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A UNIFIED CONSTRUCTION OF ADJOINT SYSTEMS AND NETWORKS

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SUMMARY

In this paper we introduce a technique for dealing with implicitly defined complementary orthogonal spaces. Using this technique we give a unified construction of various types of adjoint systems for dynamical systems defined through flow diagrams or graphs and also obtain the state and output equations of the adjoint systems in terms of the corresponding equations for the original system. Among other things we show how to construct the Kalman dual of a linear electrical network.

1. INTRODUCTION

The concept of adjoint systems (or networks) is known to have wide applications[1-3]. In this paper we describe a unified construction of the adjoints of networks and systems. This is achieved by the application of a basic theorem (see Section 3) which allows implicit construction of the complementary orthogonal space of a given vector space which is itself described implicitly. Other applications and alternative proofs of the basic theorem are given in References 4 and 5.

The conventional approach towards adjoint systems is *ad hoc*. Different techniques are used to construct adjoints of different kinds of systems. This has resulted in the word adjoint being used in different senses in control theory and network theory. In control theory, adjoints (the C-adjoints of this paper) are constructed for dynamical systems in the standard state and output equation format. Dynamical systems do not always appear in such a convenient form. Usually there are many more variables other than input, output and state variables. Further, the dynamical variables (those that occur as time derivatives) cannot always be assumed to be independent. The relation between adjoints and Kalman duals (K-adjoints) is not very clear in control theory literature. In network theory, adjoints (the R-adjoints of this paper) are based on a variation of Tellegen's Theorem.[3] The adjoint is constructed in such a way that appropriate vectors of the original network are orthogonal to the corresponding vectors of the R-adjoint. Actually, in order for the notion of adjoint to be useful for sensitivity computations more has to be shown: spaces of appropriate vectors of the adjoint have to be shown to be complementary orthogonal. Otherwise it is not clear that the inputs to the adjoint can be chosen independently (see Appendix II).

The approach adopted in this paper is based on Theorem 2 of Section 3. We define a dynamical system simply as a set of equations of the form

$$(\mathbf{N}(t))\begin{pmatrix}\dot{\mathbf{w}}\\ \mathbf{w}\\ \mathbf{u}\\ \mathbf{y}\end{pmatrix}=0$$

We next define the C-adjoint and K-adjoint. We then use the basic theorem to cast the adjoint in the appropriate form, i.e. the adjoint is based on a flow diagram or an electrical network derived from the

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original. For electrical networks we define R-adjoint as a variation of K-adjoint. An offshoot of the approach is that we obtain the state and output equations of the adjoint immediately, given those of the original.

We now discuss briefly the literature on adjoints in so far as it overlaps with the contents of this paper. The C-adjoint for a special case of dynamical systems based on flow diagrams is in fact given in Reference 6 but without formal proof. In Reference 7 a direct proof is given for the fact that inputs to R-adjoints can be chosen independently provided the original network is uniquely solvable for all the variables (not just the output variables). Such a condition may be inconvenient if the original network has a non-unique solution 'locally' at spurious devices which have been introduced for notational convenience. The basic theorem permits us to prove the same result assuming only that the output be uniquely determined for all possible inputs. In the interest of brevity, since we also wish to be able to obtain state equations of the adjoint while proving such a result, we have restricted ourselves to the case where for the original system one can write equations without involving derivatives of inputs. A direct derivation of state equations for the R-adjoint of a network, which has normal tree capacitor voltages and normal cotree inductor currents independent, is given in Reference 8. We make no such assumptions. Further our derivation is much simpler.

The organization of the paper is as follows: Section 2 deals with preliminary definitions. Section 3 contains a statement of the basic theorem and a proof of its corollary which is most pertinent to the rest of the paper. Section 4 describes the state and output equations of the C-adjoint and the K-adjoint in terms of those of the original system. Section 5 contains the development of the ideas of the C-adjoint, K-adjoint and R-adjoint of a time-varying linear electrical network. Section 6 contains the development of the ideas of the C-adjoint and K-adjoint of dynamical systems based on flow diagrams. Section 7 is the conclusion. Appendix I contains the proof of the basic theorem. Appendix II contains a description of sensitivity computations using adjoint systems.

2. PRELIMINARIES

We restrict ourselves to finite-dimensional real vector spaces. We usually treat vectors as column vectors. The transpose of a matrix $\mathbf{A}$ is denoted $\mathbf{A}^T$.

$\mathbf{U}$ denotes an identity matrix, whose order will be clear from the context. The inner product of two vectors $\mathbf{x}$, $\mathbf{y}$, in $\mathbb{R}^n$ is denoted $\langle \mathbf{x}, \mathbf{y}\rangle = \mathbf{x}^T\mathbf{y}$. Two vectors $\mathbf{x}, \mathbf{y}$ in $\mathbb{R}^n$ are said to be orthogonal iff $\langle \mathbf{x}, \mathbf{y}\rangle = 0$. Two subspaces of $\mathbb{R}^n$ are *complementary orthogonal* iff $\dim V_1 + \dim V_2 = n$ and every vector of V_1 is orthogonal to every vector of V_2.

Consider the equations

$$(\mathbf{A}_1 \quad \mathbf{A}_2)\begin{pmatrix}\mathbf{x}_1\\ \mathbf{x}_2\end{pmatrix} = 0 \tag{1}$$

$$(\hat{\mathbf{A}}_1 \quad \hat{\mathbf{A}}_3)\begin{pmatrix}\mathbf{x}_1\\ \mathbf{x}_3\end{pmatrix} = 0 \tag{2}$$

We will say that the equations (1) and (2) are *algebraically equivalent* in the vector variable $\mathbf{x}_1$ iff the space $V^1_{\mathbf{x}_1}$ of vectors $\mathbf{x}^1_1$ where $(\mathbf{x}^1_1 \quad \mathbf{x}^1_2)^T$ is a solution of (1) and the space $V^2_{\mathbf{x}_1}$ of vectors $\mathbf{x}^2_1$ where $(\mathbf{x}^2_1 \quad \mathbf{x}^2_3)^T$ is a solution of (2) are identical. Next consider the equations

$$(\mathbf{A})\begin{pmatrix}\mathbf{x}_1\\ \mathbf{x}_2\end{pmatrix} = 0 \tag{3}$$

$$(\mathbf{B})\begin{pmatrix}\mathbf{y}_1\\ \mathbf{y}_2\end{pmatrix} = 0 \tag{4}$$

Equations (3) and (4) are *complementary orthogonal* in the variables $\mathbf{x}_1$, $\mathbf{y}_1$ iff the space $V_{\mathbf{x}_1}$ of vectors $\mathbf{x}'_1$, where $(\mathbf{x}'_1 \quad \mathbf{x}'_2)^T$ is a solution of (3), and the space $V_{\mathbf{y}_1}$ of vectors $\mathbf{y}'_1$, where $(\mathbf{y}'_1 \quad \mathbf{y}'_2)^T$ is a solution of (4), are complementary orthogonal. Both in the case of algebraic equivalence and in the case of complementary orthogonality we will not mention the reference variables if they are clear from context. We use $\mathbf{A}^{\perp}$ to denote a matrix whose rows span the space complementary orthogonal to the space spanned by the rows of $\mathbf{A}$. It is a familiar fact of linear algebra that the equations $\mathbf{Ax}=\mathbf{0}$ and $\mathbf{By}=\mathbf{0}$ are complementary orthogonal in the variables $\mathbf{x}, \mathbf{y}$ iff $\mathbf{B}$ can be wirtten as $\mathbf{A}^{\perp}$.

