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June 11, 2009

Foreword

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Preface

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Chapter 1

Matroids

1.1 Introduction

Matroids are important combinatorial structures both from the point of view of theory and from that of applications. Hassler Whitney [Whitney35] introduced matroids as a generalization of the concept of linear independence in the context of matrices. The idea was arrived at independently also by Van der Waerden in [Van der Waerden37]. Matroid theory is one of the areas that straddles across several branches of discrete mathematics such as Combinatorics, Graph Theory, Finite Fields, Algebra and Coding Theory. One of the subjects to which applications were found early was electrical network theory [Seshu+Reed61]. In this chapter we give a brief sketch of the theory with electrical networks in mind.

1.2 Axiom systems for Matroids

A **matroid** can be defined in several equivalent ways. Each of these is based on an axiom system. The primitive objects of each axiom system can be identified with either the primitive or some derived objects of every other axiom system. We restrict ourselves to finite underlying sets even though it is possible to define infinite matroids. The concept of maximality and minimality (with respect to set inclusion) is often used in Matroid theory. We may note that, in general, maximal and minimal members of a collection of sets may not be largest or smallest in terms of size.
Example: Consider the collection of sets $\{\{a, b, c\}, \{g\}, \{e, f\}, \{a, b, c, d, e, f\}\}$. The minimal members of this collection are $\{a, b, c\}, \{g\}, \{e, f\}$, i.e., these do not contain proper subsets which are members of this collection. The maximal members of this collection are $\{g\}, \{a, b, c, d, e, f\}$, i.e., these are not proper subsets of other sets which are members of this collection.

However, in the key concept of collection of independent sets in matroids, maximality implies the maximum size property.

1.2.1 Independence and Base Axioms

Independent sets of a matroid correspond to subforests (or, dually, subcoforests) of graphs and to independent sets of columns of matrices.
I. **Independence Axioms:**

Definition 1.2.1 *(***Matroid: Independence Axioms***) A* **matroid** $\mathcal{M}$ *on* S *is a pair* $(S, \mathcal{I})$*, where* S *is a finite set and* $\mathcal{I}$ *is a family of subsets of* S*, always containing the empty subset, called* **independent sets***, that satisfies the following:*

I1 *if* $I_1 \in \mathcal{I}$ *and* $I_2 \subseteq I_1$*, then* $I_2 \in \mathcal{I}$*.*

I2 *maximally independent sets contained in a subset of* S *have the same cardinality.*

(equivalently axiom **I2** *can be replaced by the following)*

I2' *if* X, X' *are independent and* $|X'| > |X|$*, then there exists* $e' \in (X' - X)$ *such that* $X \cup e'$ *is independent.*

A **base** *of the matroid* $\mathcal{M} \equiv (S, \mathcal{I})$*, is a maximally independent subset of* $\mathcal{M}$ *contained in* S*. The complement, relative to* S*, of a base is called a* **cobase** *of* $\mathcal{M}$*.*

It follows immediately from the above that all bases of a matroid have the same cardinality. Indeed, as we shall soon see, matroids can also be defined using an axiom system that describes the collection of "bases".

1.2.2 Examples of Matroids

Theorem 1.2.1 *(***Polygon and Bond Matroids***) Let* $\mathcal{G}(V, E)$ *be an undirected graph. Let* $\mathcal{I}_f$ *be the collection of subforests and let* $\mathcal{I}_c$ *be the collection of subcoforests of* $\mathcal{G}$ *(denoting by forest, a spanning forest of the graph). Then* $(E, \mathcal{I}_f), (E, \mathcal{I}_c)$ *are matroids. Further, the bases of either matroid are cobases of the other.*

Proof : We will only prove that the collection $\mathcal{I}_f$ satisfies independence axioms. The proofs of the statements for $\mathcal{I}_c$ will follow directly from the definition of dual matroid that we will encounter later. That the independence axioms are satisfied by $\mathcal{I}_f$, follows from the facts below

- maximal intersection of a forest of $\mathcal{G}$ with $T, T \subseteq E$, is a forest of the subgraph of $\mathcal{G}$ on T.
- all forests of the subgraph of $\mathcal{G}$ on T have the same cardinality.

■

The matroid $(E, \mathcal{I}_f)$ is called the *polygon matroid* of $\mathcal{G}$, and is denoted by $\mathcal{M}(\mathcal{G})$. The matroid $(E, \mathcal{I}_c)$ is called the *bond matroid* of $\mathcal{G}$, and is denoted by $\mathcal{M}^*(\mathcal{G})$.

Let $\mathcal{V}$ be a vector space over the field of real numbers with components indexed by the elements of S. Let $\mathbf{R}, \mathbf{R}^*$ be representative matrices (i.e., their rows constitute bases) of $\mathcal{V}, \mathcal{V}^{\perp}$, respectively. Since the column dependence structure of all representative matrices of a vector space is the same (due to non-singularity of elementary row operations), it makes sense to consider an arbitrary representative matrix to study linear independence of columns. Let $\mathcal{I}$ be the collection of independent column sets of $\mathbf{R}$ (identified with corresponding subsets of S) and let $\mathcal{I}^*$ be the collection of independent column sets of $\mathbf{R}^*$.

Theorem 1.2.2 *(**Vectorial Matroid**) $(S, \mathcal{I}), (S, \mathcal{I}^*)$ are matroids. Further, bases of each matroid are cobases of the other.*

Proof : $(S, \mathcal{I})$ satsfies independence axioms follows from the fact that maximally independent subsets of columns of any submatrix of $\mathbf{R}$ $(\mathbf{R}^*)$ have the same cardinality. Further if $(\mathbf{I} : \mathbf{K})$ is a standard representative matrix of $\mathcal{V}$ then $(-\mathbf{K}^T : \mathbf{I})$ is a standard representative matrix of $\mathcal{V}^{\perp}$. Since there is a standard representative matrix corresponding to each maximally independent subset of columns, we conclude that bases of either of $(S, \mathcal{I}), (S, \mathcal{I}^*)$ are cobases of the other. ∎

We say that $(S, \mathcal{I})$ $((S, \mathcal{I}^*))$ is the *matroid (dual matroid) associated with* $\mathcal{V}$ and denote it by $\mathcal{M}(\mathcal{V})$ $(\mathcal{M}^*(\mathcal{V}))$.

Theorem 1.2.3 *(**Matroid Union**) [Edmonds68][Nash-Williams67] Let $\mathcal{G}(V, E)$ be an undirected graph and let k be a positive integer. Let $\mathcal{I}_{k\cup}$ be the collection of unions of k forests (not necessarily edge-disjoint) of $\mathcal{G}$. Then $(E, \mathcal{I}_{k\cup})$ is a matroid.*

Proof : Let $\mathcal{I}_{12}$ be the collection of all sets of the form $X \cup Y$, where X_1, X_2 are independent respectively in the matroids $\mathcal{M}_1$, $\mathcal{M}_2$. We prove that $\mathcal{M}_{12} \equiv (S, \mathcal{I}_{12})$ is a matroid. The hereditary axiom Axiom I1 is clearly satisfied. Therefore it suffices to prove Axiom I2' to conclude that $\mathcal{M}_{12} \equiv (S, \mathcal{I}_{12})$ is a matroid. (This result (as Theorem 1.7.1), is discussed in detail later. The present proof follows the one in [Mirsky71].) The theorem would then follow immediately since by Theorem 1.2.1 the collection of forests of a graph constitute the independent sets of a matroid. Let $Z_a = X_1 \cup X_2$ and let $Z_b = Y_1 \cup Y_2$ with X_1, Y_1 independent in $\mathcal{M}_1$ and X_2 and Y_2 independent in $\mathcal{M}_2$. Further let the division of Z_a into $X_1 \cup X_2$ be such that the 'cross sum' $|X_1 \cap Y_2| + |X_2 \cap Y_1|$ is a minimum among all such divisions. Now if $|Z_b| > |Z_a|$, either $|Y_1| > |X_1|$ or $|Y_2| > |X_2|$. Wlog let $|Y_2| > |X_2|$. Then there exists $e \in (Y_2 - X_2)$ such that $e \cup X_2$ is independent in $\mathcal{M}_2$. If $e \in X_1$ then the division of Z_a as $(X_1 - e) \cup (X_2 \cup e)$ would yield a lower cross sum than the division $X_1 \cup X_2$, a contradiction. Hence $e \notin X_1$ and therefore $e \in (Z_b - Z_a)$. Thus we have $e \cup Z_a = X_1 \cup (X_2 \cup e)$ is in the collection $\mathcal{I}_{12}$ confirming Axiom I2'. ∎

Let $\mathcal{G}(V, E)$ be an undirected graph. A matching of $\mathcal{G}$ is a subset of edges no two of which have a common endpoint. We say that a set of vertices U and a matching X **meet iff** the set of the endpoints of the edges in X include all the vertices of U. Let $\mathcal{I}_m$ be the collection of subsets of those vertices which meet a matching.

Theorem 1.2.4 *[Edmonds+Fulkerson65] (***Matching Matroid***) $(V, \mathcal{I}_m)$ is a matroid.*

Proof : Let $T \subseteq V(\mathcal{G})$ and let I_1, I_2 be two maximal subsets of T in the collection $\mathcal{I}_m$. We will show that $| I_2 |=| I_1 |$. There exist matchings M_1, M_2 which meet I_1, I_2. Consider the subgraph of $\mathcal{G}$ on $M_1 \cup M_2$ (the vertex set of this subgraph may contain vertices outside T). Each component of this subgraph is either a circuit or a path. If a node of T has degree two in the above subgraph, then it must belong to both I_1 and I_2 (using the fact that both these are maximal subsets of T in the collection $\mathcal{I}_m$). So vertices in T, which are in components which are circuits, are in $I_1 \cap I_2$. Now let $| I_2 |>| I_1 |$. Then in one of the components, that is a path, the subset of nodes of I_2 is of larger size than the subset of nodes of I_1 in the same component. Once again, the 'middle' T nodes, in such components, must belong to $I_1 \cap I_2$. This means one of the end nodes, say v, of the path is in $I_2 - I_1$ and the other end node is not in $I_1 - I_2$. If we modify M_1 by dropping the edges of M_1 in this path and adding the edges of M_2, the new matching M_1' will meet T in $I_1 \cup v$. Hence, $I_1 \cup v \in \mathcal{I}_m$, a contradiction. Therefore $| I_1 |=| I_2 |$. ∎

1.2.3 Base axioms

In this subsection, we characterize matroids through an axiom system for *bases* (that is, *maximally independent sets*). We also consider an abstraction of the concept of a circuit. Note that a circuit of a graph is not contained in any forest and is the minimal such subset of edges. This motivates us to define a *circuit* of a matroid $(S, \mathcal{I})$ to be a minimal subset of S not contained in any independent set, equivalently, we could say that a circuit is a *minimal dependent (or non-independent)* subset of the matroid.

We will now arrive at a characterizing property of bases of a matroid.

Let B_1, B_2 be two bases of the matroid $\mathcal{M} \equiv (S, \mathcal{I})$. Let $e \in B_2 - B_1$. Therefore $B_1 \cup e$, not being independent, contains a circuit (a minimal dependent set). We will prove uniqueness of this circuit. Suppose the contrary, let C_1, C_2 be two circuits contained in $B_1 \cup e$. Clearly $e \in C_1 \cap C_2$ and $\{e\}$ is not a circuit as e is an element inside a base. A maximally independent subset of $C_1 \cup C_2$ containing e, cannot have the cardinality exceed $| C_1 \cup C_2 | -2$. On the other hand $C_1 \cup C_2 - e$ is stated to be independent. This violates the Axiom I2 for $C_1 \cup C_2$. Therefore $C_1 = C_2$. (The unique circuit contained in $e \cup B_1$ is referred to as the *fundamental circuit (f-circuit) of e with respect to* B_1 and denoted by $L(e, B_1)$.)

Now $L(e, B_1)$ has a nonempty intersection with $B_1 - B_2$ since $\{e\}$ itself is not a circuit and since B_2 is independent, $L(e, B_1)$ cannot be contained in it. Let $e' \in L(e, B_1) \cap (B_1 - B_2)$. Let ${B_1}' \equiv e \cup (B_1 - e')$. Clearly ${B_1}'$ is also a base as it contains no circuit and has the same cardinality as B_1. On the other hand for any $e" \in B_1$, $(B_1 - e") \cup e$ is independent provided $e" \in L(e, B_1)$. We therefore have the following theorem.

Theorem 1.2.5 *Let B_1, B_2 be bases of a matroid $\mathcal{M}$ on S. Let $e \in B_2 - B_1$. Then,*

1. *$e \cup B_1$ contains a unique circuit $L(e, B_1)$. This circuit has a nonempty intersection with $B_1 - B_2$.*

2. *If $e' \in B_1$, then $(B_1 - e') \cup e$ is a base of $\mathcal{M}$ iff $e' \in L(e, B_1)$.*

This consequence of independence axioms for collection of bases would, as we shall soon see, indeed characterize the collection of bases of a matroid. In fact, the following "dual" observation also leads to an alternative axiomatic characterization for the collection of bases.

Let B_1, B_2 be bases of a matroid and let $e_1 \in B_1 - B_2$. B_2 is a maximally independent subset of $B_1 \cup B_2 - e_1$. Further as $|B_1| = |B_2|$, $B_1 - e_1$ is not maximally independent in this set. Hence, by Axiom I2', there exists $e_2 \in B_2 - B_1$ s.t. $(B_1 - e_1) \cup e_2$ is independent and, therefore, a base of $\mathcal{M}$. We therefore have the following.

Theorem 1.2.6 *Let B_1, B_2 be bases of $\mathcal{M}$ and let $e_1 \in B_1 - B_2$. Then there exists $e_2 \in B_2 - B_1$, such that $(B_1 - e_1) \cup e_2$ is a base of $\mathcal{M}$.*

As remarked earlier, Theorems 1.2.5 and 1.2.6 can be used to generate axiom systems for matroids.

