# Partial Pole Placement and Controller Norm Optimization over Polynomial Stability Region

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**Abstract:** An arbitrary subset $(n - m)$ of the $n$ closed loop eigenvalues of an $n^{th}$ order continuous time single input linear time invariant (LTI) system is to be placed using full state feedback, at pre-specified locations in the complex plane. The remaining $m$ closed loop eigenvalues can be placed anywhere inside a pre-defined region in the complex plane. This region constraint on the unspecified poles is translated into an ellipsoidal constraint on the characteristic polynomial coefficients through a convex inner approximation for polynomial stability regions. The closed loop locations for these $m$ eigenvalues are chosen through an explicit minimization of the feedback gain vector norm leading to an efficiently solvable semidefinite program. The required controller effort is thus minimized leading to less expensive actuators.

*Keywords:* Linear Systems, Time Invariant Systems, LMIs, Convex Optimization, Power System Stability.

## 1. INTRODUCTION

All the closed loop poles of a controllable single input linear time invariant (LTI) systems can be assigned arbitrary locations in the complex plane using full state feedback. However, in many applications, such as the power system example considered in this article, the control engineer is concerned with only a subset of the open loop poles (possibly because of their instability, low damping or associated oscillations), and would like to move these undesired poles (henceforth called critical poles) to precise pre-specified locations inside the stability region. In these applications, typically, the remaining (non-critical) open loop poles are already stable and well damped and there is no obvious desired closed loop location for them. It is considered enough if these well-behaved open loop poles do not lose their desirable properties in closed loop, or in other words, if these non-critical open loop poles lie within some desired region of the complex plane in the closed loop.

It is well known that if the desired locations of *all* the closed loop poles are specified then for a single-input LTI system the required feedback gain vector is unique (Kailath (1980)). However if only a subset of the closed loop poles are specified, the extra degrees of freedom associated with the unspecified non-critical poles can be utilized to minimize the control effort associated with the controller. It is proposed in this article that the location of these non-critical poles be chosen with the explicit objective of minimizing the controller norm, while assigning the critical poles their designed closed loop locations.

The situation described above is typical for power oscillation damping controller design problem where electromechanical oscillations (0.1-0.8 Hz) are damped through expensive actuators (Kundur (1994)). State feedback approach has been used in the past to damp oscillations following large and small disturbances in power systems where the oscillatory behavior is dominated by a few poorly damped electromechanical modes with very little to zero influence from the other modes (Chaudhuri and Pal (2004)). Hence, it is important to carefully place only those critical poles to ensure desired performance following disturbances. There is no need to worry about the remaining non-critical poles as long as their settling times do not exceed those in open loop. In fact, it often turns out to be counter productive to relocate the non-critical poles or even force them to their open-loop position. Due to the very nature of the non-critical modes, higher control efforts are required unless they are left alone to take their natural course. This results in an overall increase in the norm of the feedback gain vector and hence, costlier actuators.

In this work, we propose that the feedback gain vector norm be minimized while ensuring (i) the critical poles are moved to desired (precise) closed loop locations, and (ii) the non-critical poles remain stable in closed loop. Additionally, it is often required that all closed loop poles should have a minimum settling time which implies that they should be located to the left of a given vertical line in the left half of the complex plane. It is shown that such an explicit minimization problem can be posed by translating the stability or minimum settling time requirements on the closed loop non-critical poles into constraints in the coefficient space of the characteristic polynomial corresponding to the non-critical poles. Such a translation is achieved through an inner convex approximation of the stability region of a polynomial proposed in Henrion et al. (2003). These requirements define a quadratic constraint on the subsequent minimization problem. Thus the feedback gain vector norm is minimized with two types of constraints: (i) linear equality constraints arising out of the precise placement requirement of the critical closed loop poles, and (ii) quadratic inequality constraints arising out of the regional placement requirement of the closed loop non-critical poles. By standard results on semi-definite programming discussed in Boyd and Vandenberghe (2004) and Boyd et al. (1994), it is shown that this problem has a unique minimum which in turn can be computed by semi-definite programming methods.

The problem of minimization of the feedback gain vector with partial pole placement was introduced in Datta et al. (2010) where a maximal sphere due to Bhattacharyya et al. (1995) was used to define the region constraint on the non-critical poles. The results reported in Datta et al. (2010) are improved in this work through less conservative ellipsoidal estimates of the stability domain of a polynomial.

## 2. NOTATION AND PROBLEM FORMULATION

Let us consider a continuous time LTI single-input system, with full state feedback control, defined by the following state space equations

$$\dot{x} = Ax + bu; \quad u = -k^T x \tag{1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}$, $A \in \mathbb{R}^{n\times n}$, $b \in \mathbb{R}^n$ and $k := [k_1\ k_2 \cdots k_n]^T \in \mathbb{R}^n$. Assume that the pair $(A, b)$ is controllable; then *all* the eigenvalues of the closed loop system

$$\dot{x} = (A - bk^T)x \tag{2}$$

can be placed at any arbitrary location of the complex plane $\mathbb{C}$ through a unique choice of $k$.

