EE 640-5 Observability

Note Title 22-06-2008

Review: Span, Basis, Rank, Null Space

Recall, we learnt to implement a differential equation:

$$y^{(n)}(t) + a_1 y^{(n-1)} + \cdots + a_n y(t)$$
$$= b_0 u^{(m)}(t) + b_1 u^{(m-1)}(t) + \cdots + b_m u(t)$$

We built an ANALOG COMP. simulation with n integrators and named the output of the integrators as $(x_1, \ldots, x_n)$. From there we got the state space realization

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

So can we now actually start simulating?

$\Rightarrow$ NO, because we have not calculated the initial conditions $x_1(0), x_2(0), \ldots x_n(0)$.

QUESTION: How do we determine the initial conditions

$$x_1(0^-), x_2(0^-), \ldots, x_n(0^-)$$

of the integrators from the given initial conditions of the differential equation:

$$y(0^-), \dot{y}(0^-), \ldots, y^{(n-1)}(0^-)$$

Let us look at the output equation:

$$
(*)\quad
\begin{cases}
y = Cx \\
\dot{y} = C\dot{x} = C[Ax + Bu] = CAx + CBu \\
\ddot{y} = CA\dot{x} + CB\dot{u} = CA[Ax+Bu] + CB\dot{u} \\
\quad = CA^2x + CABu + CB\dot{u} \\
\\
\dddot{y} = CA^2[Ax+Bu] + CAB\dot{u} + CB\ddot{u} \\
\quad = CA^3x + CA^2Bu + CAB\dot{u} + CB\ddot{u} \\
\vdots \\
y^{(n-1)} = CA^{n-1}x + CA^{n-2}Bu + CA^{n-3}B\dot{u} \\
\qquad + \cdots\cdots + CBu^{n-2}
\end{cases}
$$

To write in vector form, define

$$
\mathcal{Y}(t) := \begin{bmatrix} y(t) \\ \dot{y}(t) \\ \vdots \\ y^{(n-1)}(t) \end{bmatrix}_{n\times 1} ; \quad \mathcal{O} := \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}
$$

For a SISO system, y is a scalar, $\Rightarrow$ C is $1\times n$, so all the terms C, CA, ..., CA^{n-1} are $1\times n$. Then $\mathcal{O}$ is $n\times n$.

$$
\mathcal{U}(t) = \begin{bmatrix} u(t) \\ \dot{u}(t) \\ \vdots \\ u^{n-1}(t) \end{bmatrix} \quad \text{is} \quad n\times 1 .
$$

In terms of these notation; (*) looks like:

$$
Y(t) = \begin{bmatrix} y \\ \dot{y} \\ \vdots \\ y^{(n-1)} \end{bmatrix} = \underbrace{\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}}_{\mathcal{O}} x(t) + \underbrace{\begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ CB & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ CA^{n-2}B & CA^{n-3}B & \cdots & CB & 0 \end{bmatrix}}_{T} \underbrace{\begin{bmatrix} u(t) \\ \dot{u}(t) \\ \vdots \\ u^{(n-1)}(t) \end{bmatrix}}_{U(t)}
$$

Define $T := \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ CB & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ CA^{n-2}B & CA^{n-3}B & \cdots & CB & 0 \end{bmatrix}$

Using these quantities,

$$\boxed{Y(t) = \mathcal{O} x(t) + T\, U(t)} \quad \cdots (**)$$

For $t = 0^-$: $u(0^-) = 0$

$\dot{u}(0^-) = 0$

$\vdots$

$u^{(n-2)}(0^-) = 0$

So $\mathcal{U}(0^-) = 0_{n \times 1}$

$$\Rightarrow \boxed{\mathcal{Y}(0^-) = \mathcal{O}\, x(0^-)} \quad \text{------} \quad (\mathcal{X}\mathcal{X})$$

This is the direct relationship between the given initial conditions of the differential equation and that of the integrators of the implementation.

The matrix $\mathcal{O} = \underbrace{\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}}_{(n \times n)}$ is called the <u>OBSERVABILITY MATRIX</u>

Q. Given $\mathcal{Y}(0^-)$ when can $x(0^-)$ be found from $(\mathcal{X}\mathcal{X})$?

Ans : If <u>$\mathcal{O}$ is invertible</u>, then

$$x(0^-) = \mathcal{O}^{-1} \mathcal{Y}(0^-)$$

FACT: $\mathcal{O}$ is invertible $\equiv$ $\mathcal{O}$ is full rank
$\equiv \det(\mathcal{O}) \neq 0$

DEFINITION: RANK of a $m \times n$ matrix A is maximal number of linearly independent columns/rows of A.

FACT: In a n-dim vector space, any set of n lin. ind. vectors form a basis.

Hence, if $\mathcal{O}$ has rank n, i.e. there are at least n linearly independent columns $\Longleftrightarrow$ they form a basis and span the entire n-dimensional space.

So, given any $Y(0^-)$, we can find a $x(0^-)$ such that

$$Y(0^-) = \mathcal{O}x(0^-)$$

To illustrate; let $(c_1, c_2 \cdots c_n)$ be the n linearly ind. columns of $\mathcal{O}$.

$$\begin{bmatrix} y(0^-) \\ \dot{y}(0^-) \\ \vdots \\ y^{(n-1)}(0^-) \end{bmatrix} = \underbrace{\begin{bmatrix} \uparrow & & \uparrow & & \uparrow \\ c_1 & \cdots & c_2 & \cdots & c_n \\ \downarrow & & \downarrow & & \downarrow \end{bmatrix}}_{n \times n} \begin{bmatrix} x_1(0^-) \\ x_2(0^-) \\ \vdots \\ x_n(0^-) \end{bmatrix}$$

Since they form a basis, there exist scalars, $x_1(0^-), x_2(0^-) \cdots x_n(0^-)$ such that, any $y(0^-)$ can be written as:

$$y(0^-) = x_1(0^-)\, c_1 + x_2(0^-)\, c_2 + \cdots$$
$$\cdots + x_n(0^-)\, c_n$$

We call such a realization OBSERVABLE.

