EE 640-7 : Controllability

Note Title 30-06-2008

Review:

We will again use the ANALOG COMP. simulation to introduce the concept of Controllability

We have learnt
1) How to wire a diff. eqn on an ANALOG COMP.
2) How to translate the diff. eqn. initial conditions to the initial condition of the states.

But to start the simulation, we would like to set-up the calculated state initial conditions on the ANALOG COMP.

In other words, suppose we measure the output of the integrators of the ANALOG COMP. and find them to be x_0. However, the diff. eqn. that we would like to simulate has initial conditions $(y(0), \dot{y}(0), \ldots, y^{(n-1)}(0))$ which translate into the state x_1 (according to Section 5).

Q. Can we bring x_0 to x_1?

Actually, this question is fundamental in the theory of systems. Equivalent questions (in different applications) are e.g.

Q. Can we correct the course of a satellite in space (ie. bring it from the wrong coordinates to the desirable coordinate)?

OR

Q. Can we treat a sick person, so that he is transferred from a diseased state to a healthy state?

OR

Q. Can we balance 2 poles on a cart so that both of them are brought to a vertical position?

So in terms of a state space realization, we are asking:

Q. Suppose we have a realization

$$\dot{x} = Ax + Bu \qquad x(0) = x_0$$
$$y = Cx$$

Can we move $x(t)$ to an arbitrary location x_1 in the state space using appropriate $u(t)$, in finite time.

Let us first investigate this question for SISO systems with the input $u(t)$ consisting of of delta functions and their derivatives, i.e.:

$$u(t) = g_1 \delta(t) + g_2 \dot{\delta}(t) + \cdots + g_n \delta^{(n-1)}(t)$$

Recall: Given a function $f(t)$ differentiable $(n+1)$ times at the origin

$$\int_{-\infty}^{\infty} f(t)\, \delta(t)\, dt = f(0)$$

$$\int_{-\infty}^{\infty} f(t)\, \delta^{(i)}(t)\, dt = (-1)^i f^{(i)}(0)$$

Lets find the response with this input.
We have seen:

$$x(t) = e^{At} x_0 + e^{At} \int_0^t e^{-A\tau} B u(\tau)\, d\tau$$

$$= e^{At} x_0 + e^{At} \int_0^t e^{-A\tau} B \left[g_1 \delta(\tau) + g_2 \dot{\delta}(t) + \cdots + g_n \delta^{(n-1)}(t) \right] d\tau$$

Consider: $\displaystyle \int_0^t e^{-A\tau} B\, g_{i+1}\, \delta^{(i)}(\tau)\, d\tau$

$$= g_{i+1} \left[\int_0^t e^{-A\tau} \delta^{(i)}(\tau)\, d\tau \right] B$$

$$= g_{i+1} \, (-1)^i \left(\frac{d^i}{d\tau^i} e^{-A\tau} \right)_{\tau=0} \cdot B$$

$$= g_{i+1} \, (-1)^i (-1)^i A^i e^{-A\tau} \Big|_{\tau=0} \cdot B$$

$$= g_{i+1} \, A^i B$$

Hence,

$$x(t) = e^{At} x_0 + e^{At} \left[g_1 B + g_2 AB + \cdots + g_n A^{n-1} B \right]$$

Now, lets look at this expression at $t = 0^+$.

$$x(0^+) = x_0 + \left[g_1 B + g_2 AB + \cdots + g_n A^{n-1} B \right]$$

Define the matrix,

$$\mathbb{C} = \left[B \;\; AB \;\; \cdots \;\; A^{n-1} B \right]$$

Since we are talking about SISO systems here, B is $n \times 1$. So $\mathbb{C}$ is $n \times n$.

So the equation above may be written as:

$$x(0^+) = x_0 + \mathbb{C} \begin{bmatrix} g_1 \\ g_2 \\ \vdots \\ g_n \end{bmatrix}$$

or

$$x(0^+) - x_0 = \mathbb{C} \begin{bmatrix} g_1 \\ \vdots \\ g_n \end{bmatrix} \quad \text{-----} \quad (\mathfrak{X})$$

Now given any arbitrary $x(0^+)$ we can choose $(g_1, g_2, \cdots, g_n)$ so that $(*)$ is satisfied if the columns of $\mathcal{C}$ are independent.

[Review similar arguement in observability

n ind. columns $\Leftrightarrow$ they form a basis

$\Leftrightarrow$ any vector $(x(0^+) - x_0)$ can be represented as a linear combination of these n ind. columns

$\Rightarrow$ such $(g_1, \cdots, g_n)$ exist]

Hence when $\mathcal{C}$ is full rank, we say the realization is <u>controllable</u> and name $\mathcal{C}$ as the <u>controllability matrix</u>.

