ROBUST OPTIMAL CONTROL: LOW ERROR OPERATION FOR THE LONGEST TIME

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ABSTRACT. The problem of maximizing the time during which an open loop system can operate without exceeding a specified error bound is considered for linear systems that are subject to uncertainties about their parameters and their initial conditions, and whose operation is hampered by disturbance signals. The objective is to characterize an optimal input signal that keeps performance errors within specified bounds for the longest time. It is shown that such an input signal exists, and that it can be approximated by a bang-bang input signal without significantly affecting performance.

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1. INTRODUCTION

High accuracy control systems employ feedback to help reduce the effects of uncertainties and disturbances on system performance. However, disruptions of feedback channels, and the increased performance errors caused by such disruptions, cannot be completely avoided. In fact, feedback disruptions are part of the routine operating conditions in a number of applications, including digital control of continuous time systems, where feedback is obtained only at sampling times; networked control systems, where feedback channels are disrupted intermittently to reduce network traffic (e.g., ZHIVOGYLADOV and MIDDLETON [2003], MONTESTRUQUE and ANTSAKLIS [2004]); and medical applications, such as glucose control in diabetics, where feedback requires irksome biological testing and is obtained relatively infrequently (e.g. PARKER, DOYLE, and PEPPAS [2001], BELLAZZI, NUCCI, and COBELLI [2001], JAREMKO and RORSTAD [1998]). To address the demands of such applications, we develop in this paper an open loop controller that maximizes the duration of time during which a system can operate without feedback and not exceed acceptable error bounds.

The information available about the controlled system is often incomplete: there may be uncertainties about parameter values; the system's state when the loop opens may not be precisely known; and external disturbances may interfere with performance. The situation is depicted in Figure1.1, where $v(t)$ denotes a disturbance signal.

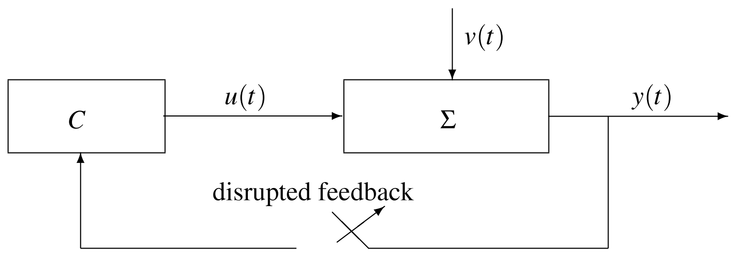

FIGURE 1.1. Basic Configuration

In technical terms, we consider a linear time-invariant system Σ whose output is its state:

$$\Sigma \ : \dot{x}(t) = A'x(t) + B'u(t) + G'v(t), \quad x(0) = x_0. \tag{1.1}$$

Here, A' is an $n \times n$ matrix, B' is an $n \times m$ matrix, and G' is an $n \times p$ matrix. Using R to denote the real numbers, the control input of the system is $u(t) \in R^m$, while $v(t) \in R^p$ is an unspecified disturbance signal. Feedback is completely lost at the time $t = 0$, and the system operates in open loop according to (1.1) for all times $t > 0$. The initial condition $x_0 \in R^n$ and the entries of the matrices A', B' and G' are not accurately specified. The only information available is the nominal version Σ_0 of the system Σ characterized by: *(i)* the nominal matrices $A' = A, B' = B$, and $G' = G$, where A, B, and G are specified; *(ii)* the nominal initial condition $x_0 = x_0^0$, where x_0^0 is specified; and *(iii)* the nominal disturbance input signal $v(t) = 0$. After possibly having applied an appropriate shift transformation on the signals, we assume that the desired state trajectory is the zero signal $x(t) = 0$ for all $t \geq 0$. Correspondingly, our objective during the open loop operation is to ensure that the state trajectory of (1.1) remains close to 0 for all $t \geq 0$ despite the presence of uncertainties and disturbances.

To describe deviations from nominality, we use the ℓ^∞-norm $\| \bullet \|$ given, for an n-dimensional vector $(c_1, ..., c_n)$ by $\|c\| := \max_{i=1,...,n} |c_i|$, and for an $n \times m$ real matrix C by $\|C\| := \max_{i=1,...,n; j=1,...,m} |c_{ij}|$; here c_{ij} is the (i, j) entry of C. The uncertainty about the initial state x_0 is characterized by a maximal deviation $\chi > 0$ from the nominal initial state, so that the set of all possible initial states is

$$X_0 := \{x_0 \in R^n : \|x_0 - x_0^0\| \leq \chi\}. \tag{1.2}$$

The uncertainties about the entries of the matrices A', B', and C' of (1.1) are characterized similarly in terms of a real number $d > 0$:

$$\|A' - A\| \leq d, \|B' - B\| \leq d, \text{ and } \|G' - G\| \leq d.$$

It is convenient to denote by Δ_A the set of all $n \times n$ matrices with entries in the interval $[-d, d]$. Analogously, Δ_B (respectively, Δ_G) is the set of all $n \times m$ (respectively, all $n \times p$) matrices with entries in the interval $[-d, d]$. Then, we can represent the perturbed matrices of (1.1) by

$$A' = A + D_A, B' = B + D_B, G' = G + D_G, \tag{1.3}$$

where $D_A \in \Delta_A, D_B \in \Delta_B$, and $D_G \in \Delta_G$. In shorthand, denote

$$D := (D_A, D_B, D_G) \text{ and } \Delta := \Delta_A \times \Delta_B \times \Delta_G, \tag{1.4}$$

so that $D \in \Delta$.

We denote by $\Sigma_{x_0,D,v}$ the system (1.1) with matrices given by (1.3), an initial condition $x_0 \in X_0$, and a disturbance signal $v(t)$. For an input signal $u(t)$, the response of the system is $x(t) = \Sigma_{x_0,D,v} u(t)$. As the nominal output signal of the system is the zero signal $x(t) = 0$ for all $t \geq 0$, we define the performance error

$$e(t) = x^T(t)x(t). \tag{1.5}$$

Our objective is to select the input signal $u(t)$ so as to keep the error $e(t)$ below a specified bound $M > 0$ for the longest time. If the error does not exceed the bound M during the time interval $[0, t_f]$, we can write

$$e(t) \leq M \text{ for all } 0 \leq t \leq t_f. \tag{1.6}$$

Then, the objective is to maximize t_f. This maximization must be performed while taking into consideration all uncertainties about the system Σ, namely, uncertainties about the entries of the matrix D of (1.4), uncertainties about the initial state x_0, and uncertainties about the disturbance signal $v(t)$. In view of (1.6), we must have the requirement

$$x_0^T x_0 \leq M, \tag{1.7}$$

as otherwise the initial error is already in excess of the permissible error.