Theorem 1.2.7 *(***Base Axioms***) Let a collection $\mathcal{B}$ of subsets of S satisfy the following equivalent axioms*

Axiom B *If B_1, B_2 are members of $\mathcal{B}$ and if $e_2 \in B_2 - B_1$, then there exists $e_1 \in B_1 - B_2$ s.t. $(B_1 - e_1) \cup e_2$ is a member of $\mathcal{B}$.*

Axiom B' *If B_1, B_2 are in $\mathcal{B}$ and if $e_1 \in B_1 - B_2$ then there exists $e_2 \in B_2 - B_1$ s.t. $(B_1 - e_1) \cup e_2$ is in $\mathcal{B}$.*

Then the collection $\mathcal{I}$ of subsets of the sets in $\mathcal{B}$ is a collection of independent subsets of a matroid $(S, \mathcal{I})$. (In other words $\mathcal{B}$ is the collection of bases of a matroid.)

Proof : When Axiom B is satisfied, to prove that $(S, \mathcal{I})$ is a matroid, we only need to prove that maximal subsets in $\mathcal{I}$ contained in a given subset T (denoted say $\mathcal{B}_T$) of S have the same cardinality. If B_{T1}, B_{T2} are two such subsets we take an element $e_2 \in B_{T2} - B_{T1}$ and add it to B_{T1} and drop a suitable element in $B_{T1} - B_{T2}$ as in the argument to prove Theorem 1.2.5 . The resulting set is also in $\mathcal{B}_T$ and has the same cardinality as B_{T1}. Repeatedly using the argument will finally give us a set in $\mathcal{B}_T$ containing B_{T2} with the same cardinality as B_{T1}. Clearly this must be B_{T2} itself.

Let us now consider the (Axiom B') case. Let $\mathcal{B}$ satisfy Base Axioms B'. We need only show that maximal subsets, from the collection $\mathcal{I}$, contained in $T \subseteq S$ have the same cardinality.

Case 1: $T = S$.

If B_1, B_2 are bases (i.e., members of $\mathcal{B}$) and $e \in B_1 - B_2$, we can find an $e' \in B_2 - B_1$ s.t. $(B_1 - e) \cup e'$ is a base. If we repeat this procedure we would finally get a base $B_k \subseteq B_2$ s.t. $\mid B_k \mid = \mid B_1 \mid$. But one base cannot properly contain another. So $B_k = B_2$ and $\mid B_2 \mid = \mid B_1 \mid$.

Case 2: $T \subset S$.

Suppose $X \equiv \{x_1, \cdots, x_k\}$ and $Y \equiv \{y_1, \cdots, y_m\}$ are maximal subsets of T, from the collection $\mathcal{I}$, with $k < m$. Let X be contained in a base B_x and Y in a base B_y. Let

$$B_x \equiv \{x_1, \cdots, x_k, p_{k+1}, \cdots, p_r\}$$

$$B_y \equiv \{y_1, \cdots, y_m, q_{m+1}, \cdots, q_r\}$$

(Note that p_i's and q_j's are outside T.) Since $k < m$, there exists $p_t \in B_x - B_y$. Therefore, there exists z in $B_y - B_x$ s.t. $(B_x - p_t) \cup z$ is a base. Clearly z cannot be one of the y_i for otherwise it would violate maximality of X. Say, $z = q_s$. We thus have a new base $B'_x \equiv (B_x - p_t) \cup q_s$. Note the progress, $(B_y - B'_x) \cap (S - T) \subset (B_y - B_x) \cap (S - T)$. Repeating this procedure we would finally arrive at a base B^f_x s.t. $B^f_x \cap (S - T) \supset B_y \cap (S - T)$.

Any further attempted exchange using one of the remaining p elements from B^f_x would result in violation of maximality of X as a subset of T from the collection $\mathcal{I}$. The only way to avoid this contradiction is to have $k = m$. Therefore we conclude that the maximal subsets of T, belonging to the collection $\mathcal{I}$, have the same cardinality. ■

1.2.4 Rank Axioms

In this subsection we discuss characterization of matroids through an axiom system for the rank function.

Let $\mathcal{M}$ be a matroid on S. The *rank* of a subset $T \subseteq S$ is defined as the cardinality of the maximally independent set contained in T. This number is well defined since all maximally independent subsets of T have the same cardinality. The rank of T is denoted by $r(T)$. $r(S)$ is also called the rank of $\mathcal{M}$ and is denoted also by $r(\mathcal{M})$.

Clearly $r(\cdot)$ takes value 0 on $\emptyset$. Moreover the rank function is clearly an integral, increasing function on subsets of S. Also $r(X \cup e) - r(X) \leq 1 \ \forall \ X \subseteq S, e \in S$. We have the following properties of rank function which will motivate axiom systems for matroids in terms of rank function.

Theorem 1.2.8 *Let $r(\cdot)$ be the rank function of a matroid on S, $X \subseteq S$ and $e_1, e_2 \in S$.*

1. $r(X \cup e_1) = r(X \cup e_2) = r(X)$*, implies* $r(X \cup e_1 \cup e_2) = r(X)$.

2. $r(X \cup e) - r(X) \geq r(Y \cup e) - r(Y)$ *whenever* $X \subseteq Y \subseteq S - e$.

3. $r(\cdot)$ *is submodular, i.e.,*

$$r(X) + r(Y) \geq r(X \cup Y) + r(X \cap Y) \ \forall \ X, Y \subseteq S.$$

Proof : i. Let B_X be a maximal independent subset of X. Clearly B_X is also a maximally independent subset of $X \cup e_1$ as well as of $X \cup e_2$ as their ranks are same. Suppose B_X is not a maximally independent subset of $X \cup e_1 \cup e_2$. This means either $B_X \cup e_1$ or $B_x \cup e_2$ must be independent. But this is a contradiction.

ii. Consider any maximally independent subset B_X of X. One can grow it into a maximally independent subset B_Y of Y. $r(X \cup e) - r(X) \geq 0$ and $r(Y \cup e) - r(Y)$ can be seen to be 0 or 1. It suffices to consider the case where $r(Y \cup e) - r(Y) = 1$. Then B_Y is not maximally independent in $Y \cup e$. Therefore $B_Y \cup e$ must be independent, which implies $B_x \cup e$ must also be independent (by Axiom I1). Therefore $r(X \cup e) - r(X) = 1$. Thus rank function always satisfies $r(X \cup e) - r(X) \geq r(Y \cup e) - r(Y)$.

iii. Let $Y - X = \{e_1, \cdots, e_k\}$. We then have

$$\begin{aligned}
r(Y) - r(X \cap Y) = \quad & r((X \cap Y) \cup e_1) - r(X \cap Y) \\
+ \quad & r((X \cap Y) \cup e_1 \cup e_2) - r((X \cap Y) \cup e_1) + \cdots \\
+ \quad & r((X \cap Y) \cup e_1 \cdots \cup e_k) - r((X \cap Y) \cup e_1 \cup \cdots \cup e_{k-1}) \\
\geq \quad & r(X \cup e_1) - r(X) + \cdots \\
+ \quad & r(X \cup e_1 \cup \cdots \cup e_k) - r(X \cup e_1 \cup \cdots \cup e_{k-1}) \\
\geq \quad & r(X \cup Y) - r(X).
\end{aligned}$$

■

Example 1.2.1 *For the polygon matroid $\mathcal{M}(\mathcal{G})$ (independent set $\equiv$ subforest), the rank function of the matroid is the same as the rank function of the graph. For the bond matroid $\mathcal{M}^*(\mathcal{G})$, the rank function is the nullity function $\nu(\cdot)$ of the graph, where $\nu(A), A \subseteq E(\mathcal{G})$, is the number of edges in a coforest of $\mathcal{G} \times A$ (the graph obtained by fusing the endpoints of edges outside A and removing them).*

An important property of matroid rank functions is that they are submodular.

Next we describe and justiify an axiom system for matroids based on the rank function. The axioms will all be properties of rank function that follow from being a matroid rank function. We show that certain combinations of these properties as axioms ensure matroid-ness.

Theorem 1.2.9 *(***Rank Axioms** *) Let S be a finite set and let $r(\cdot)$ be an integer valued submodular function on subsets of S satisfying in addition*

$$r(\emptyset) = 0$$

$$0 \leq r(X \cup e) - r(X) \leq 1 \quad \forall\ X \subseteq S, e \in S.$$

Define $\mathcal{I}$ to be the collection of all subsets X of S satisfying $r(X) =| X |$. Let members of $\mathcal{I}$ be called **independent**. *Then $(S,\mathcal{I})$ is a matroid (satisfying the Independence Axioms).*

Proof :
i. Let $Y \in \mathcal{I}$ and $X \subseteq Y$. We need to show that $X \in \mathcal{I}$.
We are given that $r(\emptyset) = 0$ and $r(A \cup e) \leq r(A) + 1\ \forall\ A \in S$. Therefore, $r(X) \leq| X |$ and $r(Y) - r(X) \leq| Y | - | X |$. So $r(X) <| X |$, implies $r(Y) <| Y |$, which is a contradiction. Thus $r(X) =| X |$, i.e., $X \in I$.
ii. Let B_1, B_2 be two maximal members of $\mathcal{I}$ contained in a subset T of S. We need to establish that $| B_1 |=| B_2 |$.
For each $e_i \in T - B_1$, $r(B_1) \leq r(B_1 \cup e_i) <| B_1 | +1$, since B_1 is a maximal subset of T such that $r(B_1) = |B_1|$. Therefore, $r(B_1 \cup e_i) = r(B_1)\ \forall\ e_i \in T - B_1$. For any U, V satisfying $r(B_1 \cup U) = r(B_1 \cup V) = r(B_1)$, we have, using submodularity of $r(\cdot)$ that

$$r(B_1 \cup U) + r(B_1 \cup V) \geq r(B_1 \cup U \cup V) + r(B_1 \cup (U \cap V)).$$

Note that LHS above is $2r(B_1)$ and RHS is greater or equal to $2r(B_1)$ ($r(\cdot)$ is an increasing function), therefore

$$r(B_1 \cup U \cup V) = r(B_1 \cup (U \cap V)) = r(B_1).$$

Using this inductively we can prove that

$$r(B_1 \cup e_1 \cup \cdots \cup e_k) = r(B_1),$$

where $\{e_1, \cdots, e_k\} = T - B_1$. Hence, $r(B_1) = r(T)$.
Similarly $r(B_2) = r(T)$.
Therefore $| B_1 |= r(B_1) = r(B_2) =| B_2 |$. ∎

We call a set function $r(\cdot)$, satisfying the properties stated in Theorem 1.2.9, a *matroid rank function.*

1.2.5 Circuit Axioms

Circuits of matroids satisfy the conditions given in the following theorem. We will use these conditions to define an axiom system for matroids using circuits as primitive objects.

Theorem 1.2.10 *Let $\mathcal{M} \equiv (S,\mathcal{I})$ be a matroid and let C_1, C_2 be non-disjoint circuits of $\mathcal{M}$ with e_c as one of the common elements. There exists an $e_1 \in C_1 - C_2$, (as circuits cannot be contained in one another, due to their minimality). Then there exists a circuit $C_3 \subseteq C_1 \cup C_2 - e_c$ s.t. $e_1 \in C_3$.*

The following lemma will be used for the proof of the theorem.

Lemma 1.2.1 *Let $\mathcal{M}, C_1, C_2$ be as in Theorem 1.2.10. Then there exists a circuit $C_3' \subseteq C_1 \cup C_2 - e_c$.*

Proof of Lemma 1.2.1: We use properties of rank function of the matroid. Recall that a set X is independent iff $r(X) = |X|$, and by the definition of a circuit, $r(C_i) = |C_i| - 1, i = 1, 2$.
$r(C_1 \cup C_2) + r(C_1 \cap C_2) \leq r(C_1) + r(C_2) = |C_1| - 1 + |C_2| - 1$. Therefore

$$r(C_1 \cup C_2) + |C_1 \cap C_2| \leq |C_1 \cup C_2| + |C_1 \cap C_2| - 2,$$

since $r(C_1 \cap C_2) = |C_1 \cap C_2|$. This implies $r(C_1 \cup C_2) \leq |C_1 \cup C_2| - 2$, therefore $r(C_1 \cup C_2 - e_c) \leq r(C_1 \cup C_2) \leq |C_1 \cup C_2 - e_c| - 1$, which proves that there exists a circuit inside $C_1 \cup C_2 - e_c$. ∎

Proof of Theorem 1.2.10: The result is clearly true when the union of the two circuits has size 3 and, trivially, when it is 2. We now use induction on the size of the union of the two circuits. Suppose that the result is true when the size of the union of the two circuits is less than n.
Let $| C_1 \cup C_2 | = n$, $e_c \in C_1 \cap C_2$ and $e_1 \in C_1 - C_2$. By the above lemma (Lemma 1.2.1), one is guaranteed existence of a circuit $C_3' \subseteq C_1 \cup C_2 - e_c$. If $e_1 \in C_3'$ we are done.

