A Dynamic Model for Lane-less (Indian) Traffic

Ameer Mulla, Rakesh Chavan, Apurva Joshi, Debraj Chakraborty,
D. Manjunath

Control and Computing Group
Department of Electrical Engineering
Indian Institute of Technology Bombay
INDIA

October 26, 2016

What?

- Popular perception is that Indian traffic is “chaotic”*
- Nobody obeys lane discipline
- The only rule is: “right of space” (as opposed to “right of way”)
- The above rule frequently interpreted as “might is right”
- We aim to build a car following model for Indian traffic

Car following model?

- Well developed microscopic model of traffic where each car is supposed to follow *a leader*.
- An early model as an example: $\ddot{x}_{n+1} = c\frac{(\dot{x}_n - \dot{x}_{n+1})}{(x_n - x_{n+1})^2}$, where x_n is the position of the n-th vehicle.

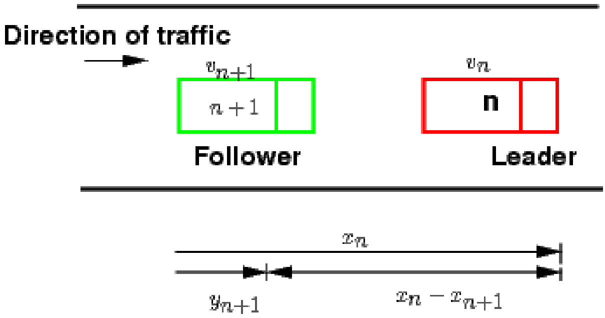


Figure: Car Following Model (from Prof T. Mathew's website)

Car following model-useful?

- Understanding vehicle level traffic/driver behaviour
- Used for traffic simulations
- For uniform velocity: such models can predict macro-level behaviour: capacity, density, interdepence between these quantities etc.
- Used to develop autonomous driverless cars

Car following in Indian Traffic?

- Does these models work for Indian Traffic?
  - Several studies exist comparing simulations using available models calibrated with Indian traffic data
  - A recent example (of course there are several more papers):
    - Gowri Asaithambi, Venkatesan Kanagaraj, Karthik K. Srinivasan & R. Sivanandan, Study of traffic flow characteristics using different vehicle-following models under mixed traffic conditions, Transportation Letters, 2016.
  - Lane shift is usually incorporated extraneously to the car following simulations
  - Numerical computation of macro-level data are accurate after complicated tuning of simulation parameters
- In our view, available models do not capture our lane-less driving behaviour.

Who follows whom?

- The right picture: laned traffic with clear validity of single leader following by each car
- Left picture: Typical Indian traffic, not clear who is following whom

Figure: Car Following? (picture from the G. Asaithambi et al. cited above)

Why?

- We do not follow a single car
  - We see every car in front, forever on the lookout for empty space
  - We see cars on both right and left, forever on ....
- We move to occupy empty space
- We “lane-change” continuously in the process
- Obstacles are everywhere
- Roads are never straight

-

In this talk

- A model which better approximates Indian driving
- Validation (to some extent) of the model with real microscopic data
- Proven stability properties (will not be usable otherwise any way)
- Use of recently developed multi-agent formation results for these purposes

Experimental "Setup"

- Video of JVLR outside IITB

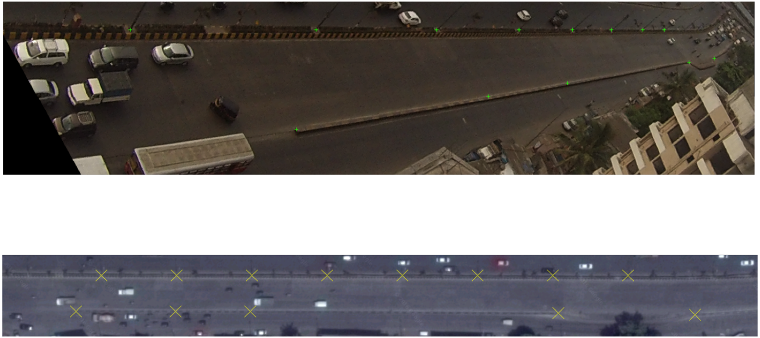


Figure: Original Image and corresponding google map image

Our Assumptions

Simplistic but some can be relaxed:

- All vehicles are identical double integrators in both directions
- Each driver uses identical driving laws
- Decoupled motion along longitudinal and lateral directions
- Cone of vision
- Road conditions create pseudo leader
- We sidestep mesh/string stability by assuming finite number of vehicles

Longitudinal Influence Graph

- Primary hypothesis
  - Each vehicle is influenced by all vehicles in the “layer” ahead of him
- How to define layers?
  - All vehicles who influence a vehicle is in one layer ahead of him
- Circular Logic - circumvented by the cone of vision argument

