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TOPOLOGICAL TRANSFORMATIONS OF ELECTRICAL NETWORKS

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SUMMARY

This paper deals with effects of modifications of network structure that may be studied without reference to the type of devices present in the network. We introduce and make systematic use of the notion of a generalized minor of a vector space. This operation generalizes the usual short and open circuit operations for a graph. Using the generalized minor operation we show how to make the equations of a given network appear to be the 'bordered version' of the equations of some other specified network. We also consider the decomposition of a network into several 'multiports' and a 'port connection diagram', and study the properties of a minimal decomposition (with port connection diagram having a minimum number of edges). In each case we present efficient algorithms wherever appropriate. Although the paper makes use of ideas from elementary matroid theory it is entirely self-contained and requires no more than the knowledge of elementary linear algebra from the reader.

1. INTRODUCTION

In this paper we introduce a generalization of the operations of open and short circuits (called 'generalized minor') and use it to study structural modifications of electrical networks. Such modifications arise, quite naturally, in both practical and theoretical network analysis. In practical network analysis it may be desirable to split the given network into several subnetworks. In theoretical network analysis splitting the original network into multiports of various types is quite common: for instance, the given non-linear network may thought of as a linear network terminated by non-linear elements, or the given dynamic network may be thought of as a resistive network terminated by reactive elements. The network theorist who is faced with such problems handles them intuitively using his experience with the *type* of network being studied. A formal study would, at the least, clarify the fundamental ideas underlying such techniques.

We use the generalized minor operation to solve the following problems:

1. Given a network N_1 on the graph G_1 on a set of edges S, what is the least number of extra variables required to 'convert' it into the network N_2 on the graph G_2 on the same set of edges and with the same device characteristic? The 'conversion' here refers to obtaining a vector space over a larger set which has the coboundary spaces of the graphs of N_1 and N_2 as generalized minors. We can use this technique to obtain the equations of N_1 as 'bordered versions' of the equations of N_2. (The 'thickness' of the border is equal to the number of extra variables). We give efficient algorithms for doing the 'conversion' in general and better algorithms for certain important special cases.
2. Given a network N with graph G on S and a partition $S_1, \ldots, S_n$ of S, how can it be decomposed into multiports on $S_1 \cup P_1, \ldots, S_n \cup P_n$ and a 'port connection diagram' on $\cup P_i$ such that $\cup P_i$ is a minimum? For this problem we give an algorithm which makes the port connection diagram into a graph, while the multiports are 'generalized'. We also show that Kron's diakoptics[1–3] is a special case of decomposition of a network into two multiports.

The results of this paper have arisen from an attempt to understand the essence of Kron's diakoptics using Tutte's algebra of minors as a tool.[4] It turns out, however, that a generalization (from the point of view of vector spaces) of Tutte's operation of 'minors' is necessary in order to carry out this program.

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Whereas Tutte's minors go over to the matroids associated with the vector spaces naturally, the generalization is no longer matroidal in the sense that the matroid of $V_S \leftarrow V_P$ cannot be uniquely reconstructed knowing only the matroids of V_S and V_P. In particular, multiport decomposition cannot be 'inverted' to define a general matroidal multiport composition. However in the case where the generalized multiports and the final generalized network are known to be binary, this is possible.[5] It is also possible in the case where the multiports are 1-ports. This latter 'composition of matroids' has been studied by Bixby.[6]

The organization of the paper is as follows.

Section 2 contains mathematical preliminaries.

Section 3 introduces the generalized minor operation.

Section 4 gives a physical interpretation for the generalized minor operation in terms of ideal transformers.

Section 5 describes the notion of mutual extension of vector spaces and gives algorithms for the construction of minimal extensions. Application to practical network analysis is outlined by considering two special topological transformations.

Section 6 describes decomposition of a vector space into several component spaces (multiports) and coupler space (port connection diagram) and the relation of these ideas to Kron's diakoptics.

2. PRELIMINARIES

We deal with finite sets throughout. If S is a set $|S|$ denotes its cardinality. A function $\mathbf{f}: S \rightarrow F$ is said to be a *vector* on S over the field F. Unless otherwise stated F would be the real field $\mathbb{R}$. If used in equations, $\mathbf{f}$ refers to a row vector and $(\mathbf{f})^T$ refers to a column vector. If T is a set, $\mathbf{f}_T$ would denote a vector on T, 0_T the zero vector on T. We define *restriction* of a vector $\mathbf{f}$ on S to a subset T of S in the usual way and denote it by f/T. *Scalar multiplication* and *addition* of two vectors on the same set are defined in the usual way. However we permit *addition* of two vectors on *different* sets as follows: let $\mathbf{f}$ be on S, $\mathbf{g}$ on T. Then $\mathbf{f} + \mathbf{g}$ is defined on $S \cup T$ and agrees with $\mathbf{f}$ on $S - T$, with $\mathbf{g}$ on $T - S$ and on $S \cap T$ with the usual addition of vectors on the same set. The Σ *notation* would be also used for the extended notion of addition. When $T \cap R = \phi$ we may write $\mathbf{f}_T \oplus \mathbf{f}_R$ to emphasize the fact that T, R are disjoint. When addition of several vectors over disjoint sets is involved we may use $\oplus_{i=1}^{n}$ in place of $\Sigma_{i=1}^{n}$. A collection of vectors on S closed under addition and scalar multiplication is called a vector space on S. $\mathbf{V}_S$ would denote a vector space on S. *Independence* and *rank* of a collection of vectors on S are defined in the usual way. The *rank* of a collection of vectors P is denoted by $r(P)$. We define $\mathbf{V}_P + \mathbf{V}_T$ in the obvious way as the collection of all sums of vectors, one in $\mathbf{V}_P$ and the other in $\mathbf{V}_T$. $\mathbf{V}_P + \mathbf{V}_T$ would therefore be on $P \cup T$. We use $\oplus$ when P, T are disjoint. $\mathbf{V}_1 - \mathbf{V}_2$ refers to the set theoretic difference. If $\mathbf{g}$ and $\mathbf{f}$ are on S, $\langle \mathbf{g}, \mathbf{f} \rangle \equiv \Sigma_{e \epsilon S} \mathbf{g}(e)\mathbf{f}(e)$. If $\langle \mathbf{g}, \mathbf{f} \rangle = 0$ then we say $\mathbf{g}$ and $\mathbf{f}$ are *orthogonal* to each other. $\mathbf{V}^*$ is the vector space of all vectors orthogonal to vectors in $\mathbf{V}$. We would call it the *dual* of $\mathbf{V}$. Let $T \subseteq S$ and let $\mathbf{V}$ be on S. Then

$$\mathbf{V} \times T \equiv \{\mathbf{g}_T = \mathbf{f}/T, \mathbf{f} \in \mathbf{V} \text{ and } \mathbf{f}(e) = 0, e \in S - T\}$$

$\mathbf{V} \cdot T$ is the collection of all restrictions of vectors of $\mathbf{V}$ to T. When $R \subseteq T \subseteq S$ we write $\mathbf{V} \times T \cdot R$ for $(\mathbf{V} \times T) \cdot R$ and $\mathbf{V} \cdot T \times R$ for $(\mathbf{V} \cdot T) \times R$. Such spaces are referred to as minors of $\mathbf{V}$. We say $T \subseteq S$ is a *separator* of $\mathbf{V}$ iff $\mathbf{V} \times T = \mathbf{V} \cdot T$. Observe that we then have $\mathbf{V} = (\mathbf{V} \times T) \oplus (\mathbf{V} \times (S - T))$. If $\mathbf{K}$ is a matrix, $(\mathbf{K})^T$ is its transpose. The symbol $[\mathbf{U}]$ refers to an identity matrix whose order would be clear from the context.

The *generator matrix* $\mathbf{A}_\mathbf{V}$ of a vector space $\mathbf{V}$ is a matrix whose rows form a maximal linearly independent set of vectors of $\mathbf{V}$ (basis of $\mathbf{V}$). We say that $\mathbf{A}_\mathbf{V}$ generates $\mathbf{V}$. Observe that $\mathbf{g} \in \mathbf{V}^*$ iff $(\mathbf{A}_\mathbf{V})(\mathbf{g})^T = 0$. If $\mathbf{A}_\mathbf{V}^1$, $\mathbf{A}_\mathbf{V}^2$ are generator matrices of $\mathbf{V}$ observe that a set of columns of $\mathbf{A}_\mathbf{V}^1$ are linearly independent iff the corresponding set of columns of $\mathbf{A}_\mathbf{V}^2$ are linearly independent. Let $\mathbf{V}$ be a vector space on S. Let $T \subseteq S$. We say that T is a *circuit* of $M(\mathbf{V})$ iff the set of columns of a generator matrix $\mathbf{A}_\mathbf{V}$ of $\mathbf{V}$ corresponding to T are minimal linearly dependent. T is a *bond of* $M(\mathbf{V})$ *iff it is a circuit of* $M(\mathbf{V}^*)$. $(M(\mathbf{V})$

stands for 'matroid associated with $\mathbf{V}$'. However we do not use the idea of a matroid explicitly anywhere in this paper.)

Let G be an oriented graph on the set of edges S. Let $T \subseteq S$. Then $G \cdot T$ is the graph obtained by deleting the edges in $S - T$ and any isolated vertices formed. $G \times T$ is the graph obtained by fusing the end vertices of each edge in $(S - T)$ and deleting it. A *coboundary* of G is a vector on S that satisfies the Kirchhoff voltage (tension) equations of G. A *cycle* of G is a vector on S that satisfies the Kirchhoff current (flow) equations of G. $\mathbf{V}_{\mathrm{cob}}(G)(\mathbf{V}_{\mathrm{cy}}(G))$ denotes the vector space of coboundaries (cycles) of G. Let G_1, G_2 be on disjoint sets of edges S_1, S_2 and vertices V_1, V_2. We construct $G_1 \oplus G_2$ on edges $S_1 \cup S_2$ and vertices $V_1' \cup V_2'$, where V_1', V_2' are disjoint copies of V_1, V_2, by making it agree with G_1 on S_1 and with G_2 on S_2. In the graph $G_1 \oplus G_2$ observe that S_1, S_2 are separators of both the coboundary as well as the cycle spaces. A vector space $\mathbf{V}_S$ is *graphic* (*cographic*) iff it is the coboundary (cycle) space of a graph.

A *generalized electrical network* $\mathbf{N}$ is a triple $(S, \mathbf{V}_S, D_S)$ where S is a finite set of 'edges', $\mathbf{V}_S$ is a vector space on S over $\mathbb{R}$ and the *device characteristic* D_S is a collection of ordered pairs $(\mathbf{v}_S(\cdot), \mathbf{i}_S(\cdot))$, where for all $t \in \mathbb{R}$, $\mathbf{v}_S(t)$ and $\mathbf{i}_S(t)$ are each vectors on S. Usually D_S will be specified informally as $D(\mathbf{v}_S, \mathbf{i}_S) = 0$. A *solution* of $\mathbf{N}$ is a pair $(\mathbf{v}_S(\cdot), \mathbf{i}_S(\cdot))$ belonging to D_S where $\mathbf{v}_S(t) \in \mathbf{V}_S$, $\mathbf{i}_S(t) \in \mathbf{V}_S^*$, for all t. We will refer to $\mathbf{V}_S$ as the coboundary space of $\mathbf{N}$ and $\mathbf{V}_S^*$ as the cycle space of $\mathbf{N}$. A generalized network is *ordinary* when $\mathbf{V}_S$ is the coboundary space of a graph. Let S be partitioned into $S_1, S_2, \ldots, S_n$. We say that $S_1, S_2, \ldots, S_n$ appear *decoupled* in D_S iff there exist collections D_{S_i} of ordered pairs $(\mathbf{v}_{S_i}(\cdot), \mathbf{i}_{S_i}(\cdot))$ such that $(\mathbf{v}_S(\cdot), \mathbf{i}_S(\cdot))$ belong to D_S if $(\mathbf{v}_S(\cdot)/S_i, \mathbf{i}_S(\cdot)/S_i)$ belong to $D_{S_i} (i = 1, 2, \ldots, n)$. A *multiport* is a generalized network with a subset P of edges specified as *ports* such that on the ports there are no device characteristic constraints.

We now present a number of results which we use freely in the rest of the paper. The results are standard in elementary linear algebra and matroid theory.[4,7] Let $\mathbf{V}_S$ be a vector space on S.

Theorem 1

(a) $(\mathbf{V}_S^*)^* = \mathbf{V}_S$

(b) $r(\mathbf{V}_S) + r(\mathbf{V}_S^*) = |S|$

Theorem 2

Let $\mathbf{V}_1$, $\mathbf{V}_2$ be vector spaces on S. Then $\mathbf{V}_1 \subseteq \mathbf{V}_2 \Leftrightarrow V_1^* \supseteq V_2^*$.

Theorem 3

Let $T \subseteq S$. Then

$$r(\mathbf{V}_S \cdot T) + r(\mathbf{V}_S \times (S - T)) = r(\mathbf{V}_S)$$

Theorem 4

Let $T \subseteq S$. Then

(a) $(\mathbf{V}_S \cdot T)^* = \mathbf{V}_S^* \times T$

(b) $(\mathbf{V}_S \times T)^* = \mathbf{V}_S^* \cdot T$

Theorem 5

$$r(\mathbf{V}_1 \cap \mathbf{V}_2) + r(\mathbf{V}_1 + \mathbf{V}_2) = r(\mathbf{V}_1) + r(\mathbf{V}_2)$$

Theorem 6

Let $T \subseteq S$. (a) The circuits of $M(\mathbf{V}_S)$ contained in T and those of $M(\mathbf{V}_S \cdot T)$ are identical. (b) The bonds of $M(\mathbf{V}_S)$ contained in T and those of $M(\mathbf{V}_S \times T)$ are identical.

Theorem 7

Let $T \subseteq S$. (a) If $M(\mathbf{V}_S)$ has circuits contained in T then $r(\mathbf{V}_S \cdot T) < |T|$. (b) If $M(\mathbf{V}_S)$ has bonds contained in T then $r(\mathbf{V}_S \times T) > 0$.