The problem of deriving an input signal that maximizes the time t_f was introduced in CHAKRABORTY [2007], CHAKRABORTY and HAMMER [2008], and CHAKRABORTY and HAMMER [2009], where the problem was considered in the absence of a disturbance signal $v(t)$ and under the assumption that the initial condition x_0 is accurately specified. The present paper extends these results to a more disturbance rich environment. Specifically, we show in section 2 that the problem of calculating an optimal signal $u(t)$ is a max-min optimization problem. In section 3 we prove that this problem has a solution, and in section 4 we show that an optimal signal $u(t)$ can be replaced by a bang-bang signal with only a negligible effect on system performance (a bang-bang signal is a signal that switches between its extremal values).

Replacement of optimal input signals by bang-bang signals leads to substantial simplifications in the computation and the implementation of the optimal solution, since bang-bang signals are completely determined by their switching times. In effect, the use of bang-bang signals amounts to transforming our dynamic optimization problem into a much simpler problem of optimization over a finite number of scalars - the switching times.

2. NOTATION AND PROBLEM FORMULATION

We start by introducing a weighted inner product over m-dimensional vector valued functions, given by

$$\langle a, b \rangle = \int_0^\infty e^{-\alpha t} a(t)^T b(t) dt, \tag{2.1}$$

where $a(t)$ and $b(t)$ are m-dimensional vectors, α is a positive real number, and the integral is taken in the Lebesgue sense. The weight function $e^{-\alpha t}$ makes it possible to include all bounded functions in the domain over which this inner product is defined. We denote by $L_2^{\alpha,m}$ the Hilbert space of all m-dimensional Lebesgue measurable functions with the inner product (2.1).

In addition to the integral norm (2.1), we use the point-wise ℓ^∞-norm, which, for a function $f(t) = (f_1(t), ..., f_m(t))$, is given at each time t by

$$\|f(t)\| := max_{i=1,...,m} |f_i(t)|.$$

Practical systems are often subject to input amplitude restrictions determined by the largest signal amplitude a system's components can tolerate. Let $K > 0$ the input amplitude bound of our system Σ. Then, the set of all permissible input functions of Σ is

$$U := \{u \in L_2^{\alpha,m} : \|u(t)\| \leq K \text{ for all } t \geq 0\}. \tag{2.2}$$

Similarly, let $L > 0$ be the bound on the amplitude of the disturbance $v(t)$ of (1.1). Then, the set of all possible disturbance signals is

$$V := \{v \in L_2^{\alpha,p} : \|v(t)\| \leq L \text{ for all } t \geq 0\}. \tag{2.3}$$

In these terms, our objective is to find an input function $u(t) \in U$ that drives Σ so as to satisfy the error bound (1.6) for the longest possible time t_f, irrespective of uncertainties and disturbances.

The state trajectory $x(t)$ of the system Σ of (1.1) depends, of course, on the initial condition x_0, on the perturbation matrix $D = (D_A, D_B, D_G)$ of (1.4), on the disturbance signal $v(t)$, and on the control input signal $u(t)$. To make these dependencies explicit, we write $x(t, x_0, D, v, u)$ instead of $x(t)$. Then, (1.6) takes the form

(2.4) $$e(t,x_0,D,v,u) := x^T(t,x_0,D,v,u)x(t,x_0,D,v,u) \leq M, 0 \leq t \leq t_f.$$

The time during which the error $e(t,x_0,D,v,u)$ does not exceed its bound M is then

(2.5) $$T(M,x_0,D,v,u) := \inf\{t \geq 0 : e(t,x_0,D,v,u) > M\},$$

where $T(M,x_0,D,v,u) := \infty$ if $e(t,x_0,D,v,u) \leq M$ for all $t \geq 0$. We have $T(M,x_0,D,v,u) \geq 0$, since the initial state satisfies $x_0^T x_0 \leq M$. We aim to select the input function u so as to obtain the longest possible duration $T(M,x_0,D,v,u)$, considering the uncertainties about the initial conditions, about the matrices A', B', G', and about the disturbance signal v.

As Σ operates without feedback, the only information available about x_0, D, and v is the a-priori information $x_0 \in X_0, D \in \Delta$, and $v \in V$. Therefore, for a given input function u, the longest time during which the error does not exceed M is given by the lowest value of $T(M,x_0,D,v,u)$ over all possible perturbations, namely, by the quantity

(2.6) $$T^*(M,u) := \inf_{(x_0,D,v)\in X_0\times\Delta\times V} T(M,x_0,D,v,u).$$

For a particular input function $u(t)$, inequality (2.4) is valid for all $t \in [0, T^*(M,u)]$, irrespective of x_0, D, or $v(t)$, as long as these are within their permissible domains.

The best input function $u(t)$ is, of course, one that maximizes the value of $T^*(M,u)$. If such an input function exists, it yields the maximal time

(2.7) $$t_f^* := sup_{u\in U} T^*(M,u)$$

during which the error remains within specified bounds, irrespective of which permissible combination of perturbations and disturbances is active. Assuming, for a moment, that such an input function exists, denote it by u^*. Then, $t_f^* = T^*(M,u^*)$, and our objectives can be phrased as follows.

Problem 2.1. Determine whether an optimal input function $u^* \in U$ exists; if such a function exists, describe a method for its computation. □

In view of (2.6) and (2.7), the calculation of an optimal input function u^* involves the solution of a max-min optimization problem. In the next section, we show that u^* exists.

3. EXISTENCE OF AN OPTIMAL SOLUTION

The existence of an optimal solution of Problem 2.1 follows from a generalized version of the Weierstrass Theorem, which, in crude terms, states that a continuous functional over a compact set achieves its maximum within the set. We start with some basic terminology (e.g., LIUSTERNIK and SOBOLEV [1982]).