So assume that $e_1 \notin C_3'$. We have, $C_3' \nsubseteq C_1$ and $C_3' \subseteq C_1 \cup C_2$. So $C_3' \cap C_2 \nsubseteq C_1 \cap C_2$. Let $e_2 \in C_3' \cap C_2 - C_1 \cap C_2$.
Consider $C_2 \cup C_3'$. We aim to use induction hypothesis on it, for which we need to show that its size is less than n. But this follows from the observation that $e_1 \notin C_2 \cup C_3'$. Further $e_2 \in C_3' \cap C_2$ and $e_c \in C_2 - C_3'$. By the induction hypothesis there is a circuit $C_2' \subseteq C_2 \cup C_3' - e_2$ s.t. $e_c \in C_2'$.
Now consider $C_1 \cup C_2'$. We have $e_c \in C_2' \cap C_1$ and $e_1 \in C_1 - C_2'$. Further $e_2 \notin C_1 \cup C_2'$ so that $| C_1 \cup C_2' | < n$ and we can apply induction hypothesis. Therefore, there exists a circuit $C_3 \subseteq C_1 \cup C_2' - e_c$ s.t. $e_1 \in C_3$. ∎

Example 1.2.2 *For the polygon matroid $\mathcal{M}(\mathcal{G})$ of $\mathcal{G}$ (independent set $\equiv$ subforest), a circuit of the matroid is the same as a circuit of the graph. For the bond matroid $\mathcal{M}^*(\mathcal{G})$ (independent set $\equiv$ subcoforest), a circuit of the matroid is the same as a cutset of the graph. For the vectorial matroid (associated with the columns of a representative matrix), a circuit is a minimal dependent set of columns.*

Theorem 1.2.11 *(***Circuit Axioms:***)*

Let S be a finite set. Let $\mathcal{C}$ denote a family of subsets (called circuits) of S satisfying
Axiom C1 *No member of $\mathcal{C}$ is a proper subset of another.*
Axiom C2 *Let $C_1, C_2 \in \mathcal{C}$ and let $e_c \in C_1 \cap C_2$ and $e_1 \in C_1 - C_2$. Then there exists $C_3 \in \mathcal{C}$ s.t. $C_3 \subseteq C_1 \cup C_2 - e_c$ and $e_1 \in C_3$. Let $\mathcal{I}$ be the class of subsets of S that do not contain a member of $\mathcal{C}$. Then $(S, \mathcal{I})$ satisfies the axioms I1 and I2 of a matroid. (This also justifies denoting a matroid $\mathcal{M}$ as a pair $(S, \mathcal{C})$ describing collection of circuits as primitive objects).*

Proof : We will show that maximal subsets of $T \subseteq S$ that are in $\mathcal{I}$ (i.e., that do not contain a circuit) have the same cardinality. For readability, during the course of the proof we will call members of $\mathcal{I}$ independent and maximal members, bases although this terminology is justified only after the proof is complete.

Let B_1, B_2 be two maximal independent sets contained in T. If $B_1 \neq B_2$, clearly $B_1 \not\supseteq B_2$. Let $e_2 \in B_2 - B_1$. Then $e_2 \cup B_1$ contains a circuit. Claim is that this circuit is unique. For otherwise if C_1, C_2 are two such circuits since both have e_2 as a member, then by the circuit axioms there exists a circuit $C_3 \subseteq C_1 \cup C_2 - e_2$. This is impossible as it would imply that a circuit C_3 is a subset of the base B_1.

Define $L(e_2, B_1)$ to be the unique circuit contained in $e_2 \cup B_1$. Since $\{e_2\}$ is not a circuit (element e_2 is inside the base B_2) we must have $L(e_2, B_1) \cap B_1$ is nonempty. Also $L(e_2, B_1)$ is not a subset of B_2. Therefore, there exists an $e_1 \in L(e_2, B_1) \cap (B_1 - B_2)$. Then $B_1' \equiv e_2 \cup B_1 - e_1$ is independent.

We claim that B_1' is also maximally independent. For, let $e' \in T - B_1'$. For the case $e' = e_1$, we note that $e' \cup B_1'$ contains $L(e_2, B_1)$, thereby not violating maximality of B_1'. So consider $e' \neq e_1$. Now $e' \cup B_1$ contains a circuit $L(e', B_1)$. If this circuit does not contain e_1, we have $L(e', B_1) \subseteq e' \cup B_1'$, again not violating maximality of B_1'. Suppose it contains e_1. Then $L(e_2, B_1)$ and $L(e', B_1)$ have the element e_1 in common. Hence, $L(e_2, B_1) \cup L(e', B_1) - e_1$ contains a circuit. This circuit is contained in $e' \cup B_1'$.
Therefore, we conclude that $e' \cup B_1'$ is not independent for every $e' \in T - B_1'$.
We now have a maximally independent subset B_1' of T which has the same cardinality as B_1. However B_1' also satisfies $\mid B_2 - B_1' \mid < \mid B_2 - B_1 \mid$. Repeating this procedure we would arrive at a maximally independent subset B_k of T that has the same cardinality as B_1 and also contains B_2. But this implies $B_k = B_2$ and hence, $\mid B_1 \mid = \mid B_2 \mid$. ∎

1.3 Dual of a Matroid

Matroids occur naturally in pairs. This pairing is analogous to that of complementary orthogonal vector spaces.

Consider a vector space $\mathcal{V}$ on S. We remind the reader that *dot product* of two vectors $\mathbf{f}, \mathbf{g}$ on S denoted by $< \mathbf{f}, \mathbf{g} >$ over a field $\mathcal{F}$ is defined by $< \mathbf{f}, \mathbf{g} > \equiv \sum_{e \in S} \mathbf{f}(e).\mathbf{g}(e)$. We say $\mathbf{f}, \mathbf{g}$ are *orthogonal* if their dot product is zero. $\mathcal{V}^{\perp}$, the space *complementary orthogonal* to $\mathcal{V}$, is the collection of all vectors on S which are orthogonal to every vector in $\mathcal{V}$. Let $\mathcal{M}(\mathcal{V})$ denote the matroid on S whose independent sets are the linearly independent column sets of a representative matrix (i.e., rows constitute a basis) of $\mathcal{V}$. Note that column dependence structure of all representative matrices of $\mathcal{V}$ is identical. Whenever we have a set of columns as a base of this matroid we know that we can build a 'standard' representative matrix with identity matrix corresponding to this set as below.

$B \qquad\qquad S - B$

$$\mathbf{R} \equiv \left[\begin{array}{ccc} \mathbf{I} & \vdots & \mathbf{K} \end{array} \right]. \tag{1.1}$$

Then we know that

$$\begin{array}{c} \\ \mathbf{R}^* \equiv \end{array} \begin{array}{c} \begin{array}{ccc} B & & S-B \end{array} \\ \left[\begin{array}{ccc} -\mathbf{K}^T & \vdots & \mathbf{I} \end{array} \right], \end{array} \tag{1.2}$$

is a representative matrix of $\mathcal{V}^{\perp}$. It is thus clear that the bases of $\mathcal{M}(\mathcal{V}^{\perp})$ are cobases of $\mathcal{M}(\mathcal{V})$ and vice versa. This situation can be shown to hold also for arbitrary matroids and the pairs of matroids are said to be dual to each other.

Theorem 1.3.1 *Let $\mathcal{M} \equiv (S, \mathcal{I})$ be a matroid. Then, $\mathcal{M}^* \equiv (S, \mathcal{I}^*)$, where $X \in \mathcal{I}^*$ iff $S - X$ contains a base of $\mathcal{M}$, is a matroid. $\mathcal{M}^*$ is called the* **dual matroid**. *Further, $(\mathcal{M}^*)^* = \mathcal{M}$.*

Proof : It suffices to prove that the collection of complements of the bases of $\mathcal{M}$ satisfy either of the equivalent Base Axioms B or B'. This is indeed easy to see due to the duality evident between the axioms B and B'. ∎

Example 1.3.1 *If $\mathcal{V}$ is a vector space on S then $(\mathcal{M}(\mathcal{V}))^* = \mathcal{M}(\mathcal{V}^{\perp})$. If $\mathcal{M}(\mathcal{G})$ is the polygon matroid associated with the graph $\mathcal{G}$, then we know that $\mathcal{M}(\mathcal{G}) = \mathcal{M}(\mathcal{V}_v(\mathcal{G}))$, where $\mathcal{V}_v(\mathcal{G})$ is the space of all vectors spanned by the rows of the incidence matrix of the directed graph $\mathcal{G}$ and $(\mathcal{M}(\mathcal{G}))^* = \mathcal{M}((\mathcal{V}_v(\mathcal{G}))^{\perp})$. Thus matroid $(\mathcal{M}(\mathcal{G}))^*$ is the bond matroid associated with $\mathcal{G}$, for which the independent subsets are subcoforests. When the graph $\mathcal{G}$ is planar there exists a graph $\mathcal{G}^*$ s.t. $\mathcal{V}_v(\mathcal{G}^*) = (\mathcal{V}_v(\mathcal{G}))^{\perp}$. It would follow that $\mathcal{M}(\mathcal{G}^*) = (\mathcal{M}(\mathcal{G}))^*$.*

The circuit of the matroid $\mathcal{M}^*$ is called a *bond* of $\mathcal{M}$. The following theorem gives some characterizations of a bond.

Theorem 1.3.2 *Let $\mathcal{M} \equiv (S, \mathcal{I})$ be a matroid. A subset $K \subseteq S$ is a bond of $\mathcal{M}$ iff any of the following equivalent conditions hold:*

1. *K is a circuit of $\mathcal{M}^*$.*

2. *K is a minimal set intersecting every base of $\mathcal{M}$.*

3. *K is a minimal set which meets no circuit of $\mathcal{M}$ in just a single element.*

We need the following lemma to prove the theorem.

Lemma 1.3.1 *Let L be independent in $\mathcal{M} \equiv (S, \mathcal{I})$ and let K be independent in $\mathcal{M}^*$. Further let $L \cap K = \emptyset$. Then there exists a base of $\mathcal{M}$ that contains L and does not intersect K.*

Proof of Lemma 1.3.1 Independence of K in $\mathcal{M}^*$ implies $S-K$ contains a base B of $\mathcal{M}$. Now, L is given to be independent in $\mathcal{M}$ and is also a subset of $S-K$. So there exists a subset B', that is maximally independent in $\mathcal{M}$ containing L and also a subset of $S-K$. Axiom I2 tells us that $\mid B' \mid = \mid B \mid$. Hence B' is the desired base of $\mathcal{M}$. ∎

Proof of Theorem 1.3.2: Condition (i) is the definition of a bond. We will show that each of the conditions (ii) and (iii) is equivalent to (i).

(i) $\Leftrightarrow$ (ii): K is a minimal set that is not contained in any base of $\mathcal{M}^*$, equivalently, that intersects every base of $\mathcal{M}$ (since bases of $\mathcal{M}$ are complements of bases of $\mathcal{M}^*$).

(i) $\Leftrightarrow$ (iii): First we will show (i) implies (iii). We begin by showing the minimality (w.r.t. the intersection property of (iii)) of a circuit of the dual. Suppose K is a circuit of $\mathcal{M}^*$. Let $X \subset K$. Then X is independent in $\mathcal{M}^*$. Let B_X^* be a base of $\mathcal{M}^*$ containing X. Let B_X be the complement of B_X^*. B_X is a base of $\mathcal{M}$ and $B_X \cap X = \emptyset$. Let $e \in X$ and let $L(e, B_X)$ be the unique circuit of $\mathcal{M}$ contained in $e \cup B_X$. This circuit intersects X in $\{e\}$. Thus every proper subset of K meets some circuit of $\mathcal{M}$ in exactly a single element.

Now we show that when a circuit of the dual meets a circuit of the original matroid, they intersect in at least 2 elements. Suppose K meets a circuit C of $\mathcal{M}$. Let $e \in C \cap K$. $C-e$ and $K-e$ are independent in the matroid and its dual respectively. Suppose C and K intersect in just the element e. This implies $(C-e) \cap (K-e) = \emptyset$. Now, by Lemma 1.3.1, there exists a base B of $\mathcal{M}$ that contains $C-e$ but does not intersect $K-e$. We must have either $e \in B$ or $e \in (S-B)$. This would imply either $C \subseteq B$ or in the latter case, $K \subseteq (S-B)$ contradicting the independence of B in $\mathcal{M}$ and $S-B$ in $\mathcal{M}^*$ respectively. Therefore $C-e$ and $K-e$ must intersect, implying $|C \cap K| > 1$.

It remains to show that (iii) implies (i). Let $K \subseteq S$ be such that it does not meet any circuit of $\mathcal{M}$ in just a single element. Such K cannot be contained in a cobase of $\mathcal{M}$, for, if $e \in S-B$, for a base B of $\mathcal{M}$, then the fundamental circuit of matroid $\mathcal{M}$, formed by e with B intersects $(S-B)$ only in e. Hence, K is dependent in $\mathcal{M}^*$. We only need to show that K is a minimally dependent subset of the dual. Suppose the contrary, that is, there exists a proper subset K' of K that is a circuit of the dual matroid. However, as already seen, any circuit K' of $\mathcal{M}^*$ does not meet any circuit of $\mathcal{M}$ in just a single element. This would contradict minimality (w.r.t. the "intersection" property) of K. Therefore K is a circuit of $\mathcal{M}^*$. ∎

1.4 Minors of Graphs, Vector spaces and Matroids

Given a matroid, there are some natural ways of deriving matroids on subsets of the underlying sets which we will call minors of the original matroid. In this section we motivate and define this notion by first studying the most important instances of graphs and vector spaces.

1.4.1 Restriction and Contraction of Graphs

Let $\mathcal{G}(V, E)$ be a graph and let $T \subseteq E$.

Definition 1.4.1 *The graph $\mathcal{G}$* **open** $(\mathbf{E} - \mathbf{T})$ *is the subgraph of $\mathcal{G}$ with T as the set of edges and the whole $V(\mathcal{G})$ as the vertex set. That is, to obtain $\mathcal{G}$ open $(E-T)$ we remove (delete) edges in $E-T$, however, leaving their endpoints in place. The* **restriction** *of $\mathcal{G}$ to T, denoted by $\mathcal{G} \cdot \mathbf{T}$, is the subgraph of $\mathcal{G}$ obtained by deleting isolated vertices from $\mathcal{G}$ open $(E-T)$. Thus, $\mathcal{G} \cdot T$ is the subgraph of $\mathcal{G}$ on T.*
In case of a directed graph $\mathcal{G}$, we retain original directions.

