Hahn-Banach Thm (Geometric form)

Note Title 21-08-2008

Def^n: A hyperplane H in linear vector space X is a maximal proper linear variety

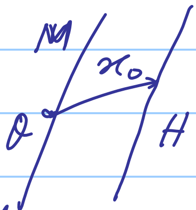
$\Updownarrow$

1) H is a linear variety
2) $H \neq X$
3) If $V \supset H$, either $V = X$ or $V = H$

FACT 1: Let H be a hyperplane in linear vector space X. Then there is a linear functional f on X and a constant c s.t.

$$H = \{x : f(x) = c\}$$

Conversely, if f is a non-zero linear functional on X, the set $\{x : f(x) = c\}$ is a hyperplane in X



Hint: $H = x_0 + M$

If $x_0 \notin M$, $[M + x_0] = X$

and any $x \in X$ can be written as

$x = \alpha x_0 + m$, for some $m \in M$

Define $f(x) = \alpha$

Then $H \overset{?}{=} \{x : f(x) = 1\}$

If $x_0 \in M$, $H = \{x : f(x) = 0\}$

Converse: do it yourself!

FACT 2: Let H be a hyperplane in a linear vector space X. If H does not contain the origin, there is a unique linear functional f on X s.t. $H = \{x : f(x) = 1\}$

Proof: By fact 1, $\exists f \in X^*$ s.t.

$$H = \{x : f(x) = 1\}$$

Let $\exists g \neq f$ s.t. $H = \{x : g(x) = 1\}$

Then $H \subset \{x : f(x) - g(x) = 0\}$

But smallest subspace of $X \supset H$ is X itself. So $f = g$.

FACT: A hyperplane H in N.L.S X

either $\overline{H} = H$ (closed)

or $\overline{H} = X$ (dense in X)

Hint: H is a maximal linear variety

FACT3: Let f be a non-zero linear functional on a N.L.S X.
Then $H = \{x : f(x) = c\}$ is closed $\forall c$ iff f is continuous (i.e. bdd).

Hint: If f continuous, let $\underset{\Downarrow}{\overset{H}{x_n}} \to x \in X$
$c = f(x_n) \to f(x) \Rightarrow x \in H \Rightarrow H$ is closed.
Converse: Exercise.

Half-spaces

$$\underbrace{\{x : f(x) \le c\}, \{x : f(x) < c\}}_{\text{-ve half space}}, \underbrace{\{x : f(x) \ge c\}, \{x : f(x) > c\}}_{\text{+ve half spaces}}$$

If f is continuous, $\left.\begin{array}{l}\{x : f(x) < c\} \\ \{x : f(x) > c\}\end{array}\right]$ are open

Other two are closed.

Hahn-Banach Thm (Geometric form)

Defn: Let K be a convex set in NLS X and let $0 \in \mathring{K}$ (int of K). Then the Minkowski functional p of K is defined on X by

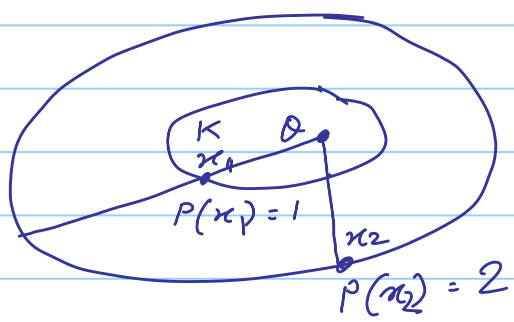
$$p(x) = \inf\left\{ r : \frac{x}{r} \in K ,\ r > 0 \right\}$$

Properties

1) $0 \leq p(x) < \infty \quad \forall x \in X$
2) $p(\alpha x) = \alpha p(x) \quad \forall \alpha > 0$
3) $p(x_1 + x_2) \leq p(x_1) + p(x_2)$
4) p is continuous
5) $\overline{K} = \{ x : p(x) \leq 1 \}$, $\overset{\circ}{K} = \{ x : p(x) < 1 \}$