Definition 3.1. Let H be a Hilbert space with the inner product $\langle \bullet, \bullet \rangle$.

(i) A sequence $\{x_n\}$ in H *converges weakly* to an element $x \in H$ if $\lim_{n\to\infty} \langle x_n, y\rangle = \langle x, y\rangle$ for every element $y \in H$.

(ii) A subset W of H is *weakly compact* if every sequence of elements of W has a subsequence that converges weakly to an element of W.

(iii) A sequence $\{z_n\} \subset H$ is *strongly convergent* if there is an element $z \in H$ such that $\lim_{n\to\infty} \langle z_n - z, z_n - z\rangle = 0$.

(iv) A set $S \subset H$ is *strongly closed* if every strongly convergent sequence of elements of S has its limit in S. □

To show the existence of an optimal input function for Problem 2.1, we show first that the set U of (2.2) has a certain compactness feature; next, we show that the function $T^*(M,u)$ of (2.6) has an appropriate continuity property; then, existence of the supremal time t_f^* of (2.7) follows then from a generalized version of the Weierstrass Theorem. We start with the following fact from CHAKRABORTY and HAMMER [2009, Lemma 3.2]).

Lemma 3.2. *The set U of (2.2) is weakly compact in the topology of the Hilbert space $L_2^{\alpha,m}$.* □

The system Σ of (1.1) is *nominally unstable* if the nominal matrix A has an eigenvalue with strictly positive real part. The state trajectory of a nominally unstable system cannot be bounded for all disturbances and uncertainties, as follows.

Lemma 3.3. *Assume that the system Σ of (1.1) is nominally unstable and recall the notation of (1.2), (1.4), and (2.5). Then, for each input function $u(t) \in U$, there is a triplet $(x_0, D, v) \in X_0 \times \Delta \times V$ for which $T(M, x_0, D, v, u) < \infty$.*

Proof. Note that the set of initial conditions X_0 includes a non-zero initial state x_0. Then, Lemma 3.3 of CHAKRABORTY and HAMMER [2009] shows that our present lemma is valid for the zero disturbance signal $v(t)=0$ for all $t \geq 0$, and this completes our proof. □

Lemma 3.3 leads to the following.

Corollary 3.4. *If the system Σ of (1.1) is nominally unstable, then $T^*(M,u)<\infty$ for every input function $u(t) \in U$ and for every disturbance range $X_0 \times \Delta \times V$.* □

Having seen in Lemma 3.2 that the set U of input functions has a compactness feature, we turn next to continuity properties of the functional $T^*(M,u)$. We start by reviewing a notion from mathematical analysis.

Definition 3.5. A functional F is *weakly upper semi-continuous* when the following is true for every weakly convergent sequence $\{z_n\}$: if $F(\lim_{n\to\infty} z_n)$ is bounded, then, for every $\varepsilon>0$, there is an integer $N>0$ such that $F(z_n)-F(\lim_{n\to\infty} z_n)<\varepsilon$ for all integers $n \geq N$. □

The next two statements show that $T^*(M,u)$ is weakly upper semi-continuous. When combined with Lemma 3.2, this property will allow us to prove the existence an optimal input function for Problem 2.1.

Lemma 3.6. *For fixed $(x_0,D,v) \in X_0 \times \Delta \times V$ and $M>0$, the functional $T(M,x_0,D,v,u)$ of (2.5) is weakly upper semi-continuous in u.*

Proof. Let $x(t,u)$ be the solution of the differential equation (1.1) for given selections of x_0, D, v, M, and input function u. Using the well known solution of (1.1), we can write

$$x(t,u)=e^{A't}\left[x_0+\int_0^t e^{-A'\tau}B'u(\tau)d\tau+\int_0^t e^{-A'\tau}G'v(\tau)d\tau\right]. \tag{3.1}$$

Consequently, at each time $t \geq 0$, the functional

$$\xi(t,u):=x(t,u)-e^{A't}\left[x_0+\int_0^t e^{-A'\tau}G'v(\tau)d\tau\right]=e^{A't}\int_0^t e^{-A'\tau}B'u(\tau)d\tau$$

is linear in u.

Now, let $u_1,u_2,... \in U$ be a weakly convergent sequence of input functions with the limit u_0. Now, weak convergence of a sequence implies convergence of any bounded linear functional of that sequence; consequently, the sequence of vectors $\{\xi(t,u_n)\}$ is convergent and $\lim_{n\to\infty}\xi(t,u_n)=\xi(t,u_0)$. But then, since

$$x(t,u)=\xi(t,u)+e^{A't}x_0+\int_0^t e^{A'(t-\tau)}G'v(\tau)d\tau,$$

where x_0 and $v(t)$ are fixed, it follows that $\lim_{n\to\infty} x(t,u_n)=x(t,u_0)$. Thus, we conclude that $\lim_{n\to\infty} x^T(t,u_n)x(t,u_n)= x^T(t,u_0)x(t,u_0)$ for all $t \geq 0$. Denoting by $e(t,u):=x^T(t,u)x(t,u)$ the value of our error criterion at the time t for the input function u, we can rewrite the last equation as $\lim_{n\to\infty} e(t,u_n)=e(t,u_0)$ for all $t \geq 0$.

Next, for a function $e(t)$, define the functional

$$\Theta(e)=\inf\{t \geq 0 : e(t)>M\}, \tag{3.2}$$

where $\Theta(e):=\infty$ if $e(t) \leq M$ for all $t \geq 0$. Let $e_1(t), e_2(t), ...$ be a sequence of functions that converges to the function $e_0(t)$ at each $t \geq 0$. Assume that there is a real number $\theta \geq 0$ such that $\Theta(e_0) \leq \theta$. Then, we show that, for every $\varepsilon>0$, there is an integer $N>0$ such that $\Theta(e_n)-\Theta(e_0)<\varepsilon$ for all integers $n>N$. To this end, we can distinguish between two cases:

CASE 1: There is an integer $N>0$ for which $\Theta(e_n) \leq \Theta(e_0)$ for all $n>N$. Then, $\Theta(e_n)-\Theta(e_0)<\varepsilon$ for any $\varepsilon>0$, and our claim is true.

