Maximal Open Loop Operation under Integral Error Constraints

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Abstract

The problem of controlling a linear time invariant system in open loop for the maximal time period is considered. The system is subjected to bounded parametric uncertainties and time varying disturbances. The open loop operation is continued as long as the L_2 norm of the states is bounded by a pre-specified threshold, for every possible value of the uncertainty and disturbance. It is shown that an optimal open loop input that achieves the maximal open loop operation, and a worst disturbance input exist. Moreover, both can be approximated for computational purposes by bang-bang functions with finite number of switches.

Index Terms

bang-bang, max-min, optimization, approximation.

I. Introduction

Feedback is essential for controlling most real world systems affected by noise and uncertainty. However, it is also quite common for the feedback signal to be temporarily unavailable due to a disruption in the feedback channel. For example, in applications such as the control and guidance of space vehicles, accidental disruptions of the line-of-sight may cause extended disruptions in feedback signal reception. In other applications, cost related factors may dictate a resource allocation policy, whereby the feedback channel is connected only when performance degrades beyond an acceptable level. Such is commonly the case in networked control systems, where

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feedback is often used only intermittently, so as to reduce network traffic ([1],[2] and [3]). Moreover in control of biological systems, it is often impractical to continuously measure the output of the controlled system. Such a situation arises, for example, in control of blood glucose concentration in diabetic patients by insulin infusion (e.g. see [4] and [5]). Insulin can be injected according to arbitrary infusion profiles using portable infusion pumps. However, measurement of blood glucose concentration is invasive (finger-prick) and non-invasive glucose monitors are still an area of active research (see [6] and references therein). So the glucose concentration measurements are necessarily intermittent, thus forcing any control algorithm to function in open-loop for the intervals between two consecutive measurements. For these and other applications, we propose to develop an open loop controller that (i) Maximizes the duration of open loop operation (ii) Guarantees that the system does not exceed pre-specified error bounds for *all* uncertainties and disturbances for this maximal duration.

In this article, we concentrate on the case of a linear time-invariant input/state system Σ with given initial conditions (see Figure 1, where C is the open loop controller). We denote by Σ_0 the nominal version of Σ with no disturbance input, and let $\Sigma_{\epsilon,v}$ be the system that results when the parameters of Σ experience a perturbation ϵ from their nominal values and simultaneously, an external disturbance input $v(t)$ is present. The exact value of the perturbation ϵ or the disturbance input $v(t)$ is not known, but it is known that ϵ does not exceed a specified bound d and the disturbance input amplitude is uniformly bounded by some known bound L at all time. After possibly having applied an appropriate shift transformation on the signals, we assume that the desired nominal output of Σ is the zero signal. A maximum *cumulative error* of magnitude $M > 0$ is permitted. Our objective is to find an input signal $u(t)$ that drives the system $\Sigma_{\epsilon,v}$ in such a way as to guarantee that the cumulative or integral error stays below M for as long as possible, irrespective of the (unknown) deviation ϵ and the (unknown) disturbance $v(t)$. Assuming that the feedback is completely disconnected at time $t = 0$, we are seeking a signal $u(t)$ and a maximal time t_f such that

$$\int_0^t \|\Sigma_{\epsilon,v}u(\tau)\|d\tau \leq M \;\; \forall\, 0 \leq t \leq t_f, \;\; \forall\, |\epsilon| \leq d, \;\; \forall\, |v(t)| \leq L \; (t \in [0, t_f]) \tag{1}$$

where $\|\cdot\|$ is some appropriate norm of the system output. Regarding the various scenarios of feedback disruption mentioned in the first paragraph, the signal $u(t)$ provides maximal time for repair, help minimize operational costs, or reduce patient discomfort and improve quality of

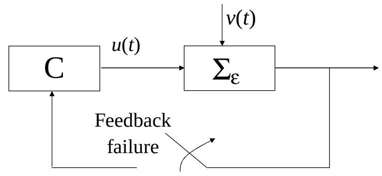


Fig. 1: Feedback Failure

life for patients by reducing the frequency of invasive measurements. Of course, at the time t_f, feedback must be restored to prevent further increase of the error. The conditions when such a feedback signal exists in the allowed set of the bounded controls, were studied in the context of constrained controllability by [7],[8] and the references therein. In this note however, we concentrate on the maximal open loop operation.

The problem of maximizing open loop operation under conditions of feedback failure was introduced for unstable systems in [9],[10] and [11], where instantaneous error constraints were treated. In this article, integral error constraints are considered and the assumption of instability is eliminated. Robustness with respect to external disturbance inputs and a characterization of the worst case noise are also introduced in this article. Moreover, it was felt that the first order necessary conditions used in [11] and elsewhere to characterize the optimal control, are too complicated for computational use. This article tries to address this issue through a novel function approximation method, as opposed to the more conventional methods of writing necessary conditions similar to the Pontryagin's Maximum Principle (PMP) [12], for solving such problems.

We show that this essentially max-min optimal control problem is guaranteed to have a solution. Moreover, this optimal input can be replaced by a bang-bang signal, with only a negligible effect on system performance. For computation and implementation, the fact that the optimal signal can be replaced by a bang-bang signal is significant. This is because a bang-bang signal is completely characterized by its switching instances, which only needs to be computed. In effect, this result transforms the dynamic optimization of (1) into a *finite dimensional* optimization problem. Similarly, we show that there is a worst case disturbance, whose effect can also be approximated by a bang-bang signal. This in turn makes the computation of the worst disturbance signal and, consequently, that of the best control input, numerically feasible.

