Bang-Bang Solutions To The Open Loop Maximal Time Problem

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***Abstract*—The problem of maximizing the duration of open loop operation of a perturbed linear time invariant system, while keeping performance errors within bound, is considered. It was shown in an earlier article that the optimal control for this problem is purely bang-bang if an associated switching function is non-zero almost everywhere. Sufficient conditions are derived in this article to guarantee this situation.**

I. Introduction

Consider a single input to state linear time invariant system given by the state space description of the form

$$\dot{x}(t) = A^{'}x(t) + B^{'}u(t), \tag{1}$$

where $x \in \mathbb{R}^n$ is the state of the system; $u \in \mathbb{R}$ is the input and $y \in \mathbb{R}^m$ is the output. Here $A^{'}$ and $B^{'}$ are uncertain system matrices of compatible dimensions. It is assumed that the nominal values of these matrices are known: let the nominal values be A and B while the uncertainties are assumed to be norm bounded and additive with the following representation: let $d > 0$ be a real number. Denote by Δ_A and Δ_B the sets of all $n \times n$ and $n \times 1$ matrices respectively with each element in the interval $[-d, d]$. Then,

$$A' := A + D_A \text{ and } B' := B + D_B \tag{2}$$

where $D_A \in \Delta_A$ and $D_B \in \Delta_B$ are unspecified matrices representing perturbations and uncertainties. In shorthand, denote $D := (D_A, D_B)$ and $\Delta := \Delta_A \times \Delta_B$ so that $D \in \Delta$. The input $u(t)$ is assumed to be Lebesgue measurable and bounded. Let $K \in \mathbb{R}^+$; then the set of possible inputs is denoted by $U = \{u(t) : |u(t)| \leq K \text{ for a.a. } t \in [0, \infty)\}$

Let $M \in \mathbb{R}^+$ be a pre-specified number. Assuming that the system is started at time $t = 0$ with initial conditions such that $x^T(0)x(0) \leq M$, we define the system performance as acceptable for as long as the following equation hold:

$$x^T(t)x(t) \leq M \text{ for all } D \in \Delta \text{ and all } t \in [0, t_f]. \tag{3}$$

In this setting, our objective can be stated as follows:

Problem 1: Find $\max_{u(t)\in U}\ t_f$ such that the solution to (1) satisfies (3). Compute the optimal input achieving the maximum (also see [2], [3]).

Such a problem arises in a variety of applications where the feedback signal is intermittently available and the system is often forced to function in open loop while maintaining acceptable performance[3]. Interruptions in the feedback signal may be caused by malfunctions or disruptions in the feedback communication link, or they may be the result of efforts to reduce operating costs. In other applications, feedback channels are opened only occasionally, when system performance degrades below an acceptable level.

Consider, for example, the medical treatment of type 1 diabetes. Individuals afflicted by this condition require periodic injections of insulin in order to control the glucose concentration in their blood. Insulin is injected when glucose concentration deviates by more than a specified amount from nominal level. Insulin injection is often done by an implanted insulin infusion pump, which allows excellent control of the infusion profile. The feedback mechanism in this case consists of periodical blood analyses, which, at the present time, require the drawing of blood through finger pricks or similar irksome procedures. In order to improve patient comfort, it would be desirable to maximize the time interval between blood samplings, while maintaining blood glucose concentration within desirable bounds. Needless to say, models of the dynamics of blood glucose concentration are subject to significant errors and depend on external interferences. In this context, the objective of the present paper is to develop techniques for the design of glucose infusion profiles that keep blood glucose concentrations within desirable bounds and allow the longest possible time interval between blood samplings.

Intermittent use of feedback is also of interest in other biomedical applications. Consider, for example, the treatment of cancer by chemotherapy. Here, it would be of advantage to maximize the time between observations of cancer status, observations that often require extensive testing. The methodology developed in the present paper can be used to design optimal chemotherapy protocols that maximize the time between subsequent tests. Such protocols will improve patient independence and reduce costs (e.g. [5] and others). Many additional potential applications in biomedicine are possible as well.

Another potential applications can be found in networked control systems, where feedback is used only intermittently so as to reduce network traffic (e.g., [6], [8], [7] and others). Here, feedback sensors and system actuators communicate through networks that are shared by a vast number of users, with only limited network capacity available for each user. To abide by network capacity limitations, feedback can only be used intermittently. Examples of applications of networked control systems include spatially distributed resource allocation networks, highway transportation control systems, power generation and distribution networks, and others. Clearly, to minimize traffic within communications networks, it is necessary to reduce feedback and actuator use. The methodology developed in the present paper can help accomplish this task by providing open loop input signals that allow operation without feedback for maximal intervals of time.

