EE 640 — Similarity Transforms

Note Title 18-06-2008

Review: Vector spaces, linear independence, basis, change of basis.

We have seen that, given an implementation / wiring diagram for a differential equation, we can write down a state representation of the form:

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

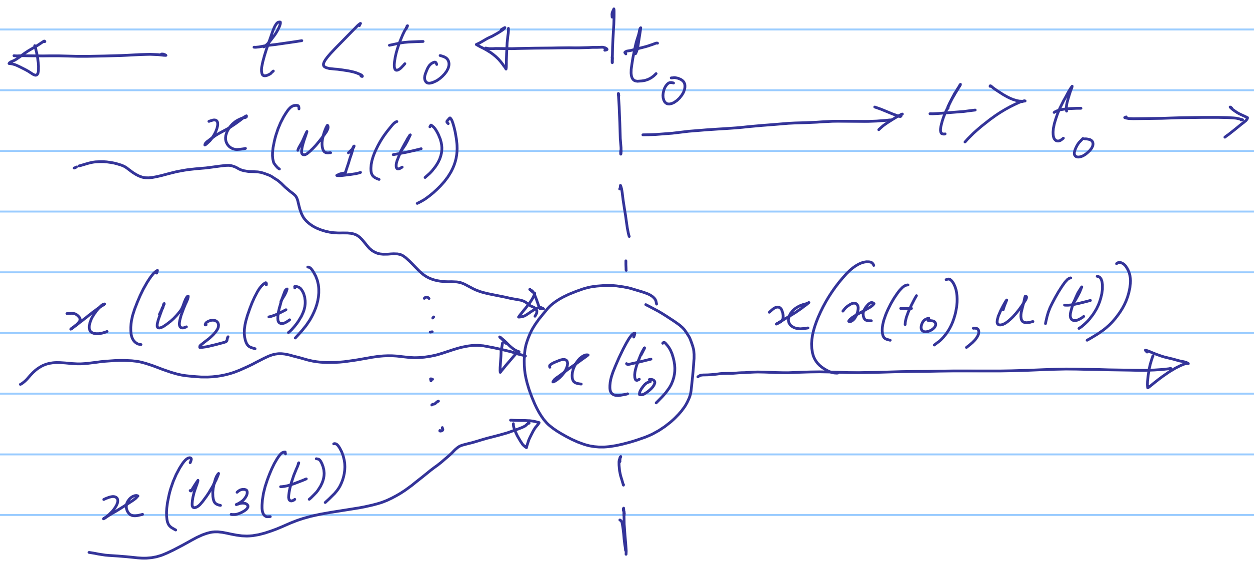
The inverse is also true: given a state representation we can draw a wiring diagram.

<u>FACT</u>: A state representation (or realization) is equivalent to an implementation diagram.

We had defined the state variables as the outputs of the integrators in a <u>particular</u> implementation.

Here is an abstract definition of "states of a system"

<u>DEFINITION</u>: The state of a system at time t_0 is the amount of information at t_0 that together with $u(t)$ for $t \geq t_0$, determines uniquely every response in the system for all $t \geq t_0$.



NOTE: 1) This definition obviously includes our earlier definition of state. (Think how the ANALOG COMP works.)

2) As we saw, a particular system or differential equation may have many different realizations. Each such realization produces its own set of state variables. Hence "states" of a system are non-unique.

So the phrase "the states of the system" is meaningless. One should instead say "the states of a realization".

The Non-Uniqueness of Realizations

We have seen that there are more than one implementation/realization for a diff. eqn.

Can we gain some general insight into this issue?

2) Can we create different realization from a given one?

One way of doing this is by using <u>Similarity Transformation</u>:

Consider the n-dim. realization

$$\left.\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned}\right\} \quad *$$

Let T be a $n \times n$ constant real matrix which is non-singular.

FACT: Non-singular $:= \det(T) \neq 0$
$\equiv T$ is full rank
$\equiv T$ is invertible $\equiv T^{-1}$ exists.

