambda_i(s)}{d(s)}\right\}$$

Next, reduce the element of the rational matrix $\frac{\Delta(s)}{d(s)}$ to lowest terms

Say, $$\frac{\lambda_i(s)}{d(s)} = \frac{\varepsilon_i(s)}{\psi_i(s)}$$

where $\varepsilon_i(s)$, $\psi_i(s)$ are coprime

$i = 1, 2, \cdots, r$

r = rank of $H(s)$

Then $$H(s) = U_1(s)\,M(s)\,U_2(s)$$

$$M(s) = \begin{bmatrix} \frac{\varepsilon_1(s)}{\psi_1(s)} & & 0 & \vert & 0 \\ & \frac{\varepsilon_2(s)}{\psi_2(s)} & & \vert & \\ & & \ddots & \vert & \\ 0 & & \frac{\varepsilon_r(s)}{\psi_r(s)} & \vert & \\ \hline & 0 & & \vert & 0 \end{bmatrix}$$

Smith-Mcmillan form (Unique)

Also,
1) $\psi_{i+1}(s) \mid \psi_i(s)$
2) $\varepsilon_i(s) \mid \varepsilon_{i+1}(s)$ $\Big\}$ $i = 1, \cdots, r-1$
3) $d(s) = \psi_1(s)$

Proof of (3): Let $\psi_1(s) \neq d(s)$. Then from

$$\frac{\varepsilon_1(s)}{\psi_1(s)} = \frac{n_1(s)}{d(s)}$$

$n_1(s)$ and $d(s)$ must have a common factor.

$\Rightarrow$ $d(s)$ and all elements of $N(s)$ have a common factor
$\rightarrow$ contradiction.

Ex: $$H(s) = \begin{bmatrix} \frac{s}{(s+1)^2(s+2)^2} & \frac{s}{(s+2)^2} \\ \frac{-s}{(s+2)^2} & \frac{-s}{(s+2)^2} \end{bmatrix}$$

$$= \frac{1}{d(s)} \begin{bmatrix} s & s(s+1)^2 \\ -s(s+1)^2 & -s(s+1)^2 \end{bmatrix}$$

$$d(s) = (s+1)^2 (s+2)^2$$

$$H(s) = \begin{bmatrix} 1 & 0 \\ -(s+1)^2 & 1 \end{bmatrix} \begin{bmatrix} \frac{s}{(s+1)^2(s+2)^2} & 0 \\ 0 & \frac{s^2}{(s+2)} \end{bmatrix} \begin{bmatrix} 1 & (s+1)^2 \\ 0 & 1 \end{bmatrix}$$

Note: $H(s)$ is strictly proper but S-M form is improper.

Mcmillan degree

$$M(s) = \left[\begin{array}{ccc|c} \frac{\varepsilon_1(s)}{\psi_1(s)} & & & \\ & \ddots & & 0 \\ & & \frac{\varepsilon_r(s)}{\psi_r(s)} & \\ \hline & 0 & & 0 \end{array}\right]$$

$$= E(s)\, \psi_R^{-1}(s) = \psi_L^{-1}(s)\, E(s)$$

where $E(s) = \left[\begin{array}{ccc|c} \varepsilon_1(s) & & & \\ & \ddots & \varepsilon_r(s) & 0 \\ \hline & 0 & & 0 \end{array}\right]_{p \times m}$

$$\Psi_R(s) = \left[\begin{array}{ccc|c} \psi_1(s) & & & \\ & \ddots & & 0 \\ & & \psi_r(s) & \\ \hline & 0 & & I_{m-r} \end{array}\right] \quad m \times m$$

$$\Psi_c(s) = \left[\begin{array}{ccc|c} \psi_1(s) & & & \\ & \ddots & & 0 \\ & & \psi_r(s) & \\ \hline & 0 & & I_{p-r} \end{array}\right] \quad p \times p$$

Claim:

$E(s)$ and $\Psi_R(s)$ are coprime

Proof: Exercise

Now, $H(s) = U_1(s) M(s) U_2(s)$

$$= \underbrace{U_1(s) E(s)}_{N_0(s)} \Psi_R^{-1}(s) U_2(s)$$

Define: $N_0(s)$ $\quad D_0(s) = U_2^{-1}(s)\Psi_R(s)$

$H(s) = N_0(s) D_0^{-1}(s) \leftarrow$ irreducible MFD

By Thm above,

$$n_{min} = \deg \det D_0(s)$$

$$= \deg \det \Psi_R(s) \quad [\because U_2 \text{ unimodular}]$$

$$= \sum \deg \psi_i(s) =: \text{the McMillan degree of } H(s)$$

FACT: The (right or left) numerators of irreducible MFDs of $H(s)$ all have same Smith form

Proof: We showed earlier that

$$N_1(s) D_1^{-1}(s) = N_2(s) D_2^{-1}(s) \quad \text{(both irreducible)}$$

implies that
$$D_1(s) = D_2(s) U(s)$$
$$N_1(s) = N_2(s) U(s)$$
with $U(s)$ unimodular

Take $N_2(s) = E(s)$. $\Rightarrow$ $E(s)$ is the Smith form of any right numerator

Similarly for left.

FACT: The denominators of irreducible MFDs of $H(s)$ all have the same non-unity invariant polynomials.

Proof: Same logic as above proof with $D(s)$ and $\Psi_R(s)$ and $\Psi_L(s)$

\# This implies that the determinants are the same

Poles and Zeros of Transfer Matrices

Defn.: Zeros are the roots of the (non-zero) numerator polynomials $\{\varepsilon_i(s)\}$ in the Smith-McMillan form of $H(s)$

Def: The poles are the roots of the denominators in the S-M form. H(s)

For the example above:

$$M(s) = \begin{bmatrix} \frac{s}{(s+1)^2(s+2)^2} & 0 \\ 0 & \frac{s^2}{s+2} \end{bmatrix}$$

$\Rightarrow H(s)$ has 3 zeros at $s = 0$
Two (2) poles at $s = -1$
3 poles at $s = -2$

Note: $\varepsilon_i(s)$ and $\psi_j(s)$ $(i \neq j)$ can have common root

Irreducible MFDs: $H(s) = N(s) D^{-1}(s)$

→ poles of $H(s)$ are the roots of $\det D(s) = 0$ where $D(s)$ is the denominator of ANY irreducible MFD

→ If $H(s)$ is square + nonsingular then its zeros are the roots of $\det N(s) = 0$, ($N(s)$ is numerator of ANY irreducible MFD

→ In general: Zeros are the roots of the invariant polynomials of the numerator of *any* irreducible MFD of $H(s)$