Linear Functionals & Hahn-Banach Thm (Extension Form)

Consider $\min f(x)$ s.t. $G(x) \leq 0$ $\quad \Big| \; x \in \mathbb{R}^n$

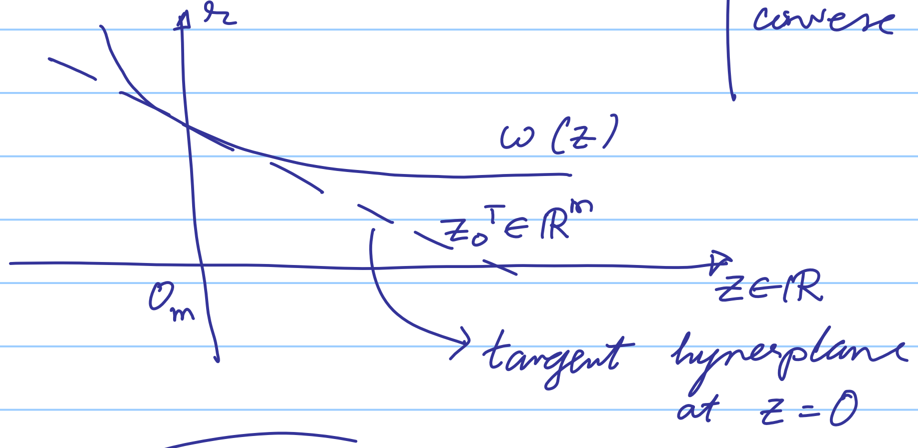
$f : \mathbb{R}^n \to \mathbb{R}$ is <u>convex</u>
$G : \mathbb{R}^n \to \mathbb{R}$ (normed space).

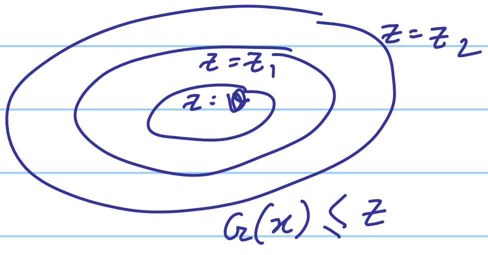
Let $\underset{\text{finite}}{\mu_0} = \inf f(x)$ s.t. $G(x) \leq 0$

$\omega(z) = \inf_{x \in \mathbb{R}^n} \{ f(x) : G(x) \leq z \}$

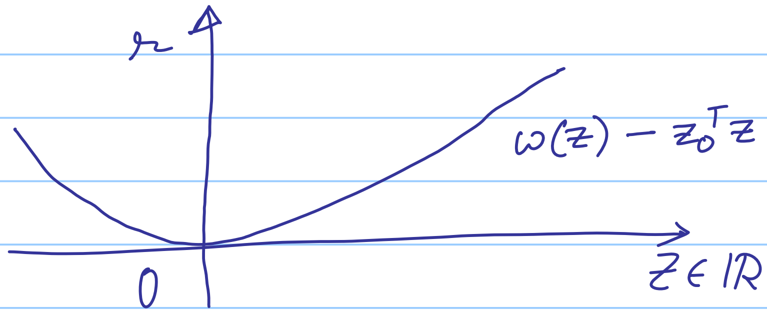
So $\mu_0 = \omega(0)$

<u>Exercise</u>: $\omega(z)$ is convex.