Definition 1.4.2 *The graph $\mathcal{G}$* **short** $(\mathbf{E} - \mathbf{T})$*, is built by first building $\mathcal{G}$ open T. We then get connected components. Let $V_1, \cdots, V_k$ be the vertex sets of the connected components of $\mathcal{G}$ open T. The set $\{V_1, \cdots, V_k\}$ is the vertex set and T is the edge set of $\mathcal{G}$ short $(E-T)$. (The reader may imagine $\{V_1, \cdots, V_k\}$ as a set of 'supernodes' enclosed by surfaces). An edge $e \in T$ would have V_i, V_j as its endpoints in $\mathcal{G}$ short $(E-T)$ iff the endpoints of e in $\mathcal{G}$ lie in V_i, V_j. If $\mathcal{G}$ is directed, V_i, V_j would be the positive and negative endpoints of e in $\mathcal{G}$ short $(E-T)$ provided the positive and negative endpoints of e in $\mathcal{G}$ lie in V_i, V_j respectively.*
(In other words, $\mathcal{G}$ short $(E-T)$ is obtained from $\mathcal{G}$ by short circuiting the edges in $(E-T)$ (fusing their end points) and removing them).
The **contraction** *of $\mathcal{G}$ to T, denoted by $\mathcal{G} \times T$, is obtained from $\mathcal{G}$ short $(E-T)$ by deleting the isolated vertices of the latter.*

An immediate consequence of the above definitions is that the union of a forest of $\mathcal{G}$ *short* $(E-T)$ (i.e., of $\mathcal{G} \times T$) and a forest of $\mathcal{G}$ *open* T (i.e., of $\mathcal{G} \cdot (E-T)$) is a forest of $\mathcal{G}$. We therefore have

Theorem 1.4.1 $r(\mathcal{G}) = r(\mathcal{G} \times T) + r(\mathcal{G} \cdot (E-T))$.

We denote $(\mathcal{G} \times T_1) \cdot T_2$, $T_2 \subseteq T_1 \subseteq E(\mathcal{G})$ by $\mathcal{G} \times T_1 \cdot T_2$ and $(\mathcal{G} \cdot T_1) \times T_2$. $T_2 \subseteq T_1 \subseteq E(\mathcal{G})$ by $\mathcal{G} \cdot T_1 \times T_2$. Graphs denoted by such expressions are called *minors* of $\mathcal{G}$. It is easy to see that when we short a set $A \subseteq E(\mathcal{G})$ and open a disjoint set $B \subseteq E(\mathcal{G})$, then the final graph does not depend on the order in which these operations are carried out. Also note that $\mathcal{G} \times T(\mathcal{G} \cdot T)$ differs from $\mathcal{G}$ *short* $(E-T)$ ($\mathcal{G}$ *open* $(E-T)$) only in that the isolated vertices are omitted. We therefore have the following theorem. (In the statement below equality refers to isomorphism.)

Theorem 1.4.2 *Let $\mathcal{G}$ be a graph and $X_2 \subseteq X_1 \subseteq E(G)$. Then*

1. $\mathcal{G} \times X_1 \times X_2 = \mathcal{G} \times X_2$,
2. $\mathcal{G} \cdot X_1 \cdot X_2 = \mathcal{G} \cdot X_2$,
3. $\mathcal{G} \times X_1 \cdot X_2 = \mathcal{G} \cdot (E - (X_1 - X_2)) \times X_2$.

Proof : To prove each statement of the above theorem we only need to note that the graph minors on both LHS and RHS are obtained by shorting and opening the same sets. In statement (i), on both sides $E - X_2$ is shorted. In the statement (ii) $E - X_2$ is opened. In the more interesting statement of (iii), on both sides, the shorted set is $E - X_1$ and the subset of edges opened is $X_1 - X_2$. ∎

The following result shows that construction of minors is essentially a two step process. The proof is by a routine application of the previous theorem.

Theorem 1.4.3 *Any minor of the form $\mathcal{G} \times X_1 \cdot X_2 \times X_3 \ldots X_n, X_1 \supseteq \ldots \supseteq X_n$ (the graph being obtained by starting from $\mathcal{G}$ and performing the operations from left to right in succession), can be simplified to a minor of the form $\mathcal{G} \cdot X^{'} \times X_n$ or $\mathcal{G} \times X^{'} \cdot X_n$.*

The next two results are about circuits and cutsets of minors in terms of the corresponding subsets in the original graph. The routine proofs are omitted.

Theorem 1.4.4

1. *$C \subseteq T$ is a circuit of $\mathcal{G} \cdot T$ iff C is a circuit of $\mathcal{G}$.*
2. *$C \subseteq T$ is a circuit of $\mathcal{G} \times T$ iff C is a minimal intersection of circuits of $\mathcal{G}$ with T (equivalently, iff C is an intersection of a circuit of $\mathcal{G}$ with T but no proper subset of C is such an intersection).*

Theorem 1.4.5

1. *$B \subseteq T$ is a cutset of $\mathcal{G} \cdot T$ iff it is a minimal intersection of cutsets of $\mathcal{G}$ with T.*
2. *A subset B of T is a cutset of $\mathcal{G} \times T$ iff it is a cutset of $\mathcal{G}$.*

1.4.2 Restriction and Contraction of Vector Spaces

There are natural operations on vector spaces which are analogous to the operations of opening and shorting edges in a graph. We describe them now.
Let $\mathcal{V}$ be a vector space on S and let $T \subseteq S$.

Definition 1.4.3 *The* **restriction** *of $\mathcal{V}$ to T, denoted by $\mathcal{V}.T$, is defined as follows:*

$$\mathcal{V}.T \equiv \{\mathbf{f}_T : \mathbf{f}_T = \mathbf{f}/T, \mathbf{f} \in \mathcal{V}\}.$$

The **contraction** *of $\mathcal{V}$ to T, denoted by $\mathcal{V} \times T$, is defined as follows:*

$$\mathcal{V} \times T \equiv \{\mathbf{f'}_T : \mathbf{f'}_T = \mathbf{f}/T, \mathbf{f} \in \mathcal{V} \text{ and } \mathbf{f}/(S - T) = \mathbf{0}\}.$$

It is easily seen that $\mathcal{V} \cdot T$, $\mathcal{V} \times T$ are vector spaces.

We denote $(\mathcal{V} \times T_1) \cdot T_2$ by $\mathcal{V} \times T_1 \cdot T_2$, as in the case of graphs. Such vector spaces are called **minors** of $\mathcal{V}$. We say we 'open' T when we restrict $\mathcal{V}$ to $(S-T)$ and say we 'short' T when we contract $\mathcal{V}$ to $(S-T)$.
The order in which we open and short disjoint sets of elements is unimportant. This is stated formally below.

Theorem 1.4.6 *Let $T_2 \subseteq T_1 \subseteq S$. Then*

1. $\mathcal{V} \cdot T_1 \cdot T_2 = \mathcal{V} \cdot T_2$,
2. $\mathcal{V} \times T_1 \times T_2 = \mathcal{V} \times T_2$,
3. $\mathcal{V} \times T_1 \cdot T_2 = \mathcal{V} \cdot (S - (T_1 - T_2)) \times T_2$.

Proof of (iii): $LHS \subseteq RHS$
Let $\mathbf{f}_{T_2} \in \mathcal{V} \times T_1 \cdot T_2$.
Then there exists a vector $\mathbf{f}_{T_1} \in \mathcal{V} \times T_1$ such that $\mathbf{f}_{T_1}/T_2 = \mathbf{f}_{T_2}$ and a vector $\mathbf{f} \in \mathcal{V}$ with $\mathbf{f}/(S - T_1) = \mathbf{0}$ such that $\mathbf{f}/T_1 = \mathbf{f}_{T_1}$.
Now let $\mathbf{f}'$ denote $\mathbf{f}/(S - (T_1 - T_2))$.
Clearly $\mathbf{f}' \in \mathcal{V} \cdot (S - (T_1 - T_2))$. Now $\mathbf{f}'/(S - T_1) = \mathbf{0}$.
Hence, $\mathbf{f}'/T_2 \in \mathcal{V} \cdot (S - (T_1 - T_2)) \times T_2$.
Thus, $\mathcal{V} \times T_1 \cdot T_2 \subseteq \mathcal{V} \cdot (S - (T_1 - T_2)) \times T_2$.
The reverse containment is similarly proved. ∎
Remark: Observe that a typical vector of both LHS and RHS is obtained by restricting a vector of $\mathcal{V}$, that takes zero value on $S - T_1$, to T_2. We now have as in the case of graphs

Theorem 1.4.7 *Any minor of the form $\mathcal{V} \times T_1 \cdot T_2 \times T_3 \ldots T_n, T_1 \supseteq T_2 \supseteq \ldots \supseteq T_n$, can be simplified to a minor of the form*

$$\mathcal{V} \cdot T' \times T_n \text{ or } \mathcal{V} \times T' \cdot T_n.$$

1.4.3 Minors of dual vector spaces

We now relate the minors of $\mathcal{V}$ to the minors of the complementary orthogonal space $\mathcal{V}^{\perp}$. We remind the reader that $\mathcal{V}^{\perp} \equiv \{\mathbf{g} :< \mathbf{g}, \mathbf{f} >= 0, \mathbf{f} \in \mathcal{V}\}$, and that for any finite dimensional vector space $\mathcal{V}'$ $(\mathcal{V}'^{\perp})^{\perp} = \mathcal{V}'$. In the following results we see that the contraction (restriction) of a vector space corresponds to the restriction (contraction) of the orthogonal complement. We say that contraction and restriction are *(orthogonal) duals* of each other.

Theorem 1.4.8 *Let $\mathcal{V}$ be a vector space on S and let $T \subseteq S$. Then,*

1. $(\mathcal{V} \cdot T)^{\perp} = \mathcal{V}^{\perp} \times T$.
2. $(\mathcal{V} \times T)^{\perp} = \mathcal{V}^{\perp} \cdot T$.

Proof : i. Let $\mathbf{g}_T \in (\mathcal{V} \cdot T)^{\perp}$. For any $\mathbf{f}$ on S let $\mathbf{f}_T$ denote $\mathbf{f}/T$. Now if $\mathbf{f} \in \mathcal{V}$, then $\mathbf{f}_T \in \mathcal{V} \cdot T$ and $< \mathbf{g}_T, \mathbf{f}_T > = 0$.
Let $\mathbf{g}$ on S be defined by $\mathbf{g}/T \equiv \mathbf{g}_T$, $\mathbf{g}/S - T \equiv \mathbf{0}$. If $\mathbf{f} \in \mathcal{V}$ we have

$$\begin{aligned} < \mathbf{f}, \mathbf{g} > &= < \mathbf{f}_T, \mathbf{g}_T > + < \mathbf{f}_{S-T}, \mathbf{g}_{S-T} > \\ &= 0 + < \mathbf{f}_{S-T}, \mathbf{0}_{S-T} > \\ &= 0. \end{aligned}$$

Thus $\mathbf{g} \in \mathcal{V}^{\perp}$ and therefore, $\mathbf{g}_T \in \mathcal{V}^{\perp} \times T$.Hence, $(\mathcal{V} \ . \ T)^{\perp} \subseteq \mathcal{V}^{\perp} \times T$.
Next let $\mathbf{g}_T \in \mathcal{V}^{\perp} \times T$.
Then there exists $\mathbf{g} \in \mathcal{V}^{\perp}$ s.t. $\mathbf{g}/S - T = \mathbf{0}$ and $\mathbf{g}/T = \mathbf{g}_T$.
Let $\mathbf{f}_T \in \mathcal{V} \cdot T$. There exists $\mathbf{f} \in \mathcal{V}$ s.t. $\mathbf{f}/T = \mathbf{f}_T$.
Now $0 =< \mathbf{f}, \mathbf{g} >=< \mathbf{f}_T, \mathbf{g}_T > + < \mathbf{f}_{S-T}, \mathbf{0}_{S-T} > =< \mathbf{f}_T, \mathbf{g}_T >$.
Hence, $\mathbf{g}_T \in (\mathcal{V} \cdot T)^{\perp}$.
We conclude that
$\mathcal{V}^{\perp} \times T \subseteq (\mathcal{V} \cdot T)^{\perp}$. This proves that $(\mathcal{V} \cdot T)^{\perp} = \mathcal{V}^{\perp} \times T$.
ii. We have $(\mathcal{V}^{\perp} \cdot T)^{\perp} = (\mathcal{V}^{\perp})^{\perp} \times T$.
For any finite dimensional vector space $\mathcal{V}'$ we know that $(\mathcal{V}'^{\perp})^{\perp} = \mathcal{V}'$. Hence, $((\mathcal{V}^{\perp} \cdot T)^{\perp})^{\perp} = \mathcal{V}^{\perp} \cdot T$ and $(\mathcal{V}^{\perp})^{\perp} = \mathcal{V}$. Hence, $\mathcal{V}^{\perp} \cdot T = (\mathcal{V} \times T)^{\perp}$. ∎
The following corollary is immediate.

Corollary 1.4.1 $(\mathcal{V} \times P \cdot T)^{\perp} = \mathcal{V}^{\perp} \cdot P \times T, T \subseteq P \subseteq S$.

1.4.4 Representative Matrices of Minors of Vector Spaces

As defined earlier, the *representative matrix* $\mathbf{R}$ of a vector space $\mathcal{V}$ on S has the vectors of a basis of $\mathcal{V}$ as its rows. We now describe how to construct a representative matrix which contains representative matrices of $\mathcal{V} \ . \ P$ and $\mathcal{V} \times (S - P)$ as its submatrices. In such a case, $\mathcal{V} \ . \ P$ and $\mathcal{V} \times (S - P)$ are said to become 'visible' in $\mathbf{R}$.

Theorem 1.4.9 *Let $\mathcal{V}$ be a vector space on S. Let $P \subseteq S$. Let $\mathbf{R}$ be a representative matrix as shown below*

$$\begin{matrix} & P & S-P \end{matrix}$$
$$\mathbf{R} = \begin{bmatrix} \mathbf{R}_{PP} & \mathbf{R}_{P2} \\ \mathbf{0} & \mathbf{R}_{22} \end{bmatrix} \qquad (1.3)$$

where the rows of $\mathbf{R}_{PP}$ are linearly independent. Then $\mathbf{R}_{PP}$ is a representative matrix for $\mathcal{V} \ . \ P$ and $\mathbf{R}_{22}$, a representative matrix for $\mathcal{V} \times (S - P)$.