II. Notation and Problem Formulation

A. Notation

Consider an uncertain linear time invariant continuous-time input/state system Σ given by the realization:

$$\Sigma : \dot{x}(t) = A^{'}x(t) + B^{'}u(t) + G^{'}v(t), \ \ x(0) = x_0 \tag{2}$$

Here, $A^{'} \in R^{n \times n}$, $B^{'} \in R^{n \times m}$, $G^{'} \in R^{n \times p}$ are uncertain system matrices, $x(t) \in R^n$ are the states of the system while $u(t) \in R^m$ and $v(t) \in R^p$ are the control and disturbance input vectors respectively. The initial state x_0 is the state of Σ at the time feedback was lost, and thus is known. We denote the standard ℓ^{∞} -norm for both matrices and vectors by $|| \cdot ||$, given, for a $q \times r$ matrix H by $||H|| := \max_{i=1,\ldots,q;j=1,\ldots,r} |h_{ij}|$, where h_{ij} is the (i,j)-element of H; and for a n-vector $v = [v_1 \ldots v_n]^T$, by $||v|| = \max_{i=1,\ldots,n} |v_i|$. Here v^T denotes the transpose of v. Now, for a real number $d > 0$, let Δ_A, Δ_B and Δ_G be the sets of all real $n \times n$, $n \times m$ and $n \times p$ matrices respectively, with each element in the interval $[-d, d]$. Then, the uncertainties in the matrices $A^{'}$, $B^{'}$ and $G^{'}$ are modeled as follows:

$$A^{'} := A + D_A, \ \ B^{'} := B + D_B, \ \ G^{'} := G + D_G \tag{3}$$

Here A, B and G are the known nominal values of the matrices $A^{'}$, $B^{'}$ and $G^{'}$ of (2), respectively, while $D_A \in \Delta_A$, $D_B \in \Delta_B$ and $D_G \in \Delta_G$ are the unknown perturbation matrices that represent uncertainties. We use the notation $D := (D_A, D_B, D_G)$ and $\Delta := \Delta_A \times \Delta_B \times \Delta_G$ so that $D \in \Delta$.

We would like to search for the solution to (1) over the largest possible input set (e.g. all uniformly bounded measurable functions), which on the other hand should be compact in the appropriate topology. For this purpose, we denote by $L_2^{\alpha,m}$ the Hilbert space of all m-dimensional Lebesgue measurable functions with the inner product $< a, b >= \int_0^{\infty} e^{-\alpha t} a(t)^T b(t) dt$, where $a(t), b(t) \in L_2^{\alpha,m}$ and $\alpha > 0$. We note here that we interpret all integrals in this article in the Lebesgue sense. We assume that the control input $u(t)$ as well as the disturbance $v(t)$ for our system Σ are uniformly bounded respectively by $K > 0$ and $L > 0$, and thus are elements of the Hilbert spaces $L_2^{\alpha,m}$ and $L_2^{\alpha,p}$ respectively. The set of all permissible input functions of Σ is defined as follows:

$$U \ := \ \{ \, u \, \in \, L_2^{\alpha,m} \, : \, \|u(t)\| \leq K \text{ for all } t \geq 0 \}, \tag{4}$$

and that of possible disturbance inputs as:

$$V := \{v \in L_2^{\alpha,p} : \|v(t)\| \leq L \text{ for all } t \geq 0\}. \tag{5}$$

We call the pair (D, v) as the *disturbance pair* and the set $\Delta \times V$ as the *disturbance range*. Finally, recalling the bound $M > 0$ of (1), and that the output of Σ is its state $x(t)$, we formulate our performance requirement as:

$$e(t) := \int_0^t x^T(\tau)x(\tau)d\tau \leq M \ \ \forall \, (t, D, v) \in [0, t_f] \times \Delta \times V. \tag{6}$$

B. Problem Statement

First, we introduce a functional that represent the time duration during which the cumulative error $e(t)$ (defined in (6) and written explicitly as $e(t; D, v, u)$) stays below or at the bound M.

$$T(M, D, v, u) := \inf\{t \geq 0 : e(t; D, v, u) > M\}, \tag{7}$$

where $T(M, D, v, u) := \infty$ if $e(t; D, v, u) \leq M$ for all $t \geq 0$. As $e(0; D, v, u) = 0$, we have $T(M, D, v, u) > 0$. Since, the entries of the matrices D and the disturbance input $v(t)$ are unknown and unpredictable, we must consider the "worst case" with respect to the uncertainty matrices D and the disturbance input $v(t)$, and this leads us to the quantity

$$T^*(M, u) := \inf_{(D,v)\in\Delta\times V} T(M, D, v, u) \tag{8}$$

Then, for a particular choice of u, inequality (6) is valid for all $t \in [0, T^*(M, u)]$, irrespective of the entries of D or the particular realization of the disturbance $v(t)$. The best choice of $u(t) \in U$ will, of course, be the one that maximizes $T^*(M, u)$, yielding the maximal duration

$$t_f^* := \sup_{u\in U} T^*(M, u). \tag{9}$$

Assuming that such an input function exists, denote it by u^*, so that $t_f^* = T^*(M, u^*)$. In this notation, our objectives can be formally phrased as follows.