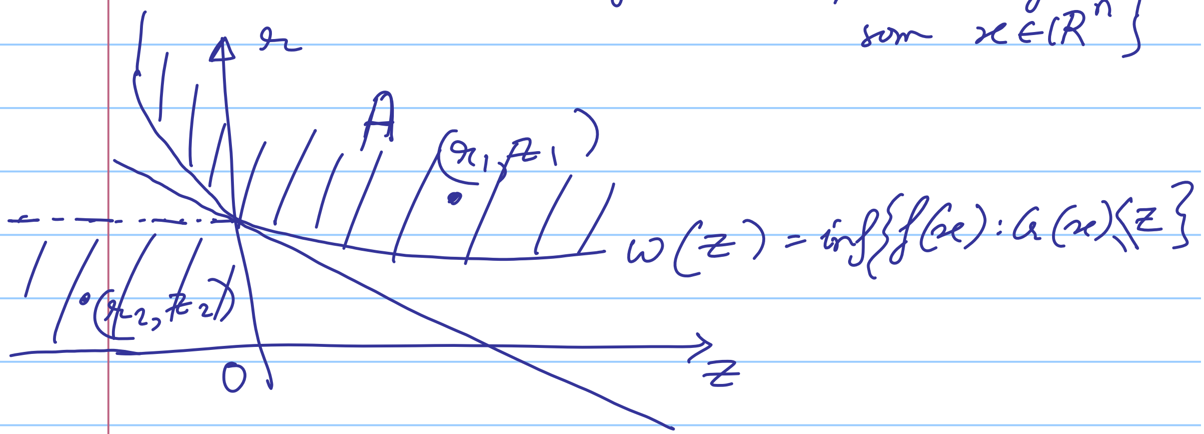




$\omega(z) - z_0^T z$



$$A = \{ (r, z) : r \geq f(x),\ z \geq G(x) \text{ for some } x \in \mathbb{R}^n \}$$



$$B = \{ (r, z) : r \leq \mu_0,\ z \leq 0 \}$$

$$[1 \quad z_0] \begin{bmatrix} r_1 \\ z_1 \end{bmatrix}_{\in A} \geq [1 \quad z_0] \begin{bmatrix} r_2 \\ z_2 \end{bmatrix}_{\in B} \quad (z_0 \geq 0)$$

Since $(\mu_0, 0) \in B$

$$r_1 + z_0 z_1 \geq \mu_0 \quad \forall (r, z_1) \in A$$

$$\Leftrightarrow \mu_0 = \inf_{(r,z) \in A} [r + z_0 z]$$

$$\leq \inf_{x \in \mathbb{R}^n} [f(x) + z_0 G(x)] \leq \underbrace{\inf_{G(x) \leq 0} f(x)}_{= \mu_0}$$

Vector space: Set X with two operations: 1) addition
2) scalar multiplication

Axioms

1) $x+y = y+x$
2) $(x+y)+z = x+(y+z)$
3) $\exists\ \theta \in X$ s.t. $x+\theta = x \quad \forall x \in X$
4) $\alpha(x+y) = \alpha x + \alpha y$
5) $(\alpha+\beta)x = \alpha x + \beta x$
6) $(\alpha\beta)x = \alpha(\beta x)$
7) $0x = \theta$, $1x = x$

Example: $C[a,b]$: $x(t), y(t) \in C[a,b]$
$x = y \iff x(t) = y(t) \ \forall t \in [a,b]$
$\theta \equiv$ identically zero function on $[a,b]$
$x+y \equiv x(t)+y(t)$
$\alpha x \equiv \alpha x(t)$

Example: $\mathbb{R}^n$!

Example: a) Infinite seq of real nos.
$x = (\xi_1, \xi_2, \dots, \xi_k, \dots)$
$= \{\xi_k\}_{k=1}^{\infty}$

b) Bounded $\{|\xi_k| < M \ \forall k\}$ infinite seq.

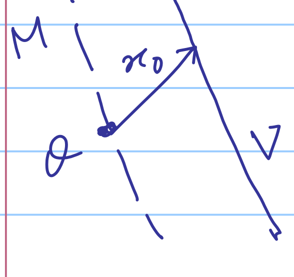
Subspace : $M \subset X$, $M \neq \phi$ is a subspace of X
$\alpha x + \beta y \in M$ whenever $x, y \in M$

Property. $M \ni \theta$, $M \cap N$ is a subspace
$M + N$ is a subspace

Def$\stackrel{n}{=}$: Let $S \subset X$. $[S] \equiv$ subspace generated by S := all vectors in X which are linear comb. of vectors in S.

Q. Why is $[S]$ a subspace?

Def$\stackrel{n}{=}$: Translation of a subspace is a linear variety (V)
$V = x_0 + M \qquad x_0 \in X$, M = subspace



linear independence → same def^n.

Def^n: Let $S \subset X$ be finite & lin. independent. Then if $[S] = X$ then X is finite dimensional. All other vector spaces are infinite dimensional.

Ex. $\mathbb{R}^n$ is finite dim, $C[a,b]$ infinite dim.

Q. What about nth deg. polynomials with complex coefficients on $[a,b]$. ?

Normed Linear Spaces (Why required?)