Theorem 8 (Tellegen)

$$(\mathbf{V}_{\text{cob}}(G))^* = \mathbf{V}_{\text{cy}}(G)$$

Theorem 9

(a) $\mathbf{V}_{\text{cob}}(G \times T) = (\mathbf{V}_{\text{cob}}(G)) \times T$
(b) $\mathbf{V}_{\text{cob}}(G \cdot T) = (\mathbf{V}_{\text{cob}}(G)) \cdot T$

Theorem 10

(a) $\mathbf{V}_{\text{cy}}(G \times T) = (\mathbf{V}_{\text{cy}}(G)) \cdot T$
(b) $\mathbf{V}_{\text{cy}}(G \cdot T) = (\mathbf{V}_{\text{cy}}(G)) \times T$

3. THE GENERALIZED MINOR

In this section we introduce an operation on vector spaces which we use throughout this paper.

Definition 1

Let $\mathbf{V}_S$ be a vector space on S. Let $P \subseteq S$. Let $\mathbf{V}_P$ be a vector space on P. Then the *generalized minor* of $\mathbf{V}_S$ with respect to $\mathbf{V}_P$ is denoted $\mathbf{V}_S \leftarrow \mathbf{V}_P$ and is defined as follows:

$$\mathbf{V}_S \leftarrow \mathbf{V}_P \equiv \{\mathbf{f}_{S-P}\text{: there exist } \mathbf{f}_S \in \mathbf{V}_S,\ \mathbf{f}_P \in \mathbf{V}_P \text{ such that } \mathbf{f}_S/P = \mathbf{f}_P,\ \mathbf{f}_S/S - P = \mathbf{f}_{S-P}\}$$

Theorem 11

$\mathbf{V}_{(S-P)}$ is a g-minor of $\mathbf{V}_S$ iff $\mathbf{V}_S \times (S-P) \subseteq \mathbf{V}_{(S-P)} \subseteq \mathbf{V}_S \cdot (S-P)$

Proof. The necessity is trivial. To see that the condition is sufficient select a maximal set of independent vectors $\mathbf{f}_P$ on P such that there exists an $\mathbf{f}_S$ in $\mathbf{V}_S$ whose restriction to P is $\mathbf{f}_P$ and whose restriction to $(S-P)$ belongs to $\mathbf{V}_{(S-P)}$. Let $\mathbf{V}_P$ be the space of vectors generated by these vectors. Clearly $\mathbf{V}_S \leftarrow \mathbf{V}_P \supseteq \mathbf{V}_{(S-P)}$. We will show that $\mathbf{V}_S \leftarrow \mathbf{V}_P \subseteq \mathbf{V}_{(S-P)}$. Let $\mathbf{f}_{(S-P)} \in \mathbf{V}_S \leftarrow \mathbf{V}_P$. Then there exists a vector $\mathbf{f}_S$ in $\mathbf{V}_S$ such that the restriction of $\mathbf{f}_S$ to $(S-P)$ is $\mathbf{f}_{S-P}$ and the restriction of $\mathbf{f}_S$ to P belongs to $\mathbf{V}_P$. By the definition of $\mathbf{V}_P$ it follows that there exists a vector $\mathbf{f}'_S$ of $\mathbf{V}_S$ such that $\mathbf{f}'_S$ and $\mathbf{f}_S$ agree over P and the restriction of $\mathbf{f}'_S$ to $(S-P)$ is in $\mathbf{V}_{(S-P)}$. Observe that the restriction of $\mathbf{f}'_S - \mathbf{f}_S$ to $S-P$ belongs to $\mathbf{V}_S \times (S-P)$. So the restriction of $\mathbf{f}_S$ to $(S-P)$ belongs to $\mathbf{V}_{S-P}$, i.e. $\mathbf{V}_S \leftarrow \mathbf{V}_P \subseteq \mathbf{V}_{(S-P)}$.

Q.E.D.

Theorem 12

$$r(\mathbf{V}_S \leftarrow \mathbf{V}_P) = r(\mathbf{V}_S \times (S-P)) + r((\mathbf{V}_S \cdot P) \cap \mathbf{V}_P) - r((\mathbf{V}_S \times P) \cap \mathbf{V}_P)$$

Proof. Let $\mathbf{V}'_S$ be the space of all vectors $\mathbf{f}_S$ of $\mathbf{V}_S$ whose restrictions belong to $\mathbf{V}_P$. Choose a basis for this vector space by first choosing a basis for all those vectors in $\mathbf{V}'_S$ with all entries zero in P. Extend this by choosing vectors with all entries zero in $(S-P)$ and finally with vectors which have non-zero entries in S as well as in P. It is easy to see that the number of vectors in this basis is

$$r[\mathbf{V}_S \times (S-P)] + r[(\mathbf{V}_S \cdot P) \cap \mathbf{V}_P]$$

Restricting these vectors to $(S - P)$ and omitting the zero vectors yields a basis for $\mathbf{V}_S \leftarrow \mathbf{V}_P$. The number of vectors in the basis of $\mathbf{V}_S'$ whose restriction to S is the zero vector is $r(\mathbf{V}_S \times (S - P))$. The theorem follows.

Q.E.D.

The following lemmas are needed for the proof of Theorem 13. Lemma 1 is a standard result from linear algebra and can be proved by routine use of Theorems 1 and 5. Lemma 2 is merely a restatement of Theorem 3.

Lemma 1

Let $\mathbf{V}_1, \mathbf{V}_2$ be on S. Then $(\mathbf{V}_1 \cap \mathbf{V}_2)^* = \mathbf{V}_1^* + \mathbf{V}_2^*$

Lemma 2

Let $\mathbf{V}_S$ be on S. Let $P \subseteq S$. Then $r(\mathbf{V}_S \cdot P) - r(\mathbf{V}_S \times P) = r(\mathbf{V}_S \cdot (S - P)) - r(\mathbf{V}_S \times (S - P))$

Theorem 13

Let $\mathbf{V}_S$, $\mathbf{V}_P$ be spaces on S, P, respectively, where $P \subseteq S$. Then $(\mathbf{V}_S \leftarrow \mathbf{V}_P)^* = \mathbf{V}_S^* \leftarrow \mathbf{V}_P^*$.

Proof. We will first show that the two spaces are orthogonal to each other and then show that their ranks add up to $|S - P|$. Let $\mathbf{f} \in \mathbf{V}_S \leftarrow \mathbf{V}_P$. Then there exist vectors $\mathbf{f}_S$, $\mathbf{f}_P$ belonging to $\mathbf{V}_S$, $\mathbf{V}_P$, respectively, such that $\mathbf{f}_S/P = \mathbf{f}_P$, $\mathbf{f} = \mathbf{f}_S/_{(S-P)}$. Let $\mathbf{g} \in \mathbf{V}_S^* \leftarrow \mathbf{V}_P^*$. Then there exist vectors $\mathbf{g}_S$, $\mathbf{g}_P$ belonging to $\mathbf{V}_S^*$, $\mathbf{V}_P^*$ such that $\mathbf{g}_S/_P = \mathbf{g}_P$, $\mathbf{g} = \mathbf{g}_S/_{(S-P)}$. We now have $\langle \mathbf{f}, \mathbf{g} \rangle = -\langle \mathbf{f}_S/_P, \mathbf{g}_S/_P \rangle = -\langle \mathbf{f}_P, \mathbf{g}_P \rangle = 0$. Next consider $r(\mathbf{V}_S \leftarrow \mathbf{V}_P) + r(\mathbf{V}_S^* \leftarrow \mathbf{V}_P^*)$. By Theorem 12,

$$r(\mathbf{V}_S \leftarrow \mathbf{V}_P) = r(\mathbf{V}_S \times (S - P)) + r((\mathbf{V}_S \cdot P) \cap \mathbf{V}_P) - r((\mathbf{V}_S \times P) \cap \mathbf{V}_P)$$

$$r(\mathbf{V}_S^* \leftarrow \mathbf{V}_P^*) = r(\mathbf{V}_S^* \times (S - P)) + r((\mathbf{V}_S^* \cdot P) \cap \mathbf{V}_P^*) - r((\mathbf{V}_S^* \times P) \cap \mathbf{V}_P^*)$$

By application of Lemma 1 and Theorems 1 and 4 we have

$$r((\mathbf{V}_S^* \cdot P) \cap \mathbf{V}_P^*) = |P| - r((\mathbf{V}_S \times P) + \mathbf{V}_P)$$

and

$$r((\mathbf{V}_S^* \times P) \cap \mathbf{V}_P^*) = |P| - r((\mathbf{V}_S \cdot P) + \mathbf{V}_P)$$

So

$$\begin{aligned} r(\mathbf{V}_S \leftarrow \mathbf{V}_P) + r(\mathbf{V}_S^* \leftarrow \mathbf{V}_P^*) &= r(\mathbf{V}_S \times (S - P)) + r(\mathbf{V}_S^* \times (S - P)) \\ &\quad + r((\mathbf{V}_S \cdot P) \cap \mathbf{V}_P) + r((\mathbf{V}_S \cdot P) + \mathbf{V}_P) \\ &\quad - |P| - r((\mathbf{V}_S \times P) \cap \mathbf{V}_P) - r((\mathbf{V}_S \times P) + \mathbf{V}_P) + |P| \\ &= r(\mathbf{V}_S \times (S - P)) + r(\mathbf{V}_S^* \times (S - P)) + r(\mathbf{V}_S \cdot P) + r(\mathbf{V}_P) - r(\mathbf{V}_S \times P) \\ &\quad - r(\mathbf{V}_P) \end{aligned}$$

by Theorem 5

$$\begin{aligned} &= |S - P| - [r(\mathbf{V}_S \cdot (S - P)) - r(\mathbf{V}_S \times (S - P))] + r(\mathbf{V}_S \cdot P) - r(\mathbf{V}_S \times P) \\ &= |S - P| \end{aligned}$$

by Lemma 2. This proves the theorem.

4. IDEAL TRANSFORMERS

In this section we dwell briefly on the concept of an ideal transformer.[8] Using ideal transformers we give a simple physical interpretation for the notion of a generalized minor. Generalized networks may be thought of as being constructed by plugging two-terminal electrical devices to the ports of ideal transformers. The g-minor operation is therefore natural for generalized networks. Ordinary networks obtained by connecting two-terminal electrical devices according to a graph are a special case of generalized networks. The g-minor operation is therefore applicable to ordinary networks also. We show in this section that the g-minor operation generalizes the short circuit and open circuit operations.

Definition 2

An ideal transformer $\mathbf{I}_S$ on S is a 'black box' with S as its set of ports, and satisfying the following condition: let $\mathbf{V}_S^v$, $\mathbf{V}_S^i$ be the sets of all voltage vectors and current vectors that can exist at S. Then $\mathbf{V}_S^v = (\mathbf{V}_S^i)^*$.

Since an ideal transformer is fully characterized by the vector space $\mathbf{V}_S^v$ on S we will identify $\mathbf{I}_S$ with the pair $(S, \mathbf{V}_S^v)$. We will refer to $\mathbf{V}_S^v$ as the space of coboundaries of $\mathbf{I}_S$ and $(\mathbf{V}_S^v)^*$ as the space of cycles of $\mathbf{I}_S$.

Definition 3

Let $\mathbf{I}_S = (S, \mathbf{V})$. Then the *dual* of $\mathbf{I}_S$, denoted $\mathbf{I}_S^*$, is the pair $(S, \mathbf{V}^*)$.

Observe that in our notation the dual of a non-planar graph would be an ideal transformer.

Definition 4

Let $\mathbf{I}_{S_1} = (S, \mathbf{V}_1), \ldots, \mathbf{I}_{S_n} = (S_n, \mathbf{V}_n)$ be ideal transformers on pairwise disjoint sets $S_1, \ldots, S_n$. Then their direct sum $\mathbf{I}_{S_1} \oplus \cdots \oplus \mathbf{I}_{S_n}$ is the ideal transformer

$$\left(\bigcup_{i=1}^{n} S_i, \mathbf{V}_1 \oplus \cdots \oplus \mathbf{V}_n\right)$$

The following simple lemma is useful. Its routine proof is omitted.

Lemma 3

$$(\mathbf{V}_{S_1} \oplus \cdots \oplus \mathbf{V}_{S_n})^* = (\mathbf{V}_{S_1}^* \oplus \cdots \oplus \mathbf{V}_{S_n}^*)$$

An immediate consequence is

Theorem 14

$$(\mathbf{I}_{S_1} \oplus \cdots \oplus \mathbf{I}_{S_n})^* = (\mathbf{I}_{S_1}^* \oplus \cdots \oplus \mathbf{I}_{S_n}^*)$$

Definition 5

Let S be a set and let $P \subseteq S$. Let $\mathbf{I}_S = (S, \mathbf{V}_S)$, $\mathbf{I}_P = (P, \mathbf{V}_P)$. Then the g-minor of $\mathbf{I}_S$ with respect to $\mathbf{I}_P$ is denoted $\mathbf{I}_S \leftarrow \mathbf{I}_P$ and is defined by $\mathbf{I}_S \leftarrow \mathbf{I}_P = ((S - P), \mathbf{V}_S \leftarrow \mathbf{V}_P)$.

The g-minor operation gives us a simple way of deriving new ideal transformers from old.

Example 1

Let $\mathbf{I}_S = (S, \mathbf{V}_S)$. Then $\mathbf{I}_{(S-P)}^1 \equiv ((S - P), \mathbf{V}_S \times (S - P)) = \mathbf{I}_S \leftarrow \mathbf{I}_P^1$, where $\mathbf{I}_P^1 = (P, 0_P)$ and $\mathbf{I}_{(S-P)}^2 \equiv ((S - P), \mathbf{V}_S \cdot (S - P)) = \mathbf{I}_S \leftarrow \mathbf{I}_P^2$, where $\mathbf{I}_P^2 = (P, \mathbb{R}^P)$, where $\mathbb{R}^P$ is the space of all vectors on P over $\mathbb{R}$. By Theorem 9, $\mathbf{V}_{\text{cob}}(G \times (S - P)) = (\mathbf{V}_{\text{cob}}(G)) \times (S - P)$ and $\mathbf{V}_{\text{cob}}(G \cdot (S - P)) = (\mathbf{V}_{\text{cob}}(G)) \cdot (S - P)$. Hence, for graphs, the operations of short circuiting edges and of open circuiting edges can be achieved by the g-minor operation. Let $\mathbf{I}_{(S-P)}^3 = ((S - P), \mathbf{V}_S \times (S - P_1) \cdot (S - P)), P_1 \subseteq P$. Then $\mathbf{I}_{(S-P)}^3 = \mathbf{I}_S \leftarrow \mathbf{I}_P^3$, where $\mathbf{I}_P^3 = (P, \{0_{P_1}\} \oplus \mathbb{R}^{(P-P_1)})$. The g-minor operation was introduced to generalize the ordinary minor operations. The above illustrate this fact.