<u>To summarize:</u>

<u>FACT</u>: Consider a realization (SISO)

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

Using the input $u(t) = g_1 \delta(t) + \cdots + g_n \delta^{(n-1)}(t)$ the following holds:

The system can reach any state at time $t = 0^+$ iff the realization is controllable, namely iff the controllability matrix $\mathcal{C} = [B \; AB \; \cdots \; A^{n-1}B]$ has full rank.

Q. Why did we use ONLY the first n derivatives of $\delta(t)$? Does it help to use higher derivatives?

A. If we use higher derivatives,

$$x(0^+) = x_0 + \big(\underbrace{B \;\; AB \;\; \cdots \;\; A^{n-1}B \,|\, A^nB \,|\, A^{n+1}B \,|\, \cdots}_{\mathcal{C}'}\big) \begin{bmatrix} g_1 \\ g_2 \\ \vdots \\ g_{n-1} \\ g_n \\ g_{n+1} \\ \vdots \\ \vdots \end{bmatrix}$$

Now recall the Cayley-Hamilton thm.:

$A^n, A^{n+1}, \cdots,$ are all linear combinations of $I, A, \cdots, A^{n-1}$

$\Rightarrow BA^n, BA^{n+1}, \cdots$ are linear comb. of $B, BA, \cdots, BA^{n-1}$. So the rank of $\mathcal{C}'$ remains equal to the rank of $\mathcal{C}$ or in other words, the columns of $\mathcal{C}'$ spans the same space as the columns of $\mathcal{C}$.

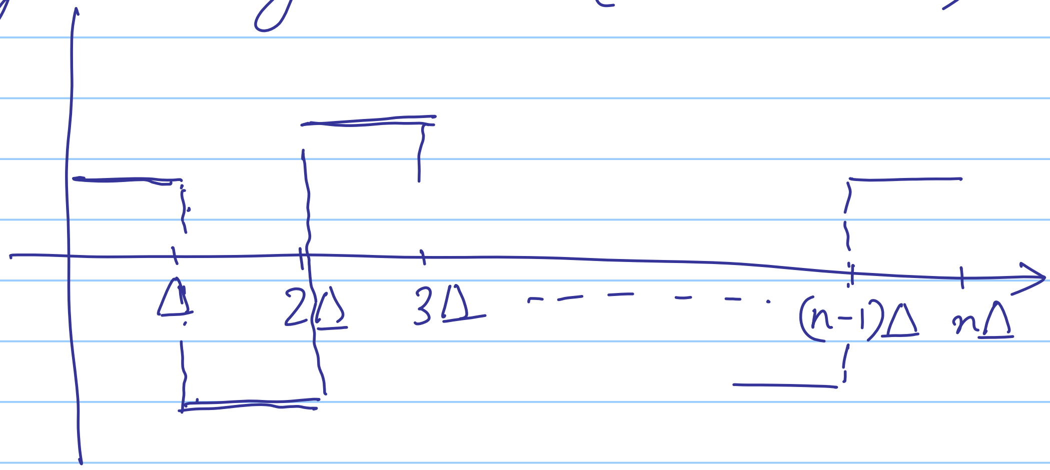
Using implementable Inputs

A similar result is valid:

For any state x^* there is a piecewise continuous input $u^*(t)$ and a finite time $t^* > 0$ such that $x(t^*) = x^*$ if and only if the realization is controllable

Furthermore, $u^*(t)$ can be chosen so that $t^* > 0$ is arbitrarily small.

We can always use piecewise constant inputs with n equal length segments ($n = \dim X$)

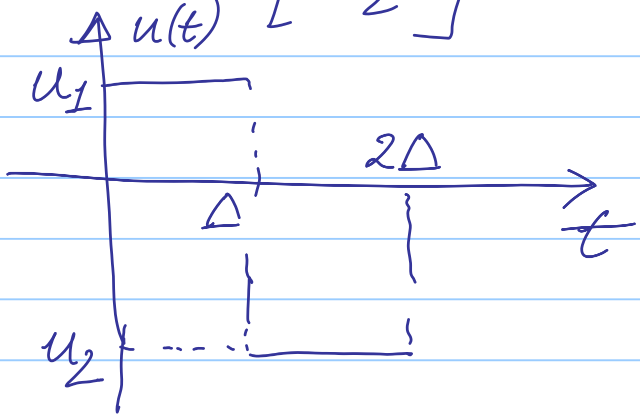


Example: $\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$

$$x_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Desired state: $x^* := \begin{bmatrix} x_1^* \\ x_2^* \end{bmatrix}$

Here $n = 2$



Find u_1, u_2 so that $x(2\Delta) = x^*$

Step 1 : Check controllability :

$$\mathcal{C} = [b, Ab] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{Hence controllable}$$

So (u_1, u_2) should exist.

