Optimizing System Performance in the Event of Feedback Failure

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The problem of maintaining acceptable performance of a perturbed control system during a disruption of the feedback signal is addressed. The objective is to maximize the time during which performance remains within desirable bounds without feedback, given that the parameters of the controlled system are within a specified neighborhood of their nominal values. The existence of an optimal open-loop controller that achieves this objective is proved. It is also shown that a bang-bang input can approximate the performance of the optimal controller.

1 Introduction

Frequently, engineering control systems use feedback to reduce the adverse effects of perturbations and uncertainties on their performance, but disruptions in the feedback signal cannot be completely avoided. For example, disruptions in feedback signal reception associated with an obstruction of the line-of-sight may occur in the control and guidance of space vehicles. In other applications, economic or convenience factors may dictate an operational policy wherein the feedback channel is connected only when performance degrades beyond an acceptable level. Control of diseases through mathematical modeling [1] is one such application, where feedback is available only when measurements on the patient can be made. Similarly, in networked control systems, feedback is often used only intermittently, so as to reduce network traffic (e.g., [2], [3] and [4]). To accommodate such and other applications, it would be beneficial to develop a controller that maximizes the duration of time during which a system can operate without feedback and not violate acceptable error bounds. Of course, in all these applications, the parameters of the controlled system are not perfectly known. The uncertainty about the system's parameter values must be considered when attempting to develop a control strategy.

In this note, we consider the case of a linear time-invariant system Σ whose states are available as output. Let Σ_0 be the nominal version of Σ, and let Σ_ϵ be the system that results when the parameters of Σ experience a perturbation ϵ from their nominal values. The exact value of the perturbation ϵ is not known, but it is known that ϵ is bounded by a pre-specified value d. After possibly having applied an appropriate shift transformation on the signals, we assume that the desired nominal output of Σ is the zero signal. We further assume that a maximal deviation of magnitude $M > 0$ from the nominal output signal is permitted. During feedback failure, a pre-compensator generates an input signal $u(t)$ that drives the system. The objective is to find an input signal $u(t)$ for which the deviation of the output of Σ_ϵ from nominality stays below M for the longest possible time, irrespective of the perturbation ϵ. In formal terms, we are seeking a signal $u(t)$ and a maximal time t_f such that

$$|\Sigma_\epsilon u(t)| \leq M \ for\ all\ 0 \leq t \leq t_f\ and\ all\ \epsilon \leq d. \tag{1}$$

The signal $u(t)$ helps maintain proper operation of Σ for the longest possible time after failure or discontinuation of the feedback channel. At the time t_f, feedback must be restored to prevent further deterioration of the output signal error. Below, we prove that an optimal solution of this problem exists. Furthermore, we show that the optimal signal $u(t)$ can be replaced by a bang-bang signal, with only a negligible effect on system performance. The fact that the optimal signal can be replaced by a bang-bang signal is significant in applications, since bang-bang signals are completely determined by their switching times, and hence their calculation and their implementation are relatively simple.

2 Notation and Problem Formulation

Consider a linear time invariant continuous-time system given by a realization of the form

$$\Sigma : \dot{x}(t) = A'x(t) + B'u(t), \ \ x(0) = x_0. \tag{2}$$

Here, $A' \in \mathcal{R}^{n\times n}$, $B' \in \mathcal{R}^{n\times m}$ and for each t, $x(t) \in \mathcal{R}^n$ and $u(t) \in \mathcal{R}^m$. The state $x(t)$ of the system is available as output; the initial state x_0 is the state of Σ at the time feedback was lost, and thus is known. An important aspect of our discussion is the fact that there are uncertainties about the entries of the matrices A' and B'. We use the $||.||$ to denote the standard l^∞ norm. Given $d > 0$, let $\Delta_A := \{D_A \in \mathcal{R}^{n\times n} : ||D_A|| \leq d\}$ and let $\Delta_B := \{D_B \in \mathcal{R}^{n\times m} : ||D_B|| \leq d\}$. Then, we set $A' := A + D_A$ and $B' := B + D_B$, where A and B are the nominal values of the matrices A' and B', respectively, while

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$D_A \in \Delta_A$ and $D_B \in \Delta_B$ are the perturbation matrices that represent uncertainties. The only information available about our system Σ are the nominal matrices A and B and the uncertainty magnitude d; the entries of the matrices D_A and D_B are not given. We use the notation $D := (D_A, D_B)$ and $\Delta := (\Delta_A \times \Delta_B)$, so that $D \in \Delta$. Needless to say, all engineering systems are subject to input amplitude restrictions determined by the largest amplitude their components can tolerate. We assume that the input amplitude of Σ must be below the bound $K > 0$. This leads us to introduce the set of input functions $U := \{u(t) : ||u(t)|| \leq K \ for\ all\ t > 0\}$, which describes all permissible input functions of Σ. Discussion of the topology of the space within which U resides are omitted in this brief note (see [5]).