For typographical convenience we write linear equations in the form

$$F_t(\boldsymbol{\rho}_1, \ldots, \boldsymbol{\rho}_k) = 0 \tag{5}$$

The coefficient matrix is a function of time t. The vector variables $\boldsymbol{\rho}_i$, $\boldsymbol{\xi}_i$ have n_i components. The n_is may not all be the same. The equation which is complementary orthogonal to (5) with respect to the variables $(\boldsymbol{\rho}_1, \ldots, \boldsymbol{\rho}_k)$, $(\boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_k)$ is written as

$$F_t^{\perp}(\boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_k) = 0 \tag{6}$$

We consider *dynamical systems* of the form

$$F_t[\dot{\mathbf{w}}(t), \mathbf{w}(t), \mathbf{u}'(t), \mathbf{y}(t)] = 0 \tag{7}$$

where $\mathbf{u}'$, $\mathbf{y}$, $\mathbf{w}$ are vector functions of time $t(\in[t_0, t_f])$; $\dot{\mathbf{w}}(t)$ denotes $\mathrm{d}\mathbf{w}/\mathrm{d}t$. For typographical convenience we do not always write variables, which are functions of time, explicitly as functions of time. The *dynamical variables* of the system are the components of $\mathbf{w}$ and may not be independent of each other. We would usually partition the input variables $\mathbf{u}'$ into ordinary inputs $\mathbf{u}$ and design parameter inputs $\mathbf{p}$. We will take (7) as our starting point even though it may have arisen by linearization of a non-linear dynamical system about some solution. The vectors of (7) would then be incremental variables of the original system.

In this paper we consider three kinds of adjoints for the system defined in (7).

(i) *The C-adjoint.* This is the type of 'adjoint system' one encounters in control theory literature. This is defined as

$$F_t^{\perp}(\mathbf{w}^{c}(t), \dot{\mathbf{w}}^{c}(t), \mathbf{y}^{c}(t), \mathbf{u}^{c}(t)) = 0 \tag{8}$$

It may be verified that the C-adjoint of

$$\begin{aligned}\dot{\mathbf{w}}(t) &= \mathbf{A}(t)\mathbf{w}(t) + \mathbf{B}(t)\mathbf{u}'(t)\\ \mathbf{y}(t) &= \mathbf{C}(t)\mathbf{w}(t) + \mathbf{D}(t)\mathbf{u}'(t)\end{aligned} \tag{9}$$

is

$$\begin{aligned}\dot{\mathbf{w}}^{c}(t) &= -\mathbf{A}^{\mathrm{T}}(t)\mathbf{w}^{c}(t) - \mathbf{C}^{\mathrm{T}}(t)\mathbf{u}^{c}(t)\\ \mathbf{y}^{c}(t) &= -\mathbf{B}^{\mathrm{T}}(t)\mathbf{w}^{c}(t) - \mathbf{D}^{\mathrm{T}}(t)\mathbf{u}^{c}(t)\end{aligned} \tag{10}$$

(ii) *The K-adjoint* (K for Kalman). This is defined as

$$F_t^{\perp}(\mathbf{w}^{k}(\tau), -\mathrm{d}\mathbf{w}^{k}/\mathrm{d}\tau, -\mathbf{y}^{k}(\tau), \mathbf{u}^{k}(\tau)) = 0 \tag{11}$$

The time variables t and τ are related by *either* $t=\tau$ (when we wish to construct Kalman-dual systems) or $t=(t_0+t_f)-\tau$ (when we wish to perform sensitivity computations). It may be verified that the K-adjoint of (9) is

$$\begin{aligned}\mathrm{d}\mathbf{w}^{k}/\mathrm{d}\tau &= \mathbf{A}^{\mathrm{T}}(t)\mathbf{w}^{k}(\tau) + \mathbf{C}^{\mathrm{T}}(t)\mathbf{u}^{k}(\tau)\\ \mathbf{y}^{k}(\tau) &= \mathbf{B}^{\mathrm{T}}(t)\mathbf{w}^{k}(\tau) + \mathbf{D}^{\mathrm{T}}(t)\mathbf{u}^{k}(\tau)\end{aligned} \tag{12}$$

(iii) *The R-adjoint* (R for Rohrer). This is defined only for the case where the original dynamical system is an electrical network. In Reference 9 this is defined as follows. The original linear network and the R-adjoint have the same graph. Let the non-dynamical devices in the network be governed by

$$(\mathbf{N}_{\mathrm{R}})\begin{pmatrix}\mathbf{i}_{\mathrm{R}}\\ \mathbf{v}_{\mathrm{R}}\end{pmatrix} = 0$$

Then in the adjoint these devices are governed by

$$(\mathbf{N}_R^{\perp})\begin{pmatrix} \mathbf{v}_R^r \\ -\mathbf{i}_R^r \end{pmatrix} = 0$$

The capactiors and mutual inductors of the original network are left unchanged in the R-adjoint. So are voltage sources and current sources.

We state the following elementary results[10] from linear algebra without proof:

Theorem 1

The following equations are complementary orthogonal in the variables $(\mathbf{x}_1 \quad \mathbf{x}_2)$, $(\mathbf{y}_1 \quad \mathbf{y}_2)$:

$$(\mathbf{U} \quad \mathbf{K})\begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{pmatrix} = 0; \qquad (-\mathbf{K}^T \quad \mathbf{U})\begin{pmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \end{pmatrix} = 0$$

Corollary 1

Let the following equations be complementary orthogonal in the variables $(\mathbf{x}_1 \quad \mathbf{x}_2)$, $(\mathbf{y}_1 \quad \mathbf{y}_2)$

$$(\mathbf{A}_1 \quad \mathbf{A}_2)\begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{pmatrix} = 0; \qquad (\mathbf{B}_1 \quad \mathbf{B}_2)\begin{pmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \end{pmatrix} = 0$$

Then $\mathbf{A}_1$ is non-singular iff $\mathbf{B}_2$ is non-singular.

3. THE BASIC THEOREM

In this section we present the basic theorem and its corollary on which most of the results of this paper are based. The theorem and its corollary allow us to speak of complementary orthogonal spaces implicitly.

Theorem 2

Let equations (13a) and (14a) be complementary orthogonal in the variables $(\mathbf{i}_s \quad \mathbf{i}_p)^T$, $(\mathbf{v}_s \quad \mathbf{v}_p)^T$, respectively, and let equations (13b) and (14b) be complementary orthogonal in the variables $(\mathbf{i}_p)$, $(\mathbf{v}_p)$, respectively:

$$(\mathbf{K}_s \quad \mathbf{K}_p)\begin{pmatrix} \mathbf{i}_s \\ \mathbf{i}_p \end{pmatrix} = 0 \tag{13a}$$

$$(\tilde{\mathbf{K}}_p)\mathbf{i}_p = 0 \tag{13b}$$

$$(\mathbf{N}_s \quad \mathbf{N}_p)\begin{pmatrix} \mathbf{v}_s \\ \mathbf{v}_p \end{pmatrix} = 0 \tag{14a}$$

$$(\tilde{\mathbf{N}}_p)\mathbf{v}_p = 0 \tag{14b}$$

Then equations (13) and (14) are complementary orthogonal in the variables $(\mathbf{i}_s)$, $(\mathbf{v}_s)$, respectively.

We relegate the proof to Appendix I.

