EE 640 – The Transfer Function

Note Title 20-06-2008

The Transfer Function of a Realization

We have seen that a LTI system (i.e. a ODE) may be represented by
(i) A transfer function
(ii) State Variable Representations

Q) How are these two related?

The transfer function of a state Representation:

Suppose we have a realization:

$$* \begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases}$$ using the usual notation.

Define $X(s) = \underbrace{\mathcal{L}\{x(t)\}}_{\text{One-sided Laplace Transform of } x(t)} = \int_{0^-}^{\infty} x(t) e^{-st} dt$

One-sided Laplace Transform of $x(t)$

Similarly $U(s) = \mathcal{L}\{u(t)\} = \int_{0^-}^{\infty} u(t) e^{-st} dt$

From $*$ we have:

$$\mathcal{L}\{\dot{x}(t)\} = A\,\mathcal{L}\{x(t)\} + B\,\mathcal{L}\{u(t)\}$$

or $sX(s) - X(0^-) = AX(s) + BU(s)$

or $\underline{(sI - A)X(s) = x(0)} + \underline{BU(s)}$

$$** \quad \therefore \boxed{X(s) = (sI-A)^{-1}x(0) + (sI-A)^{-1}BU(s)}$$

(Note that we are conveniently bypassing questions about invertibility of $(sI-A)$!)

Now consider the L.T. of the output equation:

$$Y(s) = \mathcal{L}\{y(t)\} = \mathcal{L}\{Cx(t)\} = CX(s)$$

Using (**),

$$Y(s) = C(sI-A)^{-1}x(0) + C(sI-A)^{-1}BU(s)$$

For getting the transfer function matrix or transfer matrix, set $x(0)=0$

Then $Y(s) = C(sI-A)^{-1}BU(s)$

or $Y(s) = G(s)\,U(s)$

where $G(s) := C(sI-A)^{-1}B$

For a SISO system $G(s)$ is a scalar transfer function.

It should be clear that the state space representation (*) and the equation (**) may be equivalently studied to answer any questions about the original system.
For example: if we are interested in computing the trajectories of

$x(t)$ it can be calculated directly from (*) or through (**) and then use an Inverse L.T.

- Let us consider SISO systems for the rest of this section

Then $\dfrac{Y(s)}{U(s)} = G(s) = C(sI-A)^{-1}B$

(**) $$\boxed{= \underbrace{\frac{1}{\det(sI-A)}}_{1\times 1} \cdot \underbrace{C}_{1\times n} \underbrace{\left[\operatorname{adj}(sI-A)\right]}_{n\times n} \underbrace{B}_{n\times 1}}$$

Digression :

Let B be a $n\times n$ matrix

$$B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & & & \\ b_{n1} & \cdots & \cdots & b_{nn} \end{bmatrix}$$

A minor determinant of B is the determinant of the $(n-1)\times(n-1)$ matrix obtained when one row & one column of B are deleted.

The minor determinant

$$m_{ij} = \det\left[B \text{ minus row } i, \text{column } j\right]$$

The cofactor B_{ij} of the entry b_{ij} of B is

$$B_{ij} = (-1)^{i+j} m_{ij}$$

The adjoint matrix of B is

$$\operatorname{adj}(B) = \begin{bmatrix} B_{11} & B_{12} & \cdots & B_{1n} \\ B_{21} & B_{22} & \cdots & B_{2n} \\ \vdots & & & \vdots \\ B_{n1} & B_{n2} & \cdots & B_{nn} \end{bmatrix}^T$$

Then, $$B^{-1} = \frac{1}{\det B} \operatorname{adj}(B)$$

NOTE that the term $\det(sI-A)$ occurs in the denominator of the transfer function and happens to be of special interest to us.

In fact we give it a name:

Characteristic Polynomial

Let A be a $n \times n$ matrix. The characteristic polynomial of A

is $a(s) = \det(sI - A)$

$$= s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots \cdots + a_n$$

[Exercise: This answers the doubt we had earlier about inverting $(sI-A)$. How?]

We can factor $a(s)$ over the complex numbers (but not always over the reals)

$$a(s) = (s-\lambda_1)(s-\lambda_2)\cdots\cdots(s-\lambda_n)$$

Then $(\lambda_1, \lambda_2, \cdots, \lambda_n)$ are the eigenvalues of A.

Exercise: An eigenvalue of A is also defined as a scalar corresponding to which there is a non-zero vector v so that

$$Av = \lambda v$$

Here v is an eigenvector of A corresponding to the eigenvalue λ.

How are these two definitions same?

So, we have a fairly good idea about the denominator of $(**)$.
What about the numerator.