However, in the applications of our interest, only a few critical closed loop eigenvalues are specified and the non-critical eigenvalues are allowed to assume any value in (or in a pre-specified subset of) the stable region of the complex plane. Without loss of generality, we assume that the first $m$ eigenvalues of $A$ are non-critical. The remaining $(n - m)$ eigenvalues are critical and their closed loop positions are specified. Let us denote $\{\mu_1, \mu_2, \ldots, \mu_m, \mu_{m+1}, \mu_{m+2}, \ldots, \mu_n\}$, $(m \le n)$ are the $n$ eigenvalues of $A$. Of these, $\{\mu_1, \mu_2, \ldots, \mu_m\}$ are non-critical and are not associated with any desired closed loop location whereas the remaining $(n - m)$ eigenvalues $\{\mu_{m+1}, \mu_{m+2}, \ldots, \mu_n\}$ are critical and are required to be placed at $\{-\lambda_1, -\lambda_2, \ldots, -\lambda_{n-m}\}$. In general we will assume that $m$ eigenvalues of $(A - bk^T)$ are required to be located in the stable region $S$ of the complex plane. In this article we will define $S$ as follows:

$$S = \{s \in \mathbb{C} : \ \mathrm{Re}(s) < \beta\}, \quad \beta \le 0 \tag{3}$$

Then the problem described in the introduction can be simply formulated as:

*Problem Statement 1.* Find $\inf \|k\|_2$ such that the eigenvalues of $(A - bk^T)$ have the following properties:

(1) $(n - m)$ out of the total $n$ eigenvalues are placed at $\{-\lambda_1, -\lambda_2, \ldots, -\lambda_{n-m}\}$.
(2) remaining $m$ eigenvalues are placed anywhere in $S$.

Denote the unspecified closed loop poles of the system as $\{-p_1, -p_2, \ldots, -p_m\}$. Without loss of generality, let the closed loop poles be ordered as follows

$$\{-p_1, -p_2, \ldots, -p_m, -\lambda_1, -\lambda_2, \ldots, -\lambda_{n-m}\} \tag{4}$$

Further define

$$\alpha(s) := \prod_{j=1}^{m} (s + p_j) := s^m + \alpha_{m-1}s^{m-1} + \ldots + \alpha_1 s + \alpha_0 \tag{5}$$

$$\gamma(s) := \prod_{i=1}^{n-m} (s + \lambda_i) := s^{n-m} + \gamma_{n-m-1}s^{n-m-1} + \ldots + \gamma_1 s + \gamma_0$$

where

$$\gamma_{n-m-1} = \sum_{i_1=1}^{n-m} \lambda_{i_1} \qquad \alpha_{m-1} = \sum_{i_1=1}^{m} p_{i_1}$$

$$\gamma_{n-m-2} = \sum_{i_1<i_2}^{n-m} \lambda_{i_1}\lambda_{i_2} \qquad \alpha_{m-2} = \sum_{i_1<i_2}^{m} p_{i_1}p_{i_2}$$

$$\vdots \qquad\qquad \vdots$$

$$\gamma_1 = \sum_{i_1<..<i_{n-m-1}}^{n-m} \lambda_{i_1}..\lambda_{i_{n-m-1}} \qquad \alpha_1 = \sum_{i_1<..<i_{m-1}}^{m} p_{i_1}..p_{i_{m-1}}$$

$$\gamma_0 = \lambda_1\lambda_2 \ldots \lambda_{n-m} \qquad \alpha_0 = p_1 p_2 \ldots p_m$$

Then the characteristic equation of the closed loop system will be

$$\sigma(s) = \underbrace{\left[\prod_{j=1}^{m} (s + p_j)\right]}_{\alpha(s)} \underbrace{\left[\prod_{i=1}^{n-m} (s + \lambda_i)\right]}_{\gamma(s)} \tag{6}$$

Using the above notation, the characteristic polynomial (6) is divided into two parts: $\alpha(s)$ - a monic polynomial of unknown coefficients and $\gamma(s)$ - a monic polynomial of known coefficients. Clearly the $\gamma$ polynomial is completely defined from the problem specification. However the only requirement of the $\alpha(s)$ polynomial is that its roots should be located in a pre-specified region $S \in \mathbb{C}$ defined in (3).

To pose problem 1 as a quadratic program it would be convenient to translate the requirement on the poles $(-p_i \in S,\ i = 1, .., m)$ of polynomial $\alpha(s)$ into requirements on the coefficients $(\alpha_0, \alpha_1, ..., \alpha_{m-1})$ of the polynomial $\alpha(s)$. For this purpose let us denote the set of all $m^{th}$ degree monic polynomials with real coefficients as $\mathbb{R}[s]$ and define the set

$$C_s := \{\alpha(s) \in \mathbb{R}[s] : \text{ roots of } \alpha(s) \in S\}.$$

Then problem 1 is equivalent to the following:

*Problem Statement 2.* Find $\inf \|k\|_2$ such that $(A - bk^T)$ has the following properties:

(1) $(n - m)$ out of the total $n$ eigenvalues are placed at $\{-\lambda_1, -\lambda_2, \ldots, -\lambda_{n-m}\}$.
(2) the polynomial $\alpha(s) \in C_s$ .