Def^n: NLS is a vector space X with

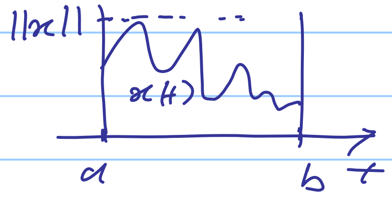
$$\|\cdot\| : \underset{X}{x} \longmapsto \underset{\mathbb{R}}{\|x\|}$$

1) $\|x\| \geq 0 \ \forall x \in X$, $\|x\| = 0$ iff $x = \theta$
2) $\|x+y\| \leq \|x\| + \|y\| \quad \forall x, y \in X$
3) $\|\alpha x\| = |\alpha| \|x\| \quad \forall \alpha, \forall x \in X$

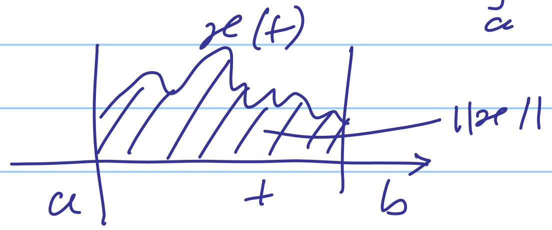
Property: $\|x\| - \|y\| \leq \|x - y\|$

Example: $C[a,b]$, $\|x\| = \max\limits_{a \leq t \leq b} |x(t)|$

Check the 3 axioms.



Ex: Continuous function on $[a,b]$ with $||x|| = \int_a^b |x(t)\,dt$



Ex: $BV[a,b]$, functions of bounded variation on $[a,b]$

Define a partition of $[a,b]$:

$a = t_0 < t_1 < t_2 < \cdots < t_n = b$.
(n finite)

$x(t) \in BV[a,b]$ if for any partition of $[a,b]$,

$$\sum_{i=1}^{n} |x(t_i) - x(t_{i-1})| \leq k < \infty$$

$$T.V.(x) = \sup_{\text{all partitions}} \sum_{i=1}^{n} |x(t_i) - x(t_{i-1})|$$

$$||x|| = x(a) + T.V.(x)$$

Q. What happened to our infinite seq vector space?

Review: Open set / closed set, Convergence, Convex set,

With the norm, it is possible to define continuity

Def$^{\underline{n}}$: $T: X \to Y$ (X, Y normed) is continuous at $x_0 \in X$ if for every $\varepsilon > 0$, $\exists\, \delta > 0$ s.t

$$\|x - x_0\| < \delta \Rightarrow \|T(x) - T(x_0)\| < \varepsilon$$

$$\Updownarrow$$

$$x_n \to x_0 \quad \Rightarrow T(x_n) \to T(x_0)$$

Banach Spaces:

Cauchy seq: $\{x_n\} \in X$ is C.S. if $\|x_n - x_m\| \to 0$ as $n, m \to \infty$

$$\Updownarrow$$

For every $\varepsilon > 0$, $\exists N$ s.t. $\|x_n - x_m\| < \varepsilon$ $\forall\, n, m > N$

In normed spaces,

\# Every convergent seq is Cauchy

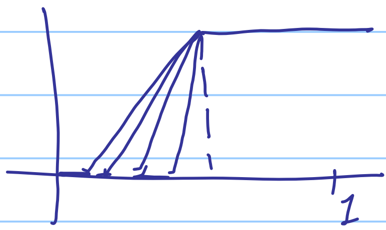
\# Cauchy is not always convergent

Def^{n}: A normed linear space is complete if every cauchy sequence from X has a limit in X.

A complete NLS is called a Banach space.

1) $C[0,1] \longrightarrow$ Banach space

2) Continuous functions on $[0,1]$ with norm $\|x\| = \int_0^1 |x(t)|\,dt$ $\rightarrow$ NOT complete (NOT B.S)



Famous Banach Spaces (without proof)

1) $C[0,1]$

2) l_p : $1 \leq p < \infty$, l_p : sequences of real nos $\{\xi_1, \xi_2, \dots\}$ for

which, $\sum_{i=1}^{\infty} |\xi_i|^p < \infty$

$$\|x\|_p = \left(\sum_{i=1}^{\infty} |\xi_i|^p\right)^{1/p}$$

$$\|x\|_\infty = \sup_i |\xi_i|$$

3) $L_p[a,b]$: real valued measurable functions x on $[a,b]$ for which $|x(t)|^p$ is Lebesgue integrable

$$\|x(t)\|_p = \left(\int_a^b |x(t)|^p dt\right)^{1/p}$$

$L_\infty[a,b]$: $\|x\|_\infty = \text{ess. sup} |x(t)|$

Defn : A transformation from a vector space X into $\mathbb{R}$ is said to be a functional on X