Define a new state vector

$$\bar{x}(t) = T^{-1}x(t) \quad \text{i.e.} \quad x(t) = T\bar{x}(t)$$

To find out the state equations for $\bar{x}$,

$$\begin{aligned} \dot{\bar{x}} &= T^{-1}\dot{x}(t) \\ &= T^{-1}[Ax + Bu] = T^{-1}Ax + T^{-1}Bu \\ &= [T^{-1}AT]\bar{x}(t) + [T^{-1}B]u(t) \end{aligned}$$

Similarly, $y = Cx = (CT)\bar{x}$

Define the new matrices:

$$\bar{A} = T^{-1}AT \qquad \bar{B} = T^{-1}B$$
$$\bar{C} = CT$$

The new realization is

$$\left.\begin{aligned} \dot{\bar{x}} &= \bar{A}\,\bar{x} + \bar{B}\,u \\ y &= \bar{C}\,\bar{x} \end{aligned}\right) \ast\ast$$

Note that y and u remain the same. But they are connected through a new "wiring diagram".

This realization is SIMILAR to the original realization and the transformation is called a SIMILARITY TRANSFORMATION.

Exercise: Check whether realization $\ast$ and $\ast\ast$ results in the same differential equation connecting y with u.

This method of creating new realization from old ones, answers our question about how many realizations are possible for a particular differential equation.

Since, in general, there are infinitely many T's available, similarity transformations with all such T's will create infinitely many realizations.

Simplification of a Realization

Q. Can we use similarity transformation to create realizations, which are simpler than the original?

E.g. We can ask:

Q. Can we diagonalize (Recall the parallel realization) any realization by similarity transformation?

A. NO in general. Let us investigate further.

This reduces to the following question:

Q Is there a matrix T such that $T^{-1}AT$ is diagonal?

FACT 1 (from Linear Algebra): A $n \times n$ matrix A is diagonalizable if and only if it has n linearly independent eigenvectors.

Example: $A = \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix}$

$$\det(\lambda I - A) = \det\begin{pmatrix} \lambda-1 & 2 \\ -1 & \lambda+1 \end{pmatrix} = \lambda^2 - 1 + 2 = \lambda^2 + 1$$

$\therefore \lambda = \pm i \implies$ Eigenvalues are distinct

Let t_1 and t_2 be the eigenvectors.

$$At_1 = (i)t_1 \iff \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} t_{11} \\ t_{12} \end{bmatrix} = \begin{bmatrix} i t_{11} \\ i t_{12} \end{bmatrix}$$

$$\left.\begin{aligned} t_{11} - 2t_{12} &= i t_{11} \\ t_{11} - t_{12} &= i t_{12} \end{aligned}\right\} t_1 = \begin{bmatrix} 0.8165 \\ 0.4082 - i\,0.4082 \end{bmatrix}$$

Similarly $t_2 = \begin{bmatrix} 0.8165 \\ 0.4082 + i\,0.4082 \end{bmatrix}$

Form $T = [t_1 \;\; t_2] = \begin{bmatrix} 0.8165 & 0.8165 \\ (0.4082 - i\,0.4082) & (0.4082 + i\,0.4082) \end{bmatrix}$

$$\therefore \Lambda = T^{-1}AT = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$$

We see that though this is diagonal, this cannot be treated as a realization (This cannot be implemented!).

The Diagonalization should be over the real numbers.

FACT 2: A $n \times n$ REAL matrix A is diagonalizable OVER THE REALS *iff* it has n linearly independent REAL eigenvectors.

An easier to check condition: (Though only sufficient)

FACT 3: A $n \times n$ real matrix A is diagonalizable over the reals *if* it has n real distinct eigenvalues.

Building the matrix T

Assume A has n linearly independent eigenvectors $t_1, t_2 \cdots t_n$

$$A t_i = \lambda_i t_i \qquad i = 1, 2, \cdots, n$$

where λ_i's are the corresponding eigenvalues. Build a matrix $T_{(n \times n)}$ with $t_1, t_2, \cdots, t_n$ as its columns

$$T = \begin{bmatrix} t_1 & t_2 & \cdots & t_n \end{bmatrix}_{n \times n}$$

Since $t_1, t_2, \cdots, t_n$ are linearly ind., T is invertible.

$$AT = A\begin{bmatrix} t_1 & t_2 & \cdots & t_n \end{bmatrix}$$

$$= \begin{bmatrix} At_1 & At_2 & \cdots & At_n \end{bmatrix}$$

$$= \begin{bmatrix} \lambda_1 t_1 & \lambda_2 t_2 & \cdots & \lambda_n t_n \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} t_1 & t_2 & \cdots & t_n \end{bmatrix}}_{T} \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}$$

$$= T \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

So $\bar{A} = T^{-1}AT = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}$

Example: $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ does not have 2 linearly independent eigen vectors

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Q. What if A does not have n real linearly ind. eigenvectors? What is the simplest (or near diagonal) realization that we can get?

