EE640-9 Uncontrollable & Unobservable Realizations

Note Title 24-07-2008

Consider the realization

$$\dot{x} = Ax + bu$$
$$y = cx$$

where, $\operatorname{rank}[\mathcal{C}(A,b)] = r < n$

FACT: Then there is a transformation matrix T such that the realization

$$\{ \bar{A} = T^{-1}AT \; , \; \bar{b} = T^{-1}b \; , \; \bar{c} = cT \}$$

has the form:

$$\bar{A} = \begin{array}{c} \left[\begin{array}{c|c} \bar{A}_c & \bar{A}_{c\bar{c}} \\ \hline 0 & \bar{A}_{\bar{c}} \end{array}\right] \end{array} \begin{array}{l} \}\, r \\ \}\, n-r \end{array} \; ; \; \bar{b} = \left[\begin{array}{c} \bar{b}_c \\ \hline 0 \end{array}\right]$$

$$\bar{c} = \left[\bar{c}_c \quad \bar{c}_{\bar{c}} \right]$$, which in turn

has the following properties:

(i) The $r \times r$ subsystem $\{\bar{A}_c, \bar{b}_c, \bar{c}_c\}$ is controllable

(ii) $\bar{c}(sI-\bar{A})^{-1}\bar{b} = \bar{c}_c(sI-\bar{A}_c)^{-1}\bar{b}_c$, i.e. the subsystem has the same transfer function as the original system.

Property (ii) is very easy to check:

$$\bar{c}(sI-\bar{A})^{-1}\bar{b} = \bar{c}\begin{bmatrix} (sI-\bar{A}_c)^{-1} & * \\ 0 & (sI-\bar{A}_{\bar{c}})^{-1} \end{bmatrix}\begin{bmatrix} \bar{b}_c \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} \bar{c}_c & \bar{c}_{\bar{c}} \end{bmatrix} \begin{bmatrix} (sI - \bar{A}_c)^{-1} \bar{b}_c \\ 0 \end{bmatrix} = \bar{c}_c (sI - \bar{A}_c)^{-1} \bar{b}_c$$

<u>Proof of (i)</u> Let us calculate

$$\mathcal{C}(\bar{A}, \bar{b}) = \begin{bmatrix} \bar{b} & \bar{A}\bar{b} & \bar{A}^2\bar{b} & \cdots & \bar{A}^{n-1}\bar{b} \end{bmatrix}$$

$$= \begin{array}{c} \left[\begin{array}{cccc} \bar{b}_c & \bar{A}_c \bar{b}_c & \cdots & \bar{A}_c^{n-1}\bar{b}_c \\ \hline 0 & 0 & \cdots & 0 \end{array}\right] \end{array} \begin{array}{l} \}\, r \\ \}\, n-r \end{array}$$

We know, that $\mathcal{C}(\bar{A}, \bar{b}) = T^{-1}\mathcal{C}(A, b)$ (by HW4)

Since, $\mathcal{C}(A,b)$ has rank r, $\mathcal{C}(\bar{A}, \bar{b})$ also has rank r and hence it can have only r lin. ind. rows and r lin. ind. columns.

<u>Claim</u>: The first r columns of $\mathcal{C}(\bar{A}, \bar{b})$ must be lin. ind.

<u>Proof</u>: Suppose $\bar{A}_c^k \bar{b}_c$ is linearly dependent on $\{\bar{A}_c^i \bar{b}_c\}, i < k\}$.

Then,
$$\bar{A}_c^{k+1}\bar{b}_c = \bar{A}_c\left[\bar{A}_c^k \bar{b}_c\right]$$
$$= \bar{A}_c\left[\alpha_1 \bar{b}_c + \alpha_2 \bar{A}_c \bar{b}_c + \cdots + \alpha_{k-1}\bar{A}_c^{k-1}\bar{b}_c\right]$$
$$= \alpha_1 \bar{A}_c \bar{b}_c + \alpha_2 \bar{A}_c^2 \bar{b}_c + \cdots + \alpha_{k-1}\bar{A}_c^k \bar{b}_c$$
$$= \alpha_1 \bar{A}_c \bar{b}_c + \alpha_2 \bar{A}_c^2 \bar{b}_c + \cdots + \alpha_{k-1}\left[\alpha_1 \bar{b}_c + \cdots\right]$$

Hence, $\bar{A}_c^{k+1}\bar{b}_c$ is also linearly dep. on the set $\underline{\{\bar{A}_c^i \bar{b}_c, i<k\}}$.