Defn : $T : X \to \mathbb{R}$ is a linear functional if

$$T(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 T(x_1) + \alpha_2 T(x_2)$$

$\forall\, x_1, x_2 \in X$, $\alpha_1, \alpha_2 \in \mathbb{R}$

Ex: On E^n : $x = (\xi_1, \xi_2, \cdots, \xi_n)$

$$f(x) = \sum_{k=1}^{n} \eta_k \xi_k \quad (\eta_k \text{ fixed})$$

(All lin. f. on E^n are of this form)

Defn: A l.f. on a N.L.S. is bounded if $\exists M$, s.t. $|f(x)| \leq M\|x\|$ $\forall x \in X$

Properties of L.F.

FACT: If a linear functional on N.L.S. X is continuous at a single point, it is continuous throughout X.

Hint: $|f(x_n) - f(x)| = |f(x_n - x + x_0) - f(x_0)|$

FACT: A L.F. on a NLS is bdd. iff it is continuous

Hint: If $x_n \to 0$, $M\|x_n\| \to 0$

The set of bdd linear functionals on X

themselve form a Normed linear space

Norm: $\|f\| = \inf \{ M : |f(x)| \leq M\|x\|, \forall x \}$

$$= \sup_{x \neq \theta} \frac{|f(x)|}{\|x\|}$$

$$= \sup_{\|x\| \leq 1} |f(x)|$$

$$= \sup_{\|x\|=1} |f(x)|$$

Why are these equal?

$$(f_1 + f_2)(x) = f_1(x) + f_2(x) \quad \forall x \in X$$

$$(\alpha f)(x) = \alpha[f(x)]$$

$$\theta := f_\theta(x) = 0 \quad \forall x \in X$$

Def^{n}: The space of all bounded linear functionals on X is called the normed dual of X and is denoted by X^*

Notation: If $x^* \in X$, the value of x^* at $x \in X$ is denoted by

$x^*(x) := \langle x, x^* \rangle$

<u>FACT</u>: X^* is a Banach Space

Hint: X^* is NLS. Prove it is complete!

<u>Examples</u>: Dual of E^n: $x = (\xi_1, \cdots, \xi_n)$
with $\|x\| = \sqrt{\sum_{i=1}^{n} (\xi_i)^2}$

$f(x) = \sum_{i=1}^{n} \eta_i \xi_i \rightarrow$ linear

$|f(x)| \leq \sqrt{\sum_{i=1}^{n} \eta_i^2} \, \|x\|$ $\qquad |ax+by| < \sqrt{(a^2+b^2)}\sqrt{x^2+y^2}$

(by Cauchy-Schwartz ineq)

So $\|f\| \leq \sqrt{\sum_{i=1}^{n} \eta_i^2}$

Since for $x = (\eta_1, \cdots, \eta_n)$, $|f(x)| = \sqrt{\cdots} \, \|x\|$

$\|f\| = \sqrt{\sum_{i=1}^{n} \eta_i^2}$

Now, let f be any l.f. on E^n.
$e_i = (0, \cdots 0, \underset{i}{1}, 0 \cdots 0)$

let $\eta_i := f(e_i)$ [η_i is defined like this (diff from 1st half)]

$$x = (\xi_1, \cdots, \xi_n) = \sum_{i=1}^{n} \xi_i e_i$$

$$f(x) = f\left(\sum \xi_i e_i\right) = \sum \xi_i f(e_i)$$

$$= \sum_{i=1}^{n} \xi_i \eta_i$$

So $X^* = E^n$.

Some famous duals (without proof)

1) Dual of l_p $(1 \le p < \infty)$

For every p, $1 \le p < \infty$, define $q = \dfrac{p}{p-1}$ s.t. $\dfrac{1}{p} + \dfrac{1}{q} = 1$

FACT: Dual of l_p is l_q

$\Updownarrow$

Every bounded l.f. on l_p is of the form

$$f(x) = \sum_{i=1}^{\infty} \eta_i \xi_i \quad (x = (\xi_1, \cdots, \xi_i, \cdots))$$

where $y_i = \{\eta_i\} \in l_q$

Moreover, every element of l_q defines an element of $(l_p)^*$:

$$\|f\| = \|y\|_q = \begin{cases} \left(\sum_{i=1}^{\infty} |\eta_i|^q\right)^{1/q} & 1<p<\infty \\ \sup_k |\eta_k| & p=1 \end{cases}$$