Problem 1: (i) Determine whether or not an input function $u^* \in U$ exists, and (ii) if there is such a function u^* , describe a method for its computation.

As we can see from (8) and (9), the calculation of the input function u^* involves the solution of a max-min optimization problem. In the next section, we show that an optimal solution u^* exists within our framework.

III. Existence of an Optimal Solution

We show that for any choice of the control input $u(t)$, the cumulative error $e(t; D, v, u)$ must escape the bound M for at least one combination of the disturbance pair $(D, v) \in \Delta \times V$.

Lemma 1: For each input function $u(t) \in U$ and for every disturbance range $\Delta \times V$, there is a disturbance pair $(D, v) \in \Delta \times V$ for which $T(M, D, v, u) < \infty$.

Proof: Consider the solution to (2): $x(t; D, v, u) = e^{A' t} x_0 + \int_0^t e^{A'(t-\tau)} B' u(\tau) d\tau + \int_0^t e^{A'(t-\tau)}$ $G' v(\tau) d\tau$. Let for some fixed $u_0 \in U$, $T(M, D, v, u_0) = \infty$ for every disturbance pair $(D, v) \in \Delta \times V$. Then it is necessary that

$$\|x(t; D, v, u_0)\| \to 0 \text{ as } t \to \infty \;\; \forall\, (D, v) \in \Delta \times V \tag{10}$$

Let $v_1(t) = 0$ for ($0 \leq t < \infty$), then $v_1(t) \in V$. By (10), for every permissible $D \in \Delta$, the j^{th} element of $x(t)$: $x_j(t; D, v_1, u_0) \to 0$ as $t \to \infty$ ($j = 1, .., n$). This in turn implies that for every $D \in \Delta$, $||e^{A' t} x_0 + \int_0^t e^{A'(t-\tau)} B' u_0(\tau) d\tau|| \to 0$ as $t \to \infty$. Using this in (10), for any $(D, v) \in \Delta \times V$, $|| \int_0^t e^{A'(t-\tau)} G' v(\tau) d\tau|| \to 0$ as $t \to \infty$. However, noting that $\exists (D_A, D_G) \in \Delta_A \times \Delta_G$ for which the pair (A', G') is controllable, it is easy to see that the last equation does not hold for all $v(t) \in V$. ∎

Clearly it follows that for each $u \in U$, $T^*(M, u) < \infty$. For solving part (i) of Problem 1, by the standard Weierstrass theorem, we just need to show that $T^*(M, u)$ is weakly upper semi-continuous.

Lemma 2: For a given disturbance pair $(D, v) \in \Delta \times V$, the function $T(M, D, v, u)$ of (7) is weakly upper semi-continuous in u.

Proof: Fix the perturbation pair (D, v). For a weakly convergent sequence of input functions $u_1, u_2, ..., \in U$, say $u_n \xrightarrow{w} u_0$, the sequence of solution to (2) : $x(t, u_1)$, $x(t, u_2)$, ... converges pointwise to the vector $x(t, u_0)$ by definition. Now recall the definition of $e(t)$ as in (6) : $e(t, u) := \int_0^t x^T(\tau; u) x(\tau; u) d\tau$. For every $\theta < \infty$, there is a $P < \infty$ such that $x^T(t; u_n) x(t; u_n) \leq P$ for $t \in [0, \theta]$ and for all n. Also $\lim_{n \to \infty} x^T(t; u_n) x(t; u_n) = x^T(t; u_0) x(t; u_0)$ for every $t \in [0, \theta]$. Interpreting the integral in (6) as a Lebesgue integral, it follows that $e(t, u_n) \to e(t, u_0)$ for each $t \in [0, \theta]$. (e.g. see [13], pg 69). Hence we can conclude that as $u_n \xrightarrow{w} u_0$, $e(t, u_n) \to e(t, u_0)$ pointwise for each $t < \theta < \infty$.

Next, consider the following functional defined over error trajectories: $\Theta(e) := \inf\{t \geq 0 : e(t) > M\}$, where $\Theta(e) := \infty$ if $e(t) \leq M$ for all $t \geq 0$. Let $e_1(t)$, $e_2(t)$, ... be a sequence of error trajectories that converges (pointwise) to the function $e_0(t)$ for each $t \geq 0$, and assume that θ is large enough such that $\Theta(e_0) < \theta < \infty$. We show that, for any $\epsilon > 0$, there is an integer $N > 0$ that satisfies the following condition: $\Theta(e_n) - \Theta(e_0) < \epsilon$ for all integers $n > N$. Clearly, if there is an integer $N > 0$ for which $\Theta(e_n) \leq \Theta(e_0)$ for all $n > N$, then our claim is true. So let us examine the case when there is no such N. In such case, there is a subsequence $n_1, n_2, \ldots$ such that $\Theta(e_{n_k}) > \Theta(e_0)$ for all integers $k > 0$. Set $T_{e_0} := \Theta(e_0)$; since $\Theta(e_0)$ is bounded by assumption, we have $T_{e_0} < \infty$. By the definition of $\Theta(e)$, the following is true for every real number $\epsilon > 0$: there is a time $t' \in [T_{e_0}, T_{e_0} + \epsilon\ )$ such that $e_0(t') > M$. Now, by assumption, we have that $e_n(t) \to e_0(t)$ pointwise for every $t \in [0, \theta]$. Therefore, setting $t = t'$, there must be an integer $N > 0$ such that for $n > N$, $|e_0(t') - e_n(t')| < [e_0(t') - M]/2$. For such n, we have $e_n(t') = e_0(t') - [e_0(t') - e_n(t')] \geq\ e_0(t') - [e_0(t') - M]/2 \geq\ e_0(t')/2 + M/2 > M$, i.e., $e_n(t') > M$. By the last inequality, $\Theta(e_n) \leq t'$; whence $\Theta(e_n) < \Theta(e) + \epsilon$ for all $n > N$, and $\Theta(e)$ is upper semi-continuous. Hence the composition $T(M, D, v, u) = \Theta(e(t; D, v, u))$ is weakly upper semi-continuous in u. ∎