For solving problem 2, we first need to express constraints (1) and (2) above in terms of the problem unknowns i.e. $k_1, k_2, ..., k_n$ which is accomplished in Section 4. It will be shown that constraint (1) is linear (and hence convex) in the unknowns. However, we note that the set $C_s$ is not a convex set for $m \ge 3$ (Henrion et al. (2003) and the references therein), and hence the optimization implied in problem 2 is not convex for $m \geq 3$. In the next section, we use a result proposed by Henrion et al. (2003) to construct a convex inner approximation of $C_s$. As demonstrated in Henrion et al. (2003), a set of linear matrix inequalities (LMI) is solved to compute a maximal ellipsoid subset of $C_s$, which is used in place of constraint (2) in problem 2.

## 3. STABLE ELLIPSOID IN THE POLYNOMIAL COEFFICIENT SPACE

### 3.1 The Bezoutian for $\alpha(s)$

The theory of Bezoutians and its implications on stability of polynomials is reviewed briefly for completeness. Let us consider the monic polynomial $\alpha(s)$ defined in (5). Define

$$\alpha := [\alpha_0\ \alpha_1\ \alpha_2\ \ldots\ \alpha_{m-1}]^T \in \mathbb{R}^m$$
$$\bar{\alpha} := [\alpha_0\ \alpha_1\ \alpha_2\ \ldots\ \alpha_{m-1}\ 1]^T \in \mathbb{R}^{m+1}$$
$$\widetilde{\alpha} := \left[\alpha_0\ -\alpha_1\ \alpha_2\ \ldots (-1)^{m-1}\alpha_{m-1}\ \ (-1)^m\right]^T \in \mathbb{R}^{m+1}$$

Corresponding to $\alpha(s)$ the Bezoutian of $\alpha(s)$ is a bivariate polynomial (Lev-Ari et al. (1991), Willems and Trentelman (1998)) and can be represented by

$$B(\zeta,\eta) := \frac{\alpha(\zeta)\alpha(\eta) - \alpha(-\zeta)\alpha(-\eta)}{\zeta+\eta} \tag{7}$$

where $\zeta, \eta$ are two indeterminates. The polynomial $B(\zeta,\eta)$ can be written in a quadratic form as follows.

$$B(\zeta,\eta) = \left[1\ \zeta\ ...\zeta^{m-1}\right] H(\alpha) \left[1\ \eta\ ...\eta^{m-1}\right]^T \tag{8}$$

where the Hermite matrix $H(\alpha) = [h_{ij}]$ $(i, j = 0, .., m-1)$ is a symmetric matrix whose $m(m+1)/2$ independent entries are polynomial functions in the coefficients of $\alpha(s)$. It is well known (Lev-Ari et al. (1991) and Henrion et al. (2003)) that all the roots of $\alpha(s)$ lie in $S$ if and only if $H(\alpha) > 0$. In our notation, we note that the set $C_s$ can also be defined in terms of $H(\alpha)$ as follows: $C_s := \{\alpha(s) \in \mathbb{R}(s) : H(\alpha) > 0\}$.

According to Lev-Ari et al. (1991) and Henrion et al. (2003), the Hermite matrix $H(\alpha)$ can be computed from the following relation

$$V + \widetilde{V} = \bar{\alpha}\bar{\alpha}^T - \widetilde{\alpha}\widetilde{\alpha}^T \tag{9}$$

where

$$V = \begin{bmatrix} \mathbf{0}_{1\times m} & 0_{1\times 1} \\ H(\alpha) & \mathbf{0}_{m\times 1} \end{bmatrix} \quad \widetilde{V} = \begin{bmatrix} \mathbf{0}_{m\times 1} & H(\alpha) \\ 0_{1\times 1} & \mathbf{0}_{1\times m} \end{bmatrix}$$

and $V$, $\widetilde{V} \in \mathbb{R}^{(m+1)\times(m+1)}$. Problem 2 can now be reformulated in terms of $H(\alpha)$.

*Problem Statement 3.* Find $\inf \|k\|_2$ such that $(A - bk^T)$ has the following properties:

(1) $(n-m)$ out of the total $n$ eigenvalues are placed at $\{-\lambda_1, -\lambda_2, \ldots, -\lambda_{n-m}\}$.
(2) the Bezoutian corresponding to the polynomial $\alpha(s)$ satisfies $H(\alpha) > 0$.

However, as mentioned earlier the set $C_s$ is not convex, and hence problem 3 do not necessarily have a computable solution. To address this issue, we find a maximal ellipsoidal subset $\mathcal{E}_s$ of $C_s$, which is convex, and can be constructed easily by solving a system of LMIs. This procedure, proposed in Henrion et al. (2003) to find $\mathcal{E}_s$, is reviewed briefly in the next section.