Theorem 13 permits us to give a simple physical interpretation of the operation of g-minor. Let $\mathbf{I}_S$, $\mathbf{I}_P$ be ideal transformers on S, P with $P \subseteq S$. Let $\mathbf{I}_S = (S, \mathbf{V}_S)$, $\mathbf{I}_P = (P, \mathbf{V}_P)$. Let us identify the ports P in both transformers as in Figure 1. Consider the current and voltage constraints on the exposed ports $(S - P)$. A

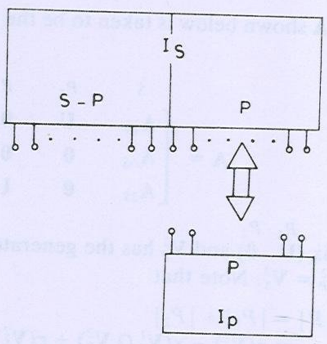

Figure 1. $I_S \leftarrow I_P$

vector $\mathbf{f}_{S-P}$ can be a voltage (current) vector on the exposed ports iff there exist voltage (current) vectors $\mathbf{f}_S \in \mathbf{V}_S(\mathbf{V}_S^*)$, $\mathbf{f}_P \in \mathbf{V}_P(\mathbf{V}_P^*)$ such that $\mathbf{f}_S/P = \mathbf{f}_P$, i.e. iff $\mathbf{f}_{S-P} \in \mathbf{V}_S \leftarrow \mathbf{V}_P(\mathbf{V}_S^* \leftarrow \mathbf{V}_P^*)$. By Theorem 13 $(\mathbf{V}_S \leftarrow \mathbf{V}_P)^* = (\mathbf{V}_S^* \leftarrow \mathbf{V}_P^*)$. It follows that on the exposed ports $(S - P)$ we have an ideal transformer. Thus if we 'plug' the ports P in $\mathbf{I}_S$ by $\mathbf{I}_P$ the ideal transformer $\mathbf{I}_S \leftarrow \mathbf{I}_P$ results.

5. EXTENSION OF VECTOR SPACES

In this section we introduce the 'inverse' operation of g-minor, namely 'extension' of a vector space. While analysing a given network we can use the topology of a different network by constructing a mutual extension of the coboundary spaces of the two networks. These ideas are detailed in subsection 5.1 and exemplified in subsection 5.2.

5.1. *Definitions and theorems on extension*

Definition 6

Let $\mathbf{V}_S$ and $\mathbf{V}_{SP}$ be vector spaces on S and $S \cup P$, respectively. We say that $\mathbf{V}_{SP}$ is an *extension* of $\mathbf{V}_S$ iff $\mathbf{V}_S$ is a g-minor of $\mathbf{V}_{SP}$.

Definition 7

Let $\mathbf{V}_S^1, \ldots, \mathbf{V}_S^n$ be vector spaces on S. $\mathbf{V}_{SP}$ is a minimal extension of $\{\mathbf{V}_S^1, \ldots, \mathbf{V}_S^n\}$ iff $\mathbf{V}_{SP}$ is an extension of $\mathbf{V}_S^i$ $(i = 1, \ldots, n)$ and if $\mathbf{V}_{SP'}^1$ is any other extension of $\mathbf{V}_S^i$ $(i = 1, \ldots, n)$ then $|P| \leq |P'|$.

Theorem 15

Let $\mathbf{V}_S^1$, $\mathbf{V}_S^2$ be vector spaces on S. Let $\mathbf{V}_{SP}$ on $S \cup P$ be an extension of $\mathbf{V}_S^1$ and $\mathbf{V}_S^2$. $\mathbf{V}_{SP}$ is a minimal extension of $\mathbf{V}_S^1$ and $\mathbf{V}_S^2$ iff

$$|P| = r(\mathbf{V}_S^1 + \mathbf{V}_S^2) - r(\mathbf{V}_S^1 \cap \mathbf{V}_S^2)$$

Proof. Let $\mathbf{V}_{SP}$ be an extension of $\mathbf{V}_S^1$, $\mathbf{V}_S^2$. Then $\mathbf{V}_{SP} \cdot S \supseteq \mathbf{V}_S^1$ and $\mathbf{V}_{SP} \cdot S \supseteq \mathbf{V}_S^2$ by Theorem 11. Hence $\mathbf{V}_{SP} \cdot S \supseteq \mathbf{V}_S^1 + \mathbf{V}_S^2$. Also $\mathbf{V}_{SP} \times S \subseteq \mathbf{V}_S^1$ and $\mathbf{V}_{SP} \times S \subseteq \mathbf{V}_S^2$, by Theorem 11. Hence $\mathbf{V}_{SP}^* \cdot S \supseteq (\mathbf{V}_S^1)^*$ and $\mathbf{V}_{SP}^* \cdot S \supseteq (\mathbf{V}_S^2)^*$ by Theorems 4 and 2. Hence $\mathbf{V}_{SP}^* \cdot S \supseteq (\mathbf{V}_S^1)^* + (\mathbf{V}_S^2)^*$ and hence $\mathbf{V}_{SP} \times S \subseteq \mathbf{V}_S^1 \cap \mathbf{V}_S^2$ by Theorems 4 and 2. By Lemma 2, $r(\mathbf{V}_{SP} \cdot S) - r(\mathbf{V}_{SP} \times S) = r(\mathbf{V}_{SP} \cdot P) - r(\mathbf{V}_{SP} \times P)$. Hence $r(\mathbf{V}_{SP} \cdot P) - r(\mathbf{V}_{SP} \times P) \geq r(\mathbf{V}_S^1 + \mathbf{V}_S^2) - r(\mathbf{V}_S^1 \cap \mathbf{V}_S^2)$. Since $0 \leq r(\mathbf{V}_{SP} \times P) \leq r(\mathbf{V}_{SP} \cdot P) \leq |P|$, it follows that if $|P| = r(\mathbf{V}_S^1 + \mathbf{V}_S^2) - r(\mathbf{V}_S^1 \cap \mathbf{V}_S^2)$, $\mathbf{V}_{SP}$ is a minimal extension of $\mathbf{V}_S^1$ and $\mathbf{V}_S^2$. We now show how to construct a minimal extension of given spaces $\mathbf{V}_S^1$ and $\mathbf{V}_S^2$. Construct a basis $B_\cap$ of $\mathbf{V}_S^1 \cap \mathbf{V}_S^2$. Extend it to a basis $\mathbf{B}_1$ of $\mathbf{V}_S^1$ and a basis $\mathbf{B}_2$ of $\mathbf{V}_S^2$. Clearly $\mathbf{B}_1 \cup \mathbf{B}_2$ is a basis of $\mathbf{V}_S^1 + \mathbf{V}_S^2$. Let $\mathbf{B}_\cap$, $\mathbf{B}_1 - \mathbf{B}_\cap$, $\mathbf{B}_2 - \mathbf{B}_\cap$ form the sets of row vectors of $\mathbf{A}_{\cap S}$, $\mathbf{A}_{1S}$,

$\mathbf{A}_{2S}$, respectively. The matrix $\mathbf{A}$ shown below is taken to be the generator matrix for $\mathbf{V}_{SP}$. (Here $P_1 \cup P_2 = P$.)

$$\mathbf{A} = \begin{array}{c} \\ \\ \\ \end{array}\begin{array}{ccc} S & P_1 & P_2 \\ \left[\begin{array}{c} \mathbf{A}_{1S} \\ \mathbf{A}_{\cap S} \\ \mathbf{A}_{2S} \end{array}\right. & \begin{array}{c} \mathbf{U} \\ \mathbf{0} \\ \mathbf{0} \end{array} & \left.\begin{array}{c} \mathbf{0} \\ \mathbf{0} \\ \mathbf{U} \end{array}\right] \end{array}$$

If $\mathbf{V}_P^1$ has the generator matrix $(\overset{P_1}{\mathbf{U}} \quad \overset{P_2}{\mathbf{0}})$ and $\mathbf{V}_P^2$ has the generator matrix $(\overset{P_1}{\mathbf{0}} \quad \overset{P_2}{\mathbf{U}})$ then it is easy to see that $\mathbf{V}_{SP} \leftarrow \mathbf{V}_P^1 = \mathbf{V}_S^1$ and $\mathbf{V}_{SP} \leftarrow \mathbf{V}_P^2 = \mathbf{V}_S^2$. Note that

$$\begin{aligned} |P| &= |P_1| + |P_2| \\ &= r(\mathbf{V}_S^1) - r(\mathbf{V}_S^1 \cap \mathbf{V}_S^2) + r(\mathbf{V}_S^2) - r(\mathbf{V}_S^1 \cap \mathbf{V}_S^2) \\ &= r(\mathbf{V}_S^1 + \mathbf{V}_S^2) - r(\mathbf{V}_S^1 \cap \mathbf{V}_S^2) \end{aligned}$$

by Theorem 5. It follows that $\mathbf{V}_{SP}$ is a minimal extension of $\mathbf{V}_S^1$ and $\mathbf{V}_S^2$.

Q.E.D.

In Theorem 15 observe that we are able to obtain $\mathbf{V}_S^1$ and $\mathbf{V}_S^2$ as ordinary minors (as opposed to g-minors) of $\mathbf{V}_{SP}$ since $\mathbf{V}_S^1 = \mathbf{V}_{SP} \cdot (S \cup P_2) \times S$ and $\mathbf{V}_S^2 = \mathbf{V}_{SP} \cdot (S \cup P_1) \times S$. If we have to construct a minimal extension of $\{\mathbf{V}_S^1 \cdots \mathbf{V}_S^n\}$ when $n > 2$, ordinary minors would be inadequate. By using the ideas of the first half of the proof of Theorem 15 we can show

$$|P| \geq r\left(\sum_{i=1}^{n} \mathbf{V}_S^i\right) - r(\mathbf{V}_S^1 \cap \cdots \cap \mathbf{V}_S^n)$$

The following matrix $\mathbf{A}$ can be taken as the generator matrix of $\mathbf{V}_{SP}$:

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_{+S} & \mathbf{U} \\ \mathbf{A}_{\cap S} & \mathbf{0} \end{bmatrix}$$

(The rows of $\mathbf{A}_{\cap S}$ form a basis for $(\mathbf{V}_S^1 \cap \cdots \cap \mathbf{V}_S^n)$, whereas the rows of

$$\begin{bmatrix} \mathbf{A}_{+S} \\ \mathbf{A}_{\cap S} \end{bmatrix}$$

form a basis for $\Sigma_{i=1}^n \mathbf{V}_S^i$.) Let $\mathbf{V}_S^i$ be generated by the matrix

$$[\mathbf{K}_+ \, \mathbf{K}_\cap] \quad \begin{bmatrix} \mathbf{A}_{+S} \\ \mathbf{A}_{\cap S} \end{bmatrix}$$

Let $\mathbf{V}_P^i$ be the space generated by the rows of $[\mathbf{K}_+]$. Clearly

$$\mathbf{V}_{SP} \leftarrow \mathbf{V}_P^i = \mathbf{V}_S^i$$

We summarize these results in Theorem 16 below:

Theorem 16

Let $\mathbf{V}_{SP}$ on $S \cup P$ be an extension of $\{\mathbf{V}_S^1 \cdots \mathbf{V}_S^n\}$. It is a minimal extension of $\{\mathbf{V}_S^1 \cdots \mathbf{V}_S^n\}$ iff $|P| = r(\Sigma_{i=1}^n \mathbf{V}_S^i) - r(\cap_{i=1}^n \mathbf{V}_S^i)$.

We next prove a simple but useful result.

Theorem 17

$\mathbf{V}_{SP}$ is a minimal extension of $\{\mathbf{V}_S^1 \cdots \mathbf{V}_S^n\}$ iff $\mathbf{V}_{SP}^*$ is a minimal extension of $\{(\mathbf{V}_S^1)^*, \cdots (\mathbf{V}_S^n)^*\}$.

Proof. By Theorem 13 $\mathbf{V}_S^i$ is a g-minor of $\mathbf{V}_{SP}$ iff $(\mathbf{V}_S^i)^*$ is a g-minor of $\mathbf{V}_{SP}^*$. We next observe that

$$r\left(\sum_{i=1}^{n} \mathbf{V}_S^i\right) = |S| - r\left(\bigcap_{i=1}^{n} (\mathbf{V}_S^i)^*\right)$$

$$r\left(\bigcap_{i=1}^{n} \mathbf{V}_S^i\right) = |S| - r\left(\sum_{i=1}^{n} (\mathbf{V}_S^i)^*\right)$$

by Theorem 1 and Lemma 1. Hence

$$|P| = r\left(\sum_{i=1}^{n} \mathbf{V}_S^i\right) - r\left(\bigcap_{i=1}^{n} \mathbf{V}_S^i\right)$$

iff

$$|P| = r\left(\sum_{i=1}^{n} (\mathbf{V}_S^i)^*\right) - r\left(\bigcap_{i=1}^{n} (\mathbf{V}_S^i)^*\right)$$

Q.E.D.

5.2. *Application to network analysis*

In this subsection we show the relevance of the notion of extension of vector spaces to network analysis. We will show that if we are allowed to increase the number of variables it is possible to solve a given network by using the topology of a different network. These ideas do not depend upon the types of devices present in the network.