Corollary 2

Let equations (15a) and (16a) be complementary orthogonal in the variables $(\mathbf{x}_1 \quad \mathbf{x}_2)^T$, $(\mathbf{y}_1 \quad \mathbf{y}_2)^T$ and let equations (15b) and (16b) be complementary orthogonal in the variables $(\mathbf{x}_2 \quad \mathbf{x}_3)^T$, $(-\mathbf{y}_2 \quad \mathbf{y}_3)^T$:

$$(\mathbf{K}_1 \quad \mathbf{K}_2)\begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{pmatrix} = 0 \tag{15a}$$

$$(\tilde{\mathbf{K}}_2 \quad \tilde{\mathbf{K}}_3)\begin{pmatrix} \mathbf{x}_2 \\ \mathbf{x}_3 \end{pmatrix} = 0 \tag{15b}$$

$$(\mathbf{N}_1 \quad \mathbf{N}_2)\begin{pmatrix}\mathbf{y}_1\\ \mathbf{y}_2\end{pmatrix} = 0 \tag{16a}$$

$$(\tilde{\mathbf{N}}_2 \quad \tilde{\mathbf{N}}_3)\begin{pmatrix}-\mathbf{y}_2\\ \mathbf{y}_3\end{pmatrix} = 0 \tag{16b}$$

Then equations (15) and (16) are complementary orthogonal in the variables $(\mathbf{x}_1 \quad \mathbf{x}_3)^{\mathrm{T}}$, $(\mathbf{y}_1 \quad \mathbf{y}_3)^{\mathrm{T}}$.

Proof. We observe that equation (15) is algebraically equivalent in the variable $(\mathbf{x}_1 \quad \mathbf{x}_3)^{\mathrm{T}}$ to

$$\begin{pmatrix}\mathbf{K}_1 & \mathbf{K}_2 & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \tilde{\mathbf{K}}_2 & \tilde{\mathbf{K}}_3\end{pmatrix}\begin{pmatrix}\mathbf{x}_1\\ \mathbf{x}_2\\ \mathbf{x}_2'\\ \mathbf{x}_3\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} \tag{17a}$$

$$(\mathbf{U} \quad -\mathbf{U})\begin{pmatrix}\mathbf{x}_2\\ \mathbf{x}_2'\end{pmatrix} = 0 \tag{17b}$$

Further equation (16) is algebraically equivalent in the variable $(\mathbf{y}_1 \quad \mathbf{y}_3)^{\mathrm{T}}$ to

$$\begin{pmatrix}\mathbf{N}_1 & \mathbf{N}_2 & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \tilde{\mathbf{N}}_2 & \tilde{\mathbf{N}}_3\end{pmatrix}\begin{pmatrix}\mathbf{y}_1\\ \mathbf{y}_2\\ \mathbf{y}_2'\\ \mathbf{y}_3\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} \tag{18a}$$

$$(\mathbf{U} \quad \mathbf{U})\begin{pmatrix}\mathbf{y}_2\\ \mathbf{y}_2'\end{pmatrix} = 0 \tag{18b}$$

From the conditions of the corollary it is clear that equations (17a) and (18a) are complementary orthogonal. It is readily verified that (17b) and (18b) are complementary orthogonal. The corollary is now immediate by applying Theorem 2.

Q.E.D.

4. STATE EQUATIONS FOR ADJOINT SYSTEMS

In this section we show that under appropriate choice of state variables for the original and the 'adjoint' the resulting state and output equations for the two systems would be 'mutually adjoint'. In the process we also show that, if the original system satisfies fairly mild conditions, the adjoint system is 'well defined' and its inputs may be chosen independently. Well defined systems are essentially those that lend themselves to being cast in the usual state and output equation format under an appropriate choice of state variables.

Definition 1

Let

$$F_t(\dot{\mathbf{w}}(t),\ \mathbf{w}(t), \mathbf{u}'(t), \mathbf{y}(t)) = 0 \tag{19}$$

be the given dynamical system.

We say that the system is *well defined* iff the following two conditions are satisfied.

(a) There exists a variable $\mathbf{x}(t)$ and a constant matrix $\mathbf{P}$ such that equation (19) is algebraically equivalent, at time t in the variables $(\dot{\mathbf{w}}, \mathbf{w}, \mathbf{u}', \mathbf{y})$, to

$$F_t[\dot{\mathbf{w}}(t), \mathbf{w}(t), \mathbf{u}'(t), \mathbf{y}(t)] = 0$$
$$(\mathbf{P})\begin{pmatrix}\mathbf{w}(t)\\ \mathbf{x}(t)\end{pmatrix} = 0 \tag{20}$$

(b) There exist matrices $\mathbf{A}(t)$, $\mathbf{B}(t)$, $\mathbf{C}(t)$, $\mathbf{D}(t)$ such that the following equations are algebraically equivalent at time t, in the variables $(\dot{\mathbf{x}}, \mathbf{x}, \mathbf{u}', \mathbf{y})$:

$$F_t[\dot{\mathbf{w}}(t), \mathbf{w}(t), \mathbf{u}'(t), \mathbf{y}(t)] = 0$$

$$(\mathbf{P})\begin{pmatrix}\mathbf{w}(t)\\ \mathbf{x}(t)\end{pmatrix} = 0 \tag{21}$$

$$(\mathbf{P})\begin{pmatrix}\dot{\mathbf{w}}(t)\\ \dot{\mathbf{x}}(t)\end{pmatrix} = 0$$

$$\begin{pmatrix}\dot{\mathbf{x}}(t)\\ \mathbf{y}(t)\end{pmatrix} = \begin{pmatrix}\mathbf{A}(t) & \mathbf{B}(t)\\ \mathbf{C}(t) & \mathbf{D}(t)\end{pmatrix}\begin{pmatrix}\mathbf{x}(t)\\ \mathbf{u}(t)\end{pmatrix} \tag{22}$$

We would call $(\mathbf{P})(\mathbf{w}(t) \quad \mathbf{x}(t))^T = 0$ the graph of the state transformation.

Remark

In order to be able to apply Theorem 2 we have to confine ourselves to algebraic operations. Definition 1 deals algebraically with dynamical systems whose input–output relationship permits being expressed in the state and output equation format of (22). Condition (a) states that the $(\mathbf{w}(t), \mathbf{x}(t))$ relationship $(\mathbf{P})(\mathbf{w}(t) \quad \mathbf{x}(t)^{\mathrm{T}} = 0)$ does not alter the constraints of equation (19) on $(\dot{\mathbf{w}}(t), \mathbf{w}(t), \mathbf{u}'(t), \mathbf{y}(t))$. Since $\mathbf{P}$ is a constant matrix (21) can be obtained from (20) by differentiating. So the constraints of (21) on the variables $(\dot{\mathbf{w}}(t), \mathbf{w}(t), \mathbf{u}'(t), \mathbf{y}(t))$ are the same as the constraints of (19). Condition (b) states that (21) and (22) are algebraically equivalent in the variables $(\dot{\mathbf{x}}(t), \mathbf{x}(t), \mathbf{u}'(t), \mathbf{y}(t))$. It follows that (19) and (22) impose the same constraints on $\mathbf{u}(\cdot)$, $\mathbf{y}(\cdot)$. Since in equation (22) the inputs are independent, a well-defined system permits an independent choice of inputs.

We now show that for the most common class of well-defined systems the C-adjoint also is well defined. Further we obtain the state and output equations of the C-adjoint in terms of the equations for the original.

