Note Title 07-07-2011

Constrained Optimization - Inequality

$$\min f(x)$$
$$\text{s.t } \quad x \in \Omega, \quad G(x) \leq \theta$$
?

$f : \Omega \to \mathbb{R}$ (convex)
$G : \Omega \to Z$ (convex?)
$\downarrow$
N.L.S.

<u>Defn</u>: A set C in a linear vector space is said to be a cone <u>with vertex at the origin</u> if
$$x \in C \Rightarrow \alpha x \in C \quad \forall \alpha \geq 0$$

\# vertex at $p \to p + C$

<u>Defn</u>: Let P be a convex cone in a vector space X. For $x, y \in X$ we write $x \geq y$ (w.r.t. P) if $x - y \in P$.

$P \to$ positive cone in X
$N = -P \to$ negative cone

$$y \leq x \quad \equiv \quad y - x \in N$$

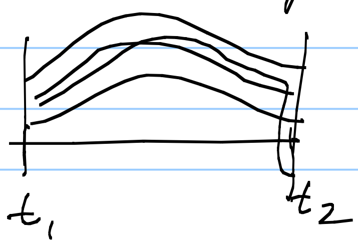
$$x \geq y, \ y \geq z \Rightarrow x \geq z$$

<u>Example</u>: In E^n
$$P = \{ x \in E^n : x = (\xi_1, \cdots \xi_n) : \xi_i > 0 \ \forall i \}$$

In <u>N.L.S</u>, cones can be closed/open.

\# $x > \theta \Leftrightarrow x$ is an int. pt. of P.

$L_1(t_1, t_2) \quad \int_{t_1}^{t_2} |f(t)| \, dt$



$$P = \{ x(t) \in L_1(t_1, t_2) : x(t) \geq 0 \ \forall t \in [t_1, t_2] \}$$

$$\overset{\circ}{P} = \phi$$

\# $$P = \{ x(t) \in C[t_1, t_2] , x(t) \geq 0 \ \forall t \in [t_1, t_1] \}$$

$$\overset{\circ}{P} \neq \phi$$

Defn: X (NLS) with positive cone $P \subset X$. Define corresponding +ve cone $P^{\oplus} \subset X^*$ by

$$P^{\oplus} = \{ x^* \in X^* : \langle x, x^* \rangle \geq 0 \ \forall x \in P \}$$

$\Rightarrow$ if $x^* \geq 0$ & $x \geq 0$, $\langle x, x^* \rangle \geq 0$
if $x^* \geq 0$ & $x \leq 0$, $\langle x, x^* \rangle \leq 0$

Defn: Let X (V.S), Z (V.S) with a cone P specified as the +ve cone.

$G : X \to Z$ is convex if the domain Ω of G is a convex set and if

$$G(\alpha x_1 + (1-\alpha) x_2) \leq \alpha G(x_1) + (1-\alpha) G(x_2)$$

$$\forall x_1, x_2 \in \Omega \quad \forall \alpha , 0 < \alpha < 1$$

w.r.t. P

Fact: If $G : X \to Z$ is a convex map then $\forall z \in Z$, the set $\{x : G(x) \le z\}$ is convex.

Problem: $\min f(x)$

s.t. $x \in \Omega$, $G(x) \le \theta$

$f : \Omega \to \mathbb{R}$, Ω convex, f convex

$G : \Omega \to Z$, Z has a +ve cone P.

Define: $\Gamma = \{ z \in Z : \exists\, x \in \Omega \text{ wit } G(x) \le z \}$

FACT: Γ is convex (Exercise)