In general terms, our objective is to address the needs exhibited by such applications and others through the development of open loop controllers that maximize the duration of time during which a perturbed system can operate without feedback and not exceed acceptable error bounds.

II. RESULTS

Assume that at least one of the eigenvalues of the nominal system matrix A has non-negative real part and that the initial condition satisfies $x(0) \neq 0$. Then the existence of a finite maximal time t_f^* and the optimal input $u^*(t)$ can be proved [2], [3]. Using the mathematical framework of [1], the optimal input function is characterized as follows [2]:

Theorem 1: Let $(u^*(t), t_f^*)$ be a solution of Problem 1. Then, there is a Lebesgue measurable function $z(t) : [0, t_f^*] \rightarrow \mathbb{R}$ not identically zero, such that $z(t)u^*(t) \leq z(t)u(t)$ for all input functions $u(t) \in U$ and for almost all times $t \in [0, t_f^*]$.

Corollary 1: [2] If the function $z(t)$ is non-zero almost everywhere in the interval $[0, t_f^*]$. Then, the optimal input function $u^*(t)$ of Problem 1 is a bang-bang function, where

$$u^*(t) := \begin{cases} -K \text{ if } z(t) > 0, \\ K \text{ if } z(t) < 0, \end{cases} \tag{4}$$

for almost all $t \in [0, t_f^*]$.

In the results above, the function $z(t)$ plays a crucial part. The expression for $z(t)$ is described next. First we introduce a few additional notations: denote

$$\Xi := \{A + \Delta_A\} \times \{B + \Delta_B\}. \tag{5}$$

Now, let ω be a Radon probability measure on the set

$$P := [0, t_f^*] \times \Xi. \tag{6}$$

Given a point $(t, A^{'}, B^{'}) \in P$, let $\omega(A^{'}, B^{'}|t)$ be the conditional probability measure induced by ω and let $\omega(t)$ be the marginal probability measure, so that

$$\omega(t, A^{'}, B^{'}) = \omega(A^{'}, B^{'}|t)\omega(t), \text{ where } (t, A^{'}, B^{'}) \in P. \tag{7}$$

Then the function $z(t) : [0, t_f^*] \rightarrow \mathbb{R}$ of theorem 1, is given by the following expression (see [2] for details):

$$\begin{aligned} & z(t) \\ & = \int_t^{t_f^*} \int_{\Xi} (x(s, A', B'; u^*))^T e^{A'(s-t)} B^{'} d\omega(A', B'|s) d\omega(s) \end{aligned} \tag{8}$$

where $\omega(t, A', B')$ is a Radon probability measure on P with the support

$$\begin{aligned} \Omega \quad = \quad & \{(t, A', B') \in [0, t_f^*] \times \Xi : \\ & x^T(t, A^{'}, B^{'}; u^*) x(t, A^{'}, B^{'}; u^*) = M\} \end{aligned} \tag{9}$$

and $x(t, A^{'}, B^{'}; u^*)$ is the solution to (1) for particular values of $(t, A^{'}, B^{'}) \in P$ and for the optimal input $u^*(t)$.

While it is important to understand the characteristics of the function $z(t)$, it is difficult to analyse except for very simple cases. For example, it is especially important to identify the cases where $z(t) \neq 0$ a.e. on $[0, t_f^*]$ since then, the optimal input is necessarily bang-bang on the entire interval of interest. Bang-bang functions are preferable for design and implementation, as they are completely determined by their switching times, i.e., the time instances when $z(t)$ changes sign. However, as the following example shows, the optimal input may not be purely bang-bang (i.e. $z(t) = 0$ over sets of non-zero measure in $[0, t_f^*]$) in some cases:

