EE640 - 11 Stabilizable Systems

Note Title 24-07-2008

We have seen that every controllable system can be stabilized by applying state feedback

Q. What if the realization is not controllable?
Can we characterize all realizations that can be stabilized by state feedback?

<u>Defn</u>: A realization $\left\{\begin{array}{l}\dot{x}=Ax+bu\\ y=cx\end{array}\right\}$ is stabilizable if there is a "static" state feedback that makes it stable.

We already know that all controllable realizations are stabilizable.

So consider a un-controllable realization. $\left\{\begin{array}{l}\dot{x}=Ax+bu\\ y=cx\end{array}\right.$

Let $\operatorname{rank}[\mathcal{C}] = r < n$. Then we have seen that there is a similarity transformation T such that

$$\bar{A} = T^{-1}AT, \quad \bar{b} = T^{-1}b, \quad \bar{c} = cT$$

with
$$\bar{A} = \begin{bmatrix} \bar{A}_c & \bar{A}_{c\bar{c}} \\ 0 & \bar{A}_{\bar{c}} \end{bmatrix} \begin{matrix} \}r \\ \}n-r \end{matrix} \quad ; \quad \bar{b} = \begin{bmatrix} \bar{b}_c \\ 0 \end{bmatrix} \begin{matrix} \}r \\ \}n-r \end{matrix}$$

and $\quad \bar{c} = [\bar{c}_c, \bar{c}_{\bar{c}}]$

where $\quad \{\bar{A}_c, \bar{b}_c, \bar{c}_c\}$ is controllable.

Lets apply a state feedback to this realization

$$\bar{k} = [\underbrace{\bar{k}_c}_{r} \,\vdots\, \underbrace{\bar{k}_{\bar{c}}}_{n-r}]$$

So the new realization

$$\dot{\bar{x}} = (\bar{A} - \bar{b}\bar{k})\bar{x} + \bar{b}u$$

$$y = \bar{c}\bar{x}$$

Check $\quad \bar{b}\bar{k} = \begin{bmatrix} \bar{b}_c \\ 0 \end{bmatrix} [\bar{k}_c \;\; \bar{k}_{\bar{c}}] = \begin{array}{c} \left[\begin{array}{c|c} \bar{b}_c\bar{k}_c & \bar{b}_c\bar{k}_{\bar{c}} \\ \hline 0 & 0 \end{array}\right] \end{array} \begin{array}{l} r \\ n-r \end{array}$

So the new characteristic polynomial

$$a_k(s) = \det[sI - (A - bk)]$$

$$= \det\left[sI - \begin{pmatrix} \bar{A}_c & \bar{A}_{c\bar{c}} \\ 0 & \bar{A}_{\bar{c}} \end{pmatrix} + \begin{pmatrix} \bar{b}_c\bar{k}_c & \bar{b}_c\bar{k}_{\bar{c}} \\ 0 & 0 \end{pmatrix}\right]$$

$$= \det\begin{bmatrix} (sI_r - \bar{A}_c + \bar{b}_c\bar{k}_c) & -\bar{A}_{c\bar{c}} + \bar{b}_c\bar{k}_{\bar{c}} \\ 0 & (sI_{n-r} - \bar{A}_{\bar{c}}) \end{bmatrix}$$

$$= \underbrace{\det(sI_r - \bar{A}_c + \bar{b}_c\bar{k}_c)}_{a_1(s)} \underbrace{\det(sI_{n-r} - \bar{A}_{\bar{c}})}_{\substack{a_2(s) \\ \text{independent of } k}}$$

$$a_1(s) = \det\left(sI_r - \bar{A}_c + \bar{b}_c \bar{k}_c\right)$$

This is the characteristic polynomial obtained when applying state feedback $\bar{k}_c$ to the realization

$$\begin{cases} \dot{z} = \bar{A}_c z + \bar{b}_c u \\ y = \bar{C}_c z \end{cases}$$

Since this realization is controllable, the roots of $a_1(s)$ can be assigned arbitrarily by properly choosing $\bar{k}_c$

The roots of $a_2(s) = \det\left(sI_{n-r} - \bar{A}_{\bar{c}}\right)$ are the uncontrollable modes of the original realization. We can see, that they are unaffected by state feedback.