Theorem 3

Let the given dynamical system be as defined in (7). Let it be well defined with respect to the state transformation graph

$$(\mathbf{U} \quad -\mathbf{K})\begin{pmatrix}\mathbf{w}(t)\\ \mathbf{x}(t)\end{pmatrix} = 0 \tag{23}$$

(a) Let the C-adjoint of the given system be as in (8). Then the C-adjoint is well defined with respect to the state transformation graph

$$(-\mathbf{K}^{\mathrm{T}} \quad \mathbf{U})\begin{pmatrix}\mathbf{w}^{\mathrm{c}}(t)\\ \mathbf{x}^{\mathrm{c}}(t)\end{pmatrix} = 0 \tag{24}$$

(b) Let the state and output equations of the dynamical system be

$$\begin{pmatrix}\dot{\mathbf{x}}(t)\\ \mathbf{y}(t)\end{pmatrix} = \begin{pmatrix}\mathbf{A}(t) & \mathbf{B}(t)\\ \mathbf{C}(t) & \mathbf{D}(t)\end{pmatrix}\begin{pmatrix}\mathbf{x}(t)\\ \mathbf{u}(t)\end{pmatrix} \tag{25}$$

Then those of the C-adjoint are

$$\begin{pmatrix}\dot{\mathbf{x}}^{\mathrm{c}}(t)\\ \mathbf{y}^{\mathrm{c}}(t)\end{pmatrix} = \begin{pmatrix}-\mathbf{A}^{\mathrm{T}}(t) & -\mathbf{C}^{\mathrm{T}}(t)\\ -\mathbf{B}^{\mathrm{T}}(t) & -\mathbf{D}^{\mathrm{T}}(t)\end{pmatrix}\begin{pmatrix}\mathbf{x}^{\mathrm{c}}(t)\\ \mathbf{u}^{\mathrm{c}}(t)\end{pmatrix} \tag{26}$$

Proof. The dynamical system of (7) is well defined with respect to the state transformation graph (23). So equations (7) and (23), together with the equation obtained by differentiating both sides of (23) with respect to t, must be algebraically equivalent at time t, in the variables $(\dot{\mathbf{x}}(t), \mathbf{x}(t), \mathbf{u}'(t), \mathbf{y}(t))$, to equation

(25) for some suitable matrices $\mathbf{A}(t)$, $\mathbf{B}(t)$, $\mathbf{C}(t)$, $\mathbf{D}(t)$. Now consider the C-adjoint system of (8). Equation (8) is (trivially) algebraically equivalent at time t in the variables $\mathbf{w}^c(t)$, $\dot{\mathbf{w}}^c(t)$, $\mathbf{y}^c(t)$, $\mathbf{u}^c(t)$ to equation (8) together with (24). Equations (8) and (24) together impose the same constraint over the interval $[t_0, t_f]$ on the variables $\mathbf{w}^c(t)$, $\dot{\mathbf{w}}^c(t)$, $\mathbf{y}^c(t)$, $\mathbf{u}^c(t)$ as equations (8) and (24) and the equation obtained by differentiating both sides of equation (24) with respect to t, all taken together.

Now observe that the set of equations (7), (23) and the derivative of (23) is complementary orthogonal, in the variables $(\dot{\mathbf{x}}(t), \mathbf{x}(t), \mathbf{u}'(t), \mathbf{y}(t))$ and the variables $(\mathbf{x}^c(t), \dot{\mathbf{x}}^c(t), \mathbf{y}^c(t), \mathbf{u}^c(t))$, to the set of equations (8), (24) and the derivative of (24). The former is algebraically equivalent to (25) in the variables $(\dot{\mathbf{x}}(t), \mathbf{x}(t), \mathbf{u}'(t), \mathbf{y}(t))$, and (25) is complementary orthogonal in the variables $(\dot{\mathbf{x}}(t), \mathbf{x}(t), \mathbf{u}'(t), \mathbf{y}(t))$ and $(\mathbf{x}^c(t), \dot{\mathbf{x}}^c(t), \mathbf{y}^c(t), \mathbf{u}^c(t))$ to equation (26). It follows that (8), (24) and the derivative of (24) are together algebraically equivalent to (26) in the variables $(\mathbf{x}^c(t), \dot{\mathbf{x}}^c(t), \mathbf{y}^c(t), \mathbf{u}^c(t))$. But this means that the C-adjoint system defined by (8) is well defined with respect to the state transformation graph (24), and its state equations are given by (26). Q.E.D.

The case of K-adjoints can be handled similarly. The K-adjoint of the system of (7) is given by

$$F_\tau^\perp\left(\mathbf{w}^k(\tau), -\frac{d\mathbf{w}^k}{d\tau}, -\mathbf{y}^k(\tau), \mathbf{u}^k(\tau)\right) = 0 \tag{27}$$

If the original system is well defined with respect to (23) the K-adjoint can be shown to be well defined with respect to

$$(-\mathbf{K}^T \quad \mathbf{U})\begin{pmatrix}\mathbf{w}^k(\tau)\\ \mathbf{x}^k(\tau)\end{pmatrix} = 0 \tag{28}$$

Further the state equations of the K-adjoint can be shown to be

$$\begin{pmatrix}\dfrac{d\mathbf{x}^k}{d\tau}\\ \mathbf{y}^k(\tau)\end{pmatrix} = \begin{pmatrix}\mathbf{A}^T(\tau) & \mathbf{C}^T(\tau)\\ \mathbf{B}^T(\tau) & \mathbf{D}^T(\tau)\end{pmatrix}\begin{pmatrix}\mathbf{x}^k(\tau)\\ \mathbf{u}^k(\tau)\end{pmatrix} \tag{29}$$

Thus the state and output equations for the K-adjoint are the Kalman duals of the state and output equations of the original system.

5. ADJOINTS FOR ELECTRICAL NETWORKS

In this section we use the basic theorem to construct three types of adjoints for electrical networks: C-adjoint, K-adjoint and R-adjoint. We use R-adjoint for the 'adjoint network' of circuit theory literature.

We would like the adjoints to look like electrical networks. To achieve this we will separate the equations of the original network into two sets:

$$\text{(topological)} \quad F_t\,(\mathbf{w}_1, \mathbf{z}, \mathbf{y}, \mathbf{u}) = 0 \tag{30a}$$

$$\text{(branch } V\text{-}i \text{ characteristics)} \quad G_t\,(\mathbf{w}_2, \mathbf{z}, \mathbf{p}) = 0 \tag{30b}$$

We will take $\mathbf{w}_1 = \dot{\mathbf{w}} = (\dot{\mathbf{q}}_c \quad \dot{\mathbf{\Phi}}_L)^T$, $\mathbf{w}_2 = \mathbf{w}$. $\mathbf{q}_c$, $\mathbf{\Phi}_L$ refer, respectively, to capacitor charges and inductor fluxes. $\mathbf{p}$ is the vector of design parameters in our control. We will assume that the components of $\mathbf{p}$ are independent of each other.