Proof : The rows of $\mathbf{R}_{PP}$ are restrictions of vectors on S to P. If $\mathbf{f}_P$ is any vector in $\mathcal{V} \ . \ P$ there exists a vector $\mathbf{f}$ in $\mathcal{V}$ s.t. $\mathbf{f}/P = \mathbf{f}_P$. Now $\mathbf{f}$ is a linear combination of the rows of $\mathbf{R}$. Hence, $\mathbf{f}/P (= \mathbf{f}_P)$ is a linear combination of the rows of $\mathbf{R}_{PP}$. Further it is given that the rows of $\mathbf{R}_{PP}$ are linearly independent. It follows that $\mathbf{R}_{PP}$ is a representative matrix of $\mathcal{V} \ . \ P$.

It is clear from the structure of $\mathbf{R}$ (the zero in the second set of rows) that any linear combination of the rows of $\mathbf{R}_{22}$ belongs to $\mathcal{V} \times (S-P)$. Further if $\mathbf{f}$ is any vector in $\mathcal{V}$ s.t. $\mathbf{f}/P = \mathbf{0}$ then $\mathbf{f}$ must be a linear combination only of the second set of rows of $\mathbf{R}$. For, if the first set of rows are involved in the linear combination, since rows of $\mathbf{R}_{PP}$ are linearly independent, $\mathbf{f}/P$ cannot be zero. We conclude that if $\mathbf{f}/(S-P)$ is a vector in $\mathcal{V} \times (S-P)$, it is linearly dependent on the rows of $\mathbf{R}_{22}$. Now rows of $\mathbf{R}$ are linearly independent. We conclude that $\mathbf{R}_{22}$ is a representative matrix of $\mathcal{V} \times P$. ∎

The following corollary is immediate

Corollary 1.4.2

$$r(\mathcal{V}) = r(\mathcal{V}\ .\ P) + r(\mathcal{V} \times (S-P))\ , P \subseteq S$$

1.4.5 Minors of Matroids

In this subsection we generalize the notion of minors of graphs and vector spaces to matroids.

Let $\mathcal{M} \equiv (S, \mathcal{I})$ be a matroid and $X \subseteq S$. The **restriction** (or **reduction**) of $\mathcal{M}$ to X, denoted by $\mathcal{M} \cdot X$, is the matroid on the ground set X whose independence family is the collection of all subsets of X which are members of $\mathcal{I}$. We define the **contraction** of $\mathcal{M}$ to X, (which we shall denote by $\mathcal{M} \times X$), as the matroid on X whose independent sets are precisely those $Y \subseteq X$ which satisfy the property that $Y \cup B_{S-X} \in \mathcal{I}$ whenever B_{S-X} is a base of $\mathcal{M} \cdot (S-X)$. Indeed it is clear from the definition that $\mathcal{M} \cdot X$ is a matroid. That $\mathcal{M} \times X$ is also a matroid needs to be proved and this we do below.

Before we proceed to its proof, we define the notion of **minor** of a matroid.

A **minor** of $\mathcal{M}$ is a matroid of the form $(\mathcal{M} \times X_1) \cdot X_2$ or $(\mathcal{M} \cdot X_1) \times X_2, X_2 \subseteq X_1 \subseteq S$. Since there is no room for confusion we omit the bracket while denoting minors.

We need the following useful lemma.

Lemma 1.4.1 *Let $\mathcal{M}$ be a matroid on S and let $Y \subseteq X \subseteq S$. Suppose B'_1, B'_2 are two bases of $\mathcal{M} \cdot (S-X)$ and $Y \cup B'_1$ is independent in $\mathcal{M}$. Then so is $Y \cup B'_2$.*

Proof : Suppose the contrary. Then there exist bases B'_1, B'_2 of $\mathcal{M} \cdot (S-X)$ s.t. $Y \cup B'_1$ is independent, but $Y \cup B'_2$ is dependent and $|B'_1 - B'_2|$ is a minimum for these conditions. For $e \in B'_2 - B'_1$, $e \cup B'_1$ contains the unique circuit $L(e, B'_1)$ of $\mathcal{M} \cdot (S-X)$. There must exist e' of $(B'_1 - B'_2)$ inside $L(e, B'_1)$. So $B'_3 = (B'_1 - e') \cup e$ is a base of $\mathcal{M} \cdot (S-X)$. There exists a base B_1 of $\mathcal{M}$ containing $Y \cup B'_1$. Therefore $e \cup B_1$ contains the unique circuit $L(e, B_1)$. Observe that $L(e, B_1)$ is the same as $L(e, B'_1)$ (this follows as circuits of $\mathcal{M} \cdot (S-X)$ are the same as circuits of $\mathcal{M}$ contained in $(S-X)$). As $L(e, B_1) = L(e, B'_1)$, it follows that $B_3 \equiv (B_1 - e') \cup e$ is a base of $\mathcal{M}$. Now, $Y \cup B'_3$ is independent in $\mathcal{M}$, being a subset of B_3. But note that B'_3 and B'_2 violate the minimum size assumption about B'_1 and B'_2 (since $|B'_3 - B'_2| < |B'_1 - B'_2|$). We conclude therefore that $Y \cup B'_2$ is independent in $\mathcal{M}$. ∎

To prove that $\mathcal{M} \times X$, given by $(S, \mathcal{I}'_X)$, is a matroid, we will verify that it satisfies the Independence Axioms. If $Y \in \mathcal{I}'_X$ and $Z \subseteq Y$ it is clear from the definition of $\mathcal{I}'_X$ that $Z \in \mathcal{I}'_X$. Therefore Axiom I1 holds. Now we prove that Axiom I2 holds. Let $X_1 \subseteq X$ and let Z_1, Z_2 be maximal members of $\mathcal{I}'_X$ which are subsets of X_1. Then Z_1, Z_2 are maximal with respect to the property that $Z_1, Z_2 \subseteq X_1$ and $Z_1 \uplus B'$, $Z_2 \uplus B'$ are independent in $\mathcal{M}$, for each base B' of $\mathcal{M} \cdot (S - X)$. We would be done if we show that $|Z_1 \uplus B'| = |Z_2 \uplus B'|$ (from which $|Z_1| = |Z_2|$ would follow). It suffices to show that $Z_1 \uplus B', Z_2 \uplus B'$ are both maximally independent subsets of $X_1 \uplus (S - X)$ in $\mathcal{M}$. Suppose not. Without loss of generality, let $Z_1 \uplus B'$ not be a maximally independent subset of $X_1 \uplus (S - X)$ in $\mathcal{M}$. Let W be a proper superset of it that is independent in $\mathcal{M}$ and and is as well a subset of $X_1 \uplus (S - X)$. Due to maximality of B' and independence of W, $W \cap (S - X) = B'$, and therefore $W \cap X_1$ would be a proper superset of Z_1. But $(W \cap X_1) \cup B'$ is independent which implies that $W \cap X_1$ is also a member of $\mathcal{I}'_X$ (note that we use Lemma 1.4.1 which assures us testing with any one base B' of $\mathcal{M} \cdot (S - X)$ is adequate). This would however violate maximality of Z_1 as a member of $\mathcal{I}'_X$. Therefore $Z_1 \uplus B'$ and similarly $Z_2 \uplus B'$ are maximally independent subsets of $X_1 \uplus (S - X)$ in $\mathcal{M}$. Thus, $|Z_1 \uplus B'| = |Z_2 \uplus B'|$ and therefore, $|Z_1| = |Z_2|$ as required.

Theorem 1.4.10 *Let $\mathcal{M}$ be a matroid on S and let $X \subseteq S$. Then*

1. *the union of a base of $\mathcal{M} \times X$ and a base of $\mathcal{M} \cdot (S - X)$ is a base of $\mathcal{M}$.*
2. *$r(\mathcal{M} \times X) + r(\mathcal{M} \cdot (S - X)) = r(\mathcal{M})$.*

Proof : i&ii. Let B_1 be a base of $\mathcal{M} \times X$ and let B_2 be a base of $\mathcal{M} \cdot (S - X)$. By the definition of $\mathcal{M} \times X$, $B_2 \cup B_1$ is independent in $\mathcal{M}$. Hence, $r(\mathcal{M} \times X) \leq r(\mathcal{M}) - r(\mathcal{M} \cdot (S - X))$.
Next, $B_2 \cup B_1$ can be extended to a base B of $\mathcal{M}$. By the definition of $\mathcal{M} \cdot (S - X)$, we must have $B_2 = B \cap (S - X)$. As $(B \cap X) \cup B_2$ is independent in $\mathcal{M}$, by the definition of $\mathcal{M} \times X$ and Lemma 1.4.1, $B \cap X$ is independent in $\mathcal{M} \times X$. Hence, $r(\mathcal{M} \times X) \geq r(\mathcal{M}) - r(\mathcal{M} \cdot (S - X))$. Therefore that $r(\mathcal{M} \times X) = r(\mathcal{M}) - r(\mathcal{M} \cdot (S - X))$ and $B_1 \cup B_2$ is a base of $\mathcal{M}$. ∎

We next study the relation between primitive notions (such as bases, circuits, bonds) associated with a matroid and those associated with restrictions and contractions of a matroid. We begin with bases.

Theorem 1.4.11 *Let $\mathcal{M}$ be a matroid on S and $X \subseteq S$. Then*

1. *B_X is a base of $\mathcal{M} \cdot X$ iff it is a maximal intersection of a base of $\mathcal{M}$ with X.*

2. *B'_X is a base of $\mathcal{M} \times X$ iff it is a minimal intersection of a base of $\mathcal{M}$ with X.*

Proof : i. B_X is a maximal intersection of a base of $\mathcal{M}$ with X iff it is a maximal subset of X that is independent in $\mathcal{M}$, i.e., iff B_X is a base of $\mathcal{M} \cdot X$.
ii. Let B_{S-X} be a base of $\mathcal{M} \cdot (S - X)$. By the definition of $\mathcal{M} \times X$ and Lemma 1.4.1, B_X is a base of $\mathcal{M} \times X$ iff $B_X \cup B_{S-X}$ is a base of $\mathcal{M}$, i.e., iff a base B of $\mathcal{M}$ intersects X in B_X and intersects $(S - X)$ maximally among all bases of $\mathcal{M}$, i.e., iff a base B of $\mathcal{M}$ intersects X in B_X and this intersection is minimal among all bases of $\mathcal{M}$. ∎

We next characterize circuits of minors.

Theorem 1.4.12 *Let $\mathcal{M}$ be a matroid on S and let $X \subseteq S$. Then*

1. *C_X is a circuit of $\mathcal{M} \cdot X$ iff it is a circuit of $\mathcal{M}$ contained in X.*

2. *C_X is circuit of $\mathcal{M} \times X$ iff it is a minimal nonvoid intersection of a circuit of $\mathcal{M}$ with X.*

Proof : i. Independent sets of $\mathcal{M} \cdot X$ are just the independent sets of $\mathcal{M}$ contained in X. Hence, C_X is a minimal dependent set of $\mathcal{M} \cdot X$ iff it is a minimal dependent set of $\mathcal{M}$ contained in X.
ii. We use the following observation: If C is a circuit of $\mathcal{M}$ intersecting X, then $C \cap X$ is dependent in $\mathcal{M} \times X$. (This follows as otherwise, the disjoint union $(C \cap X) \cup (C \cap (S - X))$ which is indeed C, would be independent in $\mathcal{M}$).

Let C_X be a circuit of $\mathcal{M} \times X$. Let B' be a base of $\mathcal{M} \cdot (S - X)$. Therefore $C_X \cup B'$ is dependent in $\mathcal{M}$, but $(C_X - e) \cup B'$ is independent in $\mathcal{M}$ for any $e \in C_X$. Let C be the unique circuit of $\mathcal{M}$ containing e, contained in $e \cup (C_X - e) \cup B'$. Note that $C \cap X \subseteq C_X$. But $C \cap X$ is dependent in $\mathcal{M} \times X$ by the above observation. Therefore due to C_X being a circuit of $\mathcal{M} \times X$, we conclude that $C_X = C \cap X$.

We still need to prove that C_X is a *minimal* nonvoid intersection of a circuit of $\mathcal{M}$ with X. If $C_X = C \cap X$ were not minimal such, then there would exist nonvoid $C'_X = C' \cap X$, C' a circuit of $\mathcal{M}$ such that $C'_X \subset C_X$ (proper!). But then C'_X would be dependent in $\mathcal{M} \times X$ by the above observation, contradicting that C_X is a circuit of $\mathcal{M} \times X$.

Next we prove the other implication, that is, a minimal nonvoid intersection $C \cap X$ of a circuit C of $\mathcal{M}$ with X is a circuit of $\mathcal{M} \times X$. Suppose the contrary, then there exists a circuit C'_X of $\mathcal{M} \times X$, that is properly contained in $C \cap X$. But then by the above there exists a circuit C' of $\mathcal{M}$ such that $C' \cap X = C'_X$, contradicting that $C \cap X$ is a minimal nonvoid intersection of a circuit of $\mathcal{M}$ with X. ∎

The next result speaks of the rank function of minors. The routine proof is omitted (the expression for rank function of contraction follows from Lemma 1.4.1).

Theorem 1.4.13 *Let $\mathcal{M}$ be a matroid on S and let $X \subseteq S$. Let $r(\cdot), r_r(\cdot), r_c(\cdot)$ be the rank functions of $\mathcal{M}, \mathcal{M} \cdot X, \mathcal{M} \times X$ respectively. Then*

1. $r_r(Y) = r(Y), Y \subseteq X$,

2. $r_c(Y) = r(Y \cup (S - X)) - r(S - X), Y \subseteq X$.

A minor of a general form could be obtained from the original matroid by a sequence of restrictions and contractions. As in the case of graphs we can simplify these operations to a single contraction followed by a single restriction or vice versa. The following result is needed for such simplification.

Theorem 1.4.14 *Let $\mathcal{M}$ be a matroid on S and let $Y \subseteq X \subseteq S$. Then,*

1. $\mathcal{M} \cdot X \cdot Y = \mathcal{M} \cdot Y$.

2. $\mathcal{M} \times X \times Y = \mathcal{M} \times Y$.

