EE 640 : 13 - Discrete Systems

Note Title 24-07-2008

Most of the results derived until now hold also for Discrete realizations. We point out the similarities and differences:

A Discrete time realization

$$x_{k+1} = A x_k + b u_k \qquad x_0 \equiv \text{initial condition}$$
$$y_k = c x_k$$

The Z-transform

Given a sequence x_k,

$$X(z) = Z[x_k] := x_0 + z^{-1} x_1 + z^{-2} x_2 + \cdots \cdots$$
$$= \sum_{k=0}^{\infty} x_k z^{-k}$$

Shift : $Z(x_{k+1}) = z\left[X(z) - x_0\right]$

For $x_0 = 0$, $Z(x_{k+i}) = z^i X(z)$

Using this property, we can calculate the transfer function of the realization.

$$Y(z) = c(zI-A)^{-1} z x_0 + c(zI-A)^{-1} b U(z)$$

With $x_0 = 0$, $\dfrac{Y(z)}{U(z)} = c(zI-A)^{-1} b$

Difference Equations :

$$y_{k+n} + a_1 y_{k+n-1} + \cdots + a_n y_k$$
$$= b_0 u_{k+m} + b_1 u_{k+m-1} + \cdots + b_m u_k$$

T.F. from Diff. Eqn

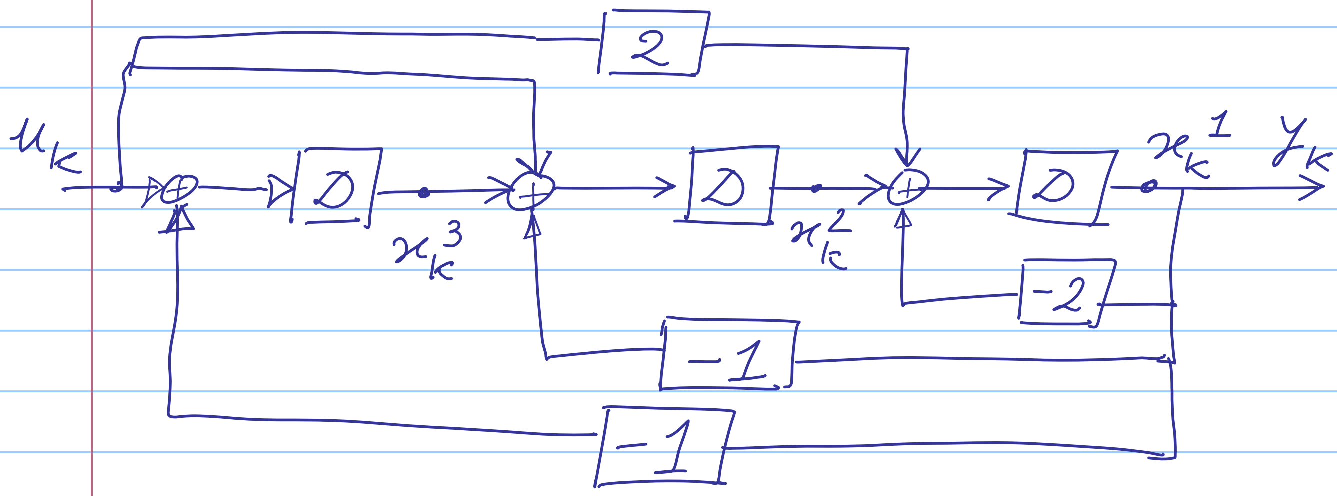
$$Z[y_{k+n} + a_1 y_{k+n-1} + \cdots + a_n y_k] = Z[b_0 u_{k+n} + \cdots + b_m u_k]$$

$$H(z) = \frac{b_0 z^m + b_1 z^{m-1} + \cdots + b_m}{z^n + a_1 z^{n-1} + \cdots + a_n}$$