2) Dual of $L_p[0,1]$ is $L_q[0,1]$ where $\frac{1}{p}+\frac{1}{q}=1$

[One to one correspondence between bdd. l.f's f and elements $y \in L_q$

$$f(x) = \int_0^1 x(t)\,y(t)\,dt$$

and $\|f\| = \|y\|_q$

3) Dual of $C[a,b]$: (Riesz Rep. Thm)
Let f be a bdd. linear function on $X = C[a,b]$. Then there is a function v of bdd. variation on $[a,b]$ s.t. $\forall x \in X$

$$f(x) = \int_a^b x(t)\,dv(t)$$

$$\|f\| = T.V.(v)$$

Conversely, every function of B.V. on $[a,b]$ defines a bdd. lin. functional on X in this way.

Hahn - Banach Theorem (Extension form)

Def^{n}: Let f be a L.F. on subspace M of vector space X

A lin. func. F is an extension of f if 1) F is defined on $N \supsetneq M$ (proper)

2) $F \equiv f$ on M

Q) For a bdd f (on M) what is the minimum norm extension to X?

Ans (H-B thm) $\exists$ a) Min norm extn exists.

b) $\|F\| = \|f\|_M = \sup\limits_{m \in M} \dfrac{|f(m)|}{\|m\|}$

Def$\underline{\underline{n}}$: $p : X \to \mathbb{R}$ is a sublinear functional on X if

1) $p(x_1 + x_2) \leq p(x_1) + p(x_2)$ $\forall x_1, x_2 \in X$
2) $p(\alpha x) = \alpha p(x)$ $\forall \alpha \geq 0, x \in X$

Examples : 1) All linear functionals are sub-li:

2) If $f(x) \in X^*$, $p(x) = |f(x)|$ is sub-linear

3) $\|x\|$ is sublinear

H-B Thm : $X :=$ N.L.S

$p :=$ continuous sublinear f$\underline{\underline{n}}$ on X

$f :=$ lin functional on M (subspace of X)

AND $f(m) \leq p(m)$ $\forall m \in M$

[Note : This implies f is continuous and hence bounded]

Hint : $-p(-x) \leq f(x) \leq p(x)$

Then $\exists$ an extension F of f from M to X s.t. $F(x) \leq p(x)$ on X

[F is hence continuous + bdd]

Corollary: Let f b.L.F on M (subspace of X). Then $\exists$ b.L.F F on X which is an extension of f and

$$\|F\|_X = \|f\|_M$$

Proof: $$p(x) = \|f\|_M \|x\|$$

$$f(x) \leqslant M\|x\| = p(x)$$

$$F(x) \leqslant \|f\|_M \|x\| \quad \forall x \in X$$

Example: In $\mathbb{R}^2$ (with Euclidean norm), consider the lin. sub.

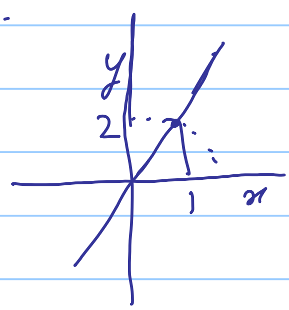
$$M = \{(x,y) : 2x - y = 0\}$$

Let $f : M \to \mathbb{R} := f(x,y) = x$

Then $F(x,y) = \dfrac{x}{5} + \dfrac{2y}{5}, \ \forall (x,y) \in \mathbb{R}^2$

is the unique H-B. extension of f from M to $\mathbb{R}^2$.

Hint: $$\|f\|_M = \sup_{(x,y)\in M} \frac{|x|}{\sqrt{x^2+y^2}}$$



$$= \frac{1}{\sqrt{5}} \Big|_{x=1,\, y=2}$$

$$\|F\|_M = \sup_{(x,y) \in \mathbb{R}^2} \frac{\left| \frac{x}{5} + \frac{2y}{5} \right|}{\sqrt{x^2+y^2}}$$

$$F(x,y)\Big|_M$$

$$= \sup_{(x,y):\, x^2+y^2=1} \left| \frac{x}{5} + \frac{2y}{5} \right|$$

$$= x$$

$$= \left| \frac{x}{5} + \frac{2y}{5} \right|_{x=\frac{1}{\sqrt{5}},\, y=\frac{2}{\sqrt{5}}}$$

$$= \frac{1}{\sqrt{5}}$$