The next result resolves Problem 1(i).

Theorem 1: Let $T^*(M, u)$ be given by (8). Then, the following are valid.

(i) There is a maximal time $t_f^* := \sup_{u \in U} T^*(M, u) < \infty$, and

(ii) There is an input function $u^* \in U$ satisfying $t_f^* = T^*(M, u^*)$.

Proof: The set U is weakly compact in $L_2^{\alpha,m}$ (see [11], [9]). Moreover, by Lemma 1 and 2, $T^*(M, u)$ of (8) is weakly upper semi-continuous in $u(t)$ (e.g., [14], p. 49). Hence the result follows from the generalized Weierstrass Theorem (e.g., [15], pg. 152). ∎

IV. Bang-bang Approximation

We define a bang-bang input as some $u(t) \in U$, whose each component assumes only the extreme values $\{+K, -K\}$ for all except a finite number of time instances in $[0, t_f^*]$. In this section we show that the *effect* (to be made precise in Theorem 2) of the optimal input $u^*(t)$ can be approximated by a bang-bang signal over the entire disturbance range. Compare with [11].

Theorem 2: Let Σ be the system of Theorem 1 and let t_f^* be the optimal time of Theorem 1(i). Then, for every $\epsilon > 0$, there is a bang-bang input function $u^{\pm}(t) \in U$ for which the following are true.

(i) $u^{\pm}(t)$ has only a finite number of switches, and

(ii) The error trajectory $e(t; D, v, u^{\pm})$ of Σ created by $u^{\pm}$ satisfies $|e(t; D, v, u^{*}) - e(t; D, v, u^{\pm})| < \epsilon$ for all $t \in [0, t_f^*]$ and all $(D, v) \in \Delta \times V$.

Proof: Fix a real number $\epsilon > 0$. Recall that all input functions $u(t)$ of Σ are bounded by K, that $t_f^* < \infty$ by Theorem 1, and that all perturbation matrices $D \in \Delta$ have elements of magnitude not exceeding $d > 0$. Let $\eta > 0$ be a real number (to be chosen later), and recall that $A' = A + D_A$ and $B' = B + D_B$, where $D_A \in \Delta_A$ and $D_B \in \Delta_B$. Due to the uniform continuity of the function $e^{A' t}$, there is a real number $\delta(\eta) > 0$ such that the function $\mu(t', t) := e^{-A' t'} - e^{-A' t}$ satisfies $||\mu(t', t)|| \leq \eta$ for all $t', t \in [0, t_f^*]$ satisfying $|t' - t| < \delta(\eta)$. Also, let $\beta := \sup\{||B + D_B|| : D_B \in \Delta_B\}$ and let $N := \sup_{D_A \in \Delta_A, t \in [0, t_f^*]} \max\{\|e^{A' t}\|, \|e^{-A' t}\|\}$; here, N exists due the fact that all involved quantities are bounded. Let $0 < \gamma \leq \delta(\eta)$ be any number for which t_f^*/γ is an integer. Denote $t_f^*/\gamma =: Q$. We build a partition of the interval $[0, t_f^*]$ into segments of length γ , namely, the partition determined by the points $0, \gamma, 2\gamma, .., Q\gamma$. Recalling that the input function $u(t)$ of Σ is an m-dimensional vector with each component bounded by K, we define a bang-bang input function $u^{\pm}(t)$ through its components $u_1^{\pm}(t), u_2^{\pm}(t), ..., u_m^{\pm}(t)$ as follows: for each component $i = 1, 2, ..., m$, we select in each interval $[q\gamma, (q+1)\gamma]$ a switching time θ_{qi} , where $q = 0, 1, 2, .., Q$ and $i = 1, 2, ..., m$. Now set

$$u_i^{\pm}(t) = \begin{cases} +K & \text{for } t \in [q\gamma, \theta_{qi}) \\ -K & \text{for } t \in [\theta_{qi}, (q+1)\gamma), \end{cases}$$

where the value of θ_{qi} is selected to satisfy the equality $\int_{q\gamma}^{(q+1)\gamma} u_i^*(\tau) d\tau = K \int_{q\gamma}^{\theta_{qi}} d\tau - K \int_{\theta_{qi}}^{(q+1)\gamma} d\tau$ $= K[2(\theta_{qi} - q\gamma) - \gamma]$. Note that a solution θ_{qi} exists for all $q = 1, 2, .., Q$ and all $i = 1, 2, ..., m$ due to the fact that $|u_i^*(t)| \leq K$ for all $t \geq 0$. Then, we obtain the equality

$$\int_{q\gamma}^{(q+1)\gamma} [u_i^*(\tau) - u_i^{\pm}(\tau)] d\tau = 0, \quad q = 1, 2, .., Q \tag{11}$$