CASE 2: There is no N that satisfies CASE 1. Then, there is a subsequence $n_1,n_2,...$ of integers such that $\Theta(e_{n_k})>\Theta(e_0)$ for all integers $k>0$. Also, by (3.2) there is, for every real number $\varepsilon>0$, a time $t' \in [\Theta(e_0),\Theta(e_0)+\varepsilon)$ such that $e_0(t')>M$. Now, as the sequence $\{e_n(t)\}$ converges to $e_0(t)$ at every $t \geq 0$, we have $e_n(t') \to e_0(t')$. Choosing $\varepsilon:=[e_0(t')-M]/2$, it follows that there is an integer $N>0$ such that $|e_0(t')-e_n(t')|<[e_0(t')-M]/2$ for all $n>N$. For such n, we have

$$\begin{aligned} e_n(t') &= e_0(t')-[e_0(t')-e_n(t')] \geq e_0(t')-\left|e_0(t')-e_n(t')\right| \\ &\geq e_0(t')-[e_0(t')-M]/2 \geq e_0(t')/2+M/2>M. \end{aligned}$$

Thus, $e_n(t') > M$, which implies that $\Theta(e_n) \leq t'$ for all integers $n > N$. By the selection of t', this yields $\Theta(e_n) < \Theta(e_0) + \varepsilon$, or $\Theta(e_n) - \Theta(e_0) < \varepsilon$ for all $n > N$. Combining the outcomes of CASE 1 and CASE 2, we conclude that $\Theta(e)$ is an upper semi-continuous functional of e.

Returning now to the functional $T(M, x_0, D, v, u)$ of (2.5), note the composition $T(M, x_0, D, v, u) = \Theta(e(t; x_0, D, v, u))$. As shown in the first part of the proof, the weakly convergent sequence of input functions $\{u_n\}$ yields the sequence of error functions $\{e(t; x_0, D, v, u_n)\}$ that is convergent at every $t \geq 0$. Combining this with the upper semi-continuity of Θ just shown, it follows that $T(M, x_0, D, v, u)$ is weakly upper semi-continuous in u, and our proof concludes. □

We will also need the following fact.

Lemma 3.7. *For a nominally unstable system Σ, the function $T^*(M, u)$ of (2.6) is weakly upper semi-continuous in u.*

Proof. The proof is based on the following fact (e.g., WILLARD [1970]): Let S and A be two topological spaces, and, for each element $\alpha \in A$, let f_α be a weakly upper semi-continuous real valued function over S. If $\inf_{\alpha \in A} f_\alpha(s)$ exists at each point $s \in S$, then the function $f(s) := \inf_{\alpha \in A} f_\alpha(s)$ is weakly upper semi-continuous in s.
By Lemma 3.6, the function $T(M, x_0, D, v, u)$ is weakly upper semi-continuous on U at each point $(x_0, D, v) \in X_0 \times \Delta \times V$. Furthermore, it follows by Lemma 3.3 that $\inf_{(x_0, D, v) \in X_0 \times \Delta \times V} T(M, x_0, D, v, u)$ exists for every $u \in U$. Thus, the fact quoted at the beginning of this proof implies that $T^*(M, u) = \inf_{(x_0, D, v) \in X_0 \times \Delta \times V} T(M, x_0, D, v, u)$ is weakly upper semi-continuous in u. □

We are ready now to state the main result of this section: there is an optimal input function $u^*(t)$ that maximizes the time during which a perturbed system remains within specified error bounds.

Theorem 3.8. *Assume that the system Σ of (1.1) is nominally unstable, and let U be given by (2.2). Then, using the notation of (2.7), the following are true.*
(i) There is a finite maximal time $t_f^* := sup_{u \in U} T^*(M, u)$, and
(ii) There is an input function $u^* \in U$ satisfying $t_f^* = T^*(M, u^*)$.

Proof. We use the Generalized Weierstrass Theorem, which, in our current terminology, states the following: A weakly upper semi-continuous functional attains a maximum in a weakly compact set (e.g., ZEIDLER [1985]). Presently, the set of input functions U of (2.2) is weakly compact by Lemma 3.2, and the functional $T^*(M, u)$ is weakly upper semi-continuous in u over U by Lemma 3.7. Consequently, the generalized Weierstrass theorem implies that $T^*(M, u)$ attains a maximum over U, and our proof concludes. □

To summarize, we have shown in this section that after a feedback failure occurs, there is an optimal input function $u^*(t)$ that keeps the open loop response below a specified error bound for a duration of at least t_f^*, irrespective of uncertainties about the initial condition and the system's parameters, or the presence of a disturbance signal. While driven by an optimal input function $u^*(t)$, the actual duration of time t_f during which the system's response remains below the specified error bound depends, of course, on the actual initial condition x_0, the particular perturbation matrix D, and the disturbance signal $v(t)$ active in the system. However, for any permissible selection of these quantities, the duration of time t_f during which the system error remains within specified bounds satisfies $t_f \geq t_f^*$, and t_f^* is the maximal duration that satisfies this inequality.

4. BANG-BANG APPROXIMATION

Optimal input functions $u^*(t)$ of Theorem 3.8 are often hard to calculate and implement in practice. In the present section, we show that $u^*(t)$ can always be replaced by a bang-bang function without causing significant performance deterioration. Recalling that $K > 0$ is the input amplitude bound of the controlled system Σ, a bang-bang input function of Σ consists of component functions whose values switch between K and $-K$ as necessitated by control action. Bang-bang functions, being completely determined by their switching times, are relatively easy to calculate and implement and are therefore preferable in applications. In general, a bang-bang function may not yield exactly the same performance as an optimal input function $u^*(t)$. However, as the next statement indicates, optimal performance can be approximated as closely as desired by bang-bang input functions (compare to CHAKRABORTY and HAMMER [2008], where a related result is derived under more restrictive conditions).

Theorem 4.1. *Let Σ be a nominally unstable system described by (1.1), let U be the set of input signals (2.2), and let $x(t, x_0, D, v, u)$ be the state trajectory of Σ induced by an input function u. Let t_f^* be the optimal time and let u^* be an optimal input function of Theorem 3.8. Then, for every $\varepsilon > 0$, there is a bang-bang input function $u^\pm \in U$ for which the following are true.*

(i) $u^{\pm}$ has only a finite number of switches, and

(ii) The discrepancy between the state trajectories satisfies $\|x(t,x_0,D,v,u^*)-x(t,x_0,D,v,u^{\pm})\|<\varepsilon$ for all $t\in[0,t_f^*]$ and for all $(x_0,D,v)\in X_0\times\Delta\times V$.