But for the realization to be stable, all the roots (of $a_1(s)$ & $a_2(s)$) must be stable.

This proves:

FACT: The realization is stabilizable iff all un-controllable modes are stable.

Finding the stabilizing feedback (when it exists)

1) Find T such that $\bar{A} = \left(\begin{array}{c|c} \bar{A}_c & \bar{A}_{c\bar{c}} \\ \hline 0 & \bar{A}_{\bar{c}} \end{array}\right)$, $\bar{b} = \begin{pmatrix} \bar{b}_c \\ 0 \end{pmatrix}$

$\bar{c} = \left(\bar{c}_c \quad \bar{c}_{\bar{c}} \right)$

2) Find a feedback $\bar{K}_c$ $(1 \times r)$ that stabilizes the r-dimensional realization

$$\left\{ \begin{array}{l} \dot{z} = \bar{A}_c z + \bar{b}_c u \\ y = \bar{c}_c z \end{array} \right\} \text{controllable}$$

3) Augment the $\bar{K}_c$ to dimension n

$\bar{k} = \left[\bar{K}_c \;\middle|\; \bar{k}_{\bar{c}} \right]$, where

$\bar{k}_{\bar{c}}$ is arbitrary (possibly $\bar{k}_{\bar{c}} = 0$)

4) Transform $\bar{k}$ back to the original realization

$$k = \bar{k}\, T^{-1}$$

<u>Example</u>:

$$\dot{x} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u$$

Find a stabilizing state feedback if there is one.

$$\mathcal{C} = [b \;\; Ab \;\; A^2 b] = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$\operatorname{rank}(\mathcal{C}) = 2 \;(= r) < n \;(= 3)$

Recall how to calculate the T matrix

$$T = \begin{bmatrix} T_1 & T_2 \end{bmatrix} \quad \left\{ T_1 = \begin{bmatrix} b & Ab \end{bmatrix} \ \& \ T_2 \text{ arbitrary} \right\}$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\bar{A} = T^{-1} A T = \left[\begin{array}{cc|c} \overbrace{\begin{matrix} 0 & -1 \\ 1 & 2 \end{matrix}}^{\bar{A}_c} & \overbrace{\begin{matrix} 1 \\ 0 \end{matrix}}^{\bar{A}_{c\bar{c}}} \\ \hline 0 \quad 0 & \underbrace{-1}_{\bar{A}_{c\bar{c}}} \end{array}\right] \qquad \bar{b} = \left[\begin{array}{c} \overbrace{\begin{matrix} 1 \\ 0 \end{matrix}}^{\bar{b}_c} \\ \hline 0 \end{array}\right]$$

Eigenvalue of $\bar{A}_{c\bar{c}} = -1 \equiv$ un-controllable mode

So this realization is stabilizable.

<u>Finding the stabilizing feedback</u>

$$\dot{z} = \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix} z + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

The characteristic polynomial

$$a(s) = \det(sI - \bar{A}_c) = s^2 - 2s + 1$$

$$a = \begin{bmatrix} -2 & 1 \end{bmatrix}$$

Desired eigenvalues: $\lambda_1 = -1$, $\lambda_2 = -1$

$$\alpha(s) = (s+1)^2 = s^2 + 2s + 1$$

$$\alpha = \begin{bmatrix} 2 & 1 \end{bmatrix}$$

$$\mathcal{C}_c = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; \quad a_- = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix};$$

$$\bar{k}_c = (\alpha - a)(a_-^T)^{-1} \mathcal{C}_c^{-1} = [4 \quad 8]$$

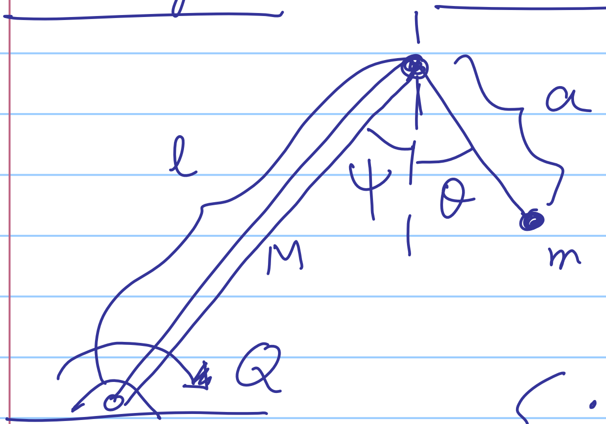
So augmenting $\bar{k}_c$ to make $\bar{k}$