Finally, let $x^{\pm}(t)$ be the state trajectory of Σ for input function $u^{\pm}(t)$, and let $x^*(t)$ be the state trajectory induced by the optimal input function $u^*(t)$. Fix $t \in [0, t_f^*]$ and let q be an integer such that $q\gamma < t \leq (q+1)\gamma$. Noting that the perturbation matrix D and the noise input $v(t)$ is the same in both cases (we are activating the same system sample with identical disturbances), one obtains (using (11))

$$\|x^*(t) - x^{\pm}(t)\|$$

$$\leq N\|\int_0^t e^{-A'\tau}B'[u^*(\tau)-u^{\pm}(\tau)]d\tau\|$$
$$=N\|\sum_{r=0}^{q-1}\int_{r\gamma}^{(r+1)\gamma} e^{-A'\tau}B'[u^*(\tau)-u^{\pm}(\tau)]d\tau+\int_{q\gamma}^{t} e^{-A'\tau}B'[u^*(\tau)-u^{\pm}(\tau)]d\tau\|$$
$$\leq N\|\sum_{r=0}^{q-1}[e^{-A'r\gamma}B'\int_{r\gamma}^{(r+1)\gamma}[u^*(\tau)-u^{\pm}(\tau)]d\tau+\int_{r\gamma}^{(r+1)\gamma}\mu(\tau,r\gamma)B'[u^*(\tau)-u^{\pm}(\tau)]d\tau]\|$$
$$+N\|\int_{q\gamma}^{t} e^{-A'\tau}B'[u^*(\tau)-u^{\pm}(\tau)]d\tau\|$$
$$\leq N\|B'\|\{\sum_{r=0}^{q-1}\int_{r\gamma}^{(r+1)\gamma}\|\mu(\tau,r\gamma)\|[\|u^*(\tau)\|+\|u^{\pm}(\tau)\|]d\tau+\int_{q\gamma}^{t}\|e^{-A'\tau}\|[\|u^*(\tau)\|+\|u^{\pm}(\tau)\|]d\tau\}$$
$$\leq 2KN\beta[\eta t_f^*+N\gamma]$$

Now, $\sup_{[0,t_f^*]}|e(t;D,v,u^*)-e(t;D,v,u^{\pm})|\leq n\sup_{[0,t_f^*]}\int_0^t[\|x^*(\tau)-x^{\pm}(\tau)\|\|x^*(\tau)+x^{\pm}(\tau)\|]d\tau$ $\leq(nt_f^*)\sup_{[0,t_f^*]}\|x^*(\tau)-x^{\pm}(\tau)\|.\sup_{[0,t_f^*]}\|x^*(\tau)+x^{\pm}(\tau)\|$. Now let $S=\sup_{[0,t_f^*]}\|x^*(\tau)+x^{\pm}(\tau)\|$. Clearly $S<\infty$. Hence $\sup_{[0,t_f^*]}|e(t;D,v,u^*)-e(t;D,v,u^{\pm})|\leq 2SKNn\beta t_f^*[\eta t_f^*+N\gamma]$.

For any choice of $\epsilon>0$, we can choose the value of η so that $2SKNn\beta\eta(t_f^*)^2<\epsilon/2$. Then, we choose γ so that $0<\gamma\leq\min\{\delta(\eta),\epsilon/(4SKN^2n\beta t_f^*)\}$ and t_f^*/γ is an integer. For these selections, we obtain $|e(t;D,v,u^*)-e(t;D,v,u^{\pm})|<\epsilon$ for all $t\in[0,t_f^*]$ and for all $(D,v)\in\Delta\times V$. ■

Note that the $u^{\pm}(t)$, while being independent of the the perturbation/disturbance, approximates the *effect* of $u^*(t)$ over all permissible perturbation matrices and incident disturbances. It does not necessarily approximate $u^*(t)$ itself. In effect, the above theorem transforms a practically infeasible dynamic optimization for $u^*(t)$ into a finite dimensional search for the best switching instances for $u^{\pm}(t)$.

Remark 1: In Theorem 2, the cost of making the error ϵ smaller is an increase in the number of switches of the bang-bang function $u^{\pm}(t)$. This can be seen by examining the inequality $0<\gamma\leq\min\{\delta(\eta),\epsilon/(4SKN^2n\beta t_f^*)\}$ and recalling that the number of switches is (in general) t_f^*/γ. Hence decrease of ϵ forces γ to decrease, leading to an increase in the number of switches. A priori estimate of the number of switches required would be evidently useful for practical application of this theory. However, it was shown in [9] that the true optimal solution itself can be bang-bang with an infinite number of switches over a finite time interval. Hence, in general, an upper bound on the number of switches is not available for this class of problems. For practical purposes, the required number of switches may be computed by repeatedly calculating the maximal time for increasing number of switches, until no appreciable improvement occurs with the increase in the number of switches.