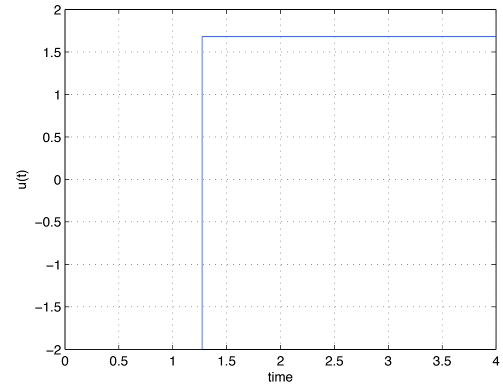

Figure 1. The Optimal Input is Not Bang-Bang

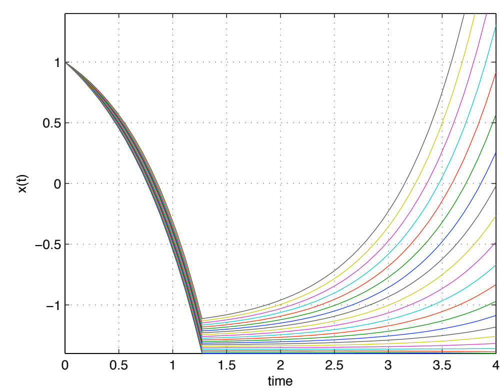

Figure 2. The Response for Different a

Example 1: Consider the system $\dot{x}(t) = ax(t) + u(t)$, where the parameter a is subject to the uncertainty $1.2 \leq a \leq 1.4$. The input is bounded: $|u(t)| \leq 2$ for all t, and $x(0) = 1$. The objective is to find an input function $u^*(t)$ that keeps the output amplitude below the bound $x^2(t) \leq 1.96$ for the longest period of time, irrespective of the value a. The optimal input is shown in the left plot, and the corresponding state trajectories for different values of a are plotted on the right with $M = 1.96$.

As can be seen from the plot in Figures1 and 2, the solution is bang-bang only over the time span $[0, 1.27]$. For the remaining time, the input switches to the value 1.67, not one of the values ± 2 that a bang-bang function would assume. The maximal time here is $t_f^* = 3.7$ seconds.

We next describe a sufficient condition for the optimal solution $u^*(t)$ to be purely bang-bang. However it turns out that this is easy to guarantee only for completely unstable systems. Moreover, the error bound M should be large enough compared to the input bound K. We make the additional (rather restrictive) assumption:

Assumption 1: All the eigenvalues of $A^{'}$ are on the right half plane.

One of the implications of the assumption above is that the quadratic form $v^T A^{'} v > 0$ for any $v \neq 0$. This fact is crucial for proving the theorem below. Denote by $\lambda_{min}(P)$ the minimum eigenvalue of a $n \times n$ matrix P.

Theorem 2: Let Assumption 1 be true and denote $\Pi := \{t \in [0, t_f^*) : z(t) = 0\}$. Define the following constants: $d = \sup_{\Delta_B} \|B + D_B\|_2$ and $g = \inf_{D_A \in \Delta_A} \lambda_{min} \left[(A^{'} + A^{'T})/2\right]$. If $\sqrt{M} > K\left(\frac{d}{g}\right)$, then following holds: (i) $z(t) \neq 0$ almost everywhere on $(0, t_f^*)$. (ii) Π do not have a limit point.

The proof of theorem 2 is divided into the following lemmas.

Lemma 1: Let $u^*(t)$ be the optimal input solving Problem 1 and let Assumption 1 be true. Define $d = \sup_{\Delta_B} \|B + D_B\|_2$ and $g = \inf_{D_A \in \Delta_A} \lambda_{min} \left[(A^{'} + A^{'T})/2\right]$. If $\sqrt{M} > K\left(\frac{d}{g}\right)$, then $x^T(t; D, u^*)x(t; D, u^*) < M$ for all $D \in \Delta$ and $t \in [0, t_f^*)$ and $x^T(t_f^*, D, u^*)x(t_f^*, D, u^*) = M$ for some $D \in \Delta$.

Proof: Let $x^T(t_0)x(t_0) = M$ for some $t_0 < \infty$ and for some $D \in \Delta$. If for every permissible $u^*(t_0)$ and every $D \in \Delta$, the derivative (alternatively the right hand derivative if $x^T(t)x(t)$ is not differentiable at t_0)

$$\frac{d}{dt}(x^T(t, D, u^*)x(t, D, u^*)|_{t=t_0} > 0, \tag{10}$$

then $x^T(t_0 + \epsilon)x(t_0 + \epsilon) > M$ for every $\epsilon > 0$ in some neighborhood of t_0. Hence by definition $t_f^* = t_0$. We derive conditions for (10) to hold.