Proof. We use the notation of (1.4), (1.6), and (2.2). As Σ is nominally unstable, it follows by Theorem 3.8 that the optimal time t_f^* is finite. Now, let $\varepsilon,\eta>0$ be two real numbers. Considering that the exponential function is uniformly continuous over any finite interval of time, there is a real number $\delta(\eta)>0$ such that the function

$$\mu(t',t):=e^{-A't'}-e^{-A't}$$

satisfies $\|\mu(t',t)\|\leq\eta$ whenever $|t'-t|<\delta(\eta)$ and $t',t\in[0,t_f^*]$. Denote $\beta:=sup\{\|B+D_B\|:D_B\in\Delta_B\}$ and $N:=\sup\left\{\left\|e^{A't}\right\|:D_A\in\Delta_A,t\in[0,t_f^*]\right\}$; here, β and N exist due the fact that all involved quantities are bounded.

Next, let $0<\gamma\leq\delta(\eta)$ be any number for which the ratio t_f^*/γ is an integer. We build a partition of the interval $[0,t_f^*]$ into segments of length γ, namely, the partition determined by the intervals $[q\gamma,(q+1)\gamma]$, $q=0,1,2,...,(t_f^*/\gamma)-1$. Recalling that input functions of Σ are m-dimensional column vectors bounded by $K>0$, we build a bang-bang input function $u^{\pm}(t)=(u_1^{\pm}(t),u_2^{\pm}(t),...,u_m^{\pm}(t))^T,0\leq t\leq t_f^*$, as follows: for the component $u_i^{\pm}(t)$, select in each interval $[q\gamma,(q+1)\gamma]$ a switching time θ_{qi} and set

$$u_i^{\pm}(t):=\begin{cases}K \text{ for } t\in[q\gamma,\theta_{qi}),\\ -K \text{ for } t\in[\theta_{qi},(q+1)\gamma),q=0,1,2,...,(t_f^*/\gamma)-1,\end{cases} \tag{4.1}$$

$i=1,2,...m$. Then, for each such component function, we have $\int_{q\gamma}^{(q+1)\gamma}u_i^{\pm}(\tau)d\tau=K\int_{q\gamma}^{\theta_{qi}}d\tau-K\int_{\theta_{qi}}^{(q+1)\gamma}d\tau=K[2(\theta_{qi}-q\gamma)-\gamma]$. Now, select θ_{qi} to satisfy the equality

$$K[2(\theta_{qi}-q\gamma)-\gamma]=\int_{q\gamma}^{(q+1)\gamma}u_i^*(\tau)d\tau.$$

Note that θ_{qi} exists due to the fact that $|u_i^*(t)|\leq K$ for all $t\geq0$. For this value of θ_{qi}, we obtain the equality

$$\int_{q\gamma}^{(q+1)\gamma}[u_i^*(\tau)-u_i^{\pm}(\tau)]d\tau=0 \tag{4.2}$$

for all $i=1,2,...m$ and all $q=0,1,2,...,(t_f^*/\gamma)-1$.

Further, let $x^{\pm}(t)$ be the state trajectory of Σ for the input function $u^{\pm}(t)$, and let $x^*(t)$ be the state trajectory induced by the optimal input function $u^*(t)$. Noting that the initial condition x_0, the perturbation matrix D, and the disturbance input $v(t)$ are all the same in both cases (we are considering the performance of the same system sample), we obtain from (3.1) and (4.2) that

$$\begin{aligned}
\|x^*(t)-x^\pm(t)\| &= \left\| e^{A't}\left[x_0+\int_0^t e^{-A'\tau}B'u^*(\tau)d\tau\right] - e^{A't}\left[x_0+\int_0^t e^{-A'\tau}B'u^\pm(\tau)d\tau\right]\right\| \\
&= \left\| e^{A't}\int_0^t e^{-A'\tau}B'[u^*(\tau)-u^\pm(\tau)]d\tau\right\| \\
&\leq N\left\|\int_0^t e^{-A'\tau}B'\left[u^*(\tau)-u^\pm(\tau)\right]d\tau\right\| \\
&= N\left\|\left[\sum_{r=0}^{q-1}\int_{r\gamma}^{(r+1)\gamma} e^{-A'\tau}B'\left[u^*(\tau)-u^\pm(\tau)\right]d\tau\right]+\int_{q\gamma}^t e^{-A'\tau}B'\left[u^*(\tau)-u^\pm(\tau)\right]d\tau\right\| \\
&\leq N\left\|\sum_{r=0}^{q-1}\left[e^{-A'r\gamma}B'\int_{r\gamma}^{(r+1)\gamma}\left[u^*(\tau)-u^\pm(\tau)\right]d\tau+\right.\right. \\
&\qquad \left.\left.\int_{r\gamma}^{(r+1)\gamma}\mu(\tau,r\gamma)B'\left[u^*(\tau)-u^\pm(\tau)\right]d\tau\right]\right\| + N\left\|\int_{q\gamma}^t e^{-A'\tau}B'\left[u^*(\tau)-u^\pm(\tau)\right]d\tau\right\| \\
&\leq N\sum_{r=0}^{q-1}\int_{r\gamma}^{(r+1)\gamma}\|\mu(\tau,r\gamma)\|\left\|B'\right\|\left[\|u^*(\tau)\|+\|u^\pm(\tau)\|\right]d\tau \\
&\quad +N\int_{q\gamma}^t\left\|e^{-A'\tau}\right\|\|B'\|\left[\|u^*(\tau)\|+\|u^\pm(\tau)\|\right]d\tau \\
&\leq 2KN\beta(\eta t_f^*+N\gamma)
\end{aligned}$$

for all $t\in[0,t_f^*]$. Finally, choose the value of η so that $2KN\beta\eta t_f^*<\varepsilon/2$. Then, choose γ so that

$$0<\gamma\leq\min\{\delta(\eta),\varepsilon/(4KN^2\beta)\}\text{ and } t_f^*/\gamma \text{ is an integer.} \tag{4.3}$$

For these selections, we obtain $\|x^*(t)-x^\pm(t)\|<\varepsilon$ for all $t\in[0,t_f^*]$, and our proof concludes. □

The bang-bang input signal $u^\pm(t)$ of Theorem 4.1 approximates optimal performance for all permissible perturbations of the initial conditions and the system matrices, as well as for all permissible disturbance signals.