V. Worst Case Disturbance

Note that we still have not completely resolved Problem 1(ii). In particular, utilizing Theorem 2, the computation of the optimal switching times of $u^{\pm}(t)$ involves (i) generating all possible bang-bang $u^{\pm}$'s (ii) calculating $T^*(M,u^{\pm}) = \inf_{(D,v)\in\Delta\times V} T(M,D,v,u^{\pm})$ for each $u^{\pm}$ (iii) finding the maximum $T^*(M,u^{\pm}) \approx t_f^*$. For step (ii) above, we still must develop a method for finding the worst case disturbance pair (D,v) corresponding to each candidate for $u^{\pm}(t)$. The situation is complicated by the fact that the functional $T(M,D,v,u)$ is not lower semi-continuous in (D,v), and hence the existence of the minimum $T^*(M,u^{\pm})$ is not guaranteed.

A. Existence of the Worst Disturbance Pair

First we show that there is a worst case disturbance pair (D_0,v_0) corresponding to any control input $u^0(t)$. For this purpose we need to introduce the following functional that can be thought of as dual to $T(M,D,v,u)$. Let

$$T_\ell(M,D,v,u) := \min\{t\geq 0 : \ e(t;D,v,u)=M\} \tag{12}$$

where $T_\ell(M,D,v,u) := \infty$ if $e(t;D,v,u) < M$ for all $t \geq 0$. Similar to (8), we introduce the notation: $T_\ell^*(M,u) := \inf_{(D,v)\in\Delta\times V} T_\ell(M,D,v,u)$. Using dual arguments of Lemma 2 for lower semi-continuity, and noting that the set V is weakly compact in $L_2^{\alpha,p}$, it follows:

Lemma 3: : For any fixed input $u^0 \in U$, the functional $T_\ell(M,D,v,u^0)$ is weakly lower semi-continuous in $(D,v)\in\Delta\times V$ and there is a $(D_0,v_0)\in\Delta\times V$ such that

$$T_\ell(M,D_0,v_0,u^0) = T_\ell^*(M,u^0). \tag{13}$$

We denote $D_0 := (D_{A0}, D_{B0}, D_{G0})$, $A_0' = A + D_{A0}$, $B_0' = B + D_{B0}$, and $G_0' = G + D_{G0}$. Next, we claim that for any u^0 , $T_\ell^*(M,u^0)$ forms the greatest lower bound for $T(M,D,v,u^0)$.

Theorem 3: : For any fixed $u^0 \in U$, $T_\ell^*(M,u^0) = \inf_{(D,v)\in\Delta\times V} T(M,D,v,u^0)$.

The proof of this theorem is divided into the following lemmas:

Lemma 4: For any $u^0 \in U$, $T_\ell^*(M,u^0) \leq \inf_{(D,v)\in\Delta\times V} T(M,D,v,u^0)$.

Proof: Let $\inf_{(D,v)\in\Delta\times V} T(M,D,v,u^0) < T_\ell^*(M,u^0)$. Then there is a $(D_1,v_1) \in \Delta \times V$ such that $T(M,D_1,v_1,u^0) < T_\ell^*(M,u^0)$. However, from (7) and (12), $T_\ell(M,D_1,v_1,u^0) \leq T(M,D_1,v_1,u^0) \Rightarrow T_\ell(M,D_1,v_1,u^0) < T_\ell^*(M,u^0)$. This contradicts the definition of $T_\ell^*(M,u^0)$. ■

Lemma 5: For any $u^0 \in U$, let $T_\ell(M,D_0,v_0,u^0)$ be as in (13). Then there is an $\epsilon > 0$ and a (D_0,v_0) satisfying (13) such that $e(t;D_0,v_0,u^0)$ is strictly increasing on $[T_\ell(M,D_0,v_0,u^0), T_\ell(M, D_0,v_0,u^0)+\epsilon]$.

Proof: : We can assume that at least one element of $G_0^{'}$ is non-zero. If every element of $G_0^{'} = 0$, then it is equivalent with having $v_0(t) = 0$ over $[0,t_f^*]$. Thus we can replace some element of $G_0^{'}$ with a permissible non-zero element while assuming $v_0(t) = 0$ over $[0,t_f^*]$. This switch would keep $T_\ell(M,D_0,v_0,u^0)$ same while simplifying some of the following arguments. Denote $S_\epsilon := [T_\ell(M,D_0,v_0,u^0), T_\ell(M,D_0,v_0,u^0)+\epsilon]$. Consider the expression: $e(t;D_0,v_0,u^0) = \int_0^t x^T(\tau;D_0,v_0,u^0)x(\tau;D_0,v_0,u^0)d\tau$. Assume that for a fixed arbitrary $\epsilon > 0$,

$$e(t;D_0,v_0,u^0) = M \text{ holds for all } t \in S_\epsilon \tag{14}$$

Noting that $x^T(\cdot)x(\cdot) \geq 0$, (14) can only hold if $x^T(t;D_0,v_0,u^0)x(t;D_0,v_0,u^0) = 0$ for all $t \in S_\epsilon$. This in turn implies that $x(t;D_0,v_0,u^0) = 0$ for all $t \in S_\epsilon$. It follows that

$$\dot{x}(t;D_0,v_0,u^0) = 0 \Rightarrow A_0^{'}x(t;.) + B_0^{'}u^0(t) + G_0^{'}v_0(t) = 0 \;\; \forall\, t \in S_\epsilon \tag{15}$$