$$\begin{aligned} &\frac{d}{dt}(x^T(t, D, u^*)x(t, D, u^*)|_{t=t_0} > 0 \ \forall D \in \Delta \\ &\qquad\qquad\qquad\qquad \text{and } \forall u^*(t_0) \in [-K, +K] \\ \Leftarrow & x^T(t_0)\left[A^{'}x(t_0) + B^{'}u^*(t_0)\right] > 0 \\ \Leftarrow & x^T(t_0)A^{'}x(t_0) > |x^T(t_0)B^{'}u^*(t_0)| \text{ with } u^*(t_0) < 0 \end{aligned} \tag{11}$$

Now, let $g = \inf_{D_A \in \Delta_A} \lambda_{min}(A^{'})$ where $\lambda_{min}(A^{'})$ is the minimum of the real parts of the eigenvalues of $A^{'}$. Then by Assumption 1, $g > 0$ and $x^T(t_0)A^{'}x(t_0) \geq g\|x(t_0)\|_2^2 = gM$. Then

$$\begin{aligned} (11) \quad &\Leftarrow \quad |x^T(t_0)B^{'}u^*(t_0)| < gM \\ &\Leftarrow \quad \|B^{'}\|\sqrt{M}K < gM \\ &\Leftarrow \quad \frac{d}{g}K < \sqrt{M} \end{aligned}$$

■

Lemma 2: Let $z(t)$ be defined as in (8) and denote $\Pi := \{t \in [0, t_f^*) : z(t) = 0\}$. If $x^T(t; D, u^*)x(t; D, u^*) < M$ for all $(D, t) \in \Delta \times [0, t_f^*)$ and $x^T(t_f^*, D, u^*)x(t_f^*, D, u^*) = M$ for some $D \in \Delta$, then the following holds: (i) $z(t) \neq 0$ almost everywhere on $(0, t_f^*)$. (ii) Π do not contain a limit point.

Proof: Consider the expression for $z(t)$ as in (8):

$$\begin{aligned} &z(t) \\ &= \int_t^{t_f^*} \int_\Xi (x(s, A', B'; v^*))^T e^{A'(s-t)} B^{'} d\omega(A', B'|s) d\omega(s), \end{aligned}$$

where the support of ω is given by $\Omega = \{(t, A', B') \in [0, t_f^*] \times \Xi : x^T(t, A^{'}, B^{'}; v^*)x(t, A^{'}, B^{'}; v^*) = M\}$. Under the hypothesis of the lemma, $\Omega = \{(t, A', B') \in \{t_f^*\} \times \Xi : x^T(t_f^*, A^{'}, B^{'}; v^*)x(t_f^*, A^{'}, B^{'}; v^*) = M\}$. Then $z(t)$ can be simplified as follows:

$$\begin{aligned} z(t) &= \int_\Xi (x(s, A', B'; v^*))^T e^{A'(t_f^* - t)} B^{'} d\omega(A', B') \\ &= \int_\Xi (x(s, A', B'; v^*))^T e^{A' t_f^*} B^{'} d\omega(A', B') \\ &- t \int_\Xi (x(s, A', B'; v^*))^T e^{A' t_f^*} A^{'} B^{'} d\omega(A', B') \\ &+ \frac{t^2}{2!} \int_\Xi (x(s, A', B'; v^*))^T e^{A' t_f^*} A^{'2} B^{'} d\omega(A', B') \\ &- \ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}$$

Recall that theorem 1 guarantees that $z(t)$ cannot be identically zero over $[0, t_f^*]$. Since, in this case $z(t)$ turns out to be a power series in t over $[0, t_f^*)$, $z(t) \neq 0$ a.e. on the interval $[0, t_f^*)$. Moreover noting that a non-zero analytic function over a open connected set cannot have a limit point in its domain of definition [4], it follows that Π do not contain a limit point. ■

Example 2: Consider the one-dimensional system

$$\dot{x}(t) = ax(t) + u(t), \tag{12}$$

where the time constant a is subject to the uncertainty $1.2 \leq a \leq 1.4$. The system has the input bound $|u(t)| \leq 2$ for all t, and the initial condition is $x(0) = 1$. We set the bound $M := 25$, so the objective is to find an input function $u^*(t)$ that keeps the state amplitude below the bound $x^2(t) \leq 25$ (i.e., $|x(t)| \leq 5$) for the longest time, irrespective of the value of a within its uncertainty range. We show next that, in this case, $z(t) \neq 0$ for almost all $t \in [0, t_f^*]$. Thus, by corollary 1, the optimal input function $u^*(t)$ is a bang-bang function, as depicted below. The maximal time during which all samples of the system can be kept below the prescribed error bound is $t_f^* = 5.08$ seconds.