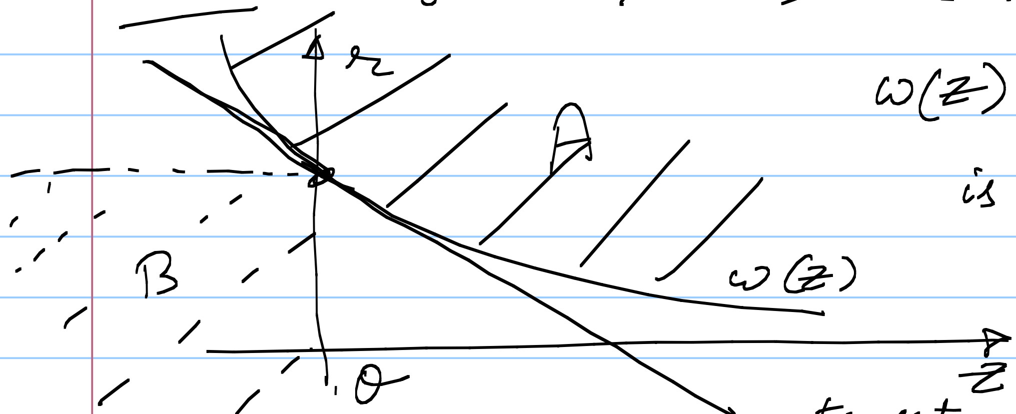
Define: $\omega(z) = \inf \{ f(x) : x \in \Omega, G(x) \le z \}$

$\omega : \Gamma \to \mathbb{R}$

FACT: ω is convex

Proof: Exercise

FACT: If $z_1 \geq z_2$, $\omega(z_1) \le \omega(z_2)$



$\omega(z) + \langle z, z_0^* \rangle$ is minimized at $z = \theta$.

$\hookrightarrow$ tangent hyperplane in $(\mathbb{R} \times Z)$ $\equiv$ linear functional of the form $(1, z_0^*)$

Thm: Let X be a linear vector space, $Z \to$ N.L.S, Ω convex subset of X, $P \to$ +ve cone in Z.

$f: \Omega \to \mathbb{R}$ (convex)

$G: \Omega \to Z$ (convex mapping)

Assume: P contains an interior pt.

Assume: $\exists\, x_1 \in \Omega$ s.t. $\underline{\underline{G(x_1) < \theta}}$

i.e. $G(x_1) \in \overset{\circ}{N}$ $\quad (N = -P)$

let

(1) $\mu_0 = \inf f(x)$ s.t. $x \in \Omega,\ G(x) \leq \theta$

and let $\mu_0 < \infty$. Then $\exists\ z_0^* \geq \theta$ in Z^* s.t.

(2) $\mu_0 = \inf\limits_{x \in \Omega} \{ f(x) + \langle G(x), z_0^* \rangle \}$

Furthermore, if infimum is achieved in (1) by some $x_0 \in \Omega$, $G(x_0) \leq \theta$ it is achieved by x_0 in (2) $\langle G(x_0), z_0^* \rangle = 0$

Proof: In the space $W = \mathbb{R} \times Z$ define

$A = \{ (r, z) : r \geq f(x),\ z \geq G(x)$ for some $x \in \Omega \}$

$B = \{ (r, z) : r \leq \mu_0,\ z \leq \theta \}$

A, B are convex sets.

$A \cap \overset{\circ}{B} = \phi$ (by def^{n}. of μ_0)

Since $\overset{\circ}{N} \neq \phi \Rightarrow \overset{\circ}{B} \neq \phi$.

Then by Sep Hyp thm: $\exists$ a nonzero element of $\omega_0^* = (r_0, z_0^*) \in W^*$ s.t.

$$r_0 r_1 + \langle z_1, z_0^* \rangle \geq r_0 r_2 + \langle z_2, z_0^* \rangle$$

$$\forall \quad (r_1, z_1) \in A, \quad (r_2, z_2) \in B$$

From the def$^{\underline{n}}$ of B, $\omega_0^* \geq \theta$ ie $r_0 \geq 0$, $z_0^* \geq \theta$

<u>Claim:</u> $r_0 > 0$.

<u>Proof:</u> Since $(\mu_0, \theta) \in B$

$$r_0 r + \langle z, z_0^* \rangle \geq r_0 \mu_0 \quad \forall (r, z) \in A$$

If $r_0 = 0$, $\langle G(x_1), z_0^* \rangle \geq 0$ with $z_0^* \neq \theta$

But $G(x_1) \in \mathring{N}$ and $z_0^* \geq \theta$

$$\Rightarrow \langle G(x_1), z_0^* \rangle < 0$$

<u>Contradiction</u>

$\Rightarrow r_0 > 0$ and without loss of generality assume $r_0 = 1$.