Let $v_0(t) = [\delta_1...\delta_p]^T$ for $t \in S_\epsilon$ where $\delta_i (i = 1,\ldots,p)$ are real constants. Then (15) has to hold for all $||\delta_i|| \leq L$ $(i = 1,\ldots,p)$. This is clearly untrue, since $G_0^{'}$ has at least one non-zero element. This argument holds for any $\epsilon > 0$. Hence (14) is false and $\exists\epsilon > 0$ such that $e(t;D_0,v_0,u^0)$ is strictly increasing in the interval $t \in [T_\ell(M,D_0,v_0,u^0), T_\ell(M,D_0,v_0,u^0)+\epsilon]$. ■

Proof of Theorem 3: By Lemma 4, $T_\ell(M,D_0,v_0,u^0) \leq \inf_{(D,v)\in\Delta\times V} T(M,D,v,u^0)$. Now assume that $T_\ell(M,D_0,v_0,u^0) < \inf_{(D,v)\in\Delta\times V} T(M,D,v,u^0)$. By Lemma 5, $e(t;D_0,v_0,u^0)$ is strictly increasing in a small enough neighborhood $[T_\ell(M,D_0,v_0,u^0), T_\ell(M,D_0,v_0,u^0)+\epsilon]$. Hence, from definitions (7) and (12), $T(M,D_0,v_0,u^0) = T_\ell(M,D_0,v_0,u^0) < \inf_{(D,v)\in\Delta\times V} T(M, D,v,u^0)$. This contradiction implies the statement of the theorem. ■

B. *Computation of the worst disturbance pair*

For an arbitrary input $u^0(t)$, we pose a standard minimum time optimal problem (e.g. see [12]) below to compute $T_\ell^*(M,u^0)$. Using the notation of (13), assume for the moment that worst case parameter disturbance matrices D_0 corresponding to $u^0(t)$ are known.

Problem 2: Find $\min_{v \in V} t_f$ such that the following constraints are satisfied: $\dot{x}(t) = A_0^{'}x(t) + B_0^{'}u^0(t) + G_0^{'}v(t)$, $x(0) = x_0$ for $0 \leq t \leq t_f$ and $\int_0^{t_f} x^T(t)x(t)dt = M$.
Let the solution of Problem 2 be $v_0(t)$ and the minimum final time $T_\ell^*(M, u^0)$. It is easy to see that $v_0(t)$ can be replaced by a bang-bang input, without appreciable effect on the final time. Using the notation of Problem 2, the following result can be derived similarly to Theorem 2.

Lemma 6: : For any $\epsilon > 0$ and $u^0(t) \in U$, there is a bang-bang function $v^{\pm}(t) \in V$ with a finite number of switches, such that the solution $y(t)$ of the state equation: $\dot{y}(t) = A_0^{'}y(t) + B_0^{'}u^0(t) + G_0^{'}v^{\pm}(t)$ with the initial condition $y(0) = x_0$, satisfies $| \int_0^{T_\ell^*(M,u^0)} y^T(t)y(t)dt - M| < \epsilon$.
This result qualifies the bang-bang function $v^{\pm}(t)$ to be an approximate solution of Problem 2. Hence, for a certain choice of the control input (say u^0), one needs to simultaneously optimize on the switching instants of $v^{\pm}(t)$ and the parameters of D to find an arbitrarily close estimate of the minimum terminal time $T_\ell^*(M, u^0)$. It may be noted that the algorithm used to perform this search can be implemented/improved by a variety of finite dimensional min-max optimization techniques developed in the literature (e.g. see [16],[17] and the references therein). The main contribution of this note is to transform the dynamic optimization problem into a finite dimensional max-min problem, which in turn makes it solvable by a wide variety of numerical procedures. We provide a simple example illustrating the main ideas of approximate bang-bang solution introduced in this work.

Example 1: Consider the system $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 + a_1 & 1 \\ 0 & 1.2 + a_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} (u + v)$ where a_1 and a_2 are uncertain. Only the ranges are known: $-0.1 \leq a_1, a_2 \leq 0.1$; The input $u(t)$ is bounded: $|u(t)| \leq 2 \ \forall t$; the disturbance $|v(t)| \leq 0.2 \ \forall t$; and the initial condition $x(0) = [-1 \ 1]^T$. We assume that the bound on the cumulative error is $M = 5$. As in (6), our objective is to find the $u(t)$ such that the inequality: $e(t) := \int_0^t (x_1^2(\tau) + x_2^2(\tau))d\tau \leq 5$ holds for all $0 \leq t \leq t_f$, for all $-0.1 \leq a_1, a_2 \leq 0.1$ and for all $|v(t)| \leq 0.2$, over the longest time t_f .
Note that even for this extremely simple case, computing the true optimal solution is numerically infeasible using brute force methods and extremely complicated, if at all possible, by shooting techniques [18] due to the complexity of the PMP conditions. However using Theorem 2 and Lemma 6, we only need to search for the solution among bang-bang inputs and disturbances. Hence we generate all possible bang-bang inputs in the interval $[0, 2.5]$. For each of them, we use a global multilevel coordinate search algorithm [19] to find the worst parameters and worst