To show that the optimal input function is a bang-bang function in this case, note first that the system cannot rebound to lower valued states after reaching the state $|x(t)| = 5$; essentially lemma 1 holds. Indeed, consider the error function

$$e(t) = x^2(t).$$

Using the system equation (12), we get

$$\dot{e}(t) = x(t)\dot{x}(t) = x^2(t)a + x(t)u(t) = x(t)[x(t)a + u(t)].$$

If $e(t) = 25$, we clearly need $\dot{e}(t) \leq 0$ for the error not to worsen. When $e(t) = 25$, we have either $x(t) = 5$ or $x(t) = -5$. For $x(t) = 5$, we obtain $\dot{e}(t) = x(t)[x(t)a + u(t)] = 5[5a + u(t)] > 0$ for all possible values of a and of $u(t)$. Also, for $x(t) = -5$, we have $\dot{e}(t) = -5[-5a + u(t)] > 0$

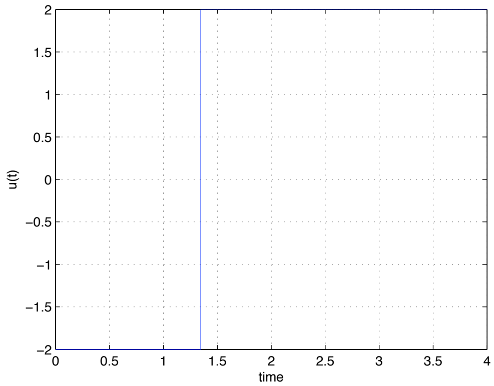

Figure 3. Optimal Input

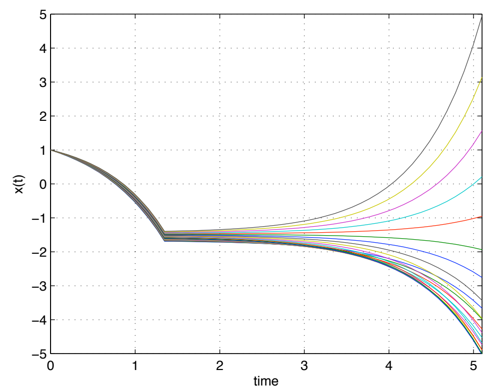

Figure 4. Optimal Response for Different a

for all possible values of a and $u(t)$. Thus, once the system reaches $e(t) = 25$, it has reached the terminal time, since the error can only continue to grow. Hence, for any value of a, the process terminates when the corresponding trajectory hits the error bound M. In other words, any trajectory meets the error bound only once: at the terminal time $t_f^* = 5.08$. Thus, in view of lemma 2, the support set of the measure ω in this case is given by the following (in this example, $B' = 1$ always).

$$\begin{aligned}\Omega \;&=\; \{(t, a', 1) \in \{5.08\} \times [1.2, 1.4] \times \{1\} : \\ &\qquad x^2(5.08, a', 1, v^*) = M\}\end{aligned} \tag{13}$$

Note that Ω cannot be empty here, since that would imply that x^2 does not meet the bound M on the time interval $[0, t_f^*]$, contradicting what we have concluded in the previous paragraph. Hence the hypotheses of theorem 2 are valid and $z(t) \neq 0$ a.e. on $[0, t_f^*)$. This can be independently verified without using theorem 2 as follows. Substituting the support set (13) into (8), we obtain

$$z(t) = \int_{1.2}^{1.4} x(5.08, a', 1, v^*) e^{a'(5.08-t)} d\omega(a').$$

Let us now expand the exponential in the integrand into a series and integrate; this yields

$$z(t) = p_0 + p_1(5.08-t) + p_2(5.08-t)^2 \ldots + p_m(5.08-t)^m + .., \tag{14}$$

where

$$p_m = \int_{1.2}^{1.4} x(5.08, a', 1, v^*) \frac{(a')^m}{m!} d\omega(a').$$

As the integrand includes the power $(a')^m$, the equality $p_m = 0$ for all $m = 0, 1, 2, ...$ would imply that $x(5.08, a', 1, v^*) = 0$ almost everywhere with respect to the measure $\omega(a')$, contradicting the support (13). Thus, at least one of the coefficients of the power series (14) is not zero, and whence $z(t) \neq 0$ almost everywhere on the interval $(0, 5.08)$. By corollary 1, this proves that the optimal input function is a bang-bang function in this case.

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