Observe that by Corollary 2, equations (30) and (31) below are complementary orthogonal in the variables $(\mathbf{w}_1, \mathbf{w}_2, \mathbf{u}, \mathbf{p}, \mathbf{y})$, $(\hat{\mathbf{w}}_1, \hat{\mathbf{w}}_2, \hat{\mathbf{y}}, \hat{\mathbf{p}}, \hat{\mathbf{u}})$:

$$F_t^\perp(\hat{\mathbf{w}}_1, \boldsymbol{\alpha}, \hat{\mathbf{u}}, \hat{\mathbf{y}}) = 0 \tag{31a}$$

$$G_t(\hat{\mathbf{w}}_2, -\boldsymbol{\alpha}, \hat{\mathbf{p}}) = 0 \tag{31b}$$

Equation (31a) will be taken as the topological equations of the adjoint. In order to make the KC and KV equations of the adjoint take the form (31a) we can either retain the current-voltage nature of the variables as in the original or make variables which are currents (voltages) in the original into voltages (currents) in the adjoint. The former would force us to make the KC and KV equations of the adjoint complementary orthogonal to the KC and KV equations, respectively, of the original which in turn implies that the networks have to be planar. We therefore adopt the latter scheme and are able to retain for the adjoint the directed graph of the original network.

Equations (31b) will be taken as the device characteristic of the adjoint. To construct the C-adjoint we will take

$$\hat{\mathbf{w}}_1 = \mathbf{w}^c = \begin{pmatrix} \mathbf{v}_C^c \\ \mathbf{i}_L^c \end{pmatrix}, \qquad \hat{\mathbf{w}}_2 = \dot{\mathbf{w}}^c, \qquad \hat{\mathbf{u}} = \mathbf{u}^c, \qquad \hat{\mathbf{y}} = \mathbf{y}^c$$

The variable $\hat{\mathbf{p}}$ would behave like an output variable. In the original network equations, whenever the vector $\mathbf{z}$ has a voltage (current) variable, $\boldsymbol{\alpha}$ would have the corresponding current (voltage) variable.

The K-adjoint is constructed by taking

$$\hat{\mathbf{w}}_1 = \mathbf{w}^k(\tau) = \begin{pmatrix} \mathbf{v}_C^k \\ \mathbf{i}_L^k \end{pmatrix}, \qquad \hat{\mathbf{w}}_2 = -\frac{d\mathbf{w}^k}{d\tau}, \qquad \hat{\mathbf{u}} = \mathbf{u}^k(\tau), \qquad \hat{\mathbf{y}} = -\mathbf{y}^k(\tau)$$

The R-adjoint is constructed from the K-adjoint by changing the signs of all the current variables and their derivatives in the equations of the K-adjoint network. Table I gives the detailed descriptions of

Table I. Transformation of devices from the original to the adjoint

Original	C-adjoint	K-adjoint	R-adjoint
$v_\gamma = Ri_\gamma + k_R p_R$	$v_\gamma^c = -Ri_\gamma^c$	$v_\gamma^k = -Ri_\gamma^k$	$v_\gamma^r = Ri_\gamma^r$
	$\hat{p}_R = -k_R i_\gamma^c$	$\hat{p}_R = -k_R i_\gamma^k$	$\hat{p}_R = k_R i_\gamma^r$
$v_\gamma = \rho v_\delta + k_\rho p_\rho$	$i_\delta^c = -\rho i_\gamma^c$	$i_\delta^k = -\rho i_\gamma^k$	$i_\delta^r = -\rho i_\gamma^r$
$i_\delta = 0$	$v_\gamma^c = 0$	$v_\gamma^k = 0$	$v_\gamma^r = 0$
	$\hat{p}_\rho = k_\rho i_\gamma^c$	$\hat{p}_\rho = k_\rho i_\gamma^k$	$\hat{p}_\rho = -k_\rho i_\gamma^r$
$i_\gamma = \mu i_\delta + k_\mu p_\mu$	$v_\delta^c = -\mu v_\gamma^c$	$v_\delta^k = -\mu v_\gamma^k$	$v_\delta^r = -\mu v_\gamma^r$
$v_\delta = 0$	$i_\gamma^c = 0$	$i_\gamma^k = 0$	$i_\gamma^r = 0$
	$\hat{p}_\mu^c = k_\mu v_\gamma^c$	$\hat{p}_\mu^k = k_\mu v_\gamma^k$	$\hat{p}_\mu^r = k_\mu v_\gamma^r$
$v_\gamma = r i_\delta + k_r p_r$	$v_\delta^c = -r i_\gamma^c$	$v_\delta^k = -r i_\gamma^k$	$v_\delta^r = r i_\gamma^r$
$v_\delta = 0$	$v_\gamma^c = 0$	$v_\gamma^k = 0$	$v_\gamma^r = 0$
	$\hat{p}_r = k_r i_\gamma^c$	$\hat{p}_r = k_r i_\gamma^k$	$\hat{p}_r = -k_r i_\gamma^r$
$i_\gamma = g v_\delta + k_g p_g$	$i_\delta^c = -g v_\gamma^c$	$i_\delta^k = -g v_\gamma^k$	$i_\delta^r = g v_\gamma^r$
$i_\delta = 0$	$i_\gamma^c = 0$	$i_\gamma^k = 0$	$i_\gamma^r = 0$
	$\hat{p}_g = k_g v_\gamma^c$	$\hat{p}_g = k_g v_\gamma^k$	$\hat{p}_g = k_g v_\gamma^r$
$i_c = C\dot{v}_c + k_c p_c$	$i_c^c = C\dot{v}_c^c$	$i_c^k = -C\dot{v}_c^k$	$i_c^r = C\dot{v}_c^r$
	$\hat{p}_c^c = -k_c \dot{v}_c^c$	$\hat{p}_c^k = k_c \dot{v}_c^k$	$\hat{p}_c^r = k_c \dot{v}_c^r$
$\mathbf{\Phi}_L = \mathbf{L}\mathbf{i}_L + \mathbf{k}_L \mathbf{p}_L$	$\mathbf{v}_L^c = \mathbf{L}\dot{\mathbf{i}}_L^c$	$\mathbf{v}_L^k = -\mathbf{L}\dot{\mathbf{i}}_L^k$	$\mathbf{v}_L^r = \mathbf{L}\dot{\mathbf{i}}_L^r$
	$\hat{\mathbf{p}}_L^c = -\mathbf{k}_L^T \dot{\mathbf{i}}_L^c$	$\hat{\mathbf{p}}_L^k = \mathbf{k}_L^T \dot{\mathbf{i}}_L^k$	$\hat{\mathbf{p}}_L^r = -\mathbf{k}_L^T \dot{\mathbf{i}}_L^r$
$u_j = v_j$	$y_j^c = i_j^c$	$y_j^k = -i_j^k$	$y_j^r = i_j^r$
	$v_j^c = 0$	$v_j^k = 0$	$v_j^r = 0$
$u_j \quad = i_j$	$y_j^c = v_j^c$	$y_j^k = -v_j^k$	$y_j^r = -v_j^r$
	$i_j^c = 0$	$i_j^k = 0$	$i_j^r = 0$
$y_j \quad = i_j$			
	$u_j^c = v_j^c$	$u_j^k = v_j^k$	$u_j^r = v_j^r$
$v_j = 0$			
$y_j = v_j$			
	$u_j^c = i_j^c$	$u_j^k = i_j^k$	$u_j^r = -i_j^r$
$i_j = 0$			

common devices in the original and in the adjoint. The parameters R, ρ, μ, g, r, C, L can be taken to be functions of time.