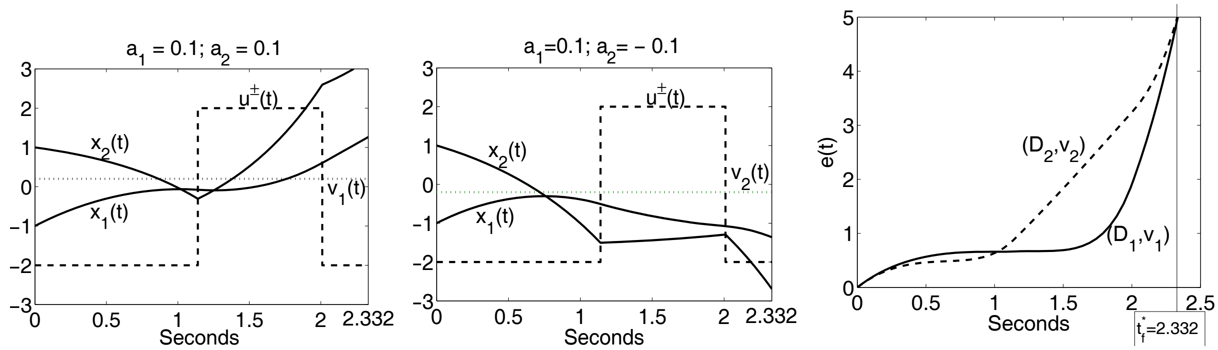
TABLE I: Max-min time vs. Number of Switches

No. of Switches	Approximate Optimal Input	Worst Disturbance Pairs	Max-min time
1	$u^{\pm}(t) = \begin{cases} -2 & \forall t \in [0, 1.166) \\ +2 & \forall t \in [1.166, \infty] \end{cases}$	$\{(a_1 = 0.1, a_2 = -0.1); v^{\pm}(t) = -0.2\ \forall t\}$. $\{(a_1 = 0.1, a_2 = 0.1); v^{\pm}(t) = 0.2\ \forall t\}$.	2.314
2	$u^{\pm}(t) = \begin{cases} -2 & \forall t \in [0, 1.14) \bigcup [2.01, \infty) \\ +2 & \forall t \in [1.14, 2.01) \end{cases}$	$\{(a_1 = 0.1, a_2 = -0.1); v^{\pm}(t) = -0.2\ \forall t\}$. $\{(a_1 = 0.1, a_2 = 0.1); v^{\pm}(t) = 0.2\ \forall t\}$.	2.332
3	$u^{\pm}(t) = \begin{cases} -2 & \forall t \in [0, 1.14) \bigcup [2.01, 2.31) \\ +2 & \forall t \in [1.14, 2.01) \bigcup [2.31, \infty) \end{cases}$	$\{(a_1 = 0.1, a_2 = -0.1); v^{\pm}(t) = -0.2\ \forall t\}$. $\{(a_1 = 0.1, a_2 = 0.1); v^{\pm}(t) = 0.2\ \forall t\}$.	2.332

switching times of the bang-bang disturbance signal. The number of allowed switches of the input and disturbance is sequentially increased until there is no further increase in the max-min time. As mentioned above, this simple scheme can be improved using numerical methods developed by other authors for finite dimensional min-max optimization. It is found that the approximate optimal (bang-bang) input (see Figure 2a and 2b) is given by:

$$u^{\pm}(t) = \begin{cases} -2 & \forall t \in [0, 1.14) \bigcup [2.01, \infty) \\ +2 & \forall t \in [1.14, 2.01) \end{cases} \tag{16}$$

The worst disturbance pairs associated with this input are found to be $(D_1, v_1) := \{(a_1 = 0.1, a_2 = 0.1); v^{\pm}(t) = 0.2\ \forall t\}$ and $(D_2, v_2) := \{(a_1 = 0.1, a_2 = -0.1); v^{\pm}(t) = -0.2\ \forall t\}$. The combination of $u^{\pm}(t)$ of (16) and each of these disturbance pairs produce the max-min open loop time $t_f^* = 2.332$ seconds. As is observed in the Table I, no appreciable improvement (less than 0.001 second) occurs by increasing the number of switches beyond two. The state trajectories corresponding to the two-switch optimal input $u^{\pm}(t)$ and the disturbance pair (D_1, v_1) are shown in Figure 2a while the state trajectories produced by $u^{\pm}(t)$ and (D_2, v_2) are depicted in Figure 2b. The cumulative error trajectories generated in these two cases: $\{u^{\pm}(t), (D_1, v_1)\}$ and $\{u^{\pm}(t), (D_2, v_2)\}$, are shown in Figure 2c from which the max-min time $t_f^* = 2.332$ seconds, is apparent.



(a) (b) (c)

Fig. 2: (a) State Trajectories generated by the combination $\{u^{\pm}(t), (D_1, v_1)\}$ (b) State Trajectories generated by the combination $\{u^{\pm}(t), (D_2, v_2)\}$ (c) Error trajectories corresponding the combination $\{u^{\pm}(t), (D_1, v_1)\}$(solid line) and $\{u^{\pm}(t), (D_2, v_2)\}$ (dashed line)

VI. Conclusion

In this note we propose an approximate solution to the robust maximal time problem under integral error constraints. While the optimal solution is hard to compute, the approximate solution is numerically computable because of its bang-bang nature; and a complete solution involves finding the optimal switching instances of $u^{\pm}(t)$. The temporary absence of feedback is quite common and hence this method has wide practical applicability.

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