Gateaux and Frechet Differentials

Note Title 20-06-2011

$X \to$ Vector Space

$Y \longrightarrow$ N.L.S.

$$T : \underset{\underset{X}{\cap}}{D} \to \underset{\underset{Y}{\cap}}{R}$$

Def^{n}: Let $x \in D \subset X$ and let $h \in X$. If the limit

$$(*) \quad \delta T(x; h) = \lim_{\alpha \to 0} \frac{1}{\alpha}\left[T(x+\alpha h) - T(x)\right]$$

exists, it is called the Gateaux differential of T at x with increment h.

If the limit in $(*)$ exists $\forall h \in X$, T is said to be Gateaux diff. at x

Note:
0> no norm on X is req. (diff to relate to cont)
1> $x + \alpha h \in D$ for small α
2> lim in the sense of $\|\cdot\|_Y$ conv in norm Y.

3> If $x = x_1 \in X$ (fixed)
$\delta T(x_1; h) : X \to Y$ (in general non-linear transform)

If however T is linear,
$$\delta T(x_1; h) = T(h)$$

4> If $Y = \mathbb{R}$, T is a functional say f

$$\delta f(x; h) = \lim_{\alpha \to 0} \frac{1}{\alpha}\left[f(x+\alpha h) - f(x + 0.h)\right]$$
$$= \frac{d}{d\alpha}\left[f(x+\alpha h)\right]\Big|_{\alpha=0}$$

Ex: 1: $X = E^n$ $\quad f : E^n \to \mathbb{R}$ with continuous partial derivatives.

$$\delta f(x; h) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} h_i$$

$$= \frac{d}{d\alpha} \left[f(x_1 + \alpha h_1, x_2 + \alpha h_2) \right] \Big|_{\alpha = 0}$$

$$= \frac{\partial f}{\partial x_1} h_1 + \frac{\partial f}{\partial x_2} h_2$$

Ex 2: $X = C[0,1]$ let $f(x) = \int_0^1 g(x(t), t)\, dt$

Let $\frac{\partial g}{\partial x}$ exists and is cont w.r.t. x & t

$$\delta f(x; h) = \frac{d}{d\alpha} \int_0^1 g(x + \alpha h, t)\, dt \Big|_{\alpha = 0}$$

$$= \int_0^1 g_x(x, t)\, h(t)\, dt$$

If X is normed, we can define Frechet differentials

Def^{n}: $T : D \to R$, $\quad D \subset$ N.L.S X
D open
$R \subset$ N.L.S. Y.

If for fixed $x \in D$ and each $h \in X$ $\exists$ $\delta T(x; h) \in Y$ which is <u>linear + cont.</u> w.r.t. h s.t.

$$\lim_{\|h\| \to 0} \frac{\| T(x+h) - T(x) - \delta T(x; h) \|}{\|h\|} = 0$$

then T is Frechet differentiable at x and $\delta T(x;h)$ is the Frechet differential of T at x with increment h.

FACT: If T has a Frechet differential it is unique.

Proof: $\delta^1 T(x;h)$ $\quad$ $\delta^2 T(x;h)$

$$
\begin{aligned}
&\| \delta^1 T(x;h) - \delta^2 T(x;h) \| \\
&= \left[\delta^1 T(x;h) - T(x+h) + T(x) \right] \\
&\qquad + \left[T(x+h) - T(x) - \delta^2 T(x;h) \right] \\
&= \left\{ - \left[T(x+h) - T(x) - \delta^1 T(x;h) \right] \right\} + \quad '' \\
&\leq \| T(x+h) - T(x) - \delta^1 T(x;h) \| + \| \cdots\cdots \| \\
&= o(\|h\|)
\end{aligned}
$$

Since $[\delta^1 T(x;h) - \delta^2 T(x;h)]$ is linear + bdd in h, it must be zero.

FACT: If the Frechet differential of T exists at x, then the Gateaux diff. exists at x and they are equal.

Proof: For any h,

$$\frac{1}{\alpha} \| T(x+\alpha h) - T(x) - \delta T(x;\alpha h) \| \to 0$$

as $\alpha \to 0$

$$\text{or} \quad \left\| \frac{T(x+\alpha h) - T(x)}{\alpha} - \delta T(x;h) \right\| \to 0$$

as $\alpha \to 0$

as $\lim_{\alpha \to 0} \left[\frac{T(x+\alpha h) - T(x)}{\alpha} \right] = \delta T(x; h)$

FACT: If T, defined on a open set $D \subset X$ has a Frechet diff at x, then T is continuous at x.

Proof: Given $\varepsilon > 0$, $\exists$ $N(x)$ s.t. for $x + h \in N(x)$

$$\|T(x+h) - T(x) - \delta T(x;h)\| < \varepsilon \|h\|$$

$$\|T(x+h) - T(x) - \delta T(x;h)\| > \|T(x+h) - T(x)\| - \|\delta T(x;h)\|$$

$$\Rightarrow \|T(x+h) - T(x)\| < \varepsilon \|h\| + \underbrace{\|\delta T(x;h)\|}_{\substack{\text{lin} \\ + \text{ cont}}}$$

$$< M\|h\|$$