Let the state transformation in the original network be

$$\begin{pmatrix}\mathbf{q}_c\\ \boldsymbol{\Phi}_L\end{pmatrix}=\begin{pmatrix}\mathbf{K}_c & \mathbf{0}\\ \mathbf{0} & \mathbf{K}_L\end{pmatrix}\begin{pmatrix}\mathbf{x}_c\\ \mathbf{x}_L\end{pmatrix}$$

Then in the K-adjoint we choose the state transformation

$$\begin{pmatrix}\mathbf{x}_c^k\\ \mathbf{x}_L^k\end{pmatrix}=\begin{pmatrix}\mathbf{K}_c^T & \mathbf{0}\\ \mathbf{0} & \mathbf{K}_L^T\end{pmatrix}\begin{pmatrix}\mathbf{v}_c^k\\ \mathbf{i}_L^k\end{pmatrix}$$

and the same transformation in the R-adjoint.

Let the state and output equations for the original network be

$$\begin{pmatrix}\dot{\mathbf{q}}_c\\ \dot{\boldsymbol{\Phi}}_L\\ \mathbf{y}_v\\ \mathbf{y}_i\end{pmatrix}=(\mathbf{T})\begin{pmatrix}\mathbf{q}_c\\ \boldsymbol{\Phi}_L\\ \mathbf{u}_v\\ \mathbf{u}_i\end{pmatrix}$$

Then by the discussion in Section 4 following Theorem 3, we know that the state and output equation for the K-adjoint would be

$$\begin{pmatrix}\dot{\mathbf{v}}_c^k\\ \dot{\mathbf{i}}_L^k\\ \mathbf{y}_i^k\\ \mathbf{y}_v^k\end{pmatrix}=(\mathbf{T}^T)\begin{pmatrix}\mathbf{v}_c^k\\ \mathbf{i}_L^k\\ \mathbf{u}_i^k\\ \mathbf{u}_v^k\end{pmatrix}$$

It follows that the state and output equation for the R-adjoint would be (changing the signs of all current variables and derivatives of current variables)

$$\begin{pmatrix}\dot{\mathbf{x}}_c^r\\ -\dot{\mathbf{x}}_L^r\\ -\mathbf{y}_i^r\\ \mathbf{y}_v^r\end{pmatrix}=(\mathbf{T}^T)\begin{pmatrix}\mathbf{x}_c^r\\ -\mathbf{x}_L^r\\ -\mathbf{u}_i^r\\ \mathbf{u}_v^r\end{pmatrix}$$

6. ADJOINTS FOR DYNAMICAL SYSTEMS BASED ON FLOW DIAGRAMS

In this section we use the basic theorem to construct the C-adjoint and K-adjoint of dynamical systems based on flow diagrams.

We would like the adjoints also to be based on flow diagrams. To achieve this we will separate the equations of the original flow diagram into two sets:

$$\text{(summer and connection point)} \quad F_t(\mathbf{z}, \mathbf{u}, \mathbf{y}) = 0 \tag{32a}$$

$$\text{(block input–output characteristics)} \quad G_t(\mathbf{w}_1, \mathbf{w}_2, \mathbf{z}, \mathbf{p}) = 0 \tag{32b}$$

Here $\mathbf{w}_1 = \dot{\mathbf{w}}$ and $\mathbf{w}_2 = \mathbf{w}$. $\mathbf{p}$ is the vector of the design parameters. We will assume that the design parameters are independent of each other. $\mathbf{z} = (\mathbf{z}_u \quad \mathbf{z}_y)^T$ where $\mathbf{z}_u$, $\mathbf{z}_y$ are all the input and all the output variables for the individual blocks. Observe that by Corollary 2, equation (32) and equation (33) below are complementary orthogonal in the variables $(\mathbf{w}_1, \mathbf{w}_2, \mathbf{u}, \mathbf{p}, \mathbf{y})$ and $(\hat{\mathbf{w}}_1, \hat{\mathbf{w}}_2, \hat{\mathbf{y}}, \hat{\mathbf{p}}, \hat{\mathbf{u}})$.

$$F_t^{\perp}(\boldsymbol{\alpha}, \hat{\mathbf{y}}, \hat{\mathbf{u}}) = 0 \tag{33a}$$

$$G_t^{\perp}(\hat{\mathbf{w}}_1, \hat{\mathbf{w}}_2, -\boldsymbol{\alpha}, \hat{\mathbf{p}}) = 0 \tag{33b}$$

Equation (33a) will be taken as the equations of the summers and connection points for the adjoint, and equation (33b) will be the block input–output characteristics. To construct the C-adjoint we will take

$$\hat{\mathbf{w}}_1 = \mathbf{w}^c, \qquad \hat{\mathbf{w}}_2 = \dot{\mathbf{w}}^c, \qquad \hat{\mathbf{y}} = \mathbf{y}^c, \qquad \hat{\mathbf{u}} = \mathbf{u}^c$$

To construct the K-adjoint we will take

$$\hat{\mathbf{w}}_1 = \mathbf{w}^k, \qquad \hat{\mathbf{w}}_2 = -\dot{\mathbf{w}}^k, \qquad \hat{\mathbf{y}} = -\mathbf{y}^k, \qquad \hat{\mathbf{u}} = \mathbf{u}^k$$

We now examine the equations of the original dynamical system in greater detail. We assume that the flow diagram of the given dynamical system is composed of a number of blocks, each of which is itself a dynamical system with inputs, outputs and dynamical variables. We assume that there are no common variables for distinct blocks, further that the overall outputs of the original dynamical system are connected only to summers and connection points and not to individual blocks. We also assume that there are no common variables between a summer (connection point) and another summer or connection point. In order to satisfy the assumptions it may be necessary to introduce dummy subsystems and extra (single-input-single-output) summers or connection points.

The jth summer has equations of the form

$$z_{j2} + \ldots + z_{jk_j} = z_{j1}$$

The rth connection point has equations of the form

$$z_{r1} = z_{r2}$$
$$\vdots$$
$$z_{r1} = z_{rk_r}$$

All the equations at the summers and connection points together have the form (32a).

Let the ith block be

$$N_{it}[\mathbf{w}_{1i}, \mathbf{w}_{2i}, \mathbf{z}_{ui}, \mathbf{p}_i, \mathbf{z}_{yi}] = 0$$

Here $\mathbf{w}_{1i} = \mathbf{w}_i$, $\mathbf{w}_{2i} = \dot{\mathbf{w}}_i$ and $\mathbf{p}_i$ are the design parameters. The equations of all the blocks put together have the form (32b). Here $\mathbf{w}_1 = \dot{\mathbf{w}}$ and $\mathbf{w}_2 = \mathbf{w}$ ($\mathbf{w}$ being the vector of dynamical variables of all the blocks), $\mathbf{p}$ is the vector of all the design parameters of all the blocks. We assume that these are all independent of each other.

We now show that the K-adjoint of the original system is constructed by the following algorithm (see Figures 1 and 2).

Algorithm I

(a) Retain the same graph for the signal flow diagram but change overall inputs to the system into overall outputs and vice versa.
(b) Replace summers by connection points and vice versa.
(c) Replace each subsystem represented by a block by its K-adjoint.

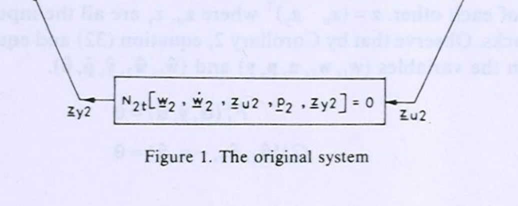

Figure 1. The original system

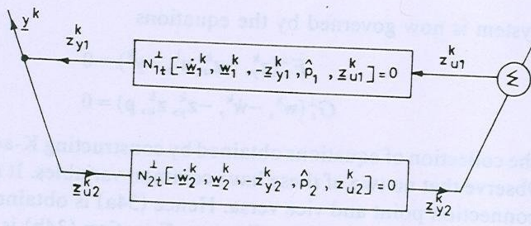

Figure 2. The K-adjoint system

The special case of a system defined through a block diagram where the blocks are either non-dynamic or simple integrators is worth considering. In this case for each integrator we introduce a connection point at the input and output of the integrator (see Figure 3). The K-adjoint of this subsystem is obtained by replacing input by output and vice versa and reversing the integrator. This may be simplified to the integrator in Figure 4.