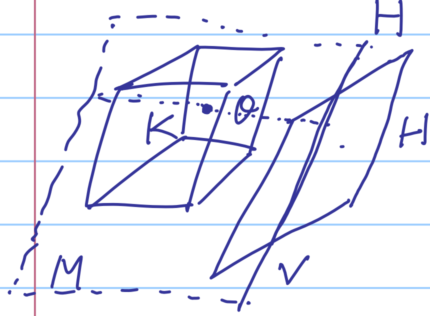
Example: $K = \{ x \in X : \|x\| \leq 1 \}$

$p(x) = \|x\|$



H-B Thm (Geometric form): Let K be a convex set with $\mathring{K} \neq \phi$ in a real N.L.S X, let V be a linear variety in X s.t. $V \cap \mathring{K} = \phi$.

Then $\exists$ a closed hyperplane H in X s.t.
$$H \supset V$$
$$H \cap \mathring{K} = \phi$$



$\Updownarrow$

$\exists\ x^* \in X^*$ and $c \in \mathbb{R}$ s.t. $\langle v, x^* \rangle = c \quad \forall v \in V$

& $\langle k, x^* \rangle < c \quad \forall k \in \mathring{K}$

<u>Proof</u>: Assume $\theta \in \mathring{K}$ (after appropriate translation)

Let $M = [V]$

Then $\exists\ f \in M^*$ s.t. $V = \{x : f(x) = 1\}$

(since V is a hyperplane in M not containing origin)

Let p be Minkowski f$\underline{n}$ on K.

Now $V \cap \mathring{K} = \phi \Rightarrow f(x) = 1 \leq p(x) \ \forall x \in V$

Then $f(\alpha x) = \alpha \leq p(\alpha x) \quad \forall x \in V, \alpha > 0$

and $f(\alpha x) \leq 0 \leq p(\alpha x)$ for $\alpha < 0$, $\forall x \in V$

Thus $f(x) \leq p(x) \quad \forall x \in M$

(since $M = [V]$)

By H-B thm, $\exists$ an extension F of f from M to X with $F(x) \leq p(x)$ $\forall x \in X$.

Let $H = \{x : F(x) = 1\}$. Since $F(x) \leq p(x)$ $(\forall x \in X$ and p is continuous

$\Rightarrow F$ is continuous

Also $F(x) \leq p(x) \leq 1$ $\forall x \in \mathring{K}$

$\Rightarrow H$ is the desired closed hyperplane.

$= \{x : f(x) = c\}$

Def^{n}: A closed hyp H in a N.L.S. X is said to be a support (supporting hyperplane) for a convex set K if

$$K \subset \{x : f(x) \leq c\}$$

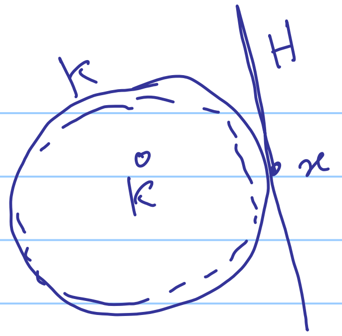
or $\quad K \subset \{x : f(x) \geq c\}$

AND $\quad H \cap \overline{K} \neq \phi$

Thm: If x is not an int. pt. of a convex set K which contains int. pts., there is a closed hyperplane H containing x s.t. K lies on one side of H.

Hint: Consider $(K - x)$ and θ.

$\theta \notin K - x$ and θ is a linear variety



For any $x \in \partial K$
$\exists H \ni x$ s.t.
H is a support for K

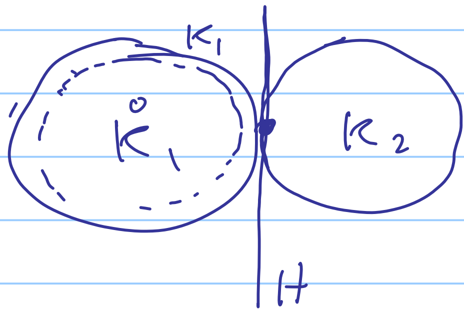
Thm: Let K_1 and K_2 be convex sets in X s.t. $\mathring{K}_1 \neq \phi$ and $K_2 \cap \mathring{K}_1 = \phi$