Remark 4.2. In Theorem 4.1, the cost of making the error ε smaller is an increase in the number of switches of the bang-bang function $u^\pm(t)$. This can be seen by examining the proof of the Theorem. Indeed, from inequality (4.3), we can see that to maintain the inequality, γ must be decreased as ε is decreased. According to (4.1), the number of switches is (in general) t_f^*/γ, so that a decrease of γ leads to an increase in the number of switches. □

4.1. **Design considerations.** According to equations (2.6) and (2.7), the calculation an optimal input function $u^*(t)$ involves finding the 'worst' selections of the initial condition x_0, of the deviation matrix D, and of the disturbance signal v, namely, the selections that create an infimum of $T(M,x_0,D,v,u)$ for a fixed input function u. Finding a worst case of the disturbance signal requires further consideration, since the disturbance signal $v(t)$ is a member of the infinite dimensional topological space V of (2.3). To simplify the selection of a worst disturbance signal, we show that it can be approximated by a bang-bang function, in close analogy to the way an optimal input function u^* can be approximated by the bang-bang function $u^\pm(t)$ of Theorem 4.1. This leads us to a situation where approximations of both signals - an optimal input signal and a worst disturbance signal - can be found by solving a finite dimensional optimization problem. The formal statement is as follows.

Theorem 4.3. *Let Σ be a nominally unstable system given by (1.1), let U be the set of input signals (2.2), and let V be the set of disturbance signals (2.3). Let $x(t,x_0,D,v,u)$ be the state trajectory induced by the input function u in the presence of the disturbance function v. Finally, let t_f^* be the optimal time and let u^* be an optimal input function of Theorem 3.8. Then, for every $\varepsilon>0$ and for every disturbance signal $v\in V$, there is a bang-bang input function $u^\pm\in U$ and a bang-bang disturbance function $v^\pm\in V$ for which the following hold true.*

(i) $u^\pm$ and $v^\pm$ have a finite number of switches, and

(ii) The state trajectory $x(t,x_0,D,v^\pm,u^\pm)$ created by $u^\pm$ and $v^\pm$ satisfies $\|x(t,x_0,D,v,u^*)-x(t,x_0,D,v^\pm,u^\pm)\|<\varepsilon$ for all $t\in[0,t_f^*]$ and all $(x_0,D)\in X_0\times\Delta$.

Proof. We use the notation of the proof of Theorem 4.1. As in that proof, the fact that Σ is nominally unstable implies, by Theorem 3.8, that the optimal time t_f^* is finite. Fix a disturbance signal $v(t) \in V$. We build a bang-bang disturbance signal $v^{\pm}(t) = (v_1^{\pm}(t), v_2^{\pm}(t), ..., v_p^{\pm}(t))^T, 0 \leq t \leq t_f^*$, that 'approximates' the effects of $v(t)$: for the component $v_i^{\pm}(t)$, select in each interval $[q\gamma, (q+1)\gamma]$ a switching time ψ_{qi} and set

$$v_i^{\pm}(t) := \begin{cases} L \text{ for } t \in [q\gamma, \psi_{qi}), \\ -L \text{ for } t \in [\psi_{qi}, (q+1)\gamma), q = 0, 1, 2, ..., (t_f^*/\gamma) - 1, \end{cases}$$

$i = 1, 2, ..., p$. Then, we have

$$\int_{q\gamma}^{(q+1)\gamma} v_i(\tau) d\tau = L \int_{q\gamma}^{\psi_{qi}} d\tau - L \int_{\psi_{qi}}^{(q+1)\gamma} d\tau = L[2(\psi_{qi} - q\gamma) - \gamma].$$

Select ψ_{qi} to satisfy the equality

$$L[2(\psi_{qi} - q\gamma) - \gamma] = \int_{q\gamma}^{(q+1)\gamma} v_i(\tau) d\tau.$$

Note that ψ_{qi} exists due to the fact that $|v_i(t)| \leq L$ for all $t \geq 0$. For this value of ψ_{qi}, we obtain

$$\int_{q\gamma}^{(q+1)\gamma} [v_i(\tau) - v_i^{\pm}(\tau)] d\tau = 0 \tag{4.4}$$

for all $i = 1, 2, ..., p$ and all $q = 0, 1, 2, ..., (t_f^*/\gamma) - 1$.

Further, let $x^{\pm}(t)$ be the state trajectory generated by the system Σ when driven by the bang-bang input function $u^{\pm}(t)$ of Theorem 4.1 in the presence of the bang-bang disturbance signal $v^{\pm}(t)$, and let $x^*(t)$ be the state trajectory induced by the optimal input function $u^*(t)$ in the presence of a worst disturbance signal $v(t)$. Noting that the initial condition x_0 and the perturbation matrix D are the same in both cases (we are considering the performance of the same system sample), we obtain from (3.1), (4.2), and (4.4) that

$$\begin{aligned} \|x^*(t) - x^{\pm}(t)\| &= \\ &= \left\| e^{A't} \left[x_0 + \int_0^t e^{-A'\tau} B' u^*(\tau) d\tau + \int_0^t e^{-A'\tau} G' v(\tau) d\tau \right] + \right. \\ &\quad \left. - e^{A't} \left[x_0 + \int_0^t e^{-A'\tau} B' u^{\pm}(\tau) d\tau + \int_0^t e^{-A'\tau} G' v^{\pm}(\tau) d\tau \right] \right\| \\ &= \left\| e^{A't} \int_0^t e^{-A'\tau} B' [u^*(\tau) - u^{\pm}(\tau)] d\tau + e^{A't} \int_0^t e^{-A'\tau} G' [v(\tau) - v^{\pm}(\tau)] d\tau \right\| \\ &\leq N \left\| \int_0^t e^{-A'\tau} B' \left[u^*(\tau) - u^{\pm}(\tau) \right] d\tau \right\| + N \left\| \int_0^t e^{-A'\tau} G' \left[v(\tau) - v^{\pm}(\tau) \right] d\tau \right\| \end{aligned} \tag{4.5}$$