If the jth non-dynamic block in the original system satisfies $z_{yj} = \mathbf{K}z_{uj}$ as in Figure 5, then the K-adjoint of this block would be $\mathbf{z}^k_{yj} = \mathbf{K}^T\mathbf{z}^k_{uj}$ as in Figure 6.

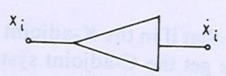

Figure 3. An integrator that is the ith block in the original system

Figure 4. The ith block simplified in the K-adjoint

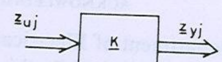

Figure 5. The jth non-dynamic block in the original system

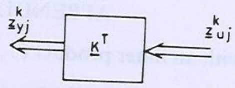

Figure 6. The jth block in the K-adjoint

The algorithm for constructing the K-adjoint when the blocks are either simple integrators or non-dynamic blocks reduces to: Steps (a), (b) as before.

(c) Reverse direction of all integrators. For each non-dynamic block, interchange input lines and output lines.

The justification for algorithm I

Observe that $\boldsymbol{\alpha}$ is a dummy variable. We are therefore free to choose convenient interpretations for it. We put $\boldsymbol{\alpha} = (\mathbf{z}^k_y \quad -\mathbf{z}^k_u)^T$ where $\mathbf{z}^k_y$, $\mathbf{z}^k_u$ are the vectors representing all the outputs of all the blocks and all the inputs to all the blocks, respectively.

The K-adjoint system is now governed by the equations

$$F_t^{\perp}(\mathbf{z}_y^k, -\mathbf{z}_u^k, \mathbf{u}^k, -\mathbf{y}^k) = 0 \tag{34a}$$

$$G_t^{\perp}(\mathbf{w}^k, -\dot{\mathbf{w}}^k, -\mathbf{z}_y^k, \mathbf{z}_u^k, \mathbf{p}) = 0 \tag{34b}$$

Equation (34a) is the collection of equations obtained by constructing K-adjoints of summer and connection point equations. Observe that no two of these have common variables. It may be verified that the K-adjoint of a summer is a connection point and vice versa. Hence (34a) is obtained by interchanging summers and connection points in the original signal flow diagram. Equation (34b) is composed of equations obtained by constructing K-adjoints of individual blocks of the original diagram. Here again no two of these have common variables. Hence (34b) is obtained by replacing each block in the original system by its K-adjoint.

The algorithm for the construction of the C-adjoint is now immediate.

Algorithm II

(a) Retain the same graph for the signal flow diagram but change overall inputs to *negative* overall outputs and overall outputs to overall inputs.
(b) Replace summers by connection points and vice versa.
(c) First replace each subsystem represented by a block by its K-adjoint. Next change the signs of the time derivative terms.

To justify algorithm II observe that if in the K-adjoint system we change the signs of the output variables and the time derivative terms we get the C-adjoint system.

7. CONCLUSION

In this paper we have introduced a new technique for dealing with implicitly defined complementary orthogonal spaces. We have illustrated this technique by applying it to the construction and study of various types of adjoint systems in a unified manner.

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APPENDIX I

We deal with real vector spaces with an inner product $\langle\,.\,,\,.\,\rangle$. We denote the space of all vectors orthogonal to a subspace V of X by V^* and we write $V \overset{\perp}{\oplus} V^* = X$. We state some elementary results from linear algebra, without proof, in Lemma 1.

Lemma 1

Let V_1, V_2 be subspaces of a finite dimensional real vector space X:
(a) $((V_1)^*)^* = V_1$
(b) $(V_1 + V_2)^* = V_1^* \cap V_2^*$
(c) $(V_1 \cap V_2)^* = V_1^* + V_2^*$.

Theorem 4

Let X_{SP} be a finite dimensional real vector space and let $X_{SP} = X_S \overset{\perp}{\oplus} X_P$. Let V_{SP}, V_P be subspaces of X_{SP}, X_P, respectively. Then $[(V_{SP} + V_P) \cap X_S]^* \cap X_S = [(V_{SP}^* + V_P^* \cap X_P) \cap X_S]$.

Proof. $[(V_{SP}+V_P)\cap X_S]^*\cap X_S=[(V_{SP}^*\cap V_P^*)+X_P]\cap X_S$ by the use of Lemma 1.

A vector $\mathbf{f}_S$ belongs to $[(V_{SP}^*\cap V_P^*)+X_P]\cap X_S$ iff $\mathbf{f}_S\in X_S$ and there exists $\mathbf{f}_P\in X_P$ such that $(\mathbf{f}_S+\mathbf{f}_P)\in V_{SP}^*\cap V_P^*$. But a vector $\mathbf{f}_P$ satisfies these conditions only if $\mathbf{f}_P\in V_P^*$. For, consider a vector $\mathbf{q}_P$ in V_P. Then $\langle\mathbf{q}_P,\mathbf{f}_S+\mathbf{f}_P\rangle=\langle\mathbf{q}_P,\mathbf{f}_S\rangle+\langle\mathbf{q}_P,\mathbf{f}_P\rangle=\langle\mathbf{q}_P,\mathbf{f}_P\rangle$. So $\mathbf{f}_S$ belongs to $[(V_{SP}^*\cap V_P^*)+X_P]\cap X_S$ iff $\mathbf{f}_S\in X_S$ and there exists $\mathbf{f}_P\in X_P$ such that $(\mathbf{f}_S+\mathbf{f}_P)\in V_{SP}^*$ and $\mathbf{f}_P\in V_P^*\cap X_P$ i.e. iff $\mathbf{f}_S\in[V_{SP}^*+V_P^*\cap X_P]\cap X_S$. Q.E.D.

Remarks

The above proof goes through unchanged in the infinite dimensional case also if we work with Hilbert spaces and take V_S and V_P to be closed subspaces of X_{SP}.

Theorem 2 is a restatement of Theorem 4. To see this, take X_{SP} to be the space of all real vectors $(\mathbf{x}_S\ \ \mathbf{x}_P)^T$, X_S to be the space of all real vectors $(\mathbf{x}_S\ \ \mathbf{0})^T$ and X_P to be the space of all real vectors $(\mathbf{0}\ \ \mathbf{x}_P)^T$. We need to show that the space of all vectors $(\mathbf{i}_S\ \ \mathbf{0})^T$ consistent with (13) and the space of all vectors $(\mathbf{v}_S\ \ \mathbf{0})^T$ of (14) are complementary orthogonal relative to X_S. Take V_{SP} to be the space of all vectors $(\mathbf{i}_S\ \ \mathbf{i}_P)^T$ satisfying (13a). Then V_{SP}^* is the space of all vectors $(\mathbf{v}_S\ \ \mathbf{v}_P)^T$ satisfying (14a). Take V_P to be the space of all vectors $(\mathbf{0}\ \ \mathbf{i}_P)^T$ relevant to (13). Then $V_P^*\cap X_P$ is the space of all vectors $(\mathbf{0}\ \ \mathbf{v}_P)^T$ relevant to (14).