### 3.2 Ellipsoidal Approximation

Recall the vector $\alpha = [\alpha_0\ \alpha_1\ \alpha_2\ \ldots\ \alpha_{m-1}]^T$ corresponding to the polynomial $\alpha(s)$ defined in (5). A set of $m^{th}$ order monic polynomials can be defined as follows:

$$\mathcal{E}_s = \{\alpha \in \mathbb{R}^m : (\alpha - \hat{\alpha})^T P(\alpha - \hat{\alpha}) \leq 1\} \tag{10}$$

where $P \in \mathbb{R}^{m\times m}$ is a positive definite symmetric matrix and $\hat{\alpha}$ is a given (nominal) $m^{th}$ order monic polynomial. The above set is an ellipsoid with center at $\widehat{\alpha}$. Let us denote

$$P_{11} = -P < 0 \qquad P_{12} = P\hat{\alpha} \tag{11}$$
$$P_{21} = P_{12}^T \qquad P_{22} = 1 - \hat{\alpha}^T P \hat{\alpha}$$

Using (11), the inequality $(\alpha - \hat{\alpha})^T P(\alpha - \hat{\alpha}) \leq 1$ can be written as

$$\left[\alpha^T\ 1\right] \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} \alpha \\ 1 \end{bmatrix} \geq 0 \tag{12}$$

with $P_{11} \in \mathbb{R}^{m\times m}$, $P_{12} \in \mathbb{R}^m$ and $P_{22} \in \mathbb{R}$. Denoting

$$\widetilde{P} := \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \text{ and } P(\alpha) := \left[\alpha^T\ 1\right] \widetilde{P} \begin{bmatrix} \alpha \\ 1 \end{bmatrix}$$

where $\widetilde{P} \in \mathbb{R}^{(m+1)\times(m+1)}$, equation (12) can be written as $P(\alpha) \geq 0$.

Recall that our objective is to find the maximal set $\mathcal{E}_s$ such that $\mathcal{E}_s \subset C_s$. Given the center (nominal) polynomial $\hat{\alpha}(s)$ in (10), the set inclusion $\mathcal{E}_s \subset C_s$ is guaranteed if and only if for all the vectors $\alpha$

$$P(\alpha) \geq 0 \quad \Rightarrow \quad H(\alpha) > 0 \tag{13}$$

The following proposition will give the sufficient LMI conditions such that the above condition (13) holds.

First, however, we note that the Hermite matrix $H(\alpha)$, discussed in the previous section, can also be written in the following form

$$H(\alpha) = [I_m \otimes \bar{\alpha}]^T H [I_m \otimes \bar{\alpha}] \tag{14}$$

where the matrix $H \in \mathbb{R}^{m(m+1)\times m(m+1)}$ consists of block matrices $H_{ij} \in \mathbb{R}^{(m+1)\times(m+1)}$ $(i, j = 1, ..., m)$, which in turn can be computed in following manner. Denote the $(i,j)^{th}$ entry in matrix $H(\alpha)$ by $[H(\alpha)]_{ij}$. Then the block $H_{ij}$ can be calculated from the equation:

$$\bar{\alpha}^T H_{ij} \bar{\alpha} = [H(\alpha)]_{ij}, \quad i, j = 1, 2, ..m \tag{15}$$

*Lemma 4.* (Henrion et al., 2003, Lemma:2) If there exists a symmetric block matrix

$$G = \begin{bmatrix} 0 & G_{21}^T & \cdots & G_{m1}^T \\ G_{21} & 0 & \cdots & G_{m2}^T \\ \vdots & \vdots & \ddots & \vdots \\ G_{m1} & G_{m2} & \cdots & 0 \end{bmatrix} \tag{16}$$

consisting of skew symmetric matrices $G_{i,j} = -G_{i,j}^T \in \mathbb{R}^{(m+1)\times(m+1)}$ for $i, j = 1, 2...m$ and a symmetric matrix $D \in \mathbb{R}^{m\times m}$ satisfying following conditions

$$D > 0$$
$$(D \otimes I_{m+1})H = H(D \otimes I_{m+1}) \tag{17}$$
$$I_m \otimes \widetilde{P} + G < (D \otimes I_{m+1})H$$

then the condition (13) holds.

It is well known (Boyd et al. (1994)) that the volume of $\mathcal{E}_s$ is proportional to the product of square roots of the reciprocals of the eigenvalues of $P$. Hence, the maximum volume $\mathcal{E}_s$ can indirectly be obtained by maximizing the trace of the matrix $-P$ under the constraint set (11), (16) and (17) with decision variables matrix $P$ and scaling matrices $D$ and $G$. Then any vector $\alpha$ such that $(\alpha - \hat{\alpha})^T P(\alpha - \hat{\alpha}) \leq 1$ will parametrize a polynomial $\alpha(s)$ with all its roots in stability region $S$. Given $\hat{\alpha}$, the maximal ellipsoid can be computed by solving the following problem

*Problem Statement 5.* (Henrion et al., 2003, Theorem 1) Maximize $tr[-P]$ over $P$, $D$ and $G$ such that (11), (16) and (17) holds.