Now, according to the proof of Theorem 4.1, we have

$$N \left\| \int_0^t e^{-A'\tau} B' \left[u^*(\tau) - u^{\pm}(\tau) \right] d\tau \right\| \leq 2KN\beta(\eta t_f^* + N\gamma). \tag{4.6}$$

Further, using the quantity $g := \sup\{\|G + D_G\| : D_G \in \Delta_G\}$, an argument similar to the one used in the proof of Theorem 4.1 yields the inequality

$$N \left\| \int_0^t e^{-A'\tau} G' \left[v(\tau) - v^{\pm}(\tau) \right] d\tau \right\| \leq 2LNg(\eta t_f^* + N\gamma). \tag{4.7}$$

Combining (4.6) and (4.7), we obtain from (4.5) that

$$\|x^*(t) - x^{\pm}(t)\| \leq 2N(K\beta + Lg)(\eta t_f^* + N\gamma).$$

Finally, choose the value of η so that $2N(K\beta + Lg)\eta t_f^* < \varepsilon/2$. Then, choose γ so that $0 < \gamma \leq \min\{\delta(\eta), \varepsilon/[4N^2(K\beta + Lg)]\}$ and t_f^*/γ is an integer. For these selections, we obtain $\|x^*(t) - x^{\pm}(t)\| < \varepsilon$ for all $t \in [0, t_f^*]$, and our proof concludes. □

The accuracy of the approximation provided by the bang-bang functions $u^{\pm} \in U$ and $v^{\pm} \in V$ of Theorem 4.3 can be improved by increasing the number of switches (see Remark 4.2).

Using Theorem 4.3, we can calculate an approximate solution to Problem 2.1 by using finite dimensional optimization techniques. The following outline describes in general terms a computational process for deriving bang-bang approximants of the control input signal and of the disturbance signal in the spirit of Theorem 4.3. These approximants yield a trajectory $x^{\pm}(t)$ that stays within the error bound M of (1.6) for a time of at least $t_f^{\pm}$, where $t_f^{\pm}$ approximates the optimal time t_f^* of Theorem 3.8. In fact, the trajectory $x^{\pm}(t)$ approximates the optimal trajectory $x^*(t)$ at all times $0 \leq t \leq t_f^{\pm}$.

Outline 4.4. ***Calculating a bang-bang approximant of an optimal input function:*** Let $u^{\pm}(t) = [u_1^{\pm}(t), u_2^{\pm}(t), ..., u_m^{\pm}(t)]^T$ be a bang-bang approximant of an optimal input function $u^*(t)$, let $v^{\pm}(t) = [v_1^{\pm}(t), v_2^{\pm}(t), ..., v_p^{\pm}(t)]^T$ be a bang-bang approximant of a 'worst' disturbance function $v^*(t)$, and let $x^{\pm}(t)$ be the state trajectory of the system (1.1) induced by $u^{\pm}$ and $v^{\pm}$. Denote by $t_f^{\pm}$ the time at which $x^{\pm}$ is about to exceed the specified error bound, i.e., $t_f^{\pm} := \inf\{t \geq 0 : [x^{\pm}(t)]^T x^{\pm}(t) > M\}$. Let μ be the largest permissible deviation between $t_f^{\pm}$ and the optimal time t_f^*, so that $\left| t_f^* - t_f^{\pm} \right| \leq \mu$. Finally, assume that a bound t_f of t_f^* is available, so that $t_f^* \leq t_f$. Let k denote the number of switches of each component of $u^{\pm}(t)$ and $v^{\pm}(t)$.

Step 1. Set $t_f^0 := 0$ and $k := 1$.

Step 2. Partition the interval $[0, t_f]$ into $Q \gg k$ equal segments. On this partition, create two families of bang-bang functions whose switching times are compatible with the partition: the family $U^{\pm}(k, Q) \subset U$ of all bang-bang functions $u(t) = [u_1(t), u_2(t), ..., u_m(t)]^T$ that have at most k switches in each component; and the family $V^{\pm}(k, Q) \subset V$ of all bang-bang functions $v(t) = [v_1(t), v_2(t), ..., v_p(t)]^T$ that have at most k switches in each component. Both families are, of course, finite.

Step 3. For each $u(t)$ created in Step 2, calculate the quantity $T(u, k) := \min_{(x_0, D, v) \in X_0 \times \Delta \times V^{\pm}(k, Q)} T(M, x_0, D, v, u)$. This is a finite dimensional minimization process.

Step 4. Let $t_f^k := \max_{u \in U^{\pm}(k, Q)} T(u, k)$ and denote by $u^k \in U^{\pm}(k, Q)$ a function that achieves this maximum. (Then, t_f^k is the longest duration that can be achieved by using bang-bang approximants with at most k switches.)

Step 5. If one of the following two conditions is satisfied, then replace k by $k+1$ and return to Step 2: (i) $k = 1$, or (ii) $k > 1$ and $t_f^k > t_f^{k-1} + \mu$.

Step 6. Otherwise, terminate the computation. The approximants are $t_f^* \approx t_f^{k-1}$ and $u^{\pm}(t) \approx u^{k-1}$. □

Outline 4.4 shows that an approximate solution of the dynamic optimization problem described in Problem 2.1 can be obtained by solving a finite dimensional min-max problem. A wide range of numerical optimization techniques for solving the latter are available in the literature (e.g., POLYAK [1988], SHEU and LIN [2004], the references mentioned in these papers, and others). The computational complexity of deriving the approximate solution will depend, of course, on the particular numerical algorithm employed to derive it, but it would be substantially lower than the computational complexity of solving Problem 2.1 directly.