Now observe that $(V_{SP}+V_P)\cap X_S$ is the space of all vectors $(\mathbf{i}_S\ \ \mathbf{0})^T$ and $(V_{SP}^*+V_P^*\cap X_P)\cap X_S$ is the space of all vectors complementary orthogonal, relative to X_S, to $(V_{SP}+V_P)\cap X_S$ and is also the space of all $(\mathbf{v}_S\ \ 0)^T$.

APPENDIX II. SENSITIVITY CALCULATIONS USING THE C-ADJOINT AND K-ADJOINT

Let the equations of the original dynamical system be

$$F_t(\dot{\mathbf{w}},\mathbf{w},\mathbf{u},\mathbf{p},\mathbf{y})=0$$

The equations of the C-adjoint system would be

$$F_t^{\perp}(\dot{\mathbf{w}}^c,\mathbf{w}^c,\mathbf{y}^c,\hat{\mathbf{p}},\mathbf{u}^c)=0$$

We then have $\langle\dot{\mathbf{w}},\mathbf{w}^c\rangle+\langle\mathbf{w},\dot{\mathbf{w}}^c\rangle+\langle\mathbf{u},\mathbf{y}^c\rangle+\langle\mathbf{y},\mathbf{u}^c\rangle+\langle\mathbf{p},\hat{\mathbf{p}}\rangle=0$, i.e.

$$\frac{\mathrm{d}}{\mathrm{d}t}(\langle\mathbf{w},\mathbf{w}^c\rangle)+\langle\mathbf{u},\mathbf{y}^c\rangle+\langle\mathbf{y},\mathbf{u}^c\rangle+\langle\mathbf{p},\hat{\mathbf{p}}\rangle=0$$

Let us compute the sensitivity of y_j with respect to the design parameter p_k. We put $u=0$, $p_i=0$, $i\neq k$, $u_i^c=0$, $i\neq j$, $p_k=1$, $u_j^c=1$. This is possible since we have assumed that p_i are independent and by Theorem 3 u_i^c can be chosen independently. We then have

$$\frac{\mathrm{d}}{\mathrm{d}t}(\langle\mathbf{w},\mathbf{w}^c\rangle)+y_ju_j^c+\hat{p}_k=0$$

Integrating both sides from t_0 to t

$$\langle\mathbf{w},\mathbf{w}^c\rangle|_{t_0}^{t}+\int_{t_0}^{t}y_j(\sigma)\,\mathrm{d}\sigma+\int_{t_0}^{t}\hat{p}_k(\sigma)\,\mathrm{d}\sigma=0$$

We take $\mathbf{w}(t_0)=0$, $\mathbf{w}^c(t)=0$. This yields

$$\int_{t_0}^{t}y_j(\sigma)\,\mathrm{d}\sigma+\int_{t_0}^{t}\hat{p}_k(\sigma)\,\mathrm{d}\sigma=0$$

Observe that $y_j(\sigma)$ does not depend on t, whereas $\hat{p}_k(\sigma)$ does. Differentiating with respect to t yields

$$y_j(t)=-\left[\hat{p}_k(t)+\int_{t_0}^{t}\frac{\mathrm{d}}{\mathrm{d}t}(\hat{p}_k(\sigma))\,\mathrm{d}\sigma\right]$$

So the sensitivity coefficient for y_j with respect to p_k is

$$-\left[\hat{p}_k(t)+\int_{t_0}^{t}\frac{\mathrm{d}}{\mathrm{d}t}(\hat{p}_k(\sigma))\,\mathrm{d}\sigma\right]$$

As the above derivation shows, in order to use the adjoint system for sensitivity computations it is necessary to be able to choose inputs to the adjoint system *independently*.

The equations of the K-adjoint system would be

$$F_t^{\perp}(\mathbf{w}^k(\tau), -\dot{\mathbf{w}}^k, -\mathbf{y}^k(\tau), \hat{\mathbf{p}}(\tau), \mathbf{u}^k(\tau))=0$$

($\dot{\mathbf{w}}^k$ refers to $\mathrm{d}\mathbf{w}^k/\mathrm{d}\tau$, $t\in[t_0, t_f]$). We then have

$$\langle\dot{\mathbf{w}}(t), \mathbf{w}^k(\tau)\rangle-\langle\mathbf{w}(t), \dot{\mathbf{w}}^k(\tau)\rangle+\langle\mathbf{u}(t), \mathbf{y}^k(\tau)\rangle+\langle\mathbf{y}(t), \mathbf{u}^k(\tau)\rangle+\langle\mathbf{p}(t), \hat{\mathbf{p}}(\tau)\rangle=0$$

In order to interpret $(\langle\dot{\mathbf{w}}, \mathbf{w}^k\rangle-\langle\mathbf{w}, \dot{\mathbf{w}}^k\rangle)$ as a total derivative we take the K-adjoint system to be moving backward in time with respect to the original system. We let $\tau=t_0+t_f-t$ so that

$$\dot{\mathbf{w}}^k=\frac{\mathrm{d}}{\mathrm{d}\tau}(\mathbf{w}^k(\tau))=-\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{w}^k(t_0+t_f-t))$$

Then

$$(\langle\dot{\mathbf{w}}, \mathbf{w}^k\rangle-\langle\mathbf{w}, \dot{\mathbf{w}}^k\rangle)=\frac{\mathrm{d}}{\mathrm{d}t}(\langle\mathbf{w}(t), \mathbf{w}^k(t_0+t_f-t)\rangle)$$

The remainder of the derivation for the expression of sensitivity coefficient is similar to the C-adjoint case and is omitted.

REFERENCES

1. P. Penfield Jr., R. Spence and S. Duinker, 'Tellegen's theorem and electrical networks', *Research Monograph No. 58*, MIT Press, Cambridge, Massachussets and London, England, 1970.
2. A. V. Oppenheim and R. W. Schafer, *Digital Signal Processing*, Prentice-Hall, Englewood Cliffs, N.J., 1975, pp. 173-182.
3. A. Fettweis, 'A general theorem for signal flow networks, with applications', *Arch. Electronik Ubertragungstechnik*, **25**, 557-561 (1971).
4. H. Narayanan, 'On the decomposition of vector spaces', *Research Report*, E. E. Department, I.I.T., Bombay, April 1983.
5. H. Narayanan, 'Topological transformations of electrical networks', U.C. Berkeley Electronics Research Laboratory, *Memorandum No. UCB/ERL M*84/109, December 1984.
6. J. H. Laning Jr. and R. H. Battin, *Random Processes in Automatic Control*, McGraw-Hill, New York, 1956.
7. C. A. Desoer, 'On the description of adjoint networks', *IEEE Transcircuited Systems*, **CAS-22**, 585-587 (1975).
8. R. N. Gadenz and G. C. Temes, 'Efficient hybrid and state space analysis of the adjoint network', *IEEE Trans. Circuit Theory*, **CT-19**, 520-521 (1972).
9. S. W. Director and R. A. Rohrer, 'The generalized adjoint network and network sensitivities', *IEEE Trans. Circuit Theory*, **CT-16**, 318-323 (1969).
10. K. Hoffman and R. Kunze, *Linear Algebra*, Prentice-Hall, Englewood Cliffs, N.J., 1971.
11. P. P. Varaiya, *Notes on Optimization*, Van Nostrand, New York, 1971, pp. 33-38.
12. C. A. Desoer, *Notes for a Second Course on Linear Systems*, Van Nostrand Reinhold, 1970.