$$\bar{k} = [\bar{k}_c \quad \bar{k}_{\bar{c}}] = [4 \quad 8 \mid 0]$$

The state feedback in the original realization:

$$k = \bar{k} T^{-1} = [4 \quad 8 \quad 0] T^{-1}$$
$$= [4 \quad 4 \quad 0]$$

Example: A double pendulum



We assume small angles. M, l, a, m have specific values so that the eqn of motion are:

$$\begin{cases} \ddot{\psi} - \frac{5}{2}\psi - \theta = Q \\ \ddot{\theta} + \theta + \ddot{\psi} = 0 \end{cases}$$

Design a feedback that stabilizes the system, if possible. Time constants after feedback ≤ 3 secs.

Step 1: Realization: $x_1 = \psi$, $x_2 = \dot{\psi}$
$x_3 = \theta$, $x_4 = \dot{\theta}$

$$\dot{x}_1 = \dot{\psi} = x_2$$

$$\dot{x}_2 = \ddot{\psi} = \frac{5}{2}\psi + \theta + Q = \frac{5}{2}x_1 + x_3 + Q$$

$$\dot{x}_3 = \dot{\theta} = x_4$$

$$\dot{x}_4 = \ddot{\theta} = -\theta - \ddot{\psi} = -\theta - \left[\frac{5}{2}\psi + \theta + Q\right]$$

$$= -\frac{5}{2}\psi - 2\theta - Q$$

$$= -\frac{5}{2}x_1 - 2x_3 - Q$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 5/2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -5/2 & 0 & -2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} Q$$

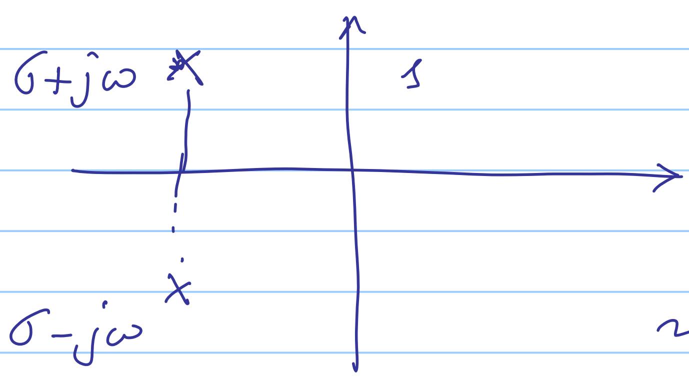
$$a(s) = s^4 - 0.5s^2 - 2.5$$

$$a = [0 \quad -0.5 \quad 0 \quad -2.5]$$

Check $\det(\mathcal{C}) \neq 0$ (MATLAB)

<u>How to choose new eigenvalues</u>:

Specs: stable + Time constant ≤ 3 secs



Contribution of such a pole to the time response

$$\sim e^{\sigma t}\sin(\omega t + \phi)$$

Time constant : $\frac{1}{|\sigma|}$. So we can take $\sigma \leq -\frac{1}{3}$

Let us choose 4 poles at :

$\lambda_{1,2} = -\frac{1}{3} \pm j\frac{3}{2}$; $s_{3,4} = -\frac{3}{2} \pm j\frac{1}{2}$

$$\alpha(s) = \left(s+\frac{1}{3}+j\frac{3}{2}\right)\left(s+\frac{1}{3}-j\frac{3}{2}\right)\left(s+\frac{3}{2}+j\frac{1}{2}\right)\left(s+\frac{3}{2}-j\frac{1}{2}\right)$$

$$= s^4 + 3.667s^3 + 6.861s^2 + 8.75s + 5.903$$

$$\alpha = [3.667 \quad 6.861 \quad 8.75 \quad 5.903]$$

The state feedback

$$k = (\alpha - a)(a_-^T)^{-1}\mathcal{C}^{-1}$$

$$= [8.75 \quad 5.083 \quad 8.403 \quad 1.042]$$