Definitions

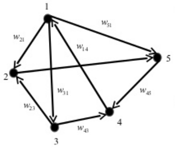
- A rooted directed tree is a digraph such that there exists a node (called root) and a directed path from that node to all other nodes in the digraph.
- A digraph $\vec{\mathscr{G}}$ is said to contain a directed spanning tree, if there exists a rooted directed tree $\vec{\mathscr{G}}_{tree} = (\mathscr{A}_{tree}, \vec{\mathscr{E}}_{tree}, w)$ such that $\mathscr{A}_{tree} = \mathscr{A}$ and $\vec{\mathscr{E}}_{tree} \subseteq \vec{\mathscr{E}}$.
- The Laplacian ($\mathscr{L}$) for a directed graph with weights w_{ij} as follows: $\ell_{ij} := -w_{ij} \quad \text{if} \quad (a_j, a_i) \in \vec{\mathscr{E}} \quad , \ell_{ij} := 0 \text{if } (a_j, a_i) \notin \vec{\mathscr{E}}$, and $\ell_{ii} := \sum_{j=1}^{n} w_{ij} :=$ cumulative weight of incoming edges.

Definitions

- For a directed graph $\vec{\mathscr{G}} = (\mathscr{A}, \vec{\mathscr{E}}, w)$, let $\mathbf{L} = \{L_0, L_1, L_2, ..., L_k\}$, $k \geq 1$ be a partition of $\mathscr{A}$ such that if $(a_i, a_j) \in \vec{\mathscr{E}}$ with $u_i \in L_p$ and $u_j \in L_q$, then $q < p$. Such $\mathbf{L}$ is called a <u>layering</u> of $\vec{\mathscr{G}}$ and L_0, L_1,...,L_k are referred to as <u>layers</u>.
- A digraph with layering is called a <u>layered digraph</u>. The index of a layer that contains a node a_i is denoted by $l(a_i, \mathbf{L})$, where $l(a_i, \mathbf{L}) := p$ if and only if $a_i \in L_p$. A layering $\mathbf{L}$ is <u>proper</u> if all edges of $\vec{\mathscr{G}}$, satisfy $s(e, \mathbf{L}) = l(a_i, \mathbf{L}) - l(a_j, \mathbf{L}) = 1$.

Definitions-Key Concepts

Graph Dynamics (Diagraph)



Adjacency Matrix	$A=\begin{array}{c}\\1\\2\\3\\4\\5\end{array}\begin{array}{c}\begin{array}{ccccc}1&2&3&4&5\end{array}\\ \begin{bmatrix}0&0&0&1&0\\1&0&1&0&0\\1&0&0&0&0\\0&0&1&0&1\\1&1&0&0&0\end{bmatrix}\end{array}$ or $A=\begin{array}{c}\\1\\2\\3\\4\\5\end{array}\begin{array}{c}\begin{array}{ccccc}1&2&3&4&5\end{array}\\ \begin{bmatrix}0&0&0&w_{14}&0\\w_{21}&0&w_{23}&0&0\\w_{31}&0&0&0&0\\0&0&w_{43}&0&w_{45}\\w_{51}&w_{52}&0&0&0\end{bmatrix}\end{array}$
Diagonal Matrix	$D=\begin{bmatrix}1&0&0&0&0\\0&2&0&0&0\\0&0&1&0&0\\0&0&0&2&0\\0&0&0&0&2\end{bmatrix}$
Laplacian matrix	$L=D-A$ $L=\begin{bmatrix}1&0&0&-1&0\\-1&2&-1&0&0\\-1&0&1&0&0\\0&0&-1&2&-1\\-1&-1&0&0&2\end{bmatrix}$

Note that $(I+L)$ is row stochastic

11/07/11 11

Figure: Laplacian of Directed Graph (from A. Das, Graph Consensus: Autonomous and Controlled, slideshare.net