3. $\mathcal{M} \times X \cdot Y = \mathcal{M} \cdot (S - (X - Y)) \times Y$.

Proof : i. Immediate from the definition of restriction.
ii. A base of $\mathcal{M} \times Y$ is a minimal intersection of a base of $\mathcal{M}$ with Y, while a base of $\mathcal{M} \times X \times Y$ is a minimal intersection of a base of $\mathcal{M} \times X$ with Y. To construct the former base, one could begin with a maximally independent set of $\mathcal{M}$ within $S - Y$ and extend it to a base of $\mathcal{M}$ using a set B_Y of elements from Y. B_Y would then be a base of $\mathcal{M} \times Y$. But this could have been done by first choosing a maximally independent set B_{S-X} within $S - X$ of $\mathcal{M}$, extending it to a maximally independent set B_{S-Y} of $\mathcal{M}$ within $S - Y$, and then growing it further using B_Y. Thus B_Y is a base of $\mathcal{M} \times X \times Y$. On the other hand, starting with a base B_Y of $\mathcal{M} \times X \times Y$, one can see that its union with a maximally independent set B_{S-Y} of $\mathcal{M}$ within $S - Y$ gives a base of $\mathcal{M}$. Therefore B_Y is a minimal intersection of a base of $\mathcal{M}$ with Y and therefore a base of $\mathcal{M} \times Y$.

A more routine proof uses Theorem 1.4.15 proved below.
$(\mathcal{M}^* \cdot Y)^* = (\mathcal{M}^* \cdot X \cdot Y)^* = (\mathcal{M}^* \cdot X)^* \times Y = (\mathcal{M}^*)^* \times X \times Y = \mathcal{M} \times X \times Y$
But $(\mathcal{M}^* \cdot Y)^* = (\mathcal{M}^*)^* \times Y = \mathcal{M} \times Y$.
We conclude that $\mathcal{M} \times X \times Y = \mathcal{M} \times Y$.
iii. Let Z be an independent set of $\mathcal{M} \times X \cdot Y$. Then, by the definition of restriction, $Z \subseteq Y$ and Z is independent in $\mathcal{M} \times X$. Let B_{S-X} be a base of $\mathcal{M} \cdot (S - X)$. Then by the definition of contraction, $Z \cup B_{S-X}$ is independent in $\mathcal{M}$. By the definition of restriction $Z \cup B_{S-X}$ must be independent in $\mathcal{M} \cdot (S - (X - Y))$. Now B_{S-X} is a base of $\mathcal{M} \cdot (S - X) = \mathcal{M} \cdot (S - (X - Y)) \cdot (S - X)$. Hence, Z is independent in $\mathcal{M} \cdot (S - (X - Y)) \times Y$ (note that $(S - X) \uplus Y = S - (X - Y)$ since $Y \subseteq X$). It is easy to see that the above sequence of implications can be reversed. Hence, if Z is independent in $\mathcal{M} \cdot (S - (X - Y)) \times Y$ then Z is also independent in $\mathcal{M} \times X \cdot Y$. Thus,

$$\mathcal{M} \times X \cdot Y = \mathcal{M} \cdot (S - (X - Y)) \times Y$$

∎

Let $\mathcal{M}$ be a matroid on S and let $S \supseteq X_1 \supseteq X_2 \supseteq \cdots \supseteq X_n$. From Theorem 1.4.14, it is clear that $\mathcal{M} \times X_1 \cdot X_2 \times X_3 \cdots \cdot X_n$ can be written in the form $\mathcal{M} \times P \cdot X_n$ for a suitable $P \supseteq X_n$.

Now we relate the minors of the dual matroid to the duals of the minors of the original matroid.

Theorem 1.4.15 *Let $\mathcal{M}$ be a matroid on S. Let $X \subseteq S$. Then*

1. $(\mathcal{M} \times X)^* = \mathcal{M}^* \cdot X$,
2. $(\mathcal{M} \cdot X)^* = \mathcal{M}^* \times X$.
3. *The rank of X in $\mathcal{M}^*$ equals* $|X| - r(\mathcal{M} \times X) = |X| - (r(\mathcal{M}) - r(\mathcal{M} \cdot (S - X)))$.

Proof : i. B_X is a base of $\mathcal{M} \times X$ iff it is a minimal intersection of a base of $\mathcal{M}$ with X, i.e., iff it is the complement of a maximal intersection of a cobase of $\mathcal{M}$ with X, i.e., iff it is the complement of a maximal intersection of a base of $\mathcal{M}^*$ with X i.e., iff it is the complement of a base of $\mathcal{M}^* \cdot X$.
ii. By the definition of dual, dual of the dual of a matroid is the original matroid itself. Hence, using (i) above, $(\mathcal{M}^* \times X)^* = (\mathcal{M}^*)^* \cdot X = \mathcal{M} \cdot X$

$$i.e., \qquad \mathcal{M}^* \times X = (\mathcal{M}^* \times X)^{**} = (\mathcal{M} \cdot X)^*.$$

iii. Immediate from **i** above. ∎

1.4.6 Notes

The reader interested in making a serious study of matroid theory would do well to begin by referring to [Kung86] where the key foundation papers in matroid theory are reproduced. A terse but readable account of matroid theory, as seen in the 60's by a master, is available in [Tutte65]. Matroids seen from the unifying perspective of combinatorial geometries is available in [Crapo+Rota70]. The first comprehensive text book on matroids is [Welsh76]. More recent books are [White86], [White87] and [Oxley92].

1.5 Convolution

1.5.1 Introduction

The convolution operation popularized by Edmonds [Edmonds70] is fundamental to the study of submodular functions and is extremely useful both for studying the structure of matroids and for generating new matroids. Indeed, the well known concept of principal partition associated with submodular functions in terms of a given weight function and the union matroid of two matroids, to name just two important instances, are best understood using convolution. In this section we begin with a description of polymatroid rank functions and use this as a framework for presenting results on convolution, and principal partition.

1.5.2 Polymatroid rank functions

It is convenient to state the basic results for convolution in terms of polymatroid rank functions which are a simple generalization of matroid rank functions.

We remind the reader that a function $f(\cdot) : 2^S \longrightarrow \Re$ is said to be *submodular* if for every pair of subsets X, Y of S we have $f(X)+f(Y) \geq f(X \cup Y)+f(X \cap Y)$. It is *supermodular* if the inequality is reversed and *modular* if the inequality is replaced by an equality. A *weight* function $g(\cdot)$ on subsets of S satisfies $g(X) \equiv \sum_{e_i \in X} g(e_i), X \subseteq S$, with $g(\emptyset) = 0$. A modular function can be seen to differ from a weight function only in that, for the latter, the value on null set is zero. Indeed if $g(\cdot)$ is modular we have $g(X) = \sum_{e \in X}(g(e) - g(\emptyset)) + g(\emptyset)$.

We saw in Section 1.2.4 that a matroid rank function $r(\cdot)$ on subsets of S is increasing, submodular with $r(\emptyset) = 0$, integral and satisfies $r(e) = 0$ or 1 for all $e \in S$. A polymatroid rank function $f(\cdot)$ on subsets of S is increasing, submodular with $r(\emptyset) = 0$, the remaining conditions being omitted.

We define restriction, contraction and dual of a polymatroid rank function essentially as in the case of a matroid rank function. The restriction $f(\cdot)/T$ of $f(\cdot)$ to $T, T \subseteq S$, is defined by $f/T(X) \equiv f(X), X \subseteq T$. The contraction $f \diamond T(\cdot)$ of $f(\cdot)$ to $T, T \subseteq S$, is defined by $f \diamond T(X) \equiv f(X \cup (S-T)) - f(S-T), X \subseteq T$.

The dual $f^*(\cdot)$ of $f(\cdot)$ on subsets of S is defined relative to a positive weight function $g(\cdot)$ on subsets of S just as $r^*(\cdot)$ is defined relative to the $|\cdot|$ function (see Theorem 1.4.15) :

$$f^*(X) \equiv g(X) - [f(S) - f(S-X)], X \subseteq S.$$

It is easily verified that $f^{**}(\cdot) = f(\cdot)$. Further, if $f(e) \leq g(e), \forall e \in S$, it will follow that $f^*(\cdot)$ is a polymatroid rank function whenever $f(\cdot)$ is.

Just as in the case of matroid rank functions, contraction and restriction operations turn out to be duals of each other for polymatroid rank functions.

We have, for $X \subseteq T$, $(f \diamond T)^*(X) = g(X) - [f \diamond T(T) - f \diamond T(T-X)]$
$= g(X) - [f(S) - f(S-T) - f((S-T) \cup (T-X)) + f(S-T)]$
$= g(X) - [f(S) - f((S-T) \cup (T-X))] = g(X) - [f(S) - f(S-X)] = f^*/T(X)$.
Since $f^{**}(\cdot) = f(\cdot)$, it will follow that $(f/T)^*(X) = (f^* \diamond T)(X)$.

1.5.3 Formal properties of the Convolution operation

Definition 1.5.1 *Let $f(\cdot), g(\cdot) : 2^S \longrightarrow \Re$. The lower convolution of $f(\cdot)$ and $g(\cdot)$, denoted by $f * g(\cdot)$, is defined by*

$$f * g(X) \equiv min_{Y \subseteq X}[f(Y) + g(X-Y)].$$

The collection of subsets Y, at which $f(Y) + g(X-Y) = f * g(X)$, is denoted by $\mathcal{B}_{f,g}(X)$. But if $X = S$, we will simply write $\mathcal{B}_{f,g}$.

It is clear that $f * g(\cdot) = g * f(\cdot)$. We now have the following elementary but important result.

Theorem 1.5.1 *If $f(\cdot)$ is submodular on subsets of S and $g(\cdot)$ is modular, then $f * g(\cdot)$ is submodular.*

We need the following lemma for the proof of the theorem.

Lemma 1.5.1 *Let $g(\cdot)$ be a modular function on the subsets of S.Let $A, B, C, D \subseteq S$ such that $A \cup B = C \cup D$, $A \cap B = C \cap D$. Then*

$$g(A) + g(B) = g(C) + g(D).$$

Proof: Both LHS and RHS in the statement of the lemma are equal to

$$\sum_{e \in A \cup B} (g(e) - g(\emptyset)) + \sum_{e \in A \cap B} (g(e) - g(\emptyset)) + 2(g(\emptyset)).$$

∎

Proof of the theorem: Let $X, Y \subseteq S$. Further let

$$f * g(X) = f(Z_X) + g(X - Z_X), f * g(Y) = f(Z_Y) + g(Y - Z_Y).$$

Then,

$$f * g(X) + f * g(Y) = f(Z_X) + g(X - Z_X) + f(Z_Y) + g(Y - Z_Y).$$

We observe that, since $Z_X \subseteq X, Z_Y \subseteq Y$,

$$(X - Z_X) \cup (Y - Z_Y) = (X \cup Y - (Z_X \cup Z_Y)) \cup (X \cap Y - (Z_X \cap Z_Y))$$

and

$$(X - Z_X) \cap (Y - Z_Y) = ((X \cup Y) - (Z_X \cup Z_Y)) \cap (X \cap Y - (Z_X \cap Z_Y)),$$

we must have, by Lemma 1.5.1,

$$g(X - Z_X) + g(Y - Z_Y) = g(X \cup Y - (Z_X \cup Z_Y)) + g(X \cap Y - (Z_X \cap Z_Y)).$$

Hence, $f * g(X) + f * g(Y)$
$\geq f(Z_X \cup Z_Y) + f(Z_X \cap Z_Y) + g(X \cup Y - (Z_X \cup Z_Y)) + g(X \cap Y - (Z_X \cap Z_Y))$.
Thus,

$$f * g(X) + f * g(Y) \geq f * g(X \cup Y) + f * g(X \cap Y),$$

which is the desired result. ∎

Remark: It is clear that if $g(\cdot)$ is not modular, but only submodular, then $g(X - Z_X) + g(Y - Z_Y)$ need not be greater or equal to $g(X \cup Y - (Z_X \cup Z_Y)) + g(X \cap Y - (Z_X \cap Z_Y))$. Thus the above proof would not hold if $g(\cdot)$ is only submodular. Indeed the following counterexample shows the convolution of two submodular functions need not be submodular. Let B_1, B_2 be bipartite graphs on $V_L \equiv \{a, b, c\}, V_R \equiv \{a', b', c', d'\}$ with adjacency functions Γ_1, Γ_2 defined as follows:

$$\Gamma_1(a) = \{a', b', d'\}, \Gamma_1(b) = \{a', b', d'\}, \Gamma_1(c) = \{b', c', d'\},$$

$$\Gamma_2(a) = \{a', b', c'\}, \Gamma_2(b) = \{a', b', d'\}, \Gamma_2(c) = \{b', c'\}.$$

It may be verified that

$$|\Gamma_1| * |\Gamma_2|(a) = 3, |\Gamma_1| * |\Gamma_2|(a,b) = 3, |\Gamma_1| * |\Gamma_2|(a,c) = 3, |\Gamma_1| * |\Gamma_2|(a,b,c) = 4.$$

Hence

$$|\Gamma_1| * |\Gamma_2|(a,b,c) - |\Gamma_1| * |\Gamma_2|(a,c) > |\Gamma_1| * |\Gamma_2|(a,b) - |\Gamma_1| * |\Gamma_2|(a).$$

This shows that $|\Gamma_1| * |\Gamma_2|(\cdot)$ is not submodular. But it is easily seen that $|\Gamma_1(\cdot)|, |\Gamma_2(\cdot)|$ are submodular.