So if we go from left to right in $\mathbb{C}(\bar{A},\bar{b})$, one we find a dependent vector, then all subsequent ones must be so too. But since the rank is r, the first r columns must be linearly independent. So we have proved property (i).

Now, the question is, can we find a T which makes the necessary transformation. Let's "back-calculate"

We require: $T\,\mathbb{C}(\bar{A},\bar{b}) = \mathbb{C}(A,b)$
(again HW 4)

i.e $$[T_1 \quad T_2]\begin{bmatrix}\mathbb{C}(\bar{A}_c,\bar{b}_c) & * \\ 0 & 0\end{bmatrix}$$

$$= [b \;\; Ab \;\; \cdots \;\; A^{r-1}b \quad *]$$

i.e $$T_1\,\mathbb{C}(\bar{A}_c,\bar{b}_c) = [b \;\; Ab \;\; \cdots \;\; A^{r-1}b]$$

i.e $$T_1 = [b \;\; Ab \;\; \cdots\cdots \;\; A^{r-1}b]\,\mathbb{C}^{-1}(\bar{A}_c,\bar{b}_c)$$

Suppose, we wish, the $\mathbb{C}(\bar{A}_c,\bar{b}_c) = I$

then, $T_1 = [b \;\; Ab \;\; \cdots \;\; A^{r-1}b]$

So the first r colums of T are known;

$$T = [\overbrace{T_1}^{r} \quad T_2]$$

Q. What about T_2?

<u>Ans:</u> We don't care, as long as T is investible. In other words, T_2 can be arbitrarily chosen, as long as the colums of T_2 are linearly ind. of each other and those of T_1.

<u>FACT:</u> $\{\bar{A}_c, \bar{b}_c, \bar{c}_c\}$ forms the largest subrealization that is reachable

<u>The Eigenvalues</u>: The eigenvalues are ~~not affected~~ not affected by a similarity transformation. Also the characteristic polynomial is not affected by similarity transformation.

<u>The characteristic polynomial</u>

$$a(s) = \det(sI - A) = \det(sI - \bar{A})$$

$$= \det\left(sI - \begin{bmatrix} \bar{A}_c & \bar{A}_{c\bar{c}} \\ 0 & \bar{A}_{\bar{c}} \end{bmatrix}\right)$$

$$= \det\begin{bmatrix} (sI_r - \bar{A}_c) & -\bar{A}_{c\bar{c}} \\ 0 & sI_{n-r} - \bar{A}_{\bar{c}} \end{bmatrix}$$

$$= \det(sI_r - \bar{A}_c)\,\det(sI_{n-r} - \bar{A}_{\bar{c}})$$

Roots of $\det(sI_r - \bar{A}_c)$ = eigenvalues of $\bar{A}_c$

Roots of $\det(sI_{n-r} - \bar{A}_{\bar{c}})$ = eigenvalues of $\bar{A}_{\bar{c}}$

<u>Definition</u>: Eigenvalues of $\bar{A}_c \equiv$ controllable modes of A

Eigenvalues of $\bar{A}_{\bar{c}} \equiv$ uncontrollable modes of A.

So : Eigenvalues of A = Controllable modes $\cup$ Uncontrollable modes

<u>Example</u> :

$$\left.\begin{aligned} \dot{x} &= \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u \\ y &= \begin{bmatrix} 0 & 1 \end{bmatrix} x \end{aligned}\right\} \quad (1)$$

The Controllability Matrix

$$\mathcal{C} = \begin{bmatrix} b & Ab \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} \Rightarrow \text{rank}(\mathcal{C}) = 1.$$

Let us build $T = \begin{bmatrix} T_1 & T_2 \end{bmatrix}$

By the derivation above $T_1 = b = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

T_2 can be arbitrarily chosen such that T is invertible

Let $T_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow T = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$

So, $\bar{A} = T^{-1} A T = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$

$$= \left[\begin{array}{c|c} 2 & 1 \\ \hline 0 & 0 \end{array}\right]$$

Hence, $\bar{A}_c = 2$, $\bar{A}_{c\bar{c}} = 1$, $\bar{A}_{\bar{c}} = 0$

$\bar{b} = T^{-1} b = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \left[\begin{array}{c} 1 \\ \hline 0 \end{array}\right] \Rightarrow \bar{b}_c = 1$

$\bar{c} = cT = [0 \quad 1] \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = [\underbrace{1}_{\bar{c}_c} \quad \underbrace{0}_{\bar{c}_{\bar{c}}}]$

Hence, the controllable subrealization

$$\left.\begin{aligned} \dot{z} &= 2z + 1.u \\ y &= 1.z \end{aligned}\right\} \quad (2)$$

Controllable mode = 2
Uncontrollable mode = 0

Ex: Check that tr. functions for (1) and (2) are the same.