Example 4.5. Consider a single state system described by the equation $\dot{x}(t) = ax(t) + u(t) + v(t)$ with the initial condition $x(0) = x_0$, the control input $u(t)$, and the disturbance signal $v(t)$. The uncertainties are described by $x_0 \in [0.9, 1.1]$, $a \in [1.2, 1.4]$, and $|v(t)| \leq 0.2$ for all $t \geq 0$ (so that $L = 0.2$ in (2.3)). The input function amplitude bound is $K = 2$ in (2.2), i.e., $|u(t)| \in [-2, 2]$ for all $t \geq 0$. We use the bound $M := 25$ in (1.6). Considering Problem 2.1, our objective is to calculate an optimal input function $u^*(t)$ that produces the maximal time t_f^*, irrespective of perturbations and disturbances. In the process, we also find worst instances of the parameters a and x_0 and of the disturbance signal $v(t)$. Specializing (2.7) to our present situation, we seek an input function $u^*(t)$ that solves the max-min problem

$$t_f^* = \sup_{\{u(t):|u(t)|\leq 2, t\geq 0\}} \left\{ \inf_{\substack{0.9\leq x_0\leq 1.1 \\ [1.2\leq a\leq 1.4] \\ \{v(t):|v(t)|\leq 0.2, t\geq 0\}}} T(25, a, x_0, v(t), u(t)) \right\}.$$

By Theorem 4.3, an approximation of the optimal time t_f^* and of the optimal input signal u^* can be obtained by using bang-bang approximants for the input signal and for the disturbance signal, following the steps of Outline 4.4. Referring to Outline 4.4, we use an error bound of $\mu = 0.01$ seconds on the estimated terminal time. To process Step 3 of Outline 4.4, we consider a bang-bang input signal $u(t)$ and, for this signal, find the lowest value $T(u, k)$ of

$T(25,a,x_0,v(t),u(t))$ as a function of the switching times of $u(t)$. Here, we implemented the latter by using a global optimization process based on multilevel coordinate search to optimize over all permissible values of a, of x_0, and of the disturbance function $v(t)$'s switching times (HUYER and NEUMAIER [1999]). In Step 4 of Outline 4.4, we search for a maximum t_f^k of $T(u,k)$ over the switching times of the bang-bang input signal $u(t)$ to find a 'best' bang-bang approximant $u^k(t)$. This process is then repeated for increasing values of k, until the improvement in the terminal time t_f^k is smaller than the prescribed error bound μ.

For the present Example, the process of Outline 4.4 ends at $k=2$, resulting in an approximate optimal terminal time of $t_f^* \approx t_f^{2-1} = t_f^1 = 2.18$ seconds and a bang-bang approximate optimal input signal

$$u^{\pm}(t) = \begin{cases} -2 & \text{for } t \leq 1.248, \\ +2 & \text{for } t > 1.248. \end{cases} \tag{4.8}$$

As we can see, this approximant has a single switch at $t = 1.248$ seconds. In this case, there are two combinations of parameter values and disturbance signals that yield the lowest terminal time for the input signal $u^{\pm}(t)$:

$$\{a = 1.4, x_0 = 1.1, \text{ and } v^{\pm}(t) = 0.2 \text{ for all } t \geq 0.\} \tag{4.9}$$

$$\{a = 1.4, x_0 = 0.9, \text{ and } v^{\pm}(t) = -0.2 \text{ for all } t \geq 0.\} \tag{4.10}$$

As we can see, the approximant $v^{\pm}(t)$ of a 'worst' disturbance signal is just a constant function in both cases here. Figure 4.1 illustrates the state trajectory $x^{\pm}(t)$, the bang-bang input function $u^{\pm}(t)$, and the bang-bang disturbance signal $v^{\pm}(t)$ obtained under the conditions of (4.9). From the figure, we can see that indeed $t_f^* \approx 2.18$ seconds.

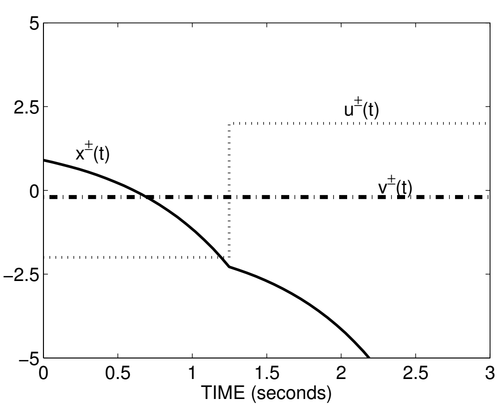

FIGURE 4.1. Disturbance set (4.9)

Similarly, Figure 4.2 displays the response under the conditions of (4.10); again, we can see that $t_f^* \approx 2.18$ seconds, as before.

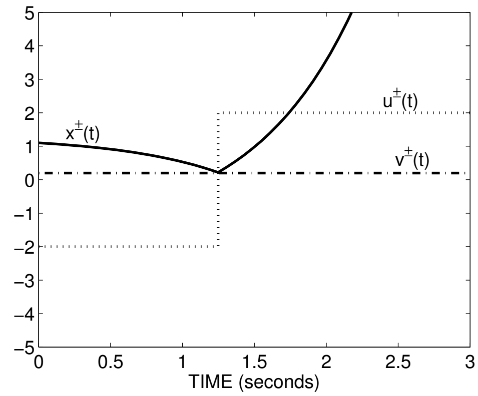

FIGURE 4.2. Disturbance set (4.10)

Note that, when using approximate bang-bang signals, Theorem 4.3 only guarantees that the terminal time $t_f^{\pm}$ is close to the optimal terminal time t_f^* and that the state trajectory $x^{\pm}(t)$ is close to an optimal state trajectory $x^*(t)$. However, the bang-bang input signal $u^{\pm}(t)$ may be entirely different from an optimal input signal $u^*(t)$, when the latter is not a bang-bang signal (see CHAKRABORTY and HAMMER [2009] for conditions under which the optimal input is a bang-bang signal for problems with specified initial state and no disturbance signals).

5. CONCLUSION

To summarize, the paper presents a general methodology for finding optimal input signals that keep performance errors below specified bounds for the longest time under a broad range of uncertainties and disturbances. The use of bang-bang functions to approximate optimal solutions provides an effective approach to finding and implementing solutions of this optimization problem.

Future directions of research include interlacing the open loop control methodology presented in this paper with bursts of feedback control, to maintain low error performance over the long term under conditions where feedback use must be limited. Another direction in which the current research can be generalized is output control, where the objective is to keep the output error (rather than the state error considered here) below a specified bound. These issues will be addressed in future reports.

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