Theorem 1.5.2 *Let $f(\cdot), g(\cdot)$ be arbitrary set functions on subsets of S.*

1. *Then $f * g(X \cup e) - f * g(X)$ $\leq min[max_{Y \subseteq X}(f(Y \cup e) - f(Y)), max_{Y \subseteq X}(g(Y \cup e) - g(Y))]$, $X \subseteq S, e \in S$.*
2. *Let $f(\cdot), g(\cdot)$ be increasing. Then $f * g(\cdot)$ is increasing.*
3. *Let $f(\cdot), g(\cdot)$ be integral. Then so is $f * g(\cdot)$.*
4. *(Edmonds [Edmonds70]) Let $f(\cdot)$ be an integral polymatroid rank function and let $g(\cdot) = |\cdot|$.Then $f * g(\cdot)$ is a matroid rank function.*

Proof : i. Let $f * g(X) = f(Z) + g(X - Z)$, where $Z \subseteq X$. Then

$$f * g(X \cup e) \leq min[f(Z \cup e) + g(X - Z), f(Z) + g((X - Z) \cup e)].$$

The proof is now immediate.
ii. Let, without loss of generality ,

$$f * g(X \cup e) = f(Z \cup e) + g(X - Z), Z \subseteq X, e \in (S - X).$$

But then

$$f * g(X) \leq f(Z) + g(X - Z) \leq f(Z \cup e) + g(X - Z).$$

iii. The proof is immediate from the definition of convolution.
iv. We need to show that $f * g(\cdot)$ is an integral polymatroid rank function that takes value atmost one on singletons. We have, $f(\cdot), g(\cdot)$ are increasing, integral, submodular, taking value zero on the null set and further $g(\cdot)$ is a weight function with $g(e) = 1 \ \ \forall e \in S$. From Theorem 1.5.1 it follows that $f * g(\cdot)$ is submodular. It is clear that $f * g(\emptyset) = 0$. The remaining properties for being a matroid rank function follow from the preceding sections of the present theorem.
■

Theorem 1.5.3 *Let $\rho(\cdot)$ be an integral polymatroid rank function on subsets of S. A set $X \subseteq S$ is independent in the matroid whose rank function is $\rho * \mid \cdot \mid$ iff $\rho(Y) \geq \mid Y \mid \quad \forall Y \subseteq X$.*

Proof : Let $r(\cdot) \equiv \rho * \mid \cdot \mid$. A set $X \subseteq S$ is independent iff $r(X) = |X|$, i.e., iff $(\rho * \mid \cdot \mid)(X) = \mid X \mid$, i.e., iff

$$min_{Y \subseteq X}(\rho(Y) + \mid X - Y \mid) = \mid X \mid .$$

Clearly this would happen iff $\rho(Y) \geq \mid Y \mid \quad \forall Y \subseteq X$. ∎

1.5.4 Connectedness for $f * g$

Let $f(\cdot)$ be submodular on subsets of S with $f(\phi) = 0$. We say that T is a separator for S iff

$$f(T) + f(S - T) = f(S).$$

We then have the following result.

Theorem 1.5.4 *Let $f(\cdot)$ be submodular on subsets of S with $f(\phi) = 0$ and let T be a separator of $f(\cdot)$. Suppose $X \subseteq T$, $Y \subseteq S - T$. Then, $f(X) + f(Y) = f(X \cup Y)$.*

By submodularity,

$$f(Y) + f(T) \geq f(Y \cup T)$$
$$i.e., f(Y) + f(S) - f(S - T) \geq f(Y \cup T)$$
$$i.e., f(Y \cup T) + f(S - T) \leq f(Y) + f(S).$$

But by submodularity again,

$$f(Y \cup T) + f(S - T) \geq f(S) + f(Y).$$

We conclude that $f(Y \cup T) + f(S - T) = f(S) + f(Y)$, i.e.,$f(Y \cup T) = f(S) - f(S - T) + f(Y) = f(T) + f(Y)$. We could now repeat the argument working with X in place of Y, Y in place of T and obtain

$$f(X \cup Y) = f(X) + f(Y).$$

∎

Thus when $f(\cdot)$ is submodular with $f(\phi) = 0$ on subsets of S and T is a separator, $f(\cdot)$ behaves as though it is the 'direct sum' of its restrictions on subsets of T and subsets of $S - T$ since

$$f(X) = f(X \cup T) + f(X \cap (S - T)).$$

The above discussion is particularly relevant when we consider $f * g$, where $f(\cdot)$ is submodular with $f(\phi) = 0$ and $g(\cdot)$ is a positive weight function on subsets of S.

Suppose

$$f * g(S) = f(T) + g(S - T).$$

We claim $f * g(S) = f * g(T) + f * g(S - T)$. To see this suppose $f * g(T) = f(T_1) + g(T - T_1)$, $T_1 \subseteq T$ and $f * g(S - T) = f(T_2) + g(S - T - T_2)$, $T_2 \subseteq S - T$. We then have

$$f(T) + g(S - T) \geq f(T_1) + g(T - T_1) + f(T_2) + g(S - T - T_2), \cdots (*)$$

By submodularity of $f(\cdot)$,

$$f(T) + g(S - T) \geq f(T_1 \cup T_2) + g((T - T_1) \cup (S - T - T_2)) = f(T_1 \cup T_2) + g(S - (T_1 \cup T_2)).$$

However,

$$f(T) + g(S - T) = \min_{X \subseteq S} f(X) + g(S - X).$$

Hence, $f(T) + g(S - T) = f(T_1 \cup T_2) + g(S - (T_1 \cup T_2))$, and the inequality $(*)$ above is an equality. The only way this can happen is if $f(T) = f(T_1) + g(T - T_1)$ and $g(S - T) = f(T_2) + g(S - T - T_2)$. It follows that $f * g(T) = f(T)$ and $f * g(S - T) = g(S - T)$ and therefore

$$f * g(S) = f * g(T) + f * g(S - T),$$

proving the claim. Thus, T, $S - T$ are separators of $f * g$.

When $f(\cdot)$ is an integral polymatroid rank function and $g(\cdot) = |\cdot|$, if $f * g(S) = f(T) + g(S - T)$, the matroid $\mathcal{M}_{f*g}$ whose rank function is $f * g$ has T,$S - T$ as separators. Further $f * g(S - T) = |S - T|$ so that $(S - T)$ is independent. Now consider any base b of $\mathcal{M}_{f*g}$. We have $b = b \cap T \cup (b \cap (S - T))$. However

$$f * g(b \cap T \cup (S - T)) = f * g(b \cap T) + f * g(S - T)) = |b \cap T| + |S - T|.$$

It is thus clear that $b \cap T \cup (S - T)$ is independent. Since b is maximally independent in $\mathcal{M}_{f*g}$, we conclude that $b \cap (S - T) = S - T$. Thus $S - T$ is a subset of every base of $\mathcal{M}_{f*g}$, i.e., $S - T$ is a set of coloops (elements which do not belong to any circuit) of $\mathcal{M}_{f*g}$.

On the other hand, suppose $(S - K)$ is the set of all coloops of $\mathcal{M}_{f*g}$. It is clear that

$$f * g(S) = f * g(K) + |S - K| = f * g(K) + f * g(S - K).$$

Thus $K, (S - K)$ are separators of $f * g$. We claim $f * g(K) = f(K)$. For, if $f * g(K) = f(K_1) + g(K - K_1)$ and $K_1 \subset K$, we have $f * g(S) = f(K_1) + g((S - K) \cup (K - K_1))$. But this means $(S - K) \cup (K - K_1)$ is a set of coloops which contradicts the fact $(S - K)$ is the set of all coloops of $\mathcal{M}_{f*g}$.

We summarize the above discussion in the following theorem.

Theorem 1.5.5 *Let* $f * g(S) = f(T) + g(S - T), T \subset S$.

1. *If* $f(\cdot)$ *is submodular on subsets of* S *with* $f(\phi) = 0$, *then* $T, S - T$ *are separators of* $f * g(\cdot)$.

2. *If $f(\cdot)$ is an integral polymatroid rank function and $g(\cdot) = |\cdot|$, the matroid $\mathcal{M}_{f*g}$ whose rank function is $f * g$ has T,$S-T$ as separators and $S-T$ as a set of coloops. Also, if $S-T$ is the set of all coloops of $\mathcal{M}_{f*g}$, then $f * g(S) = f(T) + g(S-T)$.*

1.6 The Principal Partition

1.6.1 Introduction

The 'principal partition of a graph' was defined by Kishi and Kajitani in their seminal paper [Kishi+Kajitani69] and was originally an offshoot of their work on maximally distant trees. A graph was decomposed into three minors according to how 'strongly' subsets of edges can be covered by unions of two trees or two cotrees. The extensions of this concept can be in two directions: towards making the partition finer or towards making the functions involved more general. Our present description favours the former approach and is mainly aimed at describing the principal partition of a matroid ([Narayanan74], [Tomizawa76]). However, the results are best stated in terms of convolution of polymatroid rank functions with positive weight functions. Essentially, we study the collection of all minimizing T such that $\lambda f * g(S) = \lambda f(T) + g(S-T), \lambda \geq 0$ where $f(\cdot)$ is a polymatroid rank function and $g(\cdot)$, a positive weight function on subsets of S. The algorithms for building this structure for a matroid rank function are based on the matroid union algorithm which finds the maximal union of bases from two different matroids.

1.6.2 Basic properties of Principal Partition

Definition 1.6.1 *Let $f(\cdot), g(\cdot)$ be a polymatroid rank function and a positive weight function respectively on the subsets of a set S. The collection of all sets in $\mathcal{B}_{\lambda f,g}$ (i.e., the collection of sets $X \subseteq S$ which minimize $\lambda f(X) + g(S-X)$ over subsets of S) $\forall \lambda, \lambda \geq 0$, is called the principal partition (PP) of $(f(\cdot), g(\cdot))$. We denote $\mathcal{B}_{\lambda f,g}$ by $\mathcal{B}_{\lambda}$ when $f(\cdot), g(\cdot)$ are clear from the context. We denote the maximal and minimal members of $\mathcal{B}_{\lambda}$ by X^{λ}, X_{λ}, respectively.*

We now list the important properties of the principal partition of $(f(\cdot), g(\cdot))$.

Property PP1

The collection $\mathcal{B}_{\lambda f,g}, \lambda \geq 0$, is closed under union and intersection and thus has a unique maximal and a unique minimal element.

Property PP2

If $\lambda_1 > \lambda_2 \geq 0$, then $X^{\lambda_1} \subseteq X_{\lambda_2}$.

Definition 1.6.2 *A nonnegative value λ for which $\mathcal{B}_{\lambda}$ has more than one subset as a member is called a critical value of $(f(\cdot), g(\cdot))$.*

Property PP3

the number of critical values of $(f(\cdot), g(\cdot))$ is bounded by $|S|$.

Property PP4
Let $(\lambda_i), i = 1, \cdots, t$ be the decreasing sequence of critical values of $(f(\cdot), g(\cdot))$. Then, $X^{\lambda_i} = X_{\lambda_{i+1}}$ for $i = 1, \cdots, t-1$.
Property PP5
Let (λ_i) be the decreasing sequence of critical values. Let $\lambda_i > \sigma > \lambda_{i+1}$. Then $X^{\lambda_i} = X^{\sigma} = X_{\sigma} = X_{\lambda_{i+1}}$.

Definition 1.6.3 *Let $f(\cdot)$ be a polymatroid rank function and let $g(\cdot)$ be a positive weight function on subsets of S. Let $(\lambda_i), i = 1, \cdots, t$ be the decreasing sequence of critical values of $(f(\cdot), g(\cdot))$. Then the sequence $X_{\lambda_1}, X_{\lambda_2}, \cdots, X_{\lambda_t}, X^{\lambda_t} = S$ is called the principal sequence of $(f(\cdot), g(\cdot))$. A member of $\mathcal{B}_\lambda$ would be alternatively referred to as a minimizing set corresponding to λ in the principal partition of $(f(\cdot), g(\cdot))$.*

Remark 1.6.1 *A word about the terminology is in order. For convenience, we have defined the principal partition to be the collection of all minimizing sets of the expression $\lambda f(X) + g(S - X)$. Literally speaking, this is not a partition. However, there is a natural associated partition which is simply the collection of all minimal sets of the form $X_1 - X_2$, where X_1, X_2 minimize $\lambda f(X) + g(S - X)$, for a critical value λ. The coarser partition associated with the principal sequence is $X_{\lambda_1}, X_{\lambda_2} - X_{\lambda_1}, \cdots, X^{\lambda_t} - X_{\lambda_t}$.*

Proof of the properties of the Principal Partition

i. PP1: Define $h(X) \equiv \lambda f(X) + g(S - X) \;\; \forall X \subseteq S, \lambda \geq 0$. Observe that the function $g'(\cdot)$, defined through $g'(X) \equiv g(S - X) \;\; \forall X \subseteq S$, is submodular. Thus $h(\cdot)$ is the sum of two submodular functions and is therefore submodular. The collection of sets on which this function reaches a minimum is called the principal structure of $h(\cdot)$ [Fujishige80b]. If T_1, T_2 minimize $h(\cdot)$, since $h(T_1) + h(T_2) \geq h(T_1 \cup T_2) + h(T_1 \cap T_2)$, it follows that $T_1 \cup T_2, T_1 \cap T_2$ also minimize $h(\cdot)$. Thus the principal structure of $h(\cdot)$ is closed under union and intersection and therefore has a unique minimal and a unique maximal set. The principal structure of $h(\cdot)$ is precisely the same as $\mathcal{B}_\lambda$.
ii. PP2: Observe that minimizing $\lambda_i f(X) + g(S - X) \forall X \subseteq S, \lambda_i \geq 0, i = 1, 2$, is equivalent to minimizing $f(X) + (\lambda_i)^{-1} g(S - X) \;\; \forall X \subseteq S, \lambda_i \geq 0, i = 1, 2$. (Here $+\infty \times 0$, corresponding to $\lambda_i = 0$ and $g(\emptyset) = 0$, is treated as zero). So we may take the sets which minimize the latter expression to be the sets in $\mathcal{B}_{\lambda_i}, i = 1, 2$. Define $p_i(X) \equiv f(X) + (\lambda_i)^{-1} g(S - X) \;\; \forall X \subseteq S, \lambda_i \geq 0, i = 1, 2$. As in the case of $h_i(\cdot), p_i(\cdot), i = 1, 2$ is also submodular. Let Z_1 minimize $p_1(\cdot)$. We will now show that $p_2(Z_1) < p_2(Y) \;\; \forall Y \subset Z_1$. Let $Y \subset Z_1$. We have,

$$p_2(Z_1) = p_1(Z_1) + ((\lambda_2)^{-1} - (\lambda_1)^{-1}) g(S - Z_1)$$

and

$$p_2(Y) = p_1(Y) + ((\lambda_2)^{-1} - (\lambda_1)^{-1}) g(S - Y).$$

Since $g(\cdot)$ is a positive weight function, $S - Z_1 \subset S - Y$ and $((\lambda_2)^{-1} - (\lambda_1)^{-1}) > 0$, we must have $((\lambda_2)^{-1} - (\lambda_1)^{-1}) g(S - Z_1)$

$< ((\lambda_2)^{-1} - (\lambda_1)^{-1})g(S - Y)$. Since $p_1(Y) \geq p_1(Z_1)$, it follows that $p_2(Y) > p_2(Z_1)$. Now let $Z \subseteq S$ minimize $p_2(\cdot)$. Applying the submodular inequality for p_2 on Z, Z_1, it would follow that if $Z \cup Z_1$ is not the same as Z, then $p_2(Z \cap Z_1) \leq p_2(Z_1)$, with $(Z \cap Z_1) \neq Z_1$ which would be a contradiction. It follows that $Z \supseteq Z_1$.

iii. PP3: If $\mathcal{B}_\lambda$ has more than one set as a member then $|X^\lambda| > |X_\lambda|$. So if λ_1, λ_2 are critical values and $\lambda_1 > \lambda_2$, by Property PP2, we must have $|X_{\lambda_1}| < |X_{\lambda_2}|$. Thus the sequence X_{λ_i}, where (λ_i) is the decreasing sequence of critical values cannot have more than $|S|$ elements.

iv. PP4: We need the following lemma.