Suppose we have to solve a network $\mathbf{N}_1$ on the set of edges S, i.e. solve

$$\left.\begin{array}{l} (\mathbf{A}_{1S})\mathbf{i}_S^T = 0 \\ (\mathbf{B}_{1S})\mathbf{v}_S^T = 0 \text{ or } (\mathbf{A}_{1S}^T)\mathbf{e}_n^T = \mathbf{v}_S^T \\ \mathbf{D}(\mathbf{v}_S, \mathbf{i}_S) = 0 \end{array}\right\} \tag{1}$$

We wish to use the topology of a different network, say $\mathbf{N}_2$, on S with reduced incidence matrix $\mathbf{A}_{2S}$ but the same device characteristic $\mathbf{D}(\mathbf{v}_S, \mathbf{i}_S) = 0$. Let $\mathbf{V}_S^1$, $\mathbf{V}_S^2$ be the spaces generated by the matrices $\mathbf{A}_{1S}$, $\mathbf{A}_{2S}$, respectively. We can then construct a generator matrix

$$\begin{bmatrix} \mathbf{A}'_{1S} \\ \mathbf{A}'_{\cap S} \\ \mathbf{A}'_{2S} \end{bmatrix}$$

for the space $\mathbf{V}_S^1 + \mathbf{V}_S^2$ such that

$$\begin{bmatrix} \mathbf{A}'_{1S} \\ \mathbf{A}'_{\cap S} \end{bmatrix}, \begin{bmatrix} \mathbf{A}'_{\cap S} \\ \mathbf{A}'_{2S} \end{bmatrix}$$

are generator matrices for $\mathbf{V}_S^1$, $\mathbf{V}_S^2$, respectively. We next choose the following matrix $\mathbf{A}'_{SP}$ as a generator matrix for $\mathbf{V}_{SP}$:

$$\mathbf{A}'_{SP} = \begin{bmatrix} \mathbf{A}'_{1S} & \mathbf{U} & \mathbf{0} \\ \mathbf{A}'_{\cap S} & \mathbf{0} & \mathbf{0} \\ \mathbf{A}'_{2S} & \mathbf{0} & \mathbf{U} \end{bmatrix}$$

Here rows of $\mathbf{A}'_{1S}$, $\mathbf{A}'_{2S}$ may be taken from rows of $\mathbf{A}_1$, $\mathbf{A}_2$. Then $\mathbf{V}_{SP} \leftarrow \mathbf{V}_P^1 = \mathbf{V}_S^1$ and $(\mathbf{V}_{SP} \leftarrow \mathbf{V}_P^2) = \mathbf{V}_S^2$ where $\mathbf{V}_P^1$, $\mathbf{V}_P^2$ are generated by $[\overset{P_1}{\mathbf{U}} \quad \overset{P_2}{\mathbf{0}}]$, $[\overset{P_1}{\mathbf{0}} \quad \overset{P_2}{\mathbf{U}}]$, respectively. By a suitable row transformation we can choose $\mathbf{A}_{SP}$ as a generator matrix for $\mathbf{V}_{\mathrm{SP}}$, where

$$\mathbf{A}_{SP} = \begin{bmatrix} \mathbf{U} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{K}_{22} & \mathbf{K}_{23} \end{bmatrix} \begin{bmatrix} \mathbf{A}'_{1S} & \mathbf{U} & \mathbf{0} \\ \mathbf{A}'_{\cap S} & \mathbf{0} & \mathbf{0} \\ \mathbf{A}'_{2S} & \mathbf{0} & \mathbf{U} \end{bmatrix} = \begin{bmatrix} \mathbf{A}'_{1S} & \mathbf{U} & \mathbf{0} \\ \mathbf{A}_{2S} & \mathbf{0} & \mathbf{K}_{23} \end{bmatrix}$$

Equations (1) may now be rewritten as

$$\begin{bmatrix} \mathbf{A}'_{1S} & \mathbf{U} & \mathbf{0} \\ \mathbf{A}_{2S} & \mathbf{0} & \mathbf{K}_{23} \end{bmatrix} \begin{bmatrix} \mathbf{i}_S^T \\ \mathbf{i}_{P_1}^T \\ \mathbf{i}_{P_2}^T \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{2a}$$

$$[\mathbf{U} \quad \mathbf{0}] \begin{bmatrix} \mathbf{i}_{P_1}^T \\ \mathbf{i}_{P_2}^T \end{bmatrix} = 0 \tag{2b}$$

$$\mathbf{D}(\mathbf{v}_S, \mathbf{i}_S) = 0 \tag{2c}$$

$$\begin{bmatrix} (\mathbf{A}'_{1S})^T & \mathbf{A}_{2S}^T \\ \mathbf{U} & \mathbf{0} \\ \mathbf{0} & \mathbf{K}_{23}^T \end{bmatrix} \begin{bmatrix} \mathbf{e}_1^T \\ \mathbf{e}_2^T \end{bmatrix} = \begin{bmatrix} \mathbf{v}_S^T \\ \mathbf{v}_{P_1}^T \\ \mathbf{v}_{P_2}^T \end{bmatrix} \tag{2d}$$

$$[\mathbf{0} \quad \mathbf{U}] \begin{bmatrix} \mathbf{v}_{P_1}^T \\ \mathbf{v}_{P_2}^T \end{bmatrix} = 0 \quad \text{or} \quad (\mathbf{K}_{23}^T)\mathbf{e}_2^T = 0 \tag{2e}$$

Equations (2a) and (2b) follow from $\mathbf{V}_S^1 = \mathbf{V}_{SP} \leftarrow \mathbf{V}_P^1$. Equations (2d) and (2e) follow from $(\mathbf{V}_S^1)^* = \mathbf{V}_{SP}^* \leftarrow (\mathbf{V}_P^1)^*$. Equations (2) may be rewritten as

$$(\mathbf{A}_{2S})\mathbf{i}_S^T = -(\mathbf{K}_{23})\mathbf{i}_{P_2}^T \tag{3a}$$

$$-\mathbf{v}_S^T + (\mathbf{A}_{2S}^T)\mathbf{e}_2^T = -(\mathbf{A}_{1S})^T \mathbf{v}_{P_1}^T \tag{3b}$$

$$\mathbf{D}(\mathbf{v}_S, \mathbf{i}_S) = 0 \tag{3c}$$

$$(\mathbf{K}_{23}^T)\mathbf{e}_2^T = 0 \tag{3d}$$

$$\mathbf{A}'_{1S}\mathbf{i}_S^T = 0 \tag{3e}$$

Equations (3) are equivalent to equations (2) whatever be the device characteristic $\mathbf{D}(\mathbf{v}_S, \mathbf{i}_S) = 0$. However, when networks $\mathbf{N}_1$ and $\mathbf{N}_2$ have unique solutions the following convenient procedure may be adopted for solving $\mathbf{N}_1$.

Observe that equations (3a)–(3c) would reduce to the equations of $\mathbf{N}_2$ if the right side were zero. Let us for simplicity suppose that (3c) has the form

$$\mathbf{E}\mathbf{i}_S^T + \mathbf{F}\mathbf{v}_S^T = \mathbf{s} \tag{4}$$

In order to solve $\mathbf{N}_1$ we could first solve for $\mathbf{i}_S$, $\mathbf{e}_2$, $\mathbf{v}_S$ in terms of $\mathbf{i}_{P_2}$, $\mathbf{v}_{P_1}$, and $\mathbf{s}$. In the linear steady-state case this is equivalent to solving $\mathbf{N}_2$, $|P| + 1$ times. The equations (3d) and (3e) can then be converted to an equation involving $\mathbf{i}_{P_2}$, $\mathbf{v}_{P_1}$ and $\mathbf{s}$. Solving this equation would yield $\mathbf{i}_{P_2}$, $\mathbf{v}_{P_1}$ and $\mathbf{s}$. Back substitution would yield $\mathbf{i}_S$ and $\mathbf{v}_S$. A slight modification of this technique will permit us to handle non-linear, dynamic networks also. It is clear from the construction that $\mathbf{K}_{23}$ and $(\mathbf{A}'_{1S})^T$ have linearly independent columns. It follows that by addition of a suitable subset of rows of (3a) to (3d) and a suitable subset of (3b) to (3e) we obtain the following equations, with the coefficient matrix on the right hand side being non-singular.

$$\begin{bmatrix} \mathbf{H}_{11} & \mathbf{H}_{12} & \mathbf{0} \\ \mathbf{H}_{21} & \mathbf{H}_{22} & \mathbf{H}_{23} \end{bmatrix} \begin{bmatrix} \mathbf{i}_S^T \\ \mathbf{e}_2^T \\ \mathbf{v}_S^T \end{bmatrix} = \begin{bmatrix} \mathbf{H}_{14} & \mathbf{0} \\ \mathbf{0} & \mathbf{H}_{25} \end{bmatrix} \begin{bmatrix} \mathbf{i}_{P_2}^T \\ \mathbf{v}_{P_1}^T \end{bmatrix} \tag{5}$$

Assume that we know $\mathbf{i}_{P2}^k(t_1)$, $\mathbf{v}_{P1}^k(t_1)$ and $\mathbf{s}(t_1)$ (where $\mathbf{s}$ is the source vector). We then solve the non-linear resistive network $\mathbf{N}_2^R(t_1)$ (obtained by using a non-dynamic approximation of $\mathbf{D}(\mathbf{v}_S, \mathbf{i}_S) = 0$ at t_1) iteratively in terms of $\mathbf{i}_{P2}^k(t_1)$, $\mathbf{v}_{P1}^k(t_1)$, $\mathbf{s}(t_1)$, and obtain $\mathbf{i}_S^k(t_1)$, $\mathbf{v}_S^k(t_1)$, $\mathbf{e}_2^k(t_1)$, and thence using (5) obtain $\mathbf{i}_{P_2}^{(k+1)}(t_1)$, $\mathbf{v}_{P_1}^{k+1}(t_1)$. The procedure can be repeated at $(t_1 + T)$ using $\mathbf{i}_{P2}^{\infty}(t_1)$ as $\mathbf{i}_{P2}^0(t_1 + T)$ and $\mathbf{v}_{P1}^{\infty}(t_1)$ as $\mathbf{v}_{P_1}^0(t_1 + T)$.

5.3. Two special topological transformations

The derivation of the preceding subsection glosses over practically important details such as the construction of $\mathbf{A}'_{1S}$, $\mathbf{A}'_{2S}$, $\mathbf{K}_{23}$ and the size of the border $|P|$. Our aim there has been merely to show that such a procedure is possible rather than that it is efficient. The method can compete with other methods only if the required matrices can be computed efficiently, $|P|$ is small and $\mathbf{N}_2$ has a very 'desirable' structure. We next present two cases where $\mathbf{N}_2$ has some specified sets $S_1, \ldots, S_n$ as separators. We assume that these sets have been chosen in such a way that $|P|$ would be small, i.e. we will assume that in the original network $S_1, \ldots, S_n$ are 'loosely" connected. (This problem can only be handled heuristically). In the case presented the required matrices can be computed efficiently and we present algorithms for doing so.

Case I. New graph $\equiv (G \times S_1) \oplus \cdots \oplus (G \times S_n)$

Let $\mathbf{N}_1 = (S, \mathbf{V}_S, \mathbf{D}_S)$ and let $S_1(i = 1, \ldots, n)$ be a partition of S. We will assume that $\mathbf{V}_S$ is the coboundary space of a graph G. Choose $\mathbf{N}_2 = (S, \oplus_{i=1}^{n} \mathbf{V}_S \times S_i, \mathbf{D}_S)$. In this case $\mathbf{A}'_{\cap S}$ would simply be the reduced incidence matrix of $(G \times S_1 \oplus \cdots \oplus G \times S_n)$. $\mathbf{A}'_{2S}$ would have no rows. $\mathbf{K}_{23}$ would have no columns. The following algorithm describes how to construct $\mathbf{A}'_{1S}$. In this case $P_1 = P$ and $|P| = r(G) - \sum_{i=1}^{n} r(G \times S_i)$. Observe that $\mathbf{N}_2$ is easier to solve if in the device characteristic $\mathbf{D}(\mathbf{v}_S, \mathbf{i}_S) = 0$, the sets S_i, $i = 1, \ldots, n$ appear decoupled. Solving $\mathbf{N}_2$ would then be equivalent to solving n smaller networks.

Algorithm I. To construct $\mathbf{A}'_{1S}$ *when* $\mathbf{N}_2 = (S, \oplus_{i=1}^{n} \mathbf{V}_S \times S_i, \mathbf{D}_S)$

Step I. Select trees t_i of graphs $G \times S_i (i = 1, \ldots, n)$.
Step II. Construct the reduced incidence matrix of the graph $G \times (S - (\cup_{i=1}^{n} t_i))$. Adjoin zero columns corresponding to $\cup_{i=1}^{n} t_i$. $\mathbf{A}'_{1S}$ is the resulting matrix.

Justification for Algorithm I. By Theorem 9, $\mathbf{V}_{\text{cob}}(G \times (S - \cup_{i=1}^{n} t_i)) = (\mathbf{V}_{\text{cob}}(G)) \times (S - \cup_{i=1}^{n} t_i)$. The rows of $\mathbf{A}'_{1S}$ are linearly independent and belong to $\mathbf{V}_{\text{cob}}(G)$ since they are obtained by padding vectors of $\mathbf{V}_{\text{cob}}(G) \times (S - \cup_{i=1}^{n} t_i)$ with zeros corresponding to $\cup_{i=1}^{n} t_i$. Since t_i forms a forest of $G \times S_i (i = 1, \ldots, n)$ in the reduced incidence matrix of $\oplus_{i=1}^{n} G \times S_i$ the columns corresponding to $\cup_{i=1}^{n} t_i$ form a linearly independent set. In the matrix $\mathbf{A}'_{1S}$ we have only zeros corresponding to these columns. Further the number of rows of $\mathbf{A}'_{1S}$ is $r(\mathbf{V}_{\text{cob}}(G)) - |\cup_{i=1}^{n} t_i|$. Hence the rows of $\mathbf{A}'_{1S}$ together with the rows of a reduced incidence matrix of $\oplus_{i=1}^{n} G \times S_i$ form a basis of $\mathbf{V}_{\text{cob}}(G)$.