The resolvent formula

$$\text{adj}(sI-A) = A^{n-1} + (s+a_1)A^{n-2}$$
$$+ (s^2+a_1 s+a_2)A^{n-3} + \cdots$$
$$\cdots + (s^{n-1} + a_1 s^{n-2} + \cdots + a_{n-1})I$$

OR, re-arranging the terms:

$$\text{adj}(sI-A) = s^{n-1}I + s^{n-2}(A+a_1 I)$$
$$+ s^{n-3}(A^2 + a_1 A + a_2 I) + \cdots$$
$$\cdots + (A^{n-1} + a_1 A^{n-2} + \cdots + a_{n-1} I)$$

Using this formula let us simplify equation $(**)$.

$$G(s) = \frac{1}{\det(sI-A)} C[\text{adj}(sI-A)]B$$

$$= \frac{1}{s^n + a_1 s^{n-1} + \cdots + a_n} C\Big[s^{n-1}I + s^{n-2}(A+a_1 I)$$
$$+ \cdots + (A^{n-1} + \cdots + a_{n-1} I)\Big]B$$

Note : $CB \equiv$ scalar

$CA^iB \equiv$ scalar

Hence,

$$G(s) = \frac{s^{n-1}(CB) + s^{n-2}(CAB + a_1 CB) + \cdots\cdots}{s^n + a_1 s^{n-1} + \cdots\cdots + a_n}$$

$$= \frac{\text{polynomial of degree} \leq n-1}{\text{polynomial of degree} = n}.$$

$$=: \frac{b(s)}{a(s)}$$ (i.e. name the numerator $b(s)$)

Now cancel all the common polynomial factors between $b(s)$ and $a(s)$ to get

$$G(s) = \frac{b_r(s)}{a_r(s)}$$

<u>Def$\underline{\text{n}}$</u> : The roots of $a_r(s) = 0$ are the <u>poles</u> of $G(s)$

<u>Def$\underline{\text{n}}$</u> : The roots of $b_r(s) = 0$ are the <u>zeros</u> of $G(s)$

Since $a_r(s)$ is a factor of the characteristic polynomial, <u>the poles of $G(s)$ are eigenvalues of the</u> matrix A.

If $a_r(s) \neq a(s)$, then

$$\deg(a_r(s)) < \deg(a(s))$$

The degree of $a_r(s)$ gives us the minimum number of integrators needed to implement $G_r(s)$.

So if $\deg a_r(s) < \deg a(s)$, then the same transfer function could have been implemented with less than n integrators.

If $\deg a_r(s) = \deg a(s)$ [i.e. there are no cancellations] then we say that the realization is a <u>minimal realization</u>.

<u>Impulse Response of $G_r(s)$</u>

We have seen that the transfer function $G_r(s)$ has already helped us to identify the minimum number of integrators needed to implement the underlying differential equation.

Another important use of $G_r(s)$ is to analytically calculate time domain responses of the realization.

Let us further investigate the quantity

$(sI-A)^{-1}$ in the formula for $G(s)$.

Digression: Geometric Series

$$(1-\alpha)^{-1} = 1+\alpha+\alpha^2+\alpha^3+ \cdots$$

In this form the geometric series expansion is valid for a $n \times n$ matrix.

$$(sI-A)^{-1} = \frac{1}{s}\left(I - \frac{1}{s}\cdot A\right)^{-1}$$
$$= \frac{1}{s}\left[I + \frac{1}{s}A + \frac{1}{s^2}A^2 + \cdots\right]$$
$$= \frac{1}{s}I + \frac{1}{s^2}A + \frac{1}{s^3}A^2 + \cdots$$

Now apply the inverse Laplace Transform

$$\mathcal{L}^{-1}(sI-A)^{-1} = \mathcal{L}^{-1}\left(\frac{1}{s}\right)I + \mathcal{L}^{-1}\left(\frac{1}{s^2}\right)A$$
$$+ \mathcal{L}^{-1}\left(\frac{1}{s^3}\right)A^2 + \cdots$$

[Recall that $\mathcal{L}^{-1}\left(\frac{1}{s^{n+1}}\right) = \frac{t^n}{n!}$, $t \geq 0$]

$$= I + tA + \frac{t^2}{2!}A^2 + \frac{t^3}{3!}A^3 + \cdots$$

Now where have we seen a similar form? Recall the series expansion of e^α for a scalar α.