Realization of a diffence Equation

Exactly the same as in continuous-time. Just use one-step delay D instead of integral

$$y_{k+3} + 2y_{k+2} + y_{k+1} + y_k = 2u_{k+2} + u_{k+1} + u_k$$



State Equations

$$\begin{bmatrix} x_{k+1}^1 \\ x_{k+1}^2 \\ x_{k+1}^3 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ -1 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_k^1 \\ x_k^2 \\ x_k^3 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} u_k$$

$$y_k = [1 \quad 0 \quad 0] \, x_k$$

Asymptotic Stability

$$\left.\begin{aligned} x_{k+1} &= A x_k + B u_k \\ y_k &= C x_k \\ x_0 &= x^* \end{aligned}\right\} (*)$$

$(*)$ is A.S. if the solution to

$x_{k+1} = A x_k$ satisfies $\lim_{k \to \infty} \|x_k\| = 0$
for all $x^* \in \mathbb{R}^n$

NOTE : $x_k = A^k x^*$

If A is diagonal, $\lim_{k \to \infty} \|x_k\| = 0$ iff

$$\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \qquad |\lambda_i| < 1 \text{ for all } i$$

In general:

FACT: $(*)$ is A.S. iff all eigenvalues of A lie strictly inside the unit circle on the complex plane.

Observability

Given $\begin{cases} x_{k+1} = Ax_k + bu_k \\ y_k = cx_k \end{cases}$ find x_k using only y and u

$$\begin{bmatrix} y_k \\ y_{k+1} \\ \vdots \\ y_{k+n-1} \end{bmatrix} = \underbrace{\begin{bmatrix} c \\ cA \\ \vdots \\ cA^{n-1} \end{bmatrix}}_{\mathcal{O}} x_k + \underbrace{\begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ cb & 0 & \cdots & 0 & 0 \\ & & & & \\ cA^{n-2}b & \cdots & \cdots & cb & 0 \end{bmatrix}}_{T} \begin{bmatrix} u_k \\ u_{k+1} \\ \vdots \\ u_{k+n-1} \end{bmatrix}$$

$$\boxed{\mathcal{Y}_k = \mathcal{O} x_k + T\mathcal{U}_k}$$

<u>NOTE:</u> $\mathcal{Y}_k$ and $\mathcal{U}_k$ contain <u>future</u> input/output data, so this formula cannot be used to find the present state.

Observable $\equiv$ $\mathcal{O}$ is full rank.

For an observable realization, we can calculate the state from future input/output data.

$$x_k = \mathcal{O}^{-1}\left[\underline{\mathcal{Y}_k - T\mathcal{U}_k}\right]$$

Reachability & Controllability of Discrete time systems

Given: $x_{k+1} = Ax_k + bu_k$
$y_k = cx_k$

Q. Starting from any state (x_0) can we reach any state?

$$x_1 = Ax_0 + bu_0$$
$$x_2 = A^2x_0 + Abu_0 + bu_1$$
$$\vdots$$
$$x_n = A^n x_0 + A^{n-1}bu_0 + A^{n-2}bu_1 + \cdots + bu_{n-1}$$
$$= A^n x_0 + \underbrace{\begin{bmatrix} b & Ab & \cdots & A^{n-1}b \end{bmatrix}}_{\mathbb{C}} \begin{bmatrix} u_{n-1} \\ u_{n-2} \\ \vdots \\ u_0 \end{bmatrix}$$

$$\boxed{x_n = A^n x_0 + \mathbb{C} \begin{bmatrix} u_{n-1} \\ \vdots \\ u_0 \end{bmatrix}} \quad (*)$$

From (*), we can reach any state at step n iff the controllability matrix is full rank.

Then $$\begin{bmatrix} u_{n-1} \\ \vdots \\ u_0 \end{bmatrix} = \mathbb{C}^{-1} \begin{bmatrix} x_n - A^n x_0 \end{bmatrix}$$

Important Diff from Continuous Time Case

The above requirement can be weakened if $x_n = 0$.