→ Then $\exists$ a closed hyperplane H separating K_1 and K_2

⇕

→ $\exists\ x^* \in X^*$ s.t.

$$\sup_{x \in K_1} \langle x, x^* \rangle \leq \inf_{x \in K_2} \langle x, x^* \rangle$$



Note: $\mathring{K}_2 = \phi$ is allowed

Proof: $K = K_1 - K_2 \Rightarrow \mathring{K} \neq \phi$ and $\theta \notin \mathring{K}$

Then, by last thm: $\exists\ x^* \in X^*$ s.t.

$$\langle x, x^* \rangle \leq 0 \quad \forall x \in K$$

Thus for any $x_1 \in K_1$, $x_2 \in K_2$

$$\langle (x_1 - x_2), x^* \rangle \leq 0$$

or $\quad \langle x_1, x^* \rangle \leq \langle x_2, x^* \rangle$

$\Rightarrow \exists\, c$ s.t. $\sup\limits_{x \in K_1} \langle x, x^* \rangle \leq c \leq \inf\limits_{x \in K_2} \langle x, x^* \rangle$

Hence $H = \{ x : \langle x, x^* \rangle = c \}$

Thm: If K is a closed convex set in a normed space, then K is equal to the intersection of all the closed half-spaces that contain it.

$\Updownarrow$

Thm: If K is a closed convex set and $x \notin K$, there is a closed half-space that contains K but not x.

Proof: $d = \inf\limits_{k \in K} \|x - k\|$. $d > 0$ since K closed.

Basics of Linear Operators

$$A : \underset{\substack{\uparrow \\ \text{normed} \\ \text{space}}}{X} \longrightarrow \underset{\substack{\uparrow \\ \text{normed} \\ \text{space}}}{Y}$$

$N(A) := \{x : Ax = \theta\} \rightarrow$ subspace of X

$R(A) := \{y : Ax = y,\ x \in X\}$

FACT: A linear operator on a normed L. space X is continuous at every pt in X if it is continuous at a single pt.

Def^{n}: $A : X \to Y$ is bdd if $\exists\, M < \infty$ s.t. $\|Ax\| \leq M\|x\|$

Norm: $$\|A\| = \sup_{\|x\| \leq 1} \|Ax\|$$

$$= \sup_{x \neq \theta} \frac{\|Ax\|}{\|x\|}$$

$$= \inf\{M : \|Ax\| \leq M\|x\| \ \forall x\}$$

FACT: A linear operator is bdd iff it is continuous

Def^n: The normed space of all bdd. linear operators from normed L. space X into the normed L. space Y is denoted by $B(X, Y)$

FACT: If X & Y are normed L. spaces with Y complete. Then $B(X, Y)$ is complete.

FACT: Let X, Y, Z be normed L. spaces and let $S \in B(X, Y)$, $T \in B(Y, Z)$. Then $\|TS\| \leq \|T\| \, \|S\|$

Proof: $\|TS(x)\| \leq \|T\| \|Sx\|$
$\leq \|T\| \|S\| \|x\| \quad \forall x \in X$

Examples: $X = C[0,1] \quad A : X \to X$

$$Ax = \int_0^1 k(s,t)\, x(t)\, dt \quad \left(\begin{array}{l} K \text{ cont. on} \\ 0 \leq s \leq 1 \\ 0 \leq t \leq 1 \end{array} \right)$$

$$\|A\| = \max_{0 \leq s \leq 1} \int_0^1 |K(s,t)|\, dt$$

Ex 2: $X = E^n$ $\quad A : X \to X$

A is a matrix

$$\|A\| = \sqrt{\lambda_{max}(A^T A)} = \sup_{\|x\|_2 \leq 1} x^T (A^T A) x$$

Self Reading: Banach Inverse Thm, Adjoint Operators.