$$e^{\alpha} = 1 + \alpha + \frac{\alpha^2}{2!} + \frac{\alpha^3}{3!} + \cdots$$

This looks so similar to the above formula, that we <u>DEFINE</u> the <u>matrix exponential</u>:

$$e^{At} := I + At + A^2\frac{t^2}{2!} + A^3\frac{t^3}{3!} + \cdots$$

$$= \sum_{i=0}^{\infty} \frac{t^i}{i!} A^i$$

From the above formula, for a $n \times n$ matrix A, e^{At} is also a $n \times n$ matrix.

$$\boxed{\mathcal{L}^{-1}\left\{(sI-A)^{-1}\right\} = e^{At}} \quad t \geq 0$$

<u>The Impulse Response</u>: Using the above formula for $G(s)$, the impulse response follows easily:

$$\mathcal{L}\{\delta(t)\} = 1$$

$$Y(s) = G(s) \cdot 1 = C(sI-A)^{-1}B$$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\{C(sI-A)^{-1}B\}$$

$$= C\mathcal{L}^{-1}\{(sI-A)^{-1}\}B$$

$$= Ce^{At}B$$

The impulse response is often denoted by $h(t)$. So

$$\boxed{h(t) = Ce^{At}B}$$

Invariants under Similarity Transformation

We are now in a position to answer a question we had asked in the previous chapter: namely, what remains the same for two similar realizations. We will check some of the quantities we have learned until now.

Claim 1: Similar realizations have same characteristic polynomial.

Proof: Consider the two realizations:

$$(1)\begin{cases} \dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}u \\ y = \bar{C}\bar{x} \end{cases} \qquad \begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases}(2)$$

Connected by a similarity transform T

ie $\quad x = T\bar{x} \iff \bar{A} = T^{-1}AT$

Consider the characteristic polynomial of (1):

$$c(s) = \det(sI - \bar{A})$$

$$= \det (sI - T^{-1}AT)$$
$$= \det (sT^{-1}IT - T^{-1}AT)$$
$$= \det [T^{-1}(sI - A)T]$$
$$= (\det T^{-1})(\det(sI - A))(\det T)$$
$$= \underbrace{(\det T^{-1})(\det T)}_{=1} \det(sI - A)$$
$$= \det(sI - A)$$

$= a(s) \equiv$ the characteristic polynomial of realization (2).

Claim 2: The transfer function (of the realization) is invariant under a similarity transformation

Let
$$G_1(s) = c(sI - A)^{-1}B$$
$$G_2(s) = \bar{c}(sI - \bar{A})^{-1}\bar{B}$$

where $\bar{A} = T^{-1}AT$; $\bar{B} = T^{-1}B$
$\bar{c} = cT$

$$G_2(s) = \bar{c}(sI - \bar{A})^{-1}\bar{B}$$
$$= \bar{c}\left[\frac{1}{s}I + \frac{1}{s^2}\bar{A} + \frac{1}{s^3}\bar{A}^2 + \dots\right]\bar{B}$$
$$= cT\left[\frac{1}{s}I + \frac{1}{s^2}(T^{-1}AT) + \frac{1}{s^3}(T^{-1}AT)^2 +\right.$$

$$\cdots\cdots\Big] T^{-1} B$$

<u>Note that:</u>

$$(T^{-1}AT)^i = \underbrace{(T^{-1}AT)(T^{-1}AT)\cdots\cdots(T^{-1}AT)}_{i \text{ times}}$$

$$= T^{-1}A^iT.$$

Hence

$$H(s) = CT\Big[\frac{1}{s}I + \frac{1}{s^2}(T^{-1}AT) + \frac{1}{s^3}(T^{-1}A^2T)$$
$$+ \cdots\cdots\Big] T^{-1}B$$

$$= C\Big[\frac{1}{s}(TT^{-1}) + \frac{1}{s^2}T(T^{-1}AT)T^{-1} + \cdots\cdots\Big]T^{-1}B$$

$$= C\Big[\frac{1}{s}I + \frac{1}{s^2}A + \frac{1}{s^3}A^2 + \cdots\cdots\Big]B$$

$$= C(sI-A)^{-1}B \equiv \underline{G_1(s)}$$

<u>Some Nomenclature</u> : <u>Markov Parameters</u>

Consider the formula for

$$G(s) = C\Big[\frac{1}{s}I + \frac{1}{s^2}A + \frac{1}{s^3}A^2 + \cdots\Big]B$$

$$= \frac{1}{s}(CB) + \frac{1}{s^2}(CAB) + \frac{1}{s^3}(CA^2B) + \cdots\cdots$$

The scalar sequence :

$CB, CAB, CA^2B, \ldots\ldots$

are called Markov Parameters m_i

$$\boxed{m_i = CA^{i-1}B} \quad i = 1, 2, \ldots\ldots$$

<u>Exercise</u> : Prove that the Markov parameters are invariant under similarity transformation.