Recalling the bound $M > 0$ of (1), and taking into consideration the fact that the output of our system Σ is its state $x(t)$, our performance requirement becomes $x^T(t)x(t) \leq M$, where x^T is the transpose of x. In these terms, our objective is to find an input function $u \in U$ that drives Σ so as to satisfy the state amplitude bound $x^T x \leq M$ for the longest possible time. The time duration during which the square-magnitude $x^T(t)x(t)$ stays below or at the bound M can be represented by $T(M, D, u) := \inf\{t \geq 0 : x^T(t)x(t) > M\}$, where $T(M, D, u) := \infty$ if $x^T(t)x(t) < M$ for all $t > 0$. The initial state satisfies $x_0^T x_0 < M$, since performance was within the desirable range when the feedback channel was disconnected. Hence, we have $T(M, D, u) > 0$.

In addition to its obvious dependence on the time, the state $x(\cdot)$ of Σ also depends on the entries of the pair of matrices D and on the input function $u(t)$. To emphasize these dependencies, we write $x(t, D, u)$. In particular, the entries of D are unknown and unpredictable. As there is no feedback available, the control input function u cannot depend on D. We must guarantee that the bound $x^T(t, D, u)x(t, D, u) \leq M$ is valid for all possible D. In other words, we must consider the "worst case" with respect to the pair of matrices D, and this leads us to the quantity $T^*(M, u) := \inf_{D \in \Delta} T(M, D, u)$. Then, for a particular input signal u, the bound $x^T(t, D, u)x(t, D, u) \leq M$ is satisfied for all $t \in [0, T^*(M, u)]$, irrespective of the entries of D. The duration $T^*(M, u)$ still depends on the input function u, and we can choose any input function in the set U. The best choice will, of course, be an input function u that maximizes $T^*(M, u)$, yielding the maximal duration $t_f^* := \sup_{u \in U} T^*(M, u)$. Assuming that such an input function exists, let us denote it by u^*, so that

$$t_f^* = T^*(M, u^*). \tag{3}$$

In this notation, our objectives can be formally phrased as follows.

Problem 2.1: (i)Determine whether or not an input function $u^* \in U$ exists, and (ii) if there is such a function u^*, describe a method for its computation.

3 Main Results

The following statement shows that the existence of an optimal input function u^* that satisfies (3) is guaranteed. A proof is provided in [5].

Theorem 3.1 *Assume that the system Σ of (2) is unstable and has a non-zero initial state, and let $T^*(M, u) = \inf_{D \in \Delta} T(M, u, D)$. Then, the following are valid. (i) There is a maximal time $t_f^* = \sup_{u \in U} T^*(M, u) < \infty$, and (ii) There is an input function $u^* \in U$ satisfying $t_f^* = T^*(M, u^*)$.*

Thus, we conclude that Problem 2.1 has a solution. Next, we consider the problem of computing the optimal input function u^*. A substantial simplification in the process of computing and implementing u^* is achieved by considering bang-bang functions. A bang-bang function is a function whose component values switch between the bound values $\pm K$ of the system's input; such a function is completely determined by its switching times. The following statement shows that performance close to optimal performance can be achieved by using a bang-bang input function (see [5] for proof).

Theorem 3.2 *In the notation of Theorem 3.1, let t_f^* be the optimal time and let u^* be an optimal input function. Then, for every real number $\delta > 0$, there is a bang-bang input function $u_\delta^\pm \in U$ for which the following are true. (i) $u_\delta^\pm$ has only a finite number of switches, and (ii) The state trajectory $x(t, D, u_\delta^\pm)$ of Σ induced by $u_\delta^\pm$ satisfies $||x(t, D, u^*) - x(t, D, u_\delta^\pm)|| < \delta$ for all $t \in [0, t_f^*]$ and all $D \in \Delta$.*

As the deviation δ from optimal performance can be selected as small as desired, it follows that a bang-bang input signal can always be used without incurring a significant penalty on performance. The cost of making δ smaller is an increase in the number of switches of the bang-bang input function $u_\delta^\pm$. A more detailed discussion of the results presented in this note, including proofs and numerical examples, is provided in [5].

References

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