Jordan Form: Let a real $n \times n$ matrix A have REAL but not necessarily distinct eigenvalues. Then there is a real invertible matrix T such that

$$A_J = T^{-1} A T$$

is of the following form:

$$A_J = \begin{bmatrix} A_{11} & & & 0 \\ & A_{22} & & \\ & & \ddots & \\ 0 & & & A_{kk} \end{bmatrix}$$ ← Jordan form.

where

$$A_{ii} = \begin{bmatrix} \lambda & & & & 0 \\ 1 & \lambda & & & \\ & 1 & \ddots & & \\ & & \ddots & \ddots & \\ 0 & & & 1 & \lambda \end{bmatrix}$$

where λ is an eigenvalue of A. (possibly repeated)

Example: Let A be a 9×9 real matrix with REAL but repeated eigenvalues:

λ_1 is repeated 6 times
λ_2 " " 2 "
λ_3 occurs only once

Then the corresponding Jordan form MAY look like:

$$A_J = \left[\begin{array}{c|c|c|c|c}
\begin{matrix}\lambda_1 & & 0\\ 1 & \lambda_1 & \\ 0 & 1 & \lambda_1\end{matrix} & 0 & 0 & 0 & 0 \\ \hline
0 & \begin{matrix}\lambda_1 & 0\\ 1 & \lambda_1\end{matrix} & 0 & 0 & 0 \\ \hline
0 & 0 & \lambda_1 & 0 & 0 \\ \hline
0 & 0 & 0 & \begin{matrix}\lambda_2 & 0\\ 1 & \lambda_2\end{matrix} & 0 \\ \hline
0 & 0 & 0 & 0 & \lambda_3
\end{array}\right]$$

The Real Canonical Form (This the most general form of "diagonalization" over the reals)

Let A be a real $n\times n$ matrix with eigenvalues (that may be complex and repeated), of the form λ's and $(a \pm ib)$'s.

Then there is a $n \times n$ real invertible matrix T such that $\bar{A} = T^{-1}AT$ consists of the following two type of diagonal blocks Δ_1 and Δ_2.

$$\Delta_1 = \begin{bmatrix} D & & & & 0 \\ I_2 & \ddots & & & \\ & \ddots & \ddots & & \\ & & \ddots & \ddots & \\ 0 & & & I_2 & D \end{bmatrix}$$

where $D = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ $\quad I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

and

$$\Delta_2 = \begin{bmatrix} \lambda & & & & 0 \\ 1 & \lambda & & & \\ & 1 & \ddots & & \\ & & \ddots & \ddots & \\ 0 & & & 1 & \lambda \end{bmatrix}$$

Example:

Let $$A = \begin{bmatrix} 0 & 0 & 0 & -8 \\ 1 & 0 & 0 & 16 \\ 0 & 1 & 0 & -14 \\ 0 & 0 & 1 & 6 \end{bmatrix}$$

The eigenvalues are $\{1 \pm i, 2, 2\}$

The real canonical form looks like:

$$\bar{A} = \left[\begin{array}{cc|cc} \multicolumn{2}{c|}{\overbrace{\qquad\quad}^{\Delta_2}} & & \\ 2 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ \hline 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 \\ & & \multicolumn{2}{c}{\underbrace{\qquad\quad}_{\Delta_1}} \end{array}\right]$$

Conclusion:

1) We saw how SIMILARITY TRANSFORMATION can create an infinite number of realizations from a given realization

2) Similarity transformation was used to create "simpler" realizations.
   - Under some conditions a diagonal realization was obtained
   - Otherwise "near" diagonal realizations were achieved.

3) How to compute them numerically OR how to compute the corresponding T matrices have not been studied here. The

main use for these "forms" for us are to better understand analytical properties of systems that would have been more difficult to understand with the original realizations.

Exercise: 1) "Similarity transformation is nothing but change of basis" — Explain.

2) In our effort to simplify a realization by similarity transforms, it is implied that the two related realizations have some common properties. (There is no point in simplification otherwise!). What properties are preserved?

<u>Self study</u>: The size of the largest Δ_1 and Δ_2 blocks are determined by the multiplicity of the corresponding eigenvalue in the minimal polynomial of A.