<u>Claim 3</u> : The impulse response $h(t)$ is invariant under similarity transformation.

<u>HINT</u>: The impulse response

$$h(t) = \mathcal{L}^{-1}\{G(s) \cdot 1\}$$

$$= \mathcal{L}^{-1}\left[C\left(\frac{1}{s}I + \frac{1}{s^2}A + \frac{1}{s^3}A^2 + \ldots\right)B\right]$$

<u>Some related Linear Algebra FACTS</u>

1) A (Strange) property of the characteristic polynomial.

Suppose we have a polynomial

$$p(s) = p_0 s^m + p_1 s^{m-1} + \cdots + p_{m-1} s + p_m$$

We CAN substitute the $n \times n$ matrix A for s:

$$p(A) = p_0 A^m + p_1 A^{m-1} + \cdots + p_{m-1} A + p_m I$$

FACT (Cayley Hamilton Theorem):

Let A be a $n \times n$ matrix with characteristic polynomial $a(s)$. Then

$$a(A) = O_{n \times n}$$

namely, if $a(s) = s^n + a_1 s^{n-1} + \cdots + a_n$
then $a(A) = A^n + a_1 A^{n-1} + \cdots + a_n I = O_{n \times n}$

Proof: Let $B = \mathrm{adj}(sI - A)$

$$(sI - A) \cdot B = \det(sI - A) I = a(s) I$$

We have seen that (Recall the resolvent form.)

$$B = \sum_{i=0}^{n-1} s^i B_i$$

Hence, $a(s) I = (sI - A) B$

$$= (sI - A) \sum_{i=0}^{n-1} s^i B_i$$

$$= \sum_{i=0}^{n-1} s^{i+1} B_i - \sum_{i=0}^{n-1} s^i A B_i$$

$$= s^n B_{n-1} + \sum_{i=1}^{n-1} s^i \left(B_{i-1} - AB_i\right) - AB_0$$

$$= s^n B_{n-1} + s^{n-1}\left(B_{n-2} - AB_{n-1}\right) + \cdots$$

$$+ s\left(B_0 - AB_1\right) - AB_0 \quad \text{——} \quad (1)$$

Now, $a(s) = s^n + a_1 s^{n-1} + \cdots + a_n$

or $a(s) I = s^n I + s^{n-1} a_1 I + \cdots + a_n I$ —— (2)

Matching coefficients in (1) & (2),

$$\left.\begin{array}{l} B_{n-1} = I \\ B_{n-2} - AB_{n-1} = a_1 \\ B_{n-3} - AB_{n-1} = a_2 \\ \vdots \\ B_0 - AB_1 = a_{n-1} \\ -AB_0 = a_n \end{array}\right] \begin{array}{l} \times A^n \\ \times A^{n-1} \\ \times A^{n-2} \\ \\ \times A \\ \times I \end{array}$$

$$0 = A^n + a_1 A^{n-1} + a_2 A^{n-2} + \cdots + a_n I$$

An Immediate Consequence :

ANY power of A is a linear combination of $A^{n-1}, A^{n-2}, \ldots, A, I$

Proof: $A^n = -a_1 A^{n-1} - \cdots - a_n I$

Then $A^{n+1} = A A^n$

$$= A\left[-a_1 A^{n-1} - \cdots\cdots - a_n I\right]$$

$$= -a_1 A^n - a_2 A^{n-1} - \cdots\cdots - a_n A$$

$$= -a_1\left[-a_1 A^{n-1} \cdots\cdots - a_n I\right] - a_2 A^{n-1} \cdots\cdots - \cdots\cdots - a_n A$$

Similarly for any powers of A.

2) <u>The minimal polynomial</u>

We saw that :

$$(sI - A)^{-1} = \underbrace{\frac{1}{\det(sI-A)}}_{\text{polynomial}} \underbrace{\text{adj}(sI-A)}_{\text{polynomial matrix}}$$

In general, we could have factors of the characteristic polynomial that are common to all entries of the polynomial matrix $\text{adj}(sI-A)$

Cancelling all these common factors, we get

$$(sI - A)^{-1} = \underbrace{\frac{1}{\mu(s)}}_{\text{polynomial}} \underbrace{\Gamma(s)}_{\text{polynomial matrix}}$$

$\mu(s) =:$ The minimal polynomial of A.

FACT: The minimal polynomial $\mu(s)$ is the polynomial of least degree so that

$$\mu(A) = 0_{n \times n}$$

FACT: By Cayley Hamilton thm, the minimal polynomial always a factor of the characteristic polynomial.