Calculating (u_1, u_2)

$$x(t) = e^{At}\left[x_0 + \int_0^t e^{-A\tau} \underbrace{b\,u(\tau)}_{?} d\tau\right]$$

$$= x^* \quad \text{at } t = 2\Delta$$

Now, $e^{At} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$

Let us calculate.

$$\int_0^t e^{-A\tau} b\,u(\tau)\,d\tau = \int_0^{\Delta} e^{-A\tau} b\,u_1\,d\tau + \int_{\Delta}^{2\Delta} e^{-A\tau} b\,u_2\,d\tau$$

$$= u_1 \int_0^{\Delta} e^{-A\tau} b\,d\tau + u_2 \int_{\Delta}^{2\Delta} e^{-A\tau} b\,d\tau$$

$$= u_1 \begin{bmatrix} -\Delta^2/2 \\ \Delta \end{bmatrix} + u_2 \begin{bmatrix} -3/2\,\Delta^2 \\ \Delta \end{bmatrix}$$

$$= \begin{bmatrix} -\Delta^2/2 & -\frac{3}{2}\Delta^2 \\ \Delta & \Delta \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

So $\begin{bmatrix} x_1^* \\ x_2^* \end{bmatrix} = e^{At}\left[\cancel{x_0}^{\,0} + \int_0^t e^{-A\tau} b u(\tau)\, d\tau\right]$

$$= \begin{bmatrix} 1 & 2\Delta \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -\Delta^2/2 & -\frac{3}{2}\Delta^2 \\ \Delta & \Delta \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$= \Delta \underbrace{\begin{bmatrix} \frac{3}{2}\Delta & \frac{1}{2}\Delta \\ 1 & 1 \end{bmatrix}}_{\text{invertible}} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

So, $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \frac{1}{2\Delta^2} \begin{bmatrix} 1 & -\frac{\Delta}{2} \\ -1 & \frac{3}{2}\Delta \end{bmatrix} \begin{bmatrix} x_1^* \\ x_2^* \end{bmatrix}$

So, : 1) there is an input function for any (x_1^*, x_2^*); i.e., we can get to any state at $t = 2\Delta$

2) By taking $\Delta > 0$ small enough, we can reach x^* as quickly as we like

Exercise: Suppose we have a controllable realization. So we can get to any state. But can we stop at any state?

A FORMAL DEFINITION

The state equation $\left\{\begin{array}{l}\dot{x} = Ax + Bu \\ y = Cx\end{array}\right\}$ (*)

is said to be controllable if there exists a finite $t_f > 0$ such that for any $x(0)$ and x_1 in the state space, there is an input $u_{[0, t_f]}$ that will transfer $x(0)$ to x_1 at time t_f. Otherwise the state equation is said to be uncontrollable.

Claim: The realization (*) is controllable if and only if the controllability matrix

$$\mathbb{C} = \left[B \quad AB \quad A^2B \cdots \quad A^{n-1}B\right]$$

has full rank. ($= n$ where $n = \dim X$)

Consider the solution of the state equation:

$$x(t) = e^{At}x_0 + e^{At}\int_0^t e^{-A\tau}Bu(\tau)\,d\tau$$

Q. What is $u(t)$ for $0 \le t \le t_f$ such that $x(t_f) = x_1$

Let us try to guess:

$$x_1 = e^{At_f}x_0 + e^{At_f}\int_0^{t_f} e^{-A\tau}Bu(\tau)\,d\tau$$

$$\text{or}\quad x_1 e^{-At_f} - x_0 = \underbrace{\int_0^{t_f} e^{-A\tau}Bu(\tau)\,d\tau}_{(*)}$$

Let us make a guess about $u(t)$:

$$u(t) = B^T e^{-A^T t}\left[\int_0^{t_f} e^{-At}BB^Te^{A^Tt}\,dt\right]^{-1}\left[x_1 e^{-At_f} - x_0\right]$$

Putting this $u(t)$ in the RHS of $(*)$,

$$\int_0^{t_f} e^{-A\tau}Bu(\tau)\,d\tau$$

$$= \int_0^{t_f} e^{-A\tau}B\left[B^Te^{-A^T\tau}\left\{\int_0^{t_f} e^{-At}BB^Te^{A^Tt}\,dt\right\}^{-1}\left\{x_1e^{-At_f} - x_0\right\}\right]d\tau$$

$$= \left[\int_0^{t_f} e^{-A\tau}BB^Te^{-A^T\tau}\,d\tau\right]\left[\int_0^{t_f} e^{-At}BB^Te^{A^Tt}\,dt\right]^{-1}\left\{x_1e^{-At_f} - x_0\right\}$$

$$= x_1e^{-At_f} - x_0 = \text{LHS}.$$

But the above calculations are valid $\Longleftarrow$

$$W := \int_0^{t_f} e^{-At}BB^Te^{A^Tt}\,dt$$

is invertible.