Definitions-Key Concepts

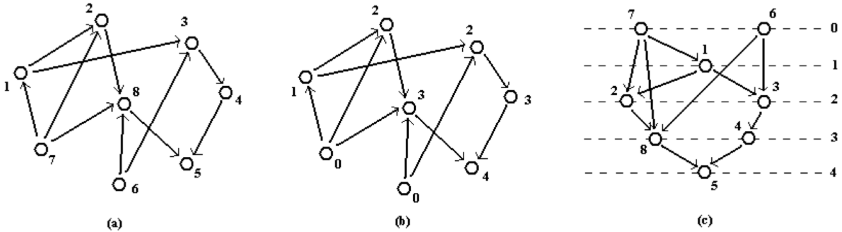
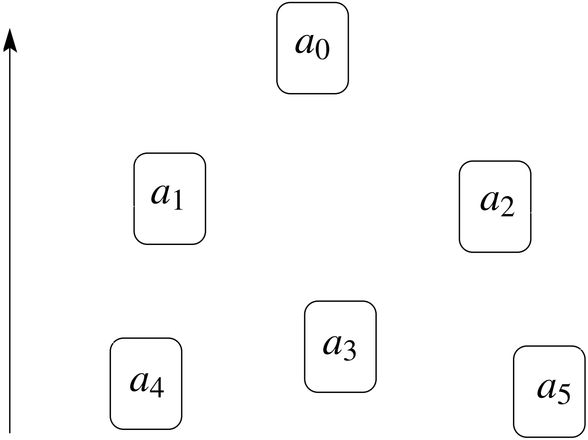
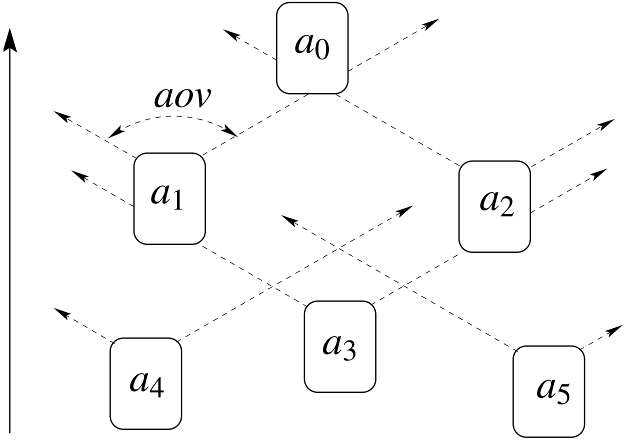


Figure: Algorithms for extracting layered graphs (from Ronald Kieft, Cross Minimization, slideshare.net)