For use in Section 4, let us denote the solution to problem 5 as $P^*_{\hat{\alpha}}$.

In the next section, we will use $\mathcal{E}_s$ to replace $C_s$ in problem (2) to get a convex optimization easily solvable by semi-definite programming. However to compute $\mathcal{E}_s$ explicitly we still need *a priori* a polynomial $\widehat{\alpha}(s) \in C_s$. For our choice of $\widehat{\alpha}(s)$, we propose to use the polynomial formed out of the open loop non-critical poles as follows:

$$\widehat{\alpha}(s) = \left[\prod_{j=1}^{m}(s - \mu_j)\right] \tag{18}$$

Usually, those open loop eigenvalues, which are stable and already have adequate damping, are classified as non-critical. Hence in most practical scenarios, $\mu_1, \mu_2, \ldots, \mu_m \in S$ and hence $\widehat{\alpha}(s) \in C_s$.

Note that, the procedure demonstrated in Section 3.1 to calculate $H(\alpha)$ from the relation (31) is only for the stability region $S$ when $\beta = 0$. However, this restrictive stability region can be relaxed by changing the Hermite matrix $H(\alpha)$. A procedure to calculate $H(\alpha)$ for the stability region $S$ when $\beta < 0$ is discussed in Lev-Ari et al. (1991) and Henrion et al. (2003).

## 4. MAIN RESULTS

In this section we will show that Problem 2 is equivalent to a quadratically constrained quadratic program. We need a few additional notations. Let the open loop characteristic polynomial of (1) be denoted by:

$$\begin{aligned} a(s) &= \det(sI - A) \\ &= s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \ldots + a_1 s + a_0, \end{aligned} \tag{19}$$

while the characteristic polynomial of the system (2) is defined as:

$$\begin{aligned} \sigma(s) &= \det(sI - A + bk^T) \\ &= s^n + \sigma_{n-1}s^{n-1} + \sigma_{n-2}s^{n-2} + \ldots + \sigma_1 s + \sigma_0 \end{aligned} \tag{20}$$

Further define

$$\begin{aligned} a &:= \left[a_0 \; a_1 \; \cdots \; a_{n-2} \; a_{n-1}\right]^T \\ \sigma &:= \left[\sigma_0 \; \sigma_1 \; \cdots \; \sigma_{n-2} \; \sigma_{n-1}\right]^T \\ \mathcal{C} &:= \left[b \; Ab \; A^2b \; \cdots \; A^{n-1}b\right], \end{aligned} \tag{21}$$

$$\text{and } \mathcal{A}^T := \begin{bmatrix} a_1 & a_2 & \cdots & a_{n-1} & 1 \\ a_2 & a_3 & \cdots & 1 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n-1} & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix} \tag{22}$$

If the system is controllable i.e. controllability matrix (21) is non-singular, the closed loop eigenvalues (of $A - bk^T$) can be placed at any arbitrary locations in $\mathbb{C}$ and, the corresponding feedback gain vector $k$ is unique (Kailath (1980)). This unique $k$ can be calculated from the following equation:

$$\mathcal{A}\mathcal{C}^T k + a = \sigma$$

If we define $\bar{k} = \mathcal{A}\mathcal{C}^T k$ where $\bar{k} = [\bar{k}_1 \;\; \bar{k}_2 ... \bar{k}_n]^T$, it follows that each $\bar{k}_i (i = 1, ..., n)$ is a linear combination of $k_1, ..., k_n$. Then:

$$\begin{aligned} \sigma_{n-1} &= \bar{k}_n + a_{n-1} \\ \sigma_{n-2} &= \bar{k}_{n-1} + a_{n-2} \\ &\vdots \\ \sigma_1 &= \bar{k}_2 + a_1 \\ \sigma_0 &= \bar{k}_1 + a_0 \end{aligned} \tag{23}$$

Recalling the expression for the required closed loop characteristic polynomial (6), and equating coefficients with (20), we get:

$$\begin{aligned} \sigma_{n-1} &= \gamma_{n-m-1} + \alpha_{m-1} \\ \sigma_{n-2} &= \gamma_{n-m-2} + \gamma_{n-m-1}\alpha_{m-1} + \alpha_{m-2} \\ &\vdots \\ \sigma_2 &= \gamma_0\alpha_2 + \gamma_1\alpha_1 + \gamma_2\alpha_0 \\ \sigma_1 &= \gamma_0\alpha_1 + \gamma_1\alpha_0 \\ \sigma_0 &= \gamma_0\alpha_0 \end{aligned} \tag{24}$$