Q. Can we reach $x_n = 0$ starting from arbitrary initial condition??

From (*) above, we require

$$A^n x_0 + \mathcal{C}\begin{bmatrix} u_{n-1} \\ \vdots \\ u_0 \end{bmatrix} = 0$$

or

$$\mathcal{C}\begin{bmatrix} u_{n-1} \\ \vdots \\ u_0 \end{bmatrix} = -A^n x_0 \quad --- \quad (**)$$

For any x_0, there is a solution $[u_0 \, --- \, u_{n-1}]^T$ to (**) iff

$$\operatorname{Im} A^n \subset \operatorname{Im} \mathcal{C}.$$

So we rename these two cases.

Reachability $\equiv$ Can we reach any state from any state?

Controllability $\equiv$ Can we reach "0", starting from any state?

$$\boxed{\begin{array}{l}\text{Reachability} \iff \mathcal{C} \text{ is full rank} \\ \text{Controllability} \iff \operatorname{Im} A^n \subset \operatorname{Im} \mathcal{C}\end{array}}$$

So it is easy to see:

Reachability $\Rightarrow$ Controllability

In general, $\not\Leftarrow$

FACT: if A is full rank, then the realization is controllable iff it is reachable

FACT: A continuous time system is reachable iff it is controllable.

Proof: Exercise (Hint: e^{At} is always invertible)

Observability & Constructibility

A realization
$$x_{k+1} = A x_k + b u_k$$
$$y_k = c x_k$$

is constructible if x_k can be calculated uniquely using past input/output data. $(\dots, y_{k-2}, y_{k-1}, y_k \mid \dots u_{k-2}, u_{k-1}, u_k)$

Linear Algebra, Given a matrix B

$$\text{Ker } B = \{ x \mid Bx = 0 \}$$

Given a system of linear equations $BX = C$ and given one sol^n x_0 ie $Bx_0 = C$, any solution is of the form $x = x_0 + \theta$ where $\theta \in \text{Ker } B$

Recall,

$$\begin{bmatrix} y_0 \\ \vdots \\ y_{n-1} \end{bmatrix} = \mathcal{O} x_0 + T \begin{bmatrix} u_{n-1} \\ \vdots \\ u_0 \end{bmatrix} \quad \text{—— (1)}$$

and

$$x_n = A^n x_0 + \mathcal{C} \begin{bmatrix} u_{n-1} \\ \vdots \\ u_0 \end{bmatrix} \quad \text{—— (2)}$$

Rewrite (1),

$$\mathcal{O} x_0 = \begin{pmatrix} y_0 \\ \vdots \\ y_{n-1} \end{pmatrix} - T \begin{pmatrix} u_{n-1} \\ \vdots \\ u_0 \end{pmatrix}$$

Let $x_0 = x_*$ be one solution, then any solution is of the form,

$x_0 = x_p - \nu$ where $\nu \in \operatorname{Ker} \mathcal{O}$

Let's substitute this x_0 in (2),

$$x_n = A^n(x_p - \nu) + \mathcal{C}\begin{bmatrix} u_{n-1} \\ \vdots \\ u_0 \end{bmatrix}$$

or $$x_n = A^n x_p - A^n \nu + \mathcal{C}\begin{bmatrix} u_{n-1} \\ \vdots \\ u_0 \end{bmatrix}$$

If $A^n \nu = 0$ for all $\nu \in \operatorname{Ker} \mathcal{O}$ then x_n is unique. (even though x_0 was not unique)

So we need, $\nu \in \operatorname{Ker} A^n$ for every $\nu \in \operatorname{Ker} \mathcal{O}$.

or $$\boxed{\operatorname{Ker} \mathcal{O} \subset \operatorname{Ker} A^n}$$

FACT: The realization is constructible iff $\operatorname{Ker} \mathcal{O} \subset \operatorname{Ker} A^n$

Observable $\equiv \operatorname{Ker} \mathcal{O} = 0$

So Observability $\Rightarrow$ constructible

In general, $\not\Leftarrow$

FACT: If A is full rank, then constructibility $\Rightarrow$ observability

FACT: For continuous time systems constructibility $\equiv$ observability.