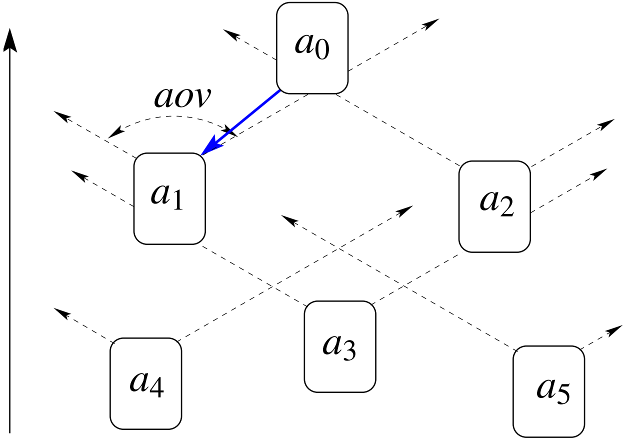
Longitudinal Influence Graph



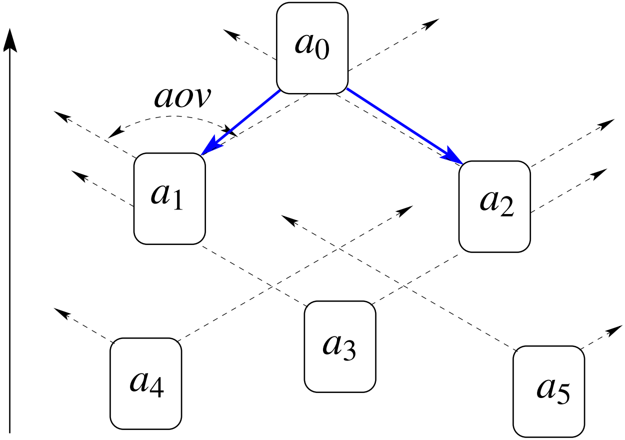
Longitudinal Influence Graph

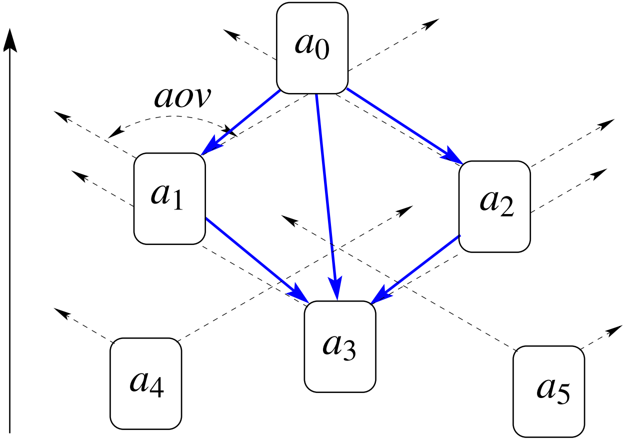


Longitudinal Influence Graph



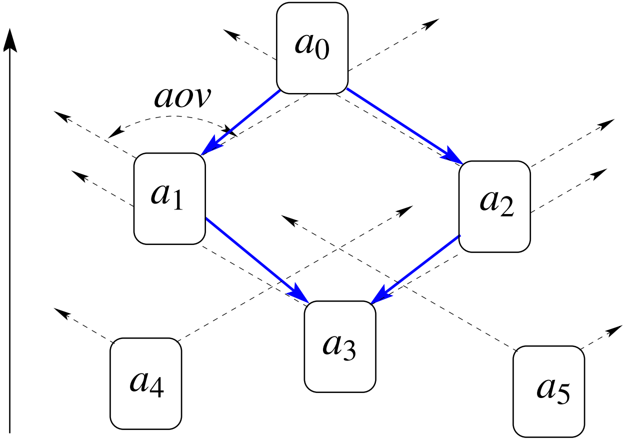
Longitudinal Influence Graph



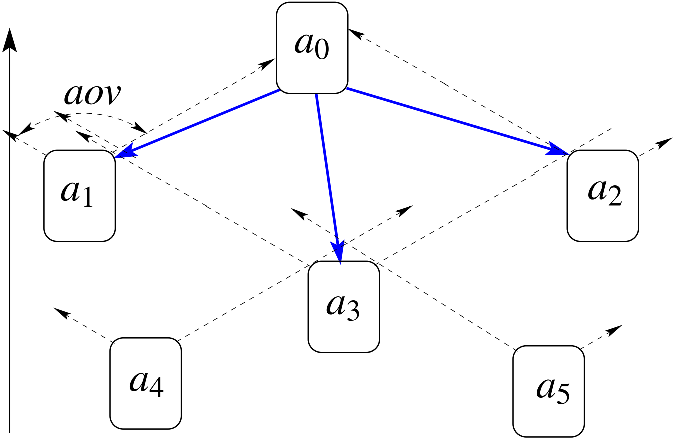


a0
aov
a1
a2
a3
a4
a5

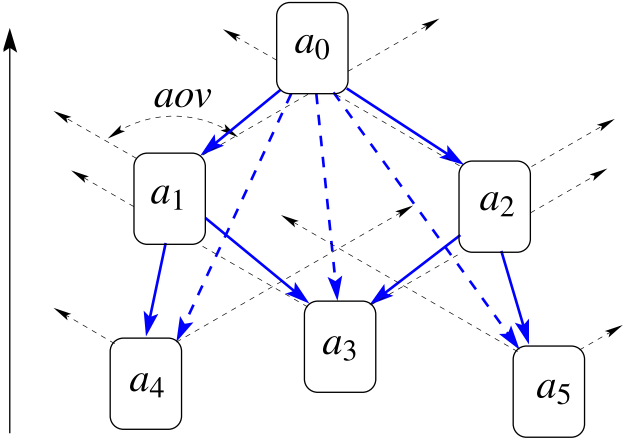
Longitudinal Influence Graph



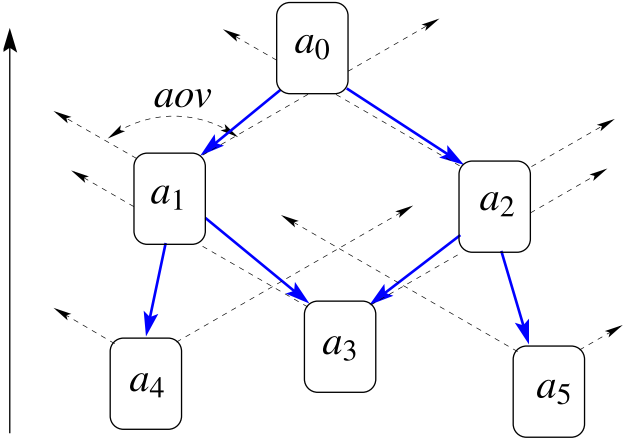
Longitudinal Influence Graph



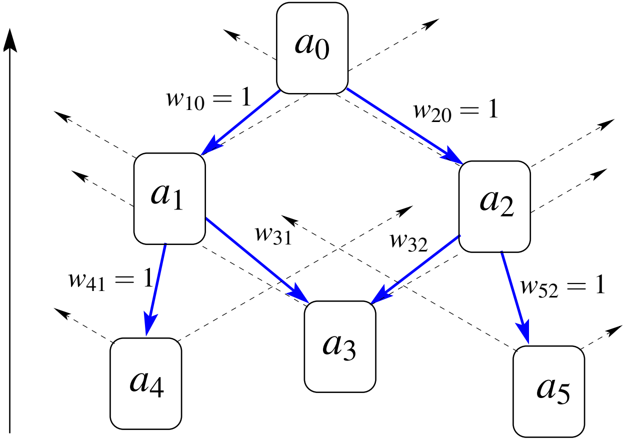
Longitudinal Influence Graph



Longitudinal Influence Graph



Longitudinal Influence Graph



Proper Layered Graph

Algorithm1: Extracting $\vec{\mathscr{G}}^y$ from $\vec{\mathscr{G}}^y_{cone}$

Assumption: $\vec{\mathscr{G}}^y_{cone}$ contains a directed spanning tree rooted at a_0^y

1. a_0^y is the leader node.
2. Number others vehicles as per their Y-coordinates, i.e. for two vehicles a_i and a_j, $i < j$ if $y_i \geq y_j$. This implies $y_0 > y_1 \geq y_2 \geq \cdots \geq y_n > 0$.
3. For vehicle a_k, $k \in \{1, ..., n\}$
    1. Calculate maximum path length l from a_0^y
    2. Assign level L_l for a_k
4. Remove all long edges from $\vec{\mathscr{E}}^y_{cone}$.