Example 2

Let G in Figure 2 be the graph of $\mathbf{N}_1$. Let $S_1 = \{1, \ldots, 5\}$, $S_2 = \{6, \ldots, 10\}$, $S_3 = \{11, \ldots, 17\}$, $S_4 = \{18, \ldots, 24\}$. The graph $\oplus G \times S_i$ is shown in Figure 3. Here $t_1 = \{1, 2\}$, $t_2 = \{6, 7\}$, $t_3 = \{11, 12, 13, 14\}$, $t_4 = \{18, 19, 20, 21\}$. $r(G) - \sum_{i=1}^{4} r(G \times S_i) = 3$. So we need three extra variables

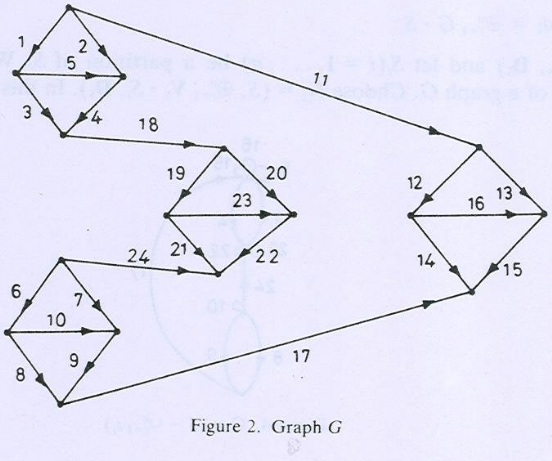

Figure 2. Graph G

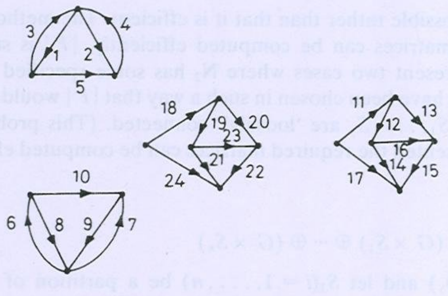

Figure 3. $\oplus_{i=1}^{n} (G \times S_i)$

for acquiring the advantages of working with S_1, S_2, S_3, S_4 as separators—$\mathbf{v}_{P_1}$ has three variables and $\mathbf{A}'_{1S}$ has three rows. The rewritten equations are as follows:

$$(\mathbf{A}_{211})\mathbf{i}^T_{S_1} = 0$$

$$(\mathbf{A}_{222})\mathbf{i}^T_{S_2} = 0$$

$$(\mathbf{A}_{233})\mathbf{i}^T_{S_3} = 0$$

$$(\mathbf{A}_{244})\mathbf{i}^T_{S_4} = 0$$

($\mathbf{A}_{2ii}$ is a reduced incidence matrix of $G \times S_i$)

$$\begin{bmatrix} -\mathbf{v}^T_{S_1} + (\mathbf{A}^T_{211})\mathbf{e}^T_{21} \\ -\mathbf{v}^T_{S_2} + (\mathbf{A}^T_{222})\mathbf{e}^T_{22} \\ -\mathbf{v}^T_{S_3} + (\mathbf{A}^T_{233})\mathbf{e}^T_{23} \\ -\mathbf{v}^T_{S_4} + (\mathbf{A}^T_{244})\mathbf{e}^T_{24} \end{bmatrix} = -\begin{bmatrix} (\mathbf{A}'_{1S})^T & \mathbf{v}^T_{P_1} \end{bmatrix}$$

$$\mathbf{D}(\mathbf{v}_S, \mathbf{i}_S) = 0$$

$$(\mathbf{A}'_{1S})\mathbf{i}^T_S = 0$$

$\mathbf{A}'_{1S}$ may be chosen as the reduced incidence matrix of the graph obtained by adding $\cup\, t_i$ as self loops to $G \times (S - \cup_{i=1}^{4} t_i)$ shown in Figure 4. In this case, since the coboundary space of the graph of $\mathbf{N}_2$ is contained in the coboundary space of the graph of $\mathbf{N}_1$, the matrix $\mathbf{K}^T_{23}$ has no rows.

Case II. New graph $= \oplus_{i=1}^{n} G \cdot S_i$

Let $\mathbf{N}_1 = (S, \mathbf{V}_S, \mathbf{D}_S)$ and let $S_i (i = 1, \ldots, n)$ be a partition of S. We will assume that $\mathbf{V}_S$ is the coboundary space of a graph G. Choose $\mathbf{N}_2 = (S, \oplus_{i=1}^{n} \mathbf{V}_S \cdot S_i, \mathbf{D}_S)$. In this case $\mathbf{A}'_{\cap S}$ would simply be the

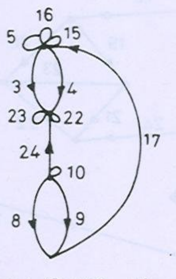

Figure 4. $G \times (S - \cup_{i=1}^{n} t_i)$

reduced incidence matrix of G. The following algorithm describes how to construct $\mathbf{K}_{23}^T$. In this case $P_2 = P$ and $|P| = (\sum_{i=1}^n r(G \cdot S_i) - r(G))$. Observe that $\mathbf{N}_2$ is easier to solve if in the device characteristic $\mathbf{D}(\mathbf{v}_S, \mathbf{i}_S) = 0$, $S_i (i = 1, \ldots, n)$ appear decoupled. Solving $\mathbf{N}_2$ would then be equivalent to solving n smaller networks.

Algorithm II. To construct $\mathbf{K}_{23}^T$ *when* $\mathbf{N}_2 = (S, \oplus_{i=1}^n \mathbf{V}_S \cdot S_i, \mathbf{D}_S)$

Let $\mathbf{n}_b$ be the set of boundary nodes of G where edges of more than one S_i are incident. Let the copy of node e_k of $\mathbf{n}_b$ in the graph $G \cdot S_i$ be named e_{ki}. Let m_i be the number of components of $G \cdot S_i (i = 1, \ldots, n)$. For $j = 1, \ldots, m_i$ and $i = 1, \ldots, n$ do the following. If the jth component of $G \cdot S_i$ has boundary nodes, select a boundary node $e_{k_{ij}i}$ as a datum node. Construct the graph G_b by adding an edge e_{ki} directed from e_k to $e_{k_{ij}}$, for each boundary node e_{ki} in the jth component of $G \cdot S_i$. Let $\mathbf{B}_b$ be a cycle matrix of G_b. To this matrix add zero columns corresponding to non-boundary nodes of each $G \cdot S_i$. The resulting matrix is $\mathbf{K}_{23}^T$.

Justification of Algorithm II. The condition $\mathbf{K}_{23}^T \mathbf{e}_2^T = 0$ represents the maximal linearly independent voltage constraints on all the nodes of $G \cdot S_i (i = 1, \ldots, n)$ in order to connect the $G \cdot S_i$ to make up the graph G. The KVL conditions of G_b also represent the voltage constraints on all the boundary nodes of $G \cdot S_i (i = 1, \ldots, n)$ in order to connect the $G \cdot S_i$ to make up the graph G. If $\mathbf{B}_b$ is a cycle matrix of G_b then $\mathbf{B}_b \mathbf{e}^T = 0$ is a maximal linearly independent set of constraints among them. Padding the matrix $\mathbf{B}_b$ with zero columns corresponding to internal nodes of $G \cdot S_i (i = 1, \ldots, n)$ therefore yields (a candidate for) $\mathbf{K}_{23}^T$.

Example 3

Let G in Figure 2 be the graph of $\mathbf{N}_1$. Let $S_1 = \{1, \ldots, 5\}$, $S_2 = \{6, \ldots, 10\}$, $S_3 = \{11, \ldots, 17\}$, $S_4 = \{18, \ldots, 24\}$. The graph $\oplus_{i=1}^4 G \cdot S_i$ is shown in Figure 5. In this case $(\sum_{i=1}^4 r(G \cdot S_i) - r(G)) = 1$. So we need one extra variable for acquiring the advantages of working with S_1, S_2, S_3, S_4 as separators. $\mathbf{i}_{P2}$ has one variable and $\mathbf{K}_{23}^T$ has one row. The rewritten equations are as follows:

$$\begin{bmatrix} (\mathbf{A}_{211})\mathbf{i}_{S_1}^T \\ (\mathbf{A}_{222})\mathbf{i}_{S_2}^T \\ (\mathbf{A}_{233})\mathbf{i}_{S_3}^T \\ (\mathbf{A}_{244})\mathbf{i}_{S_4}^T \end{bmatrix} = [\mathbf{K}_{23}] \ \mathbf{i}_{P_2}^T$$

$$-\mathbf{v}_{S_1}^T + \mathbf{A}_{211}^T \mathbf{e}_{21}^T = 0$$

$$-\mathbf{v}_{S_2}^T + \mathbf{A}_{222}^T \mathbf{e}_{22}^T = 0$$

$$-\mathbf{v}_{S_3}^T + \mathbf{A}_{233}^T \mathbf{e}_{23}^T = 0$$

$$-\mathbf{v}_{S_4}^T + \mathbf{A}_{244}^T \mathbf{e}_{24}^T = 0$$

$$\mathbf{D}(\mathbf{v}_S, \mathbf{i}_S) = 0$$

$$\mathbf{K}_{23}^T \mathbf{e}_2^T = 0$$

Here $\mathbf{A}_{2ii}$ is the reduced incidence matrix of $G \cdot S_i$. For the jth component of $G \cdot S_i$ (if it has a boundary node) the node $e_{k_{ij}i}$ must be chosen as the datum node, i.e. the datum nodes selected in the construction of G_b (see Figure 6) and the datum nodes selected for constructing the reduced incidence matrix of $G \cdot S_i$ must be the same. In this example $e_{k_{11}} = e_2, e_{k_{21}} = e_3, e_{k_{31}} = e_1$ and $e_{k_{41}} = e_3$. Different k_{ij} can turn out to be identical. Here, for instance, $k_{21} = k_{41} = 3$. It simply means that copies of the same boundary node (e_3 in this case) have been chosen as datum nodes in components of different $G \cdot S_i$. The matrix $\mathbf{K}_{23}^T$ has a single row in this example. It has entries for each non-datum node of each component of $G \cdot S_i (i = 1, \ldots, n)$. In this case it has entries corresponding to all the nodes of $\oplus_{i=1}^4 G \cdot S_i$ except e_{21}, e_{32}, e_{13} and e_{34}, i.e. it has

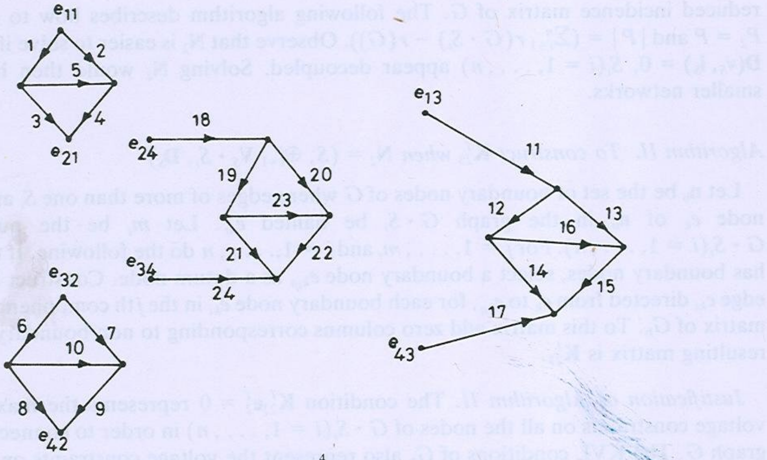

Figure 5. $\bigoplus_{i=1}^{4} (G \cdot S_i)$

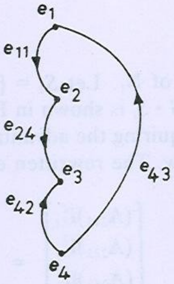

Figure 6. The graph G_b

16 entries. Of these entries all entries except those corresponding to edges of $\mathbf{G}_b$ are zero. Corresponding to edges of G_b we have the entries $\left(\cdots \begin{array}{cccc} e_{11} & e_{24} & e_{42} & e_{43} \\ 1 & 1 & -1 & 1 \end{array} \cdots\right)$ (this is a circuit vector of G_b). In this case $\mathbf{A}'_{1S}$ has no rows since $\mathbf{V}_S$ is a subspace of $\sum_{i=1}^{4}(\mathbf{V}_S \cdot S_i)$.

6. DECOMPOSITION OF VECTOR SPACES

In this section we introduce another vector space notion based on the g-minor operation, namely decomposition of a vector space. The notion of decomposition arises when we decompose an electrical network into several multiports and a port connection diagram. Practical network analysis methods such as Kron's diakoptics can be derived using the idea of decomposition of a network into two multiports. For theoretical network analysis the notion of decomposition provides a convenient means of network reduction. Properties of the original network, pertaining to interaction between different component multiports can be transferred to a reduced network based on the port connection diagram. It is therefore reasonable to study minimal decompositions where the port connection diagram has the minimum number of edges. A more detailed study of vector space decomposition is available in Reference 5.

6.1. Definitions and Theorems

Definition 8

Let $\mathbf{V}_S$ be a vector space on S. Let S be partitioned into $S_1, \ldots, S_n$. Let sets $P_1, \ldots, P_n$ be pairwise disjoint and disjoint from S. Let $P = \cup_{i=1}^{n} P_i$. Let $\mathbf{V}_{S_iP_i}(i = 1, \ldots, n)$, $\mathbf{V}_P$ be vector spaces on $S_i \cup P_i(i = 1, \ldots, n)$, P respectively. We say that $\{\mathbf{V}_P, \mathbf{V}_{S_1P_1}, \ldots, \mathbf{V}_{S_nP_n}\}$ is an n-decomposition of $\mathbf{V}_S$ with respect to $S_1, \ldots, S_n$ iff $\mathbf{V}_S = (\mathbf{V}_{S_iP_i} \oplus \cdots \oplus \mathbf{V}_{S_nP_n}) \leftarrow \mathbf{V}_P$. $\mathbf{V}_{S_iP_i}(i = 1, \ldots, n)$ will be called *components* of the decomposition. P_i will be called the *sets of ports* of the component $\mathbf{V}_{S_iP_i} \cdot \mathbf{V}_P$ will be called the *coupler* of the decomposition. The decomposition is said to be *minimal* iff whenever $\{\mathbf{V}_{P'}, \mathbf{V}_{S_iP_i'}, \ldots, \mathbf{V}_{S_nP_n'}\}$ is an n-decomposition of $\mathbf{V}_S$ with respect to $S_1, \ldots, S_n$, $|P'| \geq |P|$.

Example 4

Consider the graph G in Figure 7. Let S be the set of edges of G. Let $S_1 = \{1, \ldots, 5\}$, $S_2 = \{6, \ldots, 12\}$, $S_3 = \{13, \ldots, 19\}$, $P_1 = \{p_1\}$, $P_2 = \{p_2\}$, $P_3 = \{p_3\}$. We see (Figure 8) that the graph G has been broken up into three multiports and a port connection diagram G_p. Let $\mathbf{V}_S$, $\mathbf{V}_{S_iP_i}(i = 1, 2, 3)$, $\mathbf{V}_P$ be the coboundary spaces of G, $G_{S_iP_i}(i = 1, 2, 3)$, G_p, respectively. Then it is possible to show that $\mathbf{V}_S = (\sum_{i=1}^{3} \mathbf{V}_{S_iP_i}) \leftarrow \mathbf{V}_P$.