Since $(-\lambda_1, -\lambda_2, \ldots, -\lambda_{n-m})$ are specified by the designer, the coefficients $\gamma_0, \gamma_1, \ldots, \gamma_{n-m-1}$ in (24) are known quantities. However, the non-critical poles $-p_1, -p_2, \ldots, -p_m$ are unspecified, so that $\alpha_0, \alpha_1, \ldots, \alpha_{m-1}$ are unknown. First note that $\sigma_0, \sigma_1, ..., \sigma_{n-1}$ can be eliminated from equations (23) and (24) to get $n$ linear equations:

$$\begin{aligned} \gamma_{n-m-1} + \alpha_{m-1} &= \bar{k}_n + a_{n-1} \\ &\vdots \\ \gamma_0\alpha_1 + \gamma_1\alpha_0 &= \bar{k}_2 + a_1 \\ \gamma_0\alpha_0 &= \bar{k}_1 + a_0 \end{aligned} \tag{25}$$

From (25), $(\alpha_0, ..., \alpha_{m-1})$ can be expressed in terms of $m$ linear equations in $\bar{k}_1, \bar{k}_2, \ldots, \bar{k}_n$. A simple inductive method to get such equations is shown below:

$$\begin{aligned} \alpha_0 &= \frac{1}{\gamma_0}\left(\bar{k}_1 + a_0\right) \\ \alpha_1 &= \frac{1}{\gamma_0}\left(\left(\bar{k}_2 + a_1\right) - \frac{\gamma_1}{\gamma_0}\left(\bar{k}_1 + a_0\right)\right) \\ &\vdots \\ \alpha_{m-1} &= ..... \end{aligned} \tag{26}$$

Equations (26) can be written in matrix form as follows

$$\alpha = \mathcal{F}\bar{k} + g \tag{27}$$

where $\alpha \in \mathbb{R}^m$, $\mathcal{F} \in \mathbb{R}^{m \times n}$, $g \in \mathbb{R}^m$. Now $\alpha_0, \ldots, \alpha_{m-1}$ from (26) can be back-substituted in the set of $n$ equations (25) to get $(n-m)$ linear equations in $(\bar{k}_1, \bar{k}_2, ..., \bar{k}_n)$ which can be written in the form:

$$\mathcal{E}\bar{k} + h = \mathbf{0} \tag{28}$$

where $\mathcal{E} \in \mathbb{R}^{(n-m)\times n}$, $h \in \mathbb{R}^{n-m}$ and $\mathbf{0}$ is a zero vector of appropriate dimension. Using $\bar{k} = \mathcal{A}\mathcal{C}^T k$, and defining

$$F = \mathcal{F}\mathcal{A}\mathcal{C}^T \text{ and } E = \mathcal{E}\mathcal{A}\mathcal{C}^T \tag{29}$$

we get the following set of equations:

$$\alpha = Fk + g \quad \text{and} \quad Ek + h = \mathbf{0} \tag{30}$$

Then the following result holds:

*Theorem 1.* Let $S$ be the open left half of complex plane $\mathbb{C}$, $F$ and $E$ are as defined in (29) and $P^*_{\hat{\alpha}}$ be the solution to problem 5 for $\hat{\alpha}(s)$ defined in (18). If for some $k \in \mathbb{R}^n$, the relations $(Fk+g-\hat{\alpha})^T P^*_{\hat{\alpha}}(Fk+g-\hat{\alpha}) \leq 1$ and $Ek+h = \mathbf{0}$ holds, then the eigenvalues of the matrix $(A - bk^T)$ satisfy the following properties:

(1) $(n-m)$ out of the total $n$ eigenvalues are $\{-\lambda_1, -\lambda_2, \ldots, -\lambda_{n-m}\}$.
(2) the remaining $m$ eigenvalues $-p_i \in S$ for $i = 1, \ldots, m$.

**Proof.** Let some $k$ satisfy $(Fk+g-\hat{\alpha})^T P^*_{\hat{\alpha}}(Fk+g-\hat{\alpha}) \leq 1$ and $Ek + h = \mathbf{0}$. Then $(\alpha - \hat{\alpha})^T P^*_{\hat{\alpha}}(\alpha - \hat{\alpha}) \leq 1$. Hence the polynomial $\alpha(s) \in \mathcal{E}_s$. So we can apply Lemma 4 to guarantee that the roots of $\alpha(s)$ lie in $S$. The $(n-m)$ equations $Ek + h = \mathbf{0}$ imply that the $(n-m)$ roots of polynomial $\gamma(s)$ (see (6)) are placed at $\{-\lambda_1, \ldots, -\lambda_{n-m}\}$.