Proper Layered Graph

Algorithm1: Extracting $\vec{\mathscr{G}}^y$ from $\vec{\mathscr{G}}^y_{cone}$

Assumption: $\vec{\mathscr{G}}^y_{cone}$ contains a directed spanning tree rooted at a_0^y

1. a_0^y is the leader node.
2. Number others vehicles as per their Y-coordinates, i.e. for two vehicles a_i and a_j, $i < j$ if $y_i \geq y_j$. This implies $y_0 > y_1 \geq y_2 \geq \cdots \geq y_n > 0$.
3. For vehicle a_k, $k \in \{1, \ldots, n\}$
   1. Calculate maximum path length l from a_0^y
   2. Assign level L_l for a_k
4. Remove all long edges from $\vec{\mathscr{E}}^y_{cone}$.

Theorem

If $\vec{\mathscr{G}}^y_{cone}$ contains a directed spanning tree rooted at a_0^y, Algorithm 1 generates a proper layered graph $\vec{\mathscr{G}}^y$ with layers L_l, $l = 0, 1, 2, \ldots, m$.

Lateral Influence - Assumptions

- Either we have six pair of eyes or we have additional cow vision

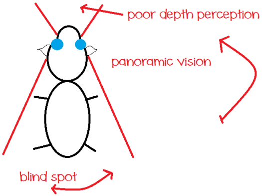
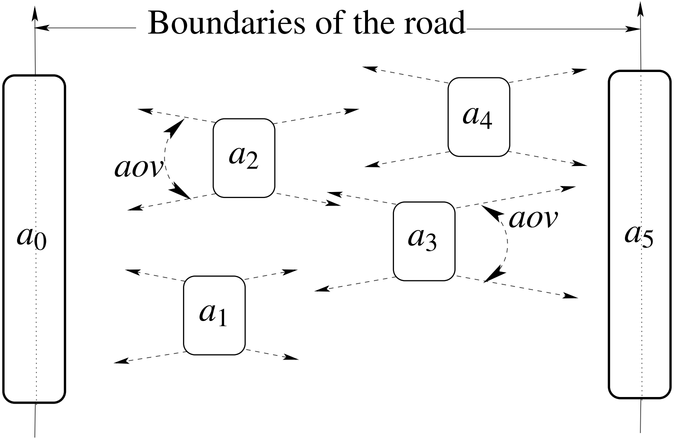


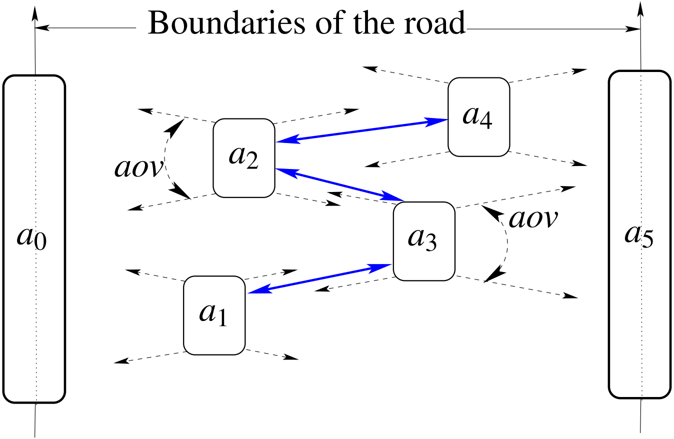
Figure: Cow Vision

- Each driver aims to position (X) himself in the middle of the nearest perceived obstacles.
- Edges of usable roads are pseudo-cars