We begin with a result which gives characteristic properties that a vector space must possess in order to be a component of a decomposition of a given space $\mathbf{V}_S$.

Theorem 18

Let $\mathbf{V}_{S_iP_i}$ $(i = 1, \ldots, n)$ be spaces on $S_i \cup P_i$, $i = 1, \ldots, n$, where $S_i \cap P_j = \phi$ for all i, j, $S_i \cap S_j = \phi$ for $i \neq j$, $P_i \cap P_j = \phi$, $i \neq j$. Let $\cup S_i = S$, $\cup P_i = P$. Then there exists $\mathbf{V}_P$ on P such that $\mathbf{V}_S = (\mathbf{V}_{S_1P_1} \oplus \cdots \oplus \mathbf{V}_{S_nP_n}) \leftarrow \mathbf{V}_P$ iff $\mathbf{V}_S \times S_i \supseteq \mathbf{V}_{S_iP_i} \times S_i$, $\mathbf{V}_S \cdot S_i \subseteq \mathbf{V}_{S_iP_i} \cdot S_i$ $(i = 1, \ldots, n)$.

Proof. $\mathbf{V}_S$ can be a g-minor of $(\mathbf{V}_{S_1P_1} \oplus \cdots \oplus \mathbf{V}_{S_nP_n})$ iff $\mathbf{V}_S \subseteq (\mathbf{V}_{S_1P_1} \oplus \cdots \oplus \mathbf{V}_{S_nP_n}) \cdot S$ *and* $\mathbf{V}_S \supseteq (\mathbf{V}_{S_1P_1} \oplus \cdots \oplus \mathbf{V}_{S_nP_n}) \times S$ by Theorem 11. We will show that these two conditions are equivalent to the ones in the statement of the theorem. If $\mathbf{V}_S \subseteq (\mathbf{V}_{S_iP_i} \oplus \cdots \oplus \mathbf{V}_{S_nP_n}) \cdot S$ then clearly $\mathbf{V}_S \cdot S_i \subseteq \mathbf{V}_{S_iP_i} \cdot S_i$ $(i = 1, \ldots, n)$. Next let $\mathbf{V}_S \cdot S_i \subseteq \mathbf{V}_{S_iP_i} \cdot S_i$ $(i = 1, \ldots, n)$. Let $\mathbf{f}_S \in \mathbf{V}_S$. Then $\mathbf{f}_S/S_i \in \mathbf{V}_{S_iP_i} \cdot S_i$ $(i = 1, \ldots, n)$. Hence $\mathbf{f}_S \in \mathbf{V}_{S_1P_1} \cdot S_1 \oplus \cdots \oplus \mathbf{V}_{S_nP_n} \cdot S_n$. But $(\mathbf{V}_{S_1P_1} \cdot S_1) \oplus \cdots \oplus (\mathbf{V}_{S_nP_n} \cdot S_n) = (\mathbf{V}_{S_1P_1} \oplus \cdots \oplus \mathbf{V}_{S_nP_n}) \cdot S$. Thus it is clear that $\mathbf{V}_S \subseteq (\mathbf{V}_{S_1P_1} \oplus \cdots \oplus \mathbf{V}_{S_nP_n}) \cdot S$, *Next* $\mathbf{V}_S \supseteq (\mathbf{V}_{S_1P_1} \oplus \cdots \oplus \mathbf{V}_{S_nP_n}) \times S$ iff $\mathbf{V}_S^* \subseteq (\mathbf{V}_{S_1P_1} \oplus \cdots \oplus \mathbf{V}_{S_nP_n})^* \cdot S$ by Theorems 2 and5, i.e. iff $\mathbf{V}_S \cdot S_i \subseteq \mathbf{V}_{S_iP_i} \cdot S_i$ $(i = 1, \ldots, n)$, i.e. iff $\mathbf{V}_S \times S_i \supseteq \mathbf{V}_{S_iP_i} \times S_i$ $(i = 1, \ldots, n)$ by Theorems 2 and 4.

Theorem 19

Let $\{\mathbf{V}_P, \mathbf{V}_{S_1P_1}, \ldots, \mathbf{V}_{S_nP_n}\}$ be a decomposition of $\mathbf{V}_S$. Then $\{\mathbf{V}_P^*, \mathbf{V}_{S_1P_1}^*, \ldots, \mathbf{V}_{S_nP_n}^*\}$ is a decomposition of $\mathbf{V}_S^*$.

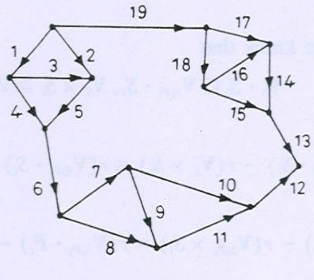

Figure 7. Graph G

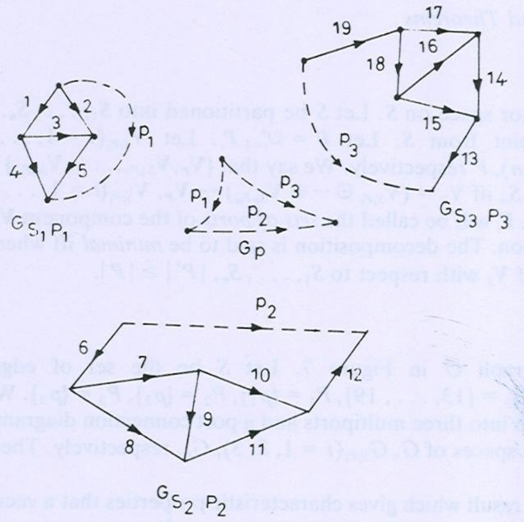

Figure 8. The decomposition of G

Proof. $\mathbf{V}_S = (\mathbf{V}_{S_1P_1} \oplus \cdots \oplus \mathbf{V}_{S_nP_n}) \leftarrow \mathbf{V}_P$ iff $\mathbf{V}_S^* = (\mathbf{V}_{S_1P_1} \oplus \cdots \oplus \mathbf{V}_{S_nP_n})^* \leftarrow \mathbf{V}_P^*$ by Theorem 13. By Lemma 3 this is equivalent to $\mathbf{V}_S^* = (\mathbf{V}_{S_1P_1}^* \oplus \cdots \oplus \mathbf{V}_{S_nP_n}^*) \leftarrow \mathbf{V}_P^*$.

Q.E.D.

An immediate consequence of Theorem 19 and the definition of minimal decomposition is the following theorem.

Theorem 20

$\{\mathbf{V}_P, \mathbf{V}_{S_iP_i}\ (i = 1, \ldots, n(\}$ is a minimal decomposition of $\mathbf{V}_S$ iff $\{\mathbf{V}_P^*, \mathbf{V}_{S_iP_i}^*\ (i = 1, \ldots, n)\}$ is a minimal decomposition of $\mathbf{V}_S^*$.

The next few theorems provide characteristic properties for a minimal decomposition.

Lemma 4

Let $\{\mathbf{V}_P, \mathbf{V}_{S_1P_1}, \ldots, \mathbf{V}_{S_nP_n}\}$ be an n-decomposition of $\mathbf{V}_S$. Then $|P_i| \geq r(\mathbf{V}_S \cdot S_i) - r(\mathbf{V}_S \times S_i)$, $i = 1, \ldots, n$.

Proof. From Theorem 18 we know that

$$\mathbf{V}_S \cdot S_i \subseteq \mathbf{V}_{S_iP_i} \cdot S_i,\ \mathbf{V}_S \times S_i \supseteq \mathbf{V}_{S_iP_i} \times S_i$$

Hence

$$r(\mathbf{V}_S \cdot S_i) - r(\mathbf{V}_S \times S_i) \leq r(\mathbf{V}_{S_iP_i} \cdot S_i) - r(\mathbf{V}_{S_iP_i} \times S_i)$$

By Lemma 2,

$$r(\mathbf{V}_{S_iP_i} \cdot S_i) - r(\mathbf{V}_{S_iP_i} \times S_i) = r(\mathbf{V}_{S_iP_i} \cdot P_i) - r(\mathbf{V}_{S_iP_i} \times P_i) \leq |P_i|$$

The Lemma follows.

Lemma 5

Let $\{\mathbf{V}_P, \mathbf{V}_{S_1P_1}, \dots, \mathbf{V}_{S_nP_n}\}$ be an n-decomposition of $\mathbf{V}_S$. Then

$$r(\mathbf{V}_S \cdot S_i) - r(\mathbf{V}_S \times S_i) \leq r(\mathbf{V}_P \cdot P_i) - r(\mathbf{V}_P \times P_i),\ i = 1, \dots, n$$

Proof. Select vectors $\mathbf{f}^1_{S_i}, \dots, \mathbf{f}^k_{S_i}$ which together with a basis of $\mathbf{V}_S \times S_i$ form a basis of $\mathbf{V}_S \cdot S_i$. Then by the definition of decomposition there exist vectors $\mathbf{f}^r_{S_iP_i}$ $(r = 1, \dots, k)$ in $\mathbf{V}_{S_iP_i}$ such that

$$\mathbf{f}^r_{S_iP_i}/_{S_i} = \mathbf{f}^r_{S_i},\ \mathbf{f}^r_{S_iP_i}/_{P_i} \in \mathbf{V}_P \cdot P_i$$

Suppose $\{\mathbf{f}^r_{S_iP_i}/_{P_i}\ (r = 1, \dots, k)\}$ does not form a linearly independent set with a basis of $\mathbf{V}_P \times P_i$. Then there exists $\mathbf{f}_{S_iP_i} \in \mathbf{V}_{S_iP_i}$ such that $\mathbf{f}_{S_iP_i}/_{P_i}$ is a non-trivial linear combination of $\mathbf{f}^r_{S_iP_i}/_{P_i}$ $(r = 1, \dots, k)$ and belongs to $\mathbf{V}_P \times \mathbf{P}_i$. This linear combination of $\mathbf{f}^r_{S_iP_i}/_{S_i}$ will however yield a vector which belongs to $\mathbf{V}_S \cdot S_i - \mathbf{V}_S \times S_i$. Thus the vector $\mathbf{f}_{S_iP_i}$ is such that $\mathbf{f}_{S_iP_i}/_{S_i} \in \mathbf{V}_S \cdot S_i - \mathbf{V}_S \times S_i$, $\mathbf{f}_{S_iP_i}/_{P_i} \in \mathbf{V}_P \times P_i$. Consider a vector $\mathbf{f}_{SP} \in \oplus \mathbf{V}_{S_iP_i}$ such that

$$\mathbf{f}_{SP}/_{S_i \cup P_i} = \mathbf{f}_{S_iP_i}$$

$$\mathbf{f}_{SP}/_{(S \cup P) - (S_i \cup P_i)} = 0$$

Next choose a vector $\mathbf{f}_P \in \mathbf{V}^P$ such that $\mathbf{f}_P$ is zero outside P_i and $\mathbf{f}_P/_{P_i} = \mathbf{f}_{S_iP_i}/_{P_i}$. This is possible since $\mathbf{f}_{S_iP_i}/_{P_i} \in \mathbf{V}_P \times P_i$. Since $\mathbf{f}_{SP}/_P = \mathbf{f}_P$ it follows that $\mathbf{f}_{SP}/_S \in \mathbf{V}_S$. But then $\mathbf{f}_{SP}/_{S_i} \in \mathbf{V}_S \times S_i$. This contradicts the fact that $\{\mathbf{f}^r_{S_iP_i}/_{S_i}\}$ forms a linearly independent set with a basis of $\mathbf{V}_S \times S_i$. We conclude that $\{\mathbf{f}^r_{S_iP_i}/_{P_i}\ (r = 1, \dots, k)\}$ forms a linearly independent set with a basis of $\mathbf{V}_P \times P_i$, i.e. $r(\mathbf{V}_P \cdot P_i) - r(\mathbf{V}_P \times P_i) \geq r(\mathbf{V}_S \cdot S_i) - r(\mathbf{V}_S \times S_i)$.

Q.E.D.

We later present an algorithm for the construction of a minimal decomposition of a vector space $\mathbf{V}_S$ that is the coboundary space of a graph and has $|P| = \Sigma(r(\mathbf{V}_S \cdot S_i) - r(\mathbf{V}_S \times S_i))$. A simple algorithm of the same kind can be given even for a general vector space $\mathbf{V}_S$.[5] We therefore assert

Lemma 6

$\{\mathbf{V}_P, \mathbf{V}_{S_1P_1}, \dots, \mathbf{V}_{S_nP_n}\}$ is a minimal n-decomposition of $\mathbf{V}_S$ iff $|P| = r(\mathbf{V}_S \cdot S_i) - r(\mathbf{V}_S \times S_i)$, $(i = 1, \dots, n)$.

Theorem 21

Let $\{\mathbf{V}_P, \mathbf{V}_{S_1P_1}, \dots, \mathbf{V}_{S_nP_n}\}$ be an n-decomposition of $\mathbf{V}_S$. Then the following statements are equivalent:

(a) It is minimal.
(b) $|P_i| = r(\mathbf{V}_S \cdot S_i) - r(\mathbf{V}_S \times S_i)$ $(i = 1, \dots, n)$.
(c) P_i has no circuits or bonds in $M(\mathbf{V}_{S_iP_i})$ $(i = 1, \dots, n)$ or $M(\mathbf{V}_P)$.