On the other hand, if $\mathcal{O}$ is of rank $m < n$ then, there are only m independent columns $(c_1, c_2, \cdots, c_m)$ $m < n$. Then: SOME $y(0^-)$ cannot be written as a linear combination of the columns of $\mathcal{O}$.

These $y(0^-)$ then cannot be simulated on the ANALOG COMP since the initial conditions $(x_1(0), \cdots, x_n(0^-))$ cannot be found. We say the realization is UNOBSERVABLE

SUMMARY:

1) We call a realization OBSERVABLE if the corresponding matrix $\mathcal{O}$ has full rank (i.e rank n). If the

rank of $\mathcal{O}$ is less than n then the realization is UNOBSERVABLE.

2) Any arbitrary initial condition $(y(0^-), \dots, y^{(n-1)}(0^-))$ can be simulated on a particular realization iff the realization is OBSERVABLE.

3) For an unobservable realization, if a particular initial condition $y^*(0^-) = (y^*(0^-), \dot{y}^*(0^-), \dots)$ can be achieved, then there are infinitely many $x(0^-)$ vectors that achieve them.

Example:
$$\dot{x} = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix} x + \begin{pmatrix} 1 \\ 2 \end{pmatrix} u$$
$$y = \begin{pmatrix} 1 & 1 \end{pmatrix} x \qquad n = 2$$

The observability matrix:

$$\mathcal{O} = \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{pmatrix}$$

For $n=2$, $\mathcal{O} = \begin{pmatrix} C \\ CA \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$

$\mathrm{rank}(\mathcal{O}) = 1$.

So (*) looks like:

$$y(0^-) = \begin{pmatrix} y(0^-) \\ \dot{y}(0^-) \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} x_1(0^-) \\ x_2(0^-) \end{pmatrix}$$

Suppose we want $y(0^-) = 1$, $\dot{y}(0^-) = 3$.

"Clearly", there are no $(x_1(0^-), x_2(0^-))$ pairs that can make $(y(0^-), \dot{y}(0^-)) = (1, 3)$.

But suppose, we want $y(0^-) = 2$, $\dot{y}(0^-) = 4$

Then, we can choose $\begin{pmatrix} x_1(0^-) \\ x_2(0^-) \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

In general, all $\begin{pmatrix} y(0^-) \\ \dot{y}(0^-) \end{pmatrix} = \alpha \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\alpha \in \mathbb{R}$

can be achieved.

<u>Exercise:</u> In cases where $\mathcal{O}$ is not full rank and hence we cannot find a solution to $(**)$, do you think it might have helped to use higher derivatives of $y(0^-)$ e.g. $y^{(n)}(0^-)$, $y^{(n+1)}(0^-)$, ... etc.? Why?

<u>Hint</u>: Cayley-Hamilton Theorem.

A RELATED PROBLEM & AN ABSTRACT DEFINITION OF OBSERVABILITY

Suppose we have a state space realization / ANALOG COMP implementation

$$(1)\quad \begin{cases} \dot{x} = Ax + Bu & t \geq 0 \\ y = Cx & x(0) = x_0 \end{cases}$$

We know (A, B, C) and can measure $y(t)$ and $u(t)$ for $t \geq 0$. However, we do not know what initial condition x_0 the realization is started from.

Q. Can we calculate $x(t)$ for $t \geq 0$?

A. We can, only if we know $\underline{x(0)}$ or equivalently $\underline{\text{any } x(t_1)}$ for some $t_1 \geq 0$.

Let us try to use equation (**) for this purpose:

$$\mathcal{Y}(t_1) = \mathcal{O}x(t_1) + T\,\mathcal{U}(t_1)$$

or $$\boxed{\mathcal{O}x(t_1) = \left[\mathcal{Y}(t_1) - T\,\mathcal{U}(t_1)\right]}$$

(1*) Since we have measured $y(t)$ & $u(t)$ for $t \geq 0$, we can calculate $\mathcal{Y}(t_1)$ and $\mathcal{U}(t_1)$. So we know the RHS.

Using a similar argument like the last problem, we can say that the equation (1*) can only be solved if $\mathcal{O}$ has full rank.

But we are doing this experiment on a physical system, obviously there is some $x(t_1)$ that satisfies equation (1*) above.

The relevant question here is can we calculate it using (1*).

$\boxed{\text{$x(t_1)$ can be calculated UNIQUELY from (1*) iff $\mathcal{O}$ has full rank.}}$

If $\mathcal{O}$ is rank deficient there is an infinite number of $x(t_1)$ satisfying (1*) and the realization could have actually started from ANY ONE AMONG THEM.

Hence:

DEFINITION: A state sp. realization of the form (1) is said to be observable if there is a finite time $t' > 0$ such that for any state $x(0)$, the knowledge of input $u_{[0,t']}$ and the output $y_{[0,t']}$ over the interval $[0,t']$ suffices to determine $x(0)$.

Otherwise the realization is said to be unobservable.

FACT: A realization

$$(2a)\quad \begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases} \Bigg| \; x(0) = x_0$$

is OBSERVABLE $\equiv$ $\mathcal{O} := \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$ is of full rank.

Otherwise (2a) is unobservable.

Exercise: What is null space of a matrix? How is it related to the above discussion?

Hint: A full rank matrix has zero null space.

Exercise: How do you think a similarity transform would affect the observability of a realization?