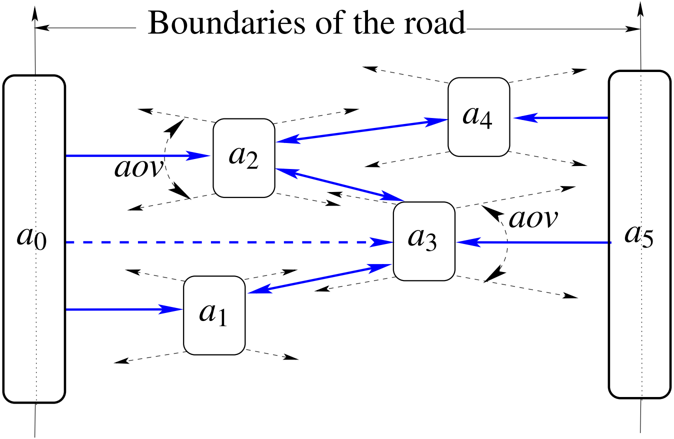
Lateral Influence Graph



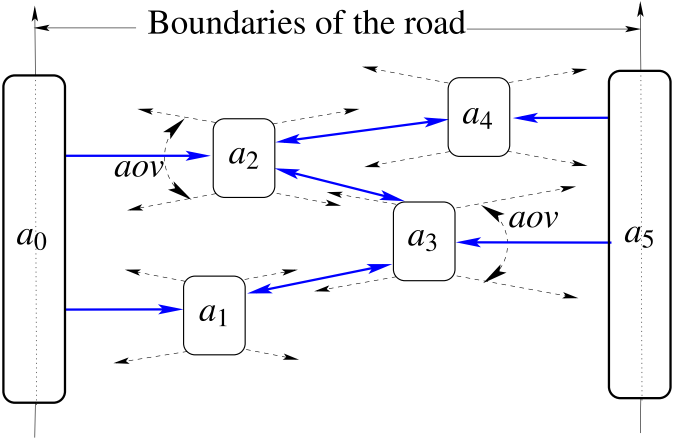
Lateral Influence Graph



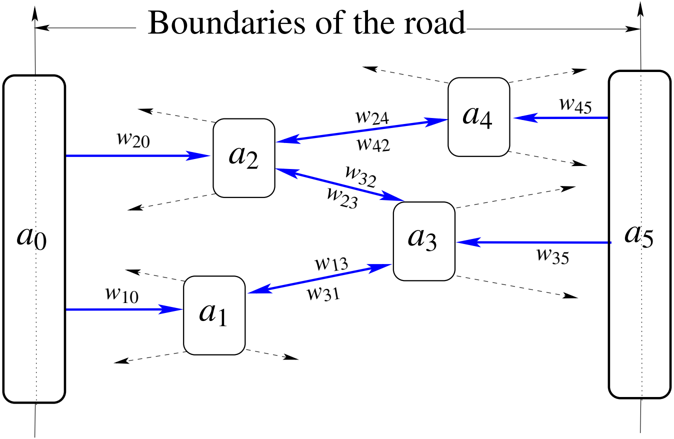
Lateral Influence Graph



Lateral Influence Graph



Lateral Influence Graph



Lane-less driving model for each car -Y direction

- Pseudo-Leader - models road, car or driver induced velocity limits and travels at constant velocity

$$
\begin{aligned}
\dot{y}_0 &= v_{yo} \\
\dot{v}_{y0} &= 0
\end{aligned}
$$

- Influencing Neighbors following model for $i = 1, ..., n$.

$$
\begin{aligned}
\dot{y}_i &= v_{yi} \\
\dot{v}_{yi} &= \sum_{j=\mathcal{N}_i} \left(b_y w_{ij}(v_{yj} - v_{yi}) + k_y \left(w_{ij}(y_j - y_i) - \frac{g_y}{|\mathcal{N}_i|} \right) \right)
\end{aligned}
$$

- Compare with classical single car following models (j is the leader for i):

$$
\dot{v}_{yi} = c \frac{(v_{yj} - v_{yi})}{(y_j - y_i - g_y)^2} \text{ with } j = i - 1
$$

Shorthand using the Laplacian of the influence graph

$$\begin{aligned} \dot{y}_i &= v_{yi} \\ \dot{v}_{yi} &= \sum_{j=\mathcal{N}_i} \left(b_y w_{ij}(v_{yj} - v_{yi}) + k_y \left(w_{ij}(y_j - y_i) - \frac{g_y}{|\mathcal{N}_i|} \right) \right) \end{aligned}$$

$$\begin{bmatrix} \dot{y} \\ \dot{v}_y \end{bmatrix} = \begin{bmatrix} \mathbf{0}_{(n+1)\times(n+1)} & I_{(n+1)\times(n+1)} \\ -k_y\, \mathscr{L}^y & -b_y\, \mathscr{L}^y \end{bmatrix} \begin{bmatrix} y \\ v_y \end{bmatrix} - k_y g_y \begin{bmatrix} \mathbf{0}_{(n+2)\times 1} \\ \mathbf{1}_{n\times 1} \end{bmatrix}$$