Unobservable Realizations

Dual statements are valid for unobservable realizations:

Consider, $\dot{x} = Ax + bu$
$y = Cx$

with $\text{rank}\left[\mathcal{O}(c, A)\right] = r < n$. We can find a non-singular matrix T such that

$$\bar{A} = T^{-1}AT, \quad \bar{b} = T^{-1}b, \quad \bar{c} = cT$$

have the form:

$$\bar{A} = \left[\begin{array}{c|c} \bar{A}_o & 0 \\ \hline \bar{A}_{\bar{o}o} & \bar{A}_{\bar{o}} \end{array}\right]\begin{array}{l} \}r \\ \}n-r \end{array} ; \quad \bar{c} = [\bar{c}_o \quad 0]$$

$$\bar{b} = \left[\begin{array}{c} \bar{b}_o \\ \hline \bar{b}_{\bar{o}} \end{array}\right] \quad \text{and}$$

(i) $\{\bar{c}_o, \bar{A}_o\}$ is observable
(ii) $c(sI - A)^{-1}b = \bar{c}_o(sI - \bar{A}_o)^{-1}\bar{b}_o$

Moreover, the sub-realization,

$$(*)\begin{cases} \dot{z} = \bar{A}_o z + \bar{b}_o u \\ y = \bar{c}_o z \end{cases}$$

is the highest dimensional observable sub-realization.

<u>Def$^{\underline{n}}$</u>: Eigenvalues of $\bar{A}_o \equiv$ observable modes of A
Eigenvalues of $\bar{A}_{\bar{o}} \equiv$ unobservable modes of A.

Eigenvalues of A = Observable modes $\cup$ Unobservable modes

General Decomposition:

Given : $\begin{cases} \dot{x} = Ax + bu \\ y = Cx \end{cases}$

Then $\exists$ a similarity tr. T s.t.

$$\bar{A} = T^{-1}AT$$
$$\bar{b} = T^{-1}b$$
$$\bar{c} = cT$$

$$\bar{A} = \left[\begin{array}{c|c|c|c} \bar{A}_{co} & 0 & \bar{A}_{13} & 0 \\ \hline \bar{A}_{21} & \bar{A}_{c\bar{o}} & \bar{A}_{23} & \bar{A}_{24} \\ \hline 0 & 0 & \bar{A}_{\bar{c}o} & 0 \\ \hline 0 & 0 & \bar{A}_{43} & \bar{A}_{\bar{c}\bar{o}} \end{array}\right] \qquad \bar{b} = \left[\begin{array}{c} \bar{b}_{co} \\ \hline \bar{b}_{c\bar{o}} \\ \hline 0 \\ \hline 0 \end{array}\right]$$

$$\bar{c} = \left[\begin{array}{c|c|c|c} \bar{c}_{co} & 0 & \bar{c}_{\bar{c}o} & 0 \end{array}\right]$$

where (1) $\{\bar{A}_{co}, \bar{b}_{co}, \bar{c}_{co}\}$ is controllable and observable, and

$$H(s) = c(sI - A)^{-1}b = \bar{c}_{co}(sI - \bar{A}_{co})^{-1}\bar{b}_{co}$$

(2) the subsystem:

$$\begin{bmatrix} \bar{A}_{co} & 0 \\ \bar{A}_{21} & \bar{A}_{c\bar{o}} \end{bmatrix}, \begin{bmatrix} \bar{b}_{co} \\ \bar{b}_{c\bar{o}} \end{bmatrix}, \begin{bmatrix} \bar{c}_{co}, & 0 \end{bmatrix}$$

is controllable

(3) the subsystem

$$\left[\begin{matrix} \bar{A}_{co} & \bar{A}_{13} \\ \bar{A}_{21} & \bar{A}_{\bar{c}o} \end{matrix}\right], \left[\begin{matrix} \bar{b}_{co} \\ 0 \end{matrix}\right], \left[\begin{matrix} \bar{c}_{co} & \bar{c}_{\bar{c}o} \end{matrix}\right]$$

is observable.

(4) The subsystem $\{ \bar{A}_{\bar{c}\bar{o}}, 0, 0 \}$ is completely un-controllable and un-observable

Important NOTE: In all the above representations, only the dimensions of the blocks and the eigenvalues of the parts are unique.