Proof. By Lemma 6 we know that (a) and (b) are equivalent. We will now show that (b) and (c) are equivalent. Let $|P_i| = r(\mathbf{V}_S \cdot S_i) - r(\mathbf{V}_S \times S_i)$ $(i = 1, \dots, n)$. Then for all i, by Theorem 18, $|P_i| \leq r(\mathbf{V}_{S_iP_i} \cdot S_i) - r(\mathbf{V}_{S_iP_i} \times S_i)$. Hence by Lemma 2, $|P_i| \leq r(\mathbf{V}_{S_iP_i} \cdot P_i) - r(\mathbf{V}_{S_iP_i} \times P_i)$. By Lemma 5, $|P_i| \leq r(\mathbf{V}_P \cdot P_i) - r(\mathbf{V}_P \times P_i)$. It follows that $|P_i| = r(V_{S_iP_i} \cdot P_i) = r(\mathbf{V}_P \cdot P_i)$ and $0 = r(\mathbf{V}_{S_iP_i} \times P_i) = r(\mathbf{V}_P \times P_i)$ for all i. Hence by Theorem 7, $M(\mathbf{V}_{S_iP_i} \cdot P_i)$ and $M(\mathbf{V}_P \cdot P_i)$ do not have circuits and $M(\mathbf{V}_{S_iP_i} \times P_i)$ and $M(\mathbf{V}_P \times P_i)$ do not have bonds. Hence P_i has no circuits or bonds in $M(\mathbf{V}_{S_iP_i})$ $(i = 1, \dots, n)$ or $M(\mathbf{V}_P)$ by Theorem 6. Conversely suppose P_i has no circuits or bonds in $M(\mathbf{V}_{S_iP_i})$, or $M(\mathbf{V}_P)$. Then for all i

$$\begin{aligned} |P_i| &= r(\mathbf{V}_{S_iP_i} \cdot P_i) = r(\mathbf{V}_P \cdot P_i) \\ 0 &= r(\mathbf{V}_{S_iP_i} \times P_i) = r(\mathbf{V}_P \times P_i) \end{aligned}$$

so,

$$\begin{aligned} |P_i| &= r(\mathbf{V}_{S_iP_i} \cdot P_i) - r(\mathbf{V}_{S_iP_i} \times P_i) \\ &= r(\mathbf{V}_{S_iP_i} \cdot S_i) - r(\mathbf{V}_{S_iP_i} \times S_i) \end{aligned}$$

$\mathbf{A}_{2S}$, respectively. The matrix $\mathbf{A}$ shown below is taken to be the generator matrix for $\mathbf{V}_{SP}$. (Here $P_1 \cup P_2 = P$.)

$$\mathbf{A} = \begin{array}{c} \\ \\ \end{array}\begin{array}{ccc} S & P_1 & P_2 \\ \end{array}$$

$$\mathbf{A} = \left[\begin{array}{ccc} \mathbf{A}_{1S} & \mathbf{U} & \mathbf{0} \\ \mathbf{A}_{\cap S} & \mathbf{0} & \mathbf{0} \\ \mathbf{A}_{2S} & \mathbf{0} & \mathbf{U} \end{array}\right]$$

If $\mathbf{V}_P^1$ has the generator matrix $\begin{array}{cc} P_1 & P_2 \\ (\mathbf{U} & \mathbf{0}) \end{array}$ and $\mathbf{V}_P^2$ has the generator matrix $\begin{array}{cc} P_1 & P_2 \\ (\mathbf{0} & \mathbf{U}) \end{array}$ then it is easy to see that $\mathbf{V}_{SP} \leftarrow \mathbf{V}_P^1 = \mathbf{V}_S^1$ and $\mathbf{V}_{SP} \leftarrow \mathbf{V}_P^2 = \mathbf{V}_S^2$. Note that

$$\begin{aligned} |P| &= |P_1| + |P_2| \\ &= r(\mathbf{V}_S^1) - r(\mathbf{V}_S^1 \cap \mathbf{V}_S^2) + r(\mathbf{V}_S^2) - r(\mathbf{V}_S^1 \cap \mathbf{V}_S^2) \\ &= r(\mathbf{V}_S^1 + \mathbf{V}_S^2) - r(\mathbf{V}_S^1 \cap \mathbf{V}_S^2) \end{aligned}$$

by Theorem 5. It follows that $\mathbf{V}_{SP}$ is a minimal extension of $\mathbf{V}_S^1$ and $\mathbf{V}_S^2$.

Q.E.D.

In Theorem 15 observe that we are able to obtain $\mathbf{V}_S^1$ and $\mathbf{V}_S^2$ as ordinary minors (as opposed to g-minors) of $\mathbf{V}_{SP}$ since $\mathbf{V}_S^1 = \mathbf{V}_{SP} \cdot (S \cup P_2) \times S$ and $\mathbf{V}_S^2 = \mathbf{V}_{SP} \cdot (S \cup P_1) \times S$. If we have to construct a minimal extension of $\{\mathbf{V}_S^1 \cdots \mathbf{V}_S^n\}$ when $n > 2$, ordinary minors would be inadequate. By using the ideas of the first half of the proof of Theorem 15 we can show

$$|P| \geq r\left(\sum_{i=1}^{n} \mathbf{V}_S^i\right) - r(\mathbf{V}_S^1 \cap \cdots \cap \mathbf{V}_S^n)$$

The following matrix $\mathbf{A}$ can be taken as the generator matrix of $\mathbf{V}_{SP}$:

$$\mathbf{A} = \left[\begin{array}{cc} \mathbf{A}_{+S} & \mathbf{U} \\ \mathbf{A}_{\cap S} & \mathbf{0} \end{array}\right]$$

(The rows of $\mathbf{A}_{\cap S}$ form a basis for $(\mathbf{V}_S^1 \cap \cdots \cap \mathbf{V}_S^n)$, whereas the rows of

$$\left[\begin{array}{c} \mathbf{A}_{+S} \\ \mathbf{A}_{\cap S} \end{array}\right]$$

form a basis for $\sum_{i=1}^n \mathbf{V}_S^i$.) Let $\mathbf{V}_S^i$ be generated by the matrix

$$[\mathbf{K}_+ \, \mathbf{K}_\cap] \left[\begin{array}{c} \mathbf{A}_{+S} \\ \mathbf{A}_{\cap S} \end{array}\right]$$

Let $\mathbf{V}_P^i$ be the space generated by the rows of $[\mathbf{K}_+]$. Clearly

$$\mathbf{V}_{SP} \leftarrow \mathbf{V}_P^i = \mathbf{V}_S^i$$

We summarize these results in Theorem 16 below:

Theorem 16

Let $\mathbf{V}_{SP}$ on $S \cup P$ be an extension of $\{\mathbf{V}_S^1 \cdots \mathbf{V}_S^n\}$. It is a minimal extension of $\{\mathbf{V}_S^1 \cdots \mathbf{V}_S^n\}$ iff $|P| = r(\sum_{i=1}^n \mathbf{V}_S^i) - r(\cap_{i=1}^n \mathbf{V}_S^i)$.

We next prove a simple but useful result.

Theorem 17

$\mathbf{V}_{SP}$ is a minimal extension of $\{\mathbf{V}_S^1 \cdots \mathbf{V}_S^n\}$ iff $\mathbf{V}_{SP}^*$ is a minimal extension of $\{(\mathbf{V}_S^1)^*, \cdots (\mathbf{V}_S^n)^*\}$.

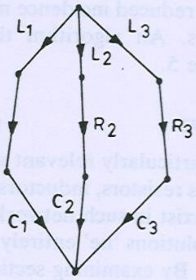

Figure 10. The graph $\hat{G}$ of N

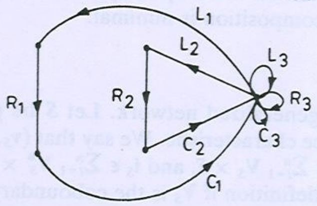

Figure 11. The graph G derived from $\hat{G}$

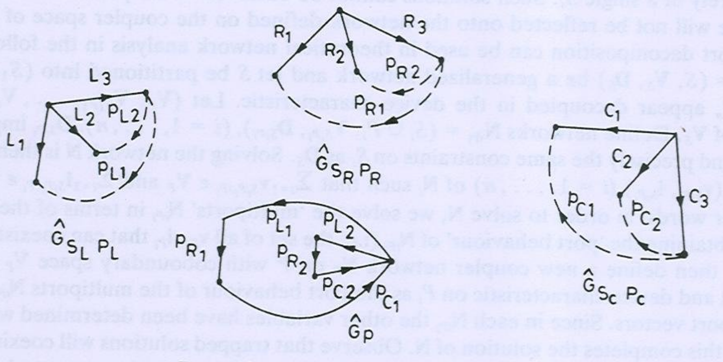

Figure 12. A minimal decomposition of $\hat{G}$

Step 5. Construct the matrix

$$\mathbf{A}_P = \begin{array}{c} \begin{array}{ccc} P_1 & & P_n \end{array} \\ [\mathbf{A}_{t_1} \vdots \cdots \mathbf{A}_{t_n}] \end{array}$$

$\mathbf{V}_P$ is the space generated by the rows of this matrix.

Step 6. Let $\mathbf{A}_{iS_i}$ be the reduced incidence matrix of $G \times S_i$ $(i = 1, \ldots, n)$. Let $\mathbf{A}_{it_i}$ be the submatrix of $\mathbf{A}_{iS_i}$ corresponding to columns t_i. Let $\hat{\mathbf{A}}_{ii}$ be the reduced incidence matrix of $\hat{G} \times S_i$ $(i = 1, \ldots, n)$. The generator matrix of $\mathbf{V}_{S_iP_i}$ $(i = 1, \ldots, n)$ is

$$\mathbf{A}_{S_iP_i} = \begin{array}{c} \begin{array}{cc} S_i & P_i \end{array} \\ \begin{bmatrix} \hat{\mathbf{A}}_{ii} & \mathbf{0} \\ \mathbf{A}_{iS_i} & \mathbf{A}_{it_i} \end{bmatrix} \end{array}$$

Remark. Observe that $\mathbf{A}_P$ is the reduced incidence matrix of a graph, whereas the spaces $\mathbf{V}_{S_iP_i}$ need not be coboundary spaces of graphs. An algorithm that makes $\mathbf{V}_{S_iP_i}$ $(i = 1, \ldots, n)$ graphic (but $\mathbf{V}_P$ non-graphic) is given in Reference 5.

6.2. Applications to network theory

The idea of decomposition is particularly relevant when the network can be naturally partitioned into different types of elements, such as resistors, inductors, capacitors or linear, non-linear or faulty and good elements etc. Solutions that can exist in such networks may be classified as of two kinds: 'trapped' and 'interactive' solutions. Trapped solutions 'lie' entirely within one type of element. A solution that is not trapped will be called interactive. By examining sections of such solutions corresponding to one part of the network, one can obtain an idea of what is happening to another part of the network. Trapped solutions can yield no such information. Interactive solutions can be studied more conveniently by working with a network defined on the coupler space of the decomposition. The new reduced network will be minimal when the decomposition is minimal.

Definition 9

Let $\mathbf{N} = (S, \mathbf{V}_S, \mathbf{D}_S)$ be a generalized network. Let S be partitioned into $S_1, \ldots, S_n$. Let these sets appear decoupled in the device characteristic. We say that $(\mathbf{v}_S, \mathbf{i}_S)$ is a *trapped solution* with respect to the partition $(S_1, \ldots, S_n)$ iff $\mathbf{v}_S \in \sum_{i=1}^{n} \mathbf{V}_S \times S_i$ and $\mathbf{i}_S \in \sum_{i=1}^{n} \mathbf{V}_S^* \times S_i$.

Observe that in the above definition if $\mathbf{V}_S$ is the coboundary space of a graph G, a solution $(\mathbf{v}_S, \mathbf{i}_S)$ is a trapped solution iff $\mathbf{v}_S$ can exist in $\oplus_{i=1}^{n} G \times S_i$ and $\mathbf{i}_S$ can exist in $\oplus_{i=1}^{n} G \cdot S_i$. One may imagine the voltages lying trapped within cutsets of G lying entirely in a single S_i and currents as trapped within circuits of G lying entirely in a single S_i. Such solutions cannot be observed at the ports of the component multiports and hence will not be reflected onto the network defined on the coupler space of the decomposition.

Multiport decomposition can be used in theoretical network analysis in the following manner.

Let $\mathbf{N} = (S, \mathbf{V}_S, \mathbf{D}_S)$ be a generalized network and let S be partitioned into $(S_1, S_2, \ldots, S_n)$. Let $S_1, S_2, \ldots, S_n$ appear decoupled in the device characteristic. Let $(\mathbf{V}_P, \mathbf{V}_{S_1P_1}, \ldots, \mathbf{V}_{S_nP_n})$ be an n-decomposition of $\mathbf{V}_S$. Define networks $\mathbf{N}_{iP_i} = (S_i \cup P_i, \mathbf{V}_{S_iP_i}, \mathbf{D}_{S_iP_i}), (i = 1, \ldots, n)$. $\mathbf{D}_{S_iP_i}$ imposes no constraint at all on P_i and precisely the same constraints on S_i as $\mathbf{D}_S$. Solving the network $\mathbf{N}$ is then equivalent to finding solutions $(\mathbf{v}_{S_iP_i}, \mathbf{i}_{S_iP_i})$ $(i = 1, \ldots, n)$ of $\mathbf{N}_i$ such that $\sum_{i=1} \mathbf{v}_{S_iP_i/P_i} \in \mathbf{V}_P$ and $\sum_{i=1} \mathbf{i}_{S_iP_i/P_i} \in \mathbf{V}_P^*$.

In other words, in order to solve $\mathbf{N}$, we solve the 'multiports' $\mathbf{N}_{iP_i}$ in terms of the port variables, in the process obtaining the 'port behaviour' of $\mathbf{N}_{iP_i}$ (i.e. the set of all $\mathbf{v}_{P_i}$, $\mathbf{i}_{P_i}$ that can coexist on P_i in the multiport $\mathbf{N}_{iP_i}$). We then define a new coupler network $\mathbf{N}_P$ on P with coboundary space $\mathbf{V}_P$ (the 'port connection diagram') and device characteristic on P_i as the port behaviour of the multiports $\mathbf{N}_{iP_i}$. Solving $\mathbf{N}_P$ gives the possible port vectors. Since in each $\mathbf{N}_{iP_i}$ the other variables have been determined with respect to the port variables this completes the solution of $\mathbf{N}$. Observe that trapped solutions will coexist with zero vectors on the port-edges, which means that the trapped solutions of the network correspond to the zero solution of $\mathbf{N}_P$. The non-zero solutions of $\mathbf{N}_P$ will correspond to the interactive solutions of $\mathbf{N}$.

This approach is quite natural in theoretical network analysis and one can use it to relate the properties of the multiports observable at their ports by projecting these properties onto the reduced network $\mathbf{N}_P$. For an RIMC network, for instance, by splitting into resistive, inductive and capacitive multiports, one can construct a minimal reduced network on $\mathbf{N}_P$ which has no zero eigenvalues and except for them has the same Jordan canonical form for its state equations as the original network.

The idea of decomposition may also be exploited for (large scale) network analysis. Assuming that $|P|$ is small the computationally crucial step in the analysis would be the solution of multiports $\mathbf{N}_{iP_i}$ in terms of the port variables. One way of doing this is to treat say $P_{ij} \subseteq P_i$ as unknown current sources of value $\mathbf{i}_{ij}$ and $\mathbf{P}_i - P_{ij} = P_{i\varepsilon}$ as unknown voltage sources of value $\mathbf{v}_{i\varepsilon}$ and solve $\mathbf{N}_{iP_i}$ in terms of these variables.