Longitudinal Motion

Shorthand using the Laplacian of the influence graph

$$
\begin{aligned}
\dot{y}_i &= v_{yi} \\
\dot{v}_{yi} &= \sum_{j=\mathcal{N}_i} \left(b_y w_{ij}(v_{yj} - v_{yi}) + k_y \left(w_{ij}(y_j - y_i) - \frac{g_y}{|\mathcal{N}_i|} \right) \right)
\end{aligned}
$$

$$
\begin{bmatrix} \dot{y} \\ \dot{v}_y \end{bmatrix} = \begin{bmatrix} \mathbf{0}_{(n+1)\times(n+1)} & I_{(n+1)\times(n+1)} \\ -k_y\ \mathcal{L}^y & -b_y\ \mathcal{L}^y \end{bmatrix} \begin{bmatrix} y \\ v_y \end{bmatrix} - k_y g_y \begin{bmatrix} \mathbf{0}_{(n+2)\times 1} \\ \mathbf{1}_{n\times 1} \end{bmatrix}
$$

Theorem

For time invariant influence graphs containing a directed spanning tree rooted at the leader, as $t \to \infty$
$v_{yi} \to v_{y0} \ \forall\, i \in \{1, \ldots n\}$
$|y_i(t) - y_j(t)| \to 0$ *for all vehicles in same layer*
$|y_i(t) - y_j(t)| \to g_y$ *for vehicles in consecutive layers*

Pros and Cons

- Pros:
  - Proof uses the fact that the Laplacian is triangular
  - Can be extended easily for influences beyond a single layer
  - Works well for relatively dense traffic where influence graphs have no chance to change
- Cons:
  - Cannot handle mesh stability (on the to do list)

Lane-less driving model for each car -X direction

$$\begin{aligned}\dot{x}_i &= v_{xi} \\ \dot{v}_{xi} &= \sum_{j\in\mathcal{N}_i} \left(b_x w_{ij}(v_{xj} - v_{xi}) + k_x w_{ij}(x_j - x_i)\right)\end{aligned}$$

Lateral Motion

Lane-less driving model for each car -X direction

$$\begin{aligned}\dot{x}_i &= v_{xi}\\ \dot{v}_{xi} &= \sum_{j\in\mathcal{N}_i}\left(b_x w_{ij}(v_{xj}-v_{xi})+k_x w_{ij}(x_j-x_i)\right)\end{aligned}$$

$$\begin{bmatrix}\dot{x}\\ \dot{v}_x\end{bmatrix}=\begin{bmatrix}\mathbf{0}_{(n+2)\times(n+2)} & I_{(n+2)\times(n+2)}\\ -k_x\,\mathscr{L}^x & -b_x\,\mathscr{L}^x\end{bmatrix}\begin{bmatrix}x\\ v_x\end{bmatrix}$$

Lateral Motion

Lane-less driving model for each car -X direction

$$\begin{aligned}\dot{x}_i &= v_{xi}\\ \dot{v}_{xi} &= \sum_{j\in\mathcal{N}_i}\left(b_x w_{ij}(v_{xj}-v_{xi}) + k_x w_{ij}(x_j - x_i)\right)\end{aligned}$$

$$\begin{bmatrix}\dot{x}\\ \dot{v}_x\end{bmatrix} = \begin{bmatrix}\mathbf{0}_{(n+2)\times(n+2)} & I_{(n+2)\times(n+2)}\\ -k_x\ \mathscr{L}^x & -b_x\ \mathscr{L}^x\end{bmatrix}\begin{bmatrix}x\\ v_x\end{bmatrix}$$

Theorem

For time invariant influence graph (with appropriate connectednedd assumptions), as $t \to \infty$

$v_{xi} \to 0 \qquad \forall\, i \in \{1,\ldots,n\}$

Position x_i of each vehicle a_i, $i = 1,\ldots,n$ converge to the weighted average of the X-positions of its neighbours

Time varying graphs

- Dwell time: τ

- Dwell time: τ
- **Longitudinal dynamics**

$$\begin{bmatrix} \dot{y} \\ \dot{v}_y \end{bmatrix} = \begin{bmatrix} \mathbf{0}_{(n+1)\times(n+1)} & I_{(n+1)\times(n+1)} \\ -k_y\, \mathscr{L}^y_{\sigma} & -b_y\, \mathscr{L}^y_{\sigma} \end{bmatrix} \begin{bmatrix} y \\ v_y \end{bmatrix} - k_y g_y \begin{bmatrix} \mathbf{0}_{(n+2)\times 1} \\ \mathbf{1}_{n\times 1} \end{bmatrix}$$

- **Lateral dynamics**

$$\begin{bmatrix} \dot{x} \\ \dot{v}_x \end{bmatrix} = \begin{bmatrix} \mathbf{0}_{(n+2)\times(n+2)} & I_{(n+2)\times(n+2)} \\ -k_x\, \mathscr{L}^x_{\sigma} & -b_x\, \mathscr{L}^x_{\sigma} \end{bmatrix} \begin{bmatrix} x \\ v_x \end{bmatrix}$$