The Popov-Belevitch-Hautus (PBH) tests for Controllability & Observability

FACT1: A pair $\{A, b\}$ will be un-controllable if and only if there exists a row vector $q \neq 0$ such that

$$qA = \lambda q \quad , \quad qb = 0$$

[i.e. $\{A, b\}$ will be controllable iff there is no row (or left) eigenvector of A that is orthogonal to b]

Similarly:

<u>FACT 2</u>: A pair $\{c, A\}$ will be un-observable iff there exists a (column) vector $p \neq 0$ such that

$$Ap = \lambda p \quad , \quad cp = 0$$

[i.e. iff some eigenvector of A is orthogonal to c]

<u>Proof</u>: "If" part: If there is $q \neq 0$ such that

$$qA = \lambda q \quad , \quad qb = 0$$

then

$$qAb = \lambda qb = 0$$
$$qA^2b = \lambda qAb = \lambda . 0 = 0$$
$$qA^3b = \lambda q A^2 b = \lambda . 0 = 0$$
$$\vdots$$
$$qA^{n-1}b = \lambda q A^{n-2}b = 0$$

This implies,

$$q\,\mathcal{C}(A, b) = q\left[b \;\; Ab \;\; \cdots \;\; A^{n-1}b\right] = 0$$

which means that the controllability matrix is singular, i.e.

$\{A, b\}$ is not controllable.

<u>The "only if" part</u>: $\{A, b\}$ uncontrollable $\Rightarrow q$ exists as above.

Let us assume that the realization has already been put into the standard un-controllable form

$$A = \left[\begin{array}{c|c} A_c & A_{c\bar{c}} \\ \hline 0 & A_{\bar{c}} \end{array}\right] \begin{matrix} \}r \\ \}n-r \end{matrix} \quad ; \quad b = \left[\begin{array}{c} b_c \\ \hline 0 \end{array}\right]$$

where $r = \text{rank } \mathbb{C}(A,b) < n$.

If we choose $q = [0 \mid z]$ it is clearly orthogonal to b.

Now, we can guess z: choose z as an eigenvector of $A_{\bar{c}}$

$$z A_{\bar{c}} = \lambda z$$

Then,
$$q A = [0 \quad z] \left[\begin{array}{c|c} A_c & A_{c\bar{c}} \\ \hline 0 & A_{\bar{c}} \end{array}\right]$$

$$= [0 \quad zA_{\bar{c}}] = [0 \quad \lambda z] = \lambda q$$

So the q of our choice satisfies the requirements

<u>Proof of FACT2 using Duality</u>

We show this in 2 steps:

(1) (c, A) is observable $\Leftrightarrow \{A^T, c^T\}$ controllable

(2) Use FACT 1 on $\{A^T, c^T\}$

(1) $\longrightarrow$ Exercise

(2) By part (1), $\{A^T, C^T\}$ is controllable iff there is a row vector p^T such that

$$p^T A^T = \lambda p^T \quad \text{and} \quad p^T C^T = 0$$

i.e.

$$Ap = \lambda p \quad \text{and} \quad Cp = 0$$

The above test gives us some intuition about the requirements on the structure of $\{A, b\}$ / $\{c, A\}$ pairs for controllability / observability.

An even easier to test, condition follows easily:

FACT (PBH Rank Tests)

1) A pair $\{A, b\}$ is controllable iff

$$\text{rank}\begin{bmatrix} sI - A & b \end{bmatrix} = n \quad \text{for all complex numbers } s.$$

2) A pair $\{c, A\}$ is observable iff

$$\text{rank}\begin{bmatrix} c \\ sI - A \end{bmatrix} = n \quad \text{for all complex numbers } s.$$

n = size of A.

NOTE: These conditions will clearly be met for all s that are not eigenvalues of A.

Why? because then $\det(sI-A) \neq 0$ for such s.

The point is that rank must be n even when s is an eigenvalue of A.

Proof: If $[sI-A \;\; b]$ has rank n, there cannot be a $q \not\equiv 0$ such that

$$q[sI-A \;\; b] = 0 \quad \text{for some } s.$$

i.e., $\quad qb = 0 \quad$ and $\quad qA = sq$

But then by FACT 1 above, $\{A, b\}$ must be controllable

Converse: Exercise

(2) $\rightarrow$ Exercise

Another Use: The rank test can be used to identify the uncontrollable/unobservable modes.

If rank $[sI-A \;\; b] < n$ for a complex number s, then s is a un-controllable mode of the realization.

Similarly, if rank $\begin{bmatrix} C \\ sI-A \end{bmatrix} < n$ for a particular complex number s, then s is a non-observable mode of the realization.

The PBH test is very useful for theoretical analysis