The form of the left hand side of the equations (after the temporary 'sources' have been shifted to the right) would be that of the network obtained from $\mathbf{N}_{iP_i}$ by contracting $P_{i\varepsilon}$ and deleting P_{ij}. One could choose P_i and this partition into 'current sources' and 'voltage sources' in such a way that this results in several decoupled pieces of equations. Kron's diakoptics[1–3] is a slight variation of a special case of this method corresponding to a number of multiports equal to two. In order to obtain the exact form of Kron's equations one may proceed as follows.

Let the given network **N** have graph G on the set of edges S. Let S be partitioned into S_1 and S_2. Let T_1 be a forest of $G \cdot S_1$ and α_2 a coforest of $G \times S_2$. Let

$$G_{S1\alpha 2} = G \times (S_1 \cup \alpha_2),\ G_{S2T_1} = G \cdot (S_2 \cup T_1)$$
$$G_{T1\alpha 2} = G \times (S_1 \cup \alpha_2) \cdot (T_1 \cup \alpha_2).$$

It may be verified that the coboundary (cycle) space of G has as its decomposition the coboundary (cycle) spaces of $G_{S1\alpha 2}$, G_{S2T_1} and $G_{T1\alpha 2}$. It can be shown that this is equivalent to: $\mathbf{i}_S(\mathbf{v}_S)$ is a cycle (coboundary) of G iff there exist cycles (coboundaries), $\mathbf{i}_{S1\alpha 2}$ $(\mathbf{v}_{S1\alpha 2})$, $\mathbf{i}_{S2T_1}$ $(\mathbf{v}_{S2T_1})$ such that

$$\begin{aligned} \mathbf{i}_{S}/_{S_j} &= \mathbf{i}_{S_j}\ \ (j = 1, 2) \\ (\mathbf{v}_{S}/_{S_j} &= \mathbf{v}_{S_j}\ \ (j = 1, 2)) \\ \mathbf{i}_{S1\alpha 2}/_{\alpha 2} &= \mathbf{i}_{S2T_1}/_{\alpha 2} \\ (\mathbf{v}_{S1\alpha 2}/_{T1} &= \mathbf{v}_{S2T_1}/_{T_1}) \end{aligned}$$

Suppose now that in the network **N**, S_1 and S_2 appear decoupled in the device characteristic. We can then define the network $\mathbf{N}_{S1\alpha 2}$ on the graph $G_{S1\alpha 2}$, and the device characteristic of S_1 as in **N** with α_2 treated as norators, the network $\mathbf{N}_{S2T_1}$ on the graph G_{S2T_1} and the device characteristic of S_2 as in **N** with T_1 treated as norators. Then to solve **N** we need only solve $\mathbf{N}_{S1\alpha 2}$ and $\mathbf{N}_{S2T_1}$ simultaneously under the condition that $\mathbf{i}_{\alpha 2}$ is the same in both networks and $\mathbf{v}_{T_1}$ is the same in both networks (this may be done by giving the corresponding variables in the two networks the same names). Kron's diakoptic equations for linear networks result if we write nodal-type equations for $\mathbf{N}_{S1\alpha 2}$ treating α_2 as current variables of value $\mathbf{i}_{\alpha 2}$ and loop-type equations for $\mathbf{N}_{S2T_1}$ treating T_1 as voltage variables. These equations are of the form

$$\begin{pmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{pmatrix} \begin{pmatrix} \mathbf{v}_{T1} \\ \mathbf{i}_{\alpha 2} \end{pmatrix} = \begin{pmatrix} \mathbf{s}_1 \\ \mathbf{s}_2 \end{pmatrix}$$

$\mathbf{A}_{11}$ would have the structure corresponding to $G \cdot S_1$ and $\mathbf{A}_{22}$ would have that of $G \times S_2$. If $G \cdot S_1$ $(G \times S_2)$ is made up of separators $S_{11}, \ldots, S_{1m}$ $(S_{21}, \ldots, S_{2k})$ and these are also decoupled in the device characteristic, $\mathbf{A}_{11}$ would appear in the block diagonal form with blocks corresponding to $T_1 \cap S_{1i}$ $(\alpha_2 \cap S_{2i})$.

The results of this paper arise in answering the following natural questions. Is it critical to diakoptics ('piecewise analysis of networks')

(a) that nodal-type equations should be written for S_1 and loop-type equations for S_2
(b) that the network be linear
(c) that we work only with $G \cdot S_1$ or $G \times S_2$?

The answer to all the above questions is 'No'.

Indeed, if we work with the network decomposed into two multiports, as already indicated, we can solve the individual multiports any way we choose; the networks can be non-linear; we can have an infinite variety of structures on S_1 and S_2—using generalized minors, on S_1 (S_2) we can have any space $\mathbf{V}_{S1}$ $(\mathbf{V}_{S2})$ such that $\mathbf{V}_{\text{cob}}(G \times S_1) \subseteq \mathbf{V}_{S1} \subseteq \mathbf{V}_{\text{cob}}(G \cdot S_1)$ $(\mathbf{V}_{\text{cob}}(G \times S_2) \subseteq \mathbf{V}_{S2} \subseteq \mathbf{V}_{\text{cob}}(G \cdot S_2))$. The cost for achieving all this is the introduction of the extra port variables. Having improved upon Kron's diakoptics thus far one is naturally led to examine the range of possible structures that can be imposed on the set of branches of the network and the minimal cost in terms of extra variables. This leads to the techniques described in Section 5 with the generalized minor unifying these techniques with those of multiport decomposition.

CONCLUSION

In this paper we have introduced and made systematic use of the generalized minor operation on vector spaces. We have shown that it arises naturally in the context of connection of ideal transformers. We have defined the notion of the minimal extension of two or more vector spaces and used it to describe a method of network analysis where one could use any desired network topology, at a certain cost, to solve a given network. We have defined the notion of decomposition of a vector space to formalize the intuitive idea of decomposition of a network into multiports and a port connection diagram. We have related this notion to Kron's diakoptics. Although all the concepts introduced in this paper have been for arbitrary vector spaces, in order to show their relevance to network theory, we have presented algorithms, wherever necessary, which are particularly appropriate to coboundary spaces of graphs.

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APPENDIX: JUSTIFICATION FOR ALGORITHM III

Observe that the rows of the following matrix generate the coboundary space of $\hat{G}$:

$$\tilde{\mathbf{A}} = \begin{bmatrix} \hat{\mathbf{A}}_{11} & \cdots & \hat{\mathbf{A}}_{nn} \\ \mathbf{A}_{S_1} & \cdots & \mathbf{A}_{S_n} \end{bmatrix}$$

where $\mathbf{A}_{S_i}$ is the submatrix of $\mathbf{A}$ corresponding to columns S_i. This is because

(a) The rows of $\mathbf{A}$ belong to the coboundary space of $\hat{G}$. (They are obtained by taking suitable vectors of $\mathbf{V}_S \times (S - \cup \hat{t}_i)$ and adding zero columns corresponding to edges in $\cup \hat{t}_i$.)

(b) The rows of $\begin{array}{c} \begin{matrix} S_1 & & S_i & & S_n \end{matrix} \\ [0 \cdots \hat{\mathbf{A}}_{ii} \cdots 0] \end{array}$ belong to the coboundary space of $\hat{G}$ since the rows of $\hat{\mathbf{A}}_{ii}$ belong to $(\mathbf{V}_{\text{cob}}(\hat{G}) \times S_i)$.

(c) The rank of $\tilde{\mathbf{A}}$ equals the rank of $\hat{G}$. [The columns $\hat{t}_i$ of $\hat{\mathbf{A}}_{ii}$ are linearly independent, whereas these columns are zero columns in the matrix $\mathbf{A}$. Hence

$$\begin{aligned} \text{rank of } \tilde{\mathbf{A}} &= \sum_{i=1}^{n} r(\hat{G} \times S_i) + r\Big(\hat{G} \times \Big(S - \bigcup_{i=1}^{n} \hat{t}_i\Big)\Big) \\ &= \Big|\bigcup_{i=1}^{n} \hat{t}_i\Big| + r\Big(\hat{G} \times \Big(S - \bigcup_{i=1}^{n} \hat{t}_i\Big)\Big) \end{aligned}$$

Since $\cup_{i=1}^{n} \hat{t}_i$ contains no circuits in $\hat{G}$

$$\begin{aligned} \text{rank of } \tilde{\mathbf{A}} &= r\Big(\hat{G} \cdot \Big(\bigcup_{i=1}^{n} \hat{t}_i\Big)\Big) + r\Big(\hat{G} \times \Big(S - \bigcup_{i=1}^{n} \hat{t}_i\Big)\Big) \\ &= r(\hat{G}) \end{aligned}$$

$\mathbf{A}_{S_i}$ can be expressed as $\mathbf{k}_i(\mathbf{A}_{iS_i})$ since $\mathbf{A}_{iS_i}$ is a reduced incidence matrix of $G \cdot S_i$. For the spaces $\oplus_{i=1}^{n}$ $\mathbf{V}_{S_iP_i}$ and $\mathbf{V}_P$ we have the generator matrices $\hat{\mathbf{A}}_{\oplus}$, $\mathbf{A}_P$ shown below:

$$\hat{\mathbf{A}}_{\oplus} = \begin{array}{c}\\ \left[\begin{array}{ccc:ccc} S_1 \cdots & S_i \cdots & S_n & P_1 \cdots & P_i \cdots & P_n \\ \hat{\mathbf{A}}_{11} & 0 & 0 & 0 & 0 & 0 \\ \mathbf{A}_{1S_1} & & & \mathbf{A}_{1t_1} & 0 & 0 \\ \ddots & \hat{\mathbf{A}}_{ii} & 0 & & \ddots & \\ 0 \quad 0 & \mathbf{A}_{iS_i} & 0 & 0 & \mathbf{A}_{it_i} & 0 \\ 0 & \ddots & \hat{\mathbf{A}}_{nn} & & \ddots & 0 \\ 0 & 0 & \mathbf{A}_{nS_n} & 0 & 0 & \mathbf{A}_{nt_n} \end{array}\right]\end{array}$$

$$\mathbf{A}_P = [\overset{P_1}{\cdots} \overset{P_i}{\mathbf{k}_i(\mathbf{A}_{it_i})} \overset{P_n}{\cdots}]$$

Consider the submatrix $\mathbf{A}_{\oplus}$ of $\hat{\mathbf{A}}_{\oplus}$ as shown below:

$$\mathbf{A}_{\oplus} = \left[\begin{array}{ccc:ccc} S_1 & S_i & S_n & P_1 & P_i & P_n \\ \mathbf{A}_{1S_1} & & & \mathbf{A}_{1t_1} & & \\ & \mathbf{A}_{iS_i} & & & \mathbf{A}_{it_i} & \\ & & \mathbf{A}_{nS_n} & & & \mathbf{A}_{nt_n} \end{array}\right]$$

If we premultiply this matrix by the matrix

$$\begin{array}{ccc} P_1 & P_i & P_n \\ [\mathbf{k}_1 \vdots \cdots & \mathbf{k}_i & \cdots \vdots \mathbf{k}_n] \end{array}$$

we obtain the matrix

$$\begin{array}{cccccc} S_1 & S_i & S_n & P_1 & P_i & P_n \\ [\mathbf{A}_{S_1} \cdots & \mathbf{A}_{S_i} \cdots & \mathbf{A}_{S_n} \vdots & \mathbf{A}_{t_1} \cdots & \mathbf{A}_{t_i} \cdots & \mathbf{A}_{t_n}] \end{array}$$

observe that $\mathbf{A}_P$ appears as the submatrix corresponding to columns P. From the definition of g-minors it follows that $\mathbf{V}_S \subseteq ((\oplus_{i=1}^{n} \mathbf{V}_{S_iP_i} \leftarrow \mathbf{V}_P)$. We will now show the reverse inequality. Let $\mathbf{f}_S$ be a vector in $(\oplus_{i=1}^{n} \mathbf{V}_{S_iP_i}) \leftarrow \mathbf{V}_P$ Then $\mathbf{f}_S$ can be written as $\mathbf{f}_S^1 + \mathbf{f}_S^2$ where $\mathbf{f}_S \epsilon \oplus_{i=1}^{n}(\mathbf{V}_S \times S_i)$ and $\mathbf{f}_S^2$ is the restriction of a vector $\mathbf{f}_{SP}^2$, which is linearly dependent on the rows of $\mathbf{A}_{\oplus}$ and further $\mathbf{f}_{SP}^2/_P \,\epsilon\, \mathbf{V}_P$. $\mathbf{f}_S$ clearly belongs to $\mathbf{V}_S$. We will show that $\mathbf{f}_S^2$ also does. We have $\mathbf{f}_{SP}^2 = (\boldsymbol{\lambda})(\mathbf{A}_{\oplus})$. Hence $\mathbf{f}_{SP}^2/_P = (\boldsymbol{\lambda})(\mathbf{A}_{\oplus P})$. We know that $\mathbf{f}_{SP}^2/_P$ is linearly dependent on the rows of

$$\begin{array}{c} P_1 \cdots P_n \\ [\mathbf{A}_{t_1} \cdots \mathbf{A}_{t_n}] \end{array} = (\mathbf{k}_1 \cdots \mathbf{k}_n) \left[\begin{array}{ccc} P_1 & P_i & P_n \\ \mathbf{A}_{1t_1} & & \\ & \mathbf{A}_{it_i} & \\ & & \mathbf{A}_{nt_n} \end{array}\right]$$

The matrix postmultiplying $(\mathbf{k}_1, \ldots, \mathbf{k}_n)$ is the matrix $\mathbf{A}_{\oplus P}$. Since this is non-singular $\boldsymbol{\lambda}$ is linearly dependent on the rows of $(\mathbf{k}_1 \cdots \mathbf{k}_n)$. Hence $\mathbf{f}_{SP}^2/_S$ is spanned by the rows of $[\overset{S_1}{\mathbf{A}_{S_1}} \cdots \overset{S_n}{\mathbf{A}_{S_n}}]$. Hence $\mathbf{f}_S^2 \,\epsilon\, \mathbf{V}_S$.

The minimality of the decomposition follows by verifying that the condition of Theorem 21(c) is satisfied in this case.

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