Now since (μ_0, θ) is arbitrarily close to both A and B,

$$\mu_0 = \inf_{(r, z) \in A} \left[r + \langle z, z_0^* \rangle \right]$$

$$\leq \inf_{x \in \Omega} \left[f(x) + \langle G(x), z_0^* \rangle \right]$$

$$\leq \inf_{\substack{x \in \Omega \\ G(x) \leq \theta}} f(x) = \mu_0$$

<u>Second Part</u>: If $\exists\ x_0$ s.t $G(x_0) \leq \theta$ and $\mu_0 = f(x_0)$

the $\mu_0 \leq f(x_0) + \langle G(x_0), z_0^* \rangle \leq \underbrace{f(x_0)}_{=\mu_0}$;

Hence $\langle G(x_0), z_0^* \rangle = 0$

Local minimum (inequality const thm)

<u>Defn</u>: Let X (V.S), Z (NLS) with the cone P ~~with~~ $\underline{\mathring{P} \neq \phi}$.

$G : X \to Z$, G with Gateaux diff. linear in its increment

$x_0 \in X$ is a <u>regular point</u> of $G(x) \leq \theta$ if $G(x_0) \leq \theta$ and $\exists\ h \in X$ s.t.

$$G(x_0) + \delta G(x_0; h) < \theta$$

(Generalized Kuhn-Tucker Thm)

Let X (V.S), Z (N.L.S) with the cone P
P.
$f : X \to R$ be Gateaux diff.
$G : X \to Z$ " " "
Let x_0 minimize $f(x)$ s.t. $G(x) \leq \theta$

Assumptions: 1> P contains an interior pt.
2> The Gateaux diff. of f and G are linear in their increments.
3> x_0 is a regular point of the inequality $G(x_0) \leq \theta$

Then $\exists$ $z_0^* \in Z^*$, $z_0^* \geq \theta$ s.t. the Lagrangian
$f(x) + \langle G(x), z_0^* \rangle$ is
stationary at x_0; also $\langle G(x_0), z_0^* \rangle = 0$

$$\delta f(x_0; h) + \langle \delta G(x_0; h), z_0^* \rangle = 0 \quad \forall h \in X.$$

Proof (Outline):

$$A = \{ (r, z) : r \geq \delta f(x_0, h), \; z \geq G(x_0) + \delta G(x_0, h) \text{ for some } h \in X \}$$

$$B = \{ (r, z) : r \leq 0, \; z \leq \theta \}$$

Claim: $A \cap \mathring{B} = \phi$.

Proof. Let $(r, z) \in A$ with $r < 0$, $z < \theta$
then $\exists$ $h \in X$ s.t

$$\delta f(x_0; h) < 0 \qquad G(x_0) + \delta G(x_0, h) < \theta$$

$G(x_0) + \delta G(x_0, h) \in \mathring{N}$ Consider a open sphere of radius ρ about $G(x_0) + \delta G(x_0, h)$
$\mathcal{B}_\rho (G(x_0) + \delta G(x_0; h)) \in N$

$$\Rightarrow OB_{(\alpha\cdot\rho)}\left(\alpha\left[G(x_0)+\delta G(x_0;h)\right]\right) \in N$$

$$\Rightarrow OB_{(\alpha\cdot\rho)}\left((1-\alpha)G(x_0)+\alpha\left[G(x_0)+\delta G(x_0;h)\right]\right) \in N$$

$$\Rightarrow OB_{(\alpha\rho)}\left(G(x_0)+\alpha\,\delta G(x_0;h)\right) \in N$$

But for fixed h,

$$\| G(x_0+\alpha h) - (G(x_0)+\alpha\,\delta G(x_0;h)) \| = o(\alpha)$$

$\Rightarrow$ For small α, $G(x_0+\alpha h) \in OB(\text{---}) \in N$

i.e, $G(x_0+\alpha h) < \theta$

Similarly, $f(x_0+\alpha h) < f(x_0)$. small α

But this contradicts the optimality of $f(x_0)$. $\Rightarrow$

$$A \cap \mathring{B} = \phi.$$

<u>Corollary</u>: Let X be NLS in last Thm, and f and G are Frechet diff. Then if the sol$^{\underline{n}}$ is a regular pt,

$$f'(x_0) + z_0^* G'(x_0) = \theta$$

$$\langle G(x_0), z_0^* \rangle = 0$$

Applications: Euler-Lag eqns, Optimal Control, Pontryagin's Max Principle.