Controllability Gramian

So let us investigate under what conditions W is invertible.

<u>DEFN</u>: Let f_i, $i=1,2,\dots,n$ be real valued functions of t. Then $\{f_i\}$ are linearly dependent on $[t_1, t_2]$ if there are scalars $\alpha_1, \dots, \alpha_n$ not all zero, such that

$$\alpha_1 f_1(t) + \alpha_2 f_2(t) + \dots + \alpha_n f_n(t) = 0$$

for all $t \in [t_1, t_2]$

<u>FACT</u>: Let f_i, for $i=1,2,\dots,n$ be n real valued continuous functions defined on $[t_1, t_2]$. Let

$$F = \begin{bmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_n(t) \end{bmatrix}_{n\times 1}$$

be a $n\times 1$ vector of functions.

Define

$$W = \int_{t_1}^{t_2} F(t) F^T(t)\, dt$$

Then $f_1, f_2, \dots, f_n$ are linearly ind. on $[t_1, t_2]$ if and only if the $n\times n$ constant matrix W is non-singular.

This means that we can form the desired control $u(t) \Longleftarrow W$ is invertible

$\Longleftrightarrow$ the rows of $e^{-At}B$ are linearly independent over $[0, t_f]$

Claim: The rows of $e^{-At}B$ are linearly independent over $[0, t_f]$ iff

$$\operatorname{rank}\left[B \quad AB \quad A^2B \cdots A^{n-1}B\right] = n.$$

Proof:

The rows of $e^{-At}B$ are linearly ind

$\Longleftrightarrow \quad q e^{-At} B \equiv 0 \quad 0 \leq t \leq t_f \Rightarrow q = 0$

$$q e^{-At} B = q\left[I - At + \frac{A^2t^2}{2!} - \cdots\right]B$$

$$= \left[qB - qABt + qA^2B\frac{t^2}{2!} - \cdots\right]$$

However this is a power series in t which is identically zero over the interval $0 \leq t \leq t_f$.

$$\Longleftrightarrow \left\{\begin{array}{l} qB = 0 \\ qAB = 0 \\ \vdots \\ qA^{n-1}B = 0 \end{array}\right. \quad \left|\quad \begin{array}{l} qA^nB = 0 \\ \vdots \\ \vdots \end{array}\right.$$

$\Longleftrightarrow q[B \;\; AB \;\; \cdots\cdots \;\; A^{n-1}B \;\vdots\; \cdots\cdots] = 0$ --- (1)

By Cayley-Hamilton thm,

(1) $\Longleftrightarrow$ rank$[B \;\; AB \cdots\cdots A^{n-1}B] = n \Rightarrow q = 0$

So we proved that

If rank $(\mathbb{C}) = n$ then realization (*) is controllable.

Let us prove the converse:

Claim:
Singular $\mathbb{C}$ implies lack of state controllability

Proof: If $\mathbb{C}$ is singular there is a vector q' such that

$$q'\mathbb{C} = 0 \quad \text{and} \quad q \not\equiv 0$$

Now suppose the realization is still controllable i.e $\exists\, u(t)$ such that $u(\cdot)$ takes $x_0 = 0$ to $x(t_f) = q$ in finite t_f.

i.e. $$q = \int_0^{t_f} e^{A(t_f - \tau)} B u(\tau)\, d\tau$$

or $$q^T q = \int_0^{t_f} q^T e^{A(t_f-\tau)} B u(\tau)\, d\tau$$

But, $q^T e^{A(t_f-\tau)} B$

$$= q^T B + q^T A B (t_f - \tau) + \cdots + q^T A^i B \frac{(t_f-\tau)^i}{i!} + \text{———} \cdots$$

But by hypothesis:

$$q^T B = 0, \quad q^T A B = 0, \quad \cdots, \quad q^T A^{n-1} B = 0$$

So by Cayley-Hamilton thm,

$$q^T q = 0,$$

But this contradicts $q \neq 0$.