Theorem 1 defines the constraint set on the feedback gain vector $k$, which can be used to pose an optimization problem that minimizes the norm of $k$. Before proceeding further let us express the constraint $(Fk+g-\hat{\alpha})^T P^*_{\hat{\alpha}}(Fk+g-\hat{\alpha}) \leq 1$ in standard form:

$$\left[Fk + (g - \widehat{\alpha})\right]^T P^*_{\hat{\alpha}} \left[Fk + (g - \widehat{\alpha})\right] \leq 1$$
$$\Rightarrow \left[(Fk)^T + (g - \widehat{\alpha})^T\right] \left[P^*_{\hat{\alpha}} Fk + P^*_{\hat{\alpha}}(g - \widehat{\alpha})\right] \leq 1$$
$$\Rightarrow (Fk)^T P^*_{\hat{\alpha}} Fk + 2(g - \widehat{\beta})^T P^*_{\hat{\alpha}} Fk + (g - \widehat{\alpha})^T P^*_{\hat{\alpha}}(g - \widehat{\alpha}) \leq 1$$

which can be written as $k^T M k + 2m^T k + c \leq 0$ where $M = F^T P^*_{\hat{\alpha}} F$, $m^T = (g - \widehat{\alpha})^T P^*_{\hat{\alpha}} F$ and $c = (g - \widehat{\alpha})^T P^*_{\hat{\alpha}}(g - \widehat{\alpha}) - 1$. Here $M$ is a positive semi-definite matrix, $m$ is a constant vector and $c \in \mathbb{R}$. The optimization problem can be formulated as follows:

*Problem Statement 6.* Find $\min_{k \in \mathbb{R}^n} \|k\|_2$ subject to

$$Ek + h = \mathbf{0}$$
$$k^T M k + 2m^T k + c \leq 0$$

It should be noted that the above constraint set is always feasible since it is known that there is always at least one $k$ which places the poles at arbitrary desired location.

It should be noted that Theorem 1 is only sufficient in guaranteeing that the corresponding eigenvalues stay in $S$. Consequently, in some cases, it might be possible to find a $k$ which preserves the pole placement requirements and but has lesser norm than the solution to problem 6. Hence, a two step design procedure is suggested below to find a controller with maximum reduction in the norm. Since, the precise location requirement on the critical eigenvalues is inflexible, the equality constraints $Ek + h = \mathbf{0}$ are assumed to be imposed on all the steps below:

(1) Solve problem 6 without considering the inequality constraints. If all the poles belong to $S$ then stop; otherwise go to step 2.
(2) Solve problem 6.

## 5. NUMERICAL EXAMPLES

*Example 7.* Consider a LTI system with

Table 1. Comparison Table

| Procedure | Closed loop poles | $\|k\|_2$ | % Red. in $\|k\|_2$ | Sys. Cond. |
|---|---|---|---|---|
| Step - 1 | $0 \pm 6.4031, -1, 1$ | 20 | 84.93 | Unstable |
| Step - 2 | $-0.1432 \pm 0.5588i$ $-11.0230, -1$ | 56.86 | 57.16 | Stable |
| Step - 3 | $-4 \pm 1i, -3, -1$ | 132.73 | – | Stable |

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 51 & -10 & -30 & -10 \end{bmatrix} \qquad b = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Eigenvalues of $A$ are at $1, -3, -4 \pm i$. Only the pole at 1 is unstable and we assume that this pole needs to be placed at $-1$ using state feedback control. The remaining 3 poles are assumed to be non-critical and are allowed to be placed arbitrarily within the open left half of complex plane. Following the discussion in Section 3, (9) takes the form:

$$\begin{bmatrix} 0 & h_{00} & h_{01} & h_{02} \\ h_{00} & 2h_{01} & h_{02} + h_{11} & h_{12} \\ h_{01} & h_{02} + h_{11} & 2h_{12} & h_{22} \\ h_{02} & h_{12} & h_{22} & 0 \end{bmatrix} = \begin{bmatrix} 0 & 2\alpha_0\alpha_1 & 0 & 2\alpha_0 \\ 2\alpha_0\alpha_1 & 0 & 2\alpha_1\alpha_2 & 0 \\ 0 & 2\alpha_1\alpha_2 & 0 & 2\alpha_2 \\ 2\alpha_0 & 0 & 2\alpha_2 & 0 \end{bmatrix} \tag{31}$$

Using (31) the Hermite matrix $H(\alpha)$ will be

$$H(\alpha) = \begin{bmatrix} 2\alpha_0\alpha_1 & 0 & 2\alpha_0 \\ 0 & 2(\alpha_1\alpha_2 - \alpha_0) & 0 \\ 2\alpha_0 & 0 & 2\alpha_2 \end{bmatrix}$$

Then matrix $H$ can be constructed using the relation given in (15). Next we are required to construct the nominal polynomial $\widehat{\alpha}(s)$ according to (18). Recalling that the open loop poles are at $-3, -4 \pm i$, the corresponding $\widehat{\alpha} = [51\ 41\ 11]^T$. Following the procedure given in Section 3, the LMI optimization problem is solved with *SeDuMi 1.05* (Sturm (2005)) and its LMI interface (Peaucelle et al. (2002)) in Matlab environment. The resulting $P^*_{\hat{\alpha}}$ matrix is

$$P^*_{\hat{\alpha}} = \begin{bmatrix} 0.0252 & -0.0311 & -0.0001 \\ -0.0311 & 0.0390 & 0.0001 \\ -0.0001 & 0.0001 & 0.0179 \end{bmatrix}$$

Following the design procedure proposed at the end of the last section, a comparison table is shown in Table 1 where Steps 1 corresponds to minimization of $\|k\|_2$with only the equality constraints $Ek + h = \mathbf{0}$, while Step 2 minimizes $\|k\|_2$with both the constraints described in problem 6. In order to compare our methods with the conventional methods of pole placement, Step - 3 evaluates $\|k\|_2$ keeping three non critical poles in their original location.