Time varying graphs

- Dwell time: τ
- **Longitudinal dynamics**

$$\begin{bmatrix} \dot{y} \\ \dot{v}_y \end{bmatrix} = \begin{bmatrix} \mathbf{0}_{(n+1)\times(n+1)} & I_{(n+1)\times(n+1)} \\ -k_y\, \mathscr{L}_{\sigma}^{y} & -b_y\, \mathscr{L}_{\sigma}^{y} \end{bmatrix} \begin{bmatrix} y \\ v_y \end{bmatrix} - k_y g_y \begin{bmatrix} \mathbf{0}_{(n+2)\times 1} \\ \mathbf{1}_{n\times 1} \end{bmatrix}$$

- **Lateral dynamics**

$$\begin{bmatrix} \dot{x} \\ \dot{v}_x \end{bmatrix} = \begin{bmatrix} \mathbf{0}_{(n+2)\times(n+2)} & I_{(n+2)\times(n+2)} \\ -k_x\, \mathscr{L}_{\sigma}^{x} & -b_x\, \mathscr{L}_{\sigma}^{x} \end{bmatrix} \begin{bmatrix} x \\ v_x \end{bmatrix}$$

Theorem

(Simplified) For each σ, assume that $\vec{\mathscr{G}}_{\sigma}$ contain a spanning tree rooted at a_0. For any $\tau > 0$, the states of the vehicles are uniformly bounded.

Changing Graphs

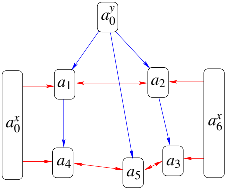
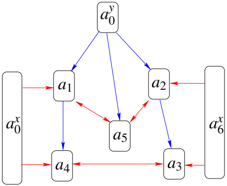
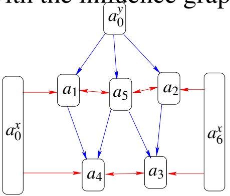


Figure: The convoy of vehicles with the influence graphs



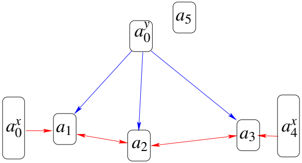
(a) Change in $\vec{\mathscr{G}}^x$



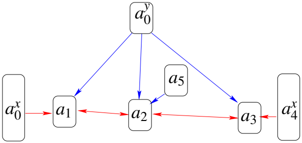
(b) Change in $\vec{\mathscr{G}}^y$ and $\vec{\mathscr{G}}^x$

Figure: Switching influence graphs

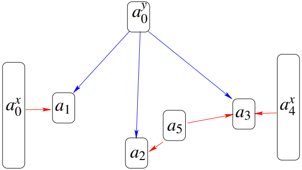
Other Complex Behaviours



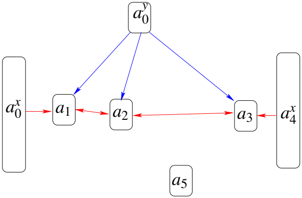
(a) Obstacle not affecting dynamics



(b) Obstacle affecting Y-dynamics of a_2



(c) Obstacle affecting X-dynamics of a_2



(d) Obstacle is crossed

Figure: Changes in influence graph due to obstacle a_4

Velocity Dependent Spacing-Time Invariant

- Inter-vehicle spacing is usuaully based on two factors:
  - Constant time headway policy distance i.e. the distance needed to decelerate to zero = $k_i v_i = \tilde{k} v_i$.
  - Desired constant spacing $g_y > 0$ so that, when $v_0 = 0$, the inter-vehicle spacing is not zero.
- Modified dynamics:

$$\begin{aligned} \dot{y}_i &= v_{yi} \\ \dot{v}_{yi} &= \sum_{j=\mathcal{N}_i} \left(b_y w_{ij}(v_{yj} - v_{yi}) + k_y \left(w_{ij}(y_j - y_i) + \frac{1}{|\mathcal{N}_i|}(g_y - \tilde{k} v_i) \right) \right) \end{aligned}$$

- Theorem: Under usual assumptions, if $\tilde{k} > 0$ and the velocity v_{y0} of a_0 is constant, then:
  1. $v_{yi} \to v_{y0}$ as $t \to \infty \; \forall \; i \in \{1, \ldots n\}$
  2. $|y_i(t) - y_j(t)| \to 0$ as $t \to \infty$ for all $a_i, a_j \in L_k$, $k = 1, 2, ..., m$.
  3. For any two vehicles a_i and a_j such that $a_i \in L_{k-1}$ and $a_j \in L_k$, $k = 1, 2, ..., m$, as $t \to \infty$, $|y_i(t) - y_j(t)| \to g_y + \tilde{k} v_{y0}$.

Switch to ppt

Conclusions

- First attempt at lane-less traffic modeling
- Stability issues were solved
- Standard questions in transportation engineering e.g. flow, capacity etc. are still open.
- Heterogeneity not addressed
- Simulation based on proposed law should be validated with macro level data
    -

Thank You!