The percentage reduction in $\|k\|_2$ in Step - 1 and Step - 2 is compared with Step - 3. It is observed that a large reduction in $\|k\|_2$ is achieved in Step-1. However, the non-critical poles are in the unstable region. Hence this example needs the design Step - 2 to ensure stability. In this step all the non-critical poles are in the stable region and hence we have achieved our goal. In this step the percentage reduction in $\|k\|$ is acceptable i.e. about 57.1595%.

Table 2. Comparison Table

| Procedure | Non critical closed loop poles | $\|k\|_2$ | % Red. in $\|k\|_2$ | Sys. Cond. |
|---|---|---|---|---|
| Step - 1 | $-10.0009 \pm 9.0498i$ $-7.0277, -0.4076$ | 2.1128 | 51.2877 | Stable |
| Step - 2 | $-6.6687 \pm 9.2732i$ $-10.4844, -0.3144$ | 2.5222 | 41.8468 | Stable |
| Step - 3 | $-5.7250 \pm 7.4088i$ $-33.5344,\ -0.3078$ | 4.3372 | – | Stable |

**Example 2:** In this example, the linearized model of a 16-generator, 68 bus bar power system (Chaudhuri and Pal (2004)) is considered around its nominal operating condition. The 133 order original model is reduced to a $10^{th}$ order equivalent without introducing much error within the frequency range of interest (0.1 to 0.8 Hz). Open loop poles of the reduced system are at are at $-33.5344, -5.7250 \pm 7.4088i, -0.1741 \pm 3.7981i, -0.1781 \pm 3.1604i, -0.1808 \pm 2.4535i, -0.3078$. It is required to place six poles at $-0.4000 \pm 3.7980i, -0.4000 \pm 3.1604i, -0.4000 \pm 2.4535i$ and remaining 4 poles are non-critical. These four non-critical poles can assume any positions in the open left half of the complex plane. According to (31) demonstrated in Section 3, the Hermite matrix $H(\alpha)$ will take the following form:

$$H(\alpha) = \begin{bmatrix} 2\alpha_0\alpha_1 & 0 & 2\alpha_0\alpha_3 & 0 \\ 0 & 2(\alpha_1\alpha_2 - \alpha_0\alpha_3) & 0 & 2\alpha_1 \\ 2\alpha_0\alpha_3 & 0 & 2(\alpha_2\alpha_3 - \alpha_1) & 0 \\ 0 & 2\alpha_1 & 0 & 2\alpha_3 \end{bmatrix}$$

Corresponding to the open-loop non-critical poles $\widehat{\alpha} = [904.87\ 3085\ 485.48\ 45.29]^T$. The resulting $P^*_{\hat{\alpha}}$ matrix obtained by solving the LMI optimization problem with *SeDuMi 1.05* and its LMI interface in Matlab environment is

$$P^*_{\hat{\alpha}} = \begin{bmatrix} 0.0063 & -0.0016 & -0.0023 & 0.0062 \\ -0.0016 & 0.0014 & -0.0061 & -0.0003 \\ -0.0023 & -0.0061 & 0.0442 & -0.0109 \\ 0.0062 & -0.0003 & -0.0109 & 0.0124 \end{bmatrix}$$

Following the design procedure described at the end of the last section, a comparison table is shown in Table 2. It can be noticed that a substantial reduction in $\|k\|_2$ i.e 51.28% is achieved in Step - 1. Furthermore all the non-critical poles have assumed their positions in the stable region and hence we have achieved our objective in this step. Moreover we have achieved the actual minimum of the $\|k\|_2$. In Step - 2 it is observed that the reduction in $\|k\|_2$ is 41.8468% and all the non-critical poles are also in the stable region. These results compare favorably with similar results in Datta et al. (2010), where the stable region was approximated by a hypersphere resulting in a corresponding reduction in $\|k\|_2$ of 5.5326%. Further comparing the $\|k\|_2$ reduction obtained in Step - 2 and Step - 1, it would be justified to say that the ellipsoidal approximation is a close approximation of the stability region $C_s$.

## 6. CONCLUSION

We have considered the problem of reducing the controller effort of a continuous time LTI single-input system. The state feedback vector norm is minimized under the constraints (a) the critical poles are placed in specified locations in the complex plane (b) the non-critical poles are placed anywhere inside a pre-specified design region $S$. Due to non-convexity of the region $C_s$ corresponding to the region $S$, a maximal ellipsoid is constructed inside $C_s$. It is demonstrated through numerical examples that the proposed ellipsoid approximation is a good approximation of the stability region $C_s$.

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