Lemma 1.6.1 *Let $\lambda > 0$. Then, for sufficiently small $\epsilon > 0$, the only set that minimizes $\lambda - \epsilon$ is X^λ.*

Proof of the Lemma: Since there are only a finite number of $(f(X), g(S-X))$ pairs, for sufficiently small $\epsilon > 0$ we must have the value of $(\lambda - \epsilon)f(X) + g(S - X)$ lower on the members of $\mathcal{B}_\lambda$ than on any other subset of S. We will now show that, among the members of $\mathcal{B}_\lambda$,X^λ takes the least value of $(\lambda - \epsilon)f(X) + g(S - X), \epsilon > 0$. This would prove the required result. If λ is not a critical value this is trivial. Let λ be a critical value and let X_1, X^λ be two distinct sets in $\mathcal{B}_\lambda$. Since $X_1 \subset X^\lambda$, we have, $g(S - X_1) > g(S - X^\lambda)$. But, $\lambda f(X_1) + g(S - X_1) = \lambda f(X^\lambda) + g(S - X^\lambda)$. So, $\lambda f(X_1) < \lambda f(X^\lambda)$. Since $\lambda > 0$, we must have, $-\epsilon f(X_1) > -\epsilon f(X^\lambda), \epsilon > 0$. It follows that, $(\lambda - \epsilon)f(X_1) + g(S - X_1) > (\lambda - \epsilon)f(X^\lambda) + g(S - X^\lambda)$. ∎

Proof of PP4: By Lemma 1.6.1, for sufficiently small values of $\epsilon > 0$, X^{λ_i} would continue to minimize $(\lambda_i - \epsilon)f(X) + g(S-X)$. As ϵ increases, because there are only a finite number of $(f(X), g(S-X))$ pairs, there would be a least value, say σ, at which X^{λ_i} and atleast one other set minimize $(\lambda_i - \sigma)f(X) + g(S - X)$. Clearly, the next critical value $\lambda_{i+1} = \lambda_i - \sigma$. Since $\lambda_i > \lambda_i - \sigma$, by Property PP2, we must have $X^{\lambda_i} \subseteq X_{\lambda_i - \sigma}$. Hence we must have, $X^{\lambda_i} = X_{\lambda_i - \sigma} = X_{\lambda_{i+1}}$, as desired.

v. **PP5:** This is clear from the above arguments.

Informally, the situation is as follows. Suppose we start with $\lambda = +\infty$. Here X_∞ would be the null set and X^∞, the set of elements e for which $f(e)$ is zero. As we reduce λ, a point would be reached, say when $\lambda = \lambda_1$, where X^{λ_1} becomes a proper superset of X^∞. This would be the next critical value. Between ∞ and λ_1, $X^\lambda = X_\lambda = X^\infty$. As we lower λ further, $X^\lambda = X_\lambda = X^{\lambda_1}$, till the next critical value λ_2 is reached. The last critical value λ_k will be such that $X^{\lambda_k} = S$. When $\lambda = 0$, it is clear that the minimum of $\lambda f(X) + g(S - X)$ is reached only at S. It follows that all critical values have to be positive.

A characterization of principal partition would be useful for justifying algorithms for its construction. We will describe one such characterization in Theorem 1.6.1 below. This is a routine restatement of the properties of PP.

Theorem 1.6.1 *Let $f(\cdot)$ be a polymatroid rank function on subsets of S and let $g(\cdot)$ be a positive weight function on subsets of S. Let $\mathcal{B}_\lambda$ denote $\mathcal{B}_{\lambda f,g}$. Let*

$\lambda_1, \cdots, \lambda_t$ be a strictly decreasing sequence of numbers such that

1. *each $\mathcal{B}_{\lambda_i}, i = 1, \cdots, t$ has atleast two members,*

2. *$\mathcal{B}_{\lambda_i}, \mathcal{B}_{\lambda_{i+1}}, i = 1, \cdots, t-1$ have atleast one common member set,*

3. *$\emptyset$ belongs to $\mathcal{B}_{\lambda_1}$, while S belongs to $\mathcal{B}_{\lambda_t}$.*

Then $\lambda_1, \cdots, \lambda_t$ is the decreasing sequence of critical values of $(f(\cdot), g(\cdot))$ and therefore the collection of all the sets which are member sets in all the $\mathcal{B}_{\lambda_i}, i = 1, \cdots, t$ is the principal partition of $(f(\cdot), g(\cdot))$.

Proof: We note that, by definition, $\lambda_1, \cdots, \lambda_t$ are some of the critical values and, in the present case, $\emptyset = X_{\lambda_1}, X_{\lambda_2}, \cdots, X_{\lambda_t}, X^{\lambda_t} = S$ is a subsequence of the principal sequence. Let $\lambda'_1, \cdots, \lambda'_k$ be the critical values and let $Y_0, \cdots, Y_k = S$ be the principal sequence of $(f(\cdot), g(\cdot))$. Since the principal sequence is increasing, it follows that $Y_0 = \emptyset$. By Property PP2 of $(f(\cdot), g(\cdot))$, the only member set in $\mathcal{B}_\lambda$, when $\lambda > \lambda'_1$, is Y_0. Further when $\lambda < \lambda'_1$, Y_0 is not in $\mathcal{B}_\lambda$. Hence $\lambda_1 = \lambda'_1$. Next by Property PP5, when $\lambda'_1 > \lambda > \lambda'_2$, the only member in $\mathcal{B}_\lambda$ is Y_1 which is the maximal set in $\mathcal{B}_{\lambda'_1}$. Since $\mathcal{B}_{\lambda_2}$ has atleast two sets we conclude that $\lambda_2 \leq \lambda'_2$. We know that $\mathcal{B}_{\lambda_1}$ and $\mathcal{B}_{\lambda_2}$ have a common member which by Property PP2 can only be Y_1. But for $\lambda < \lambda'_2$, by Property PP5, Y_1 cannot be a member of $\mathcal{B}_\lambda$. Hence $\lambda_2 = \lambda'_2$. By repeating this argument, we see that t must be equal to k and $\lambda_i = \lambda'_i, i = 1, \cdots, t$. ∎

1.6.3 Principal Partition of Contraction and Restriction

There is a simple relationship between the principal partition of a polymatroid rank function and its restrictions and contractions relative to sets in the principal partition.

If the function is $f \diamond T(\cdot), S-T \subseteq X^{\lambda_1}$ then the principal partition for $\lambda < \lambda_1$ remains essentially that of $f(\cdot)$ except that we have $X - (S-T)$ as minimizing set in place of X, a superset of $S - T$. On the other hand if the function is $f(\cdot)/T$, where $T \supseteq X^{\lambda_1}$, then the principal partition for $\lambda \geq \lambda_1$, is identical to that of $f(\cdot)$. We formalize these ideas below.

Theorem 1.6.2 *Let $f(\cdot)$ be a polymatroid rank function and $g(\cdot)$ a positive weight function on subsets of S.*

1. *Let $(S-T) \subseteq X^{\lambda_1}$. Let $\lambda < \lambda_1$. Then for $\hat{X} \supseteq X^{\lambda_1}$, $\lambda f(\hat{X}) + g(S - \hat{X}) = min_{X \subseteq S} \lambda f(X) + g(S - X)$ iff*
   $\lambda f \diamond T(\hat{X} \cap T) + g(T - \hat{X} \cap T) = min_{Y \subseteq T} \lambda f \diamond T(Y) + g(T - Y)$.

2. *Let $T \supseteq X^{\lambda_1}$. Let $\lambda \geq \lambda_1$. Then, $\lambda f(\hat{X}) + g(S - \hat{X}) = min_{X \subseteq S} \lambda f(X) + g(S - X)$ iff $\lambda f/T(\hat{X}) + g(T - \hat{X}) = min_{Y \subseteq T} \lambda f/T(Y) + g(T - Y)$.*

Proof : i. For $\lambda < \lambda_1$, we have $X_\lambda \supseteq X^{\lambda_1}$. We now have, for $\hat{X} \supseteq X^{\lambda_1} \supseteq (S-T)$,
$\lambda f \diamond T(\hat{X} \cap T) + g(T - \hat{X} \cap T) = \lambda[f(\hat{X} \cap T \cup (S-T)) - f(S-T)] + g(S - \hat{X})$
$= \lambda f(\hat{X}) + g(S - \hat{X}) - \lambda f(S-T)$.

Next we have, $min_{Y \subseteq T} \lambda f \diamond T(Y) + g(T - Y)$
$= min_{Y \subseteq T} \lambda[f(Y \cup (S-T)) - f(S-T)] + g(S - (Y \cup (S-T)))]$
$= min_{Y \subseteq T} \lambda f(Y \cup (S-T)) + g(S - (Y \cup (S-T))) - \lambda f(S-T)]$
$= min_{X \subseteq S} \lambda f(X) + g(S - X) - \lambda f(S-T)$
(noting that $\lambda < \lambda_1$ implies $X_\lambda \supseteq X^{\lambda_1} \supseteq (S-T)$).
Thus the LHS of the two equations as well as the RHS of the equations differ by $\lambda f(S-T)$, which proves the result.
ii. We note that if $\lambda \geq \lambda_1$, $X^\lambda \subseteq X^{\lambda_1}$. So the first equation holds for $\hat{X}$, only if $\hat{X} \subseteq X^{\lambda_1} \subseteq T$. The result follows by noting that the LHS of the two equations differ by $g(S-T)$ and so do the RHS of the two equations. ∎

1.6.4 Principal Partition of the Dual

We next study the principal partition of the dual.
We have the following result which summarizes the relation between the PP of $(f(\cdot), g(\cdot))$ and that of $(f^*(\cdot), g(\cdot))$. Essentially, critical values of the dual are of the form $\lambda^* \equiv (1 - (\lambda)^{-1})^{-1}$, where λ are the critical values of the original function and the minimizing sets in the dual corresponding to λ^* are complements of those corresponding to λ in the original.

Theorem 1.6.3 *Let $f(\cdot)$ be a submodular function on the subsets of S and let $g(\cdot)$ be a positive weight function on subsets of S. Let $\mathcal{B}_\lambda, \mathcal{B}^*_\lambda$ denote respectively the collection of minimizing sets corresponding to λ in the principal partitions of $(f(\cdot), g(\cdot)), (f^*(\cdot), g(\cdot))$, where*
$f^*(\cdot)$ denotes the dual of $f(\cdot)$ with respect to $g(\cdot)$. Let λ^* denote $(1 - (\lambda)^{-1})^{-1} \quad \forall \lambda \in \Re$. Then

1. *a subset X of S is in $\mathcal{B}_\lambda$ iff $S - X$ is in $\mathcal{B}^*_{\lambda^*}$,*
2. *if $\lambda_1, \cdots, \lambda_t$ is the decreasing sequence of critical values of $(f(\cdot), g(\cdot))$, then $\lambda^*_t, \cdots, \lambda^*_1$ is the decreasing sequence of critical values of $(f^*(\cdot), g(\cdot))$,*
3. *if the principal sequence of $(f(\cdot), g(\cdot))$ is $\emptyset = X_0, \cdots, X_t = S$, then the principal sequence of $(f^*(\cdot), g(\cdot))$ is $\emptyset = S - X_t, \cdots, S - X_0 = S$.*

Proof:
i. We will show that Y minimizes $\lambda f(X) + g(S-X)$ iff $S - Y$ minimizes $\lambda^* f^*(X) + g(S - X)$. We have

$$\lambda^* f^*(X) + g(S-X) = \lambda^*[g(X) - (f(S) - f(S-X))] + g(S-X)$$

$$= \lambda^* f(S-X) + (\lambda^* - 1)g(X) - \lambda^* f(S) + g(S).$$

Minimizing this expression is equivalent to minimizing the expression $\lambda^*(\lambda^* - 1)^{-1}f(S - X) + g(X)$. Noting that $\lambda^*(\lambda^* - 1)^{-1} = \lambda$ we get the desired result. (We note that when one of λ, λ^* is 1, the other is to be taken as $+\infty$.)

The remaining sections of the theorem are now straightforward. ∎

1.6.5 Principal Partition and the Density of Sets

The principal partition gives information about which subsets are densely packed relative to $(f(\cdot), g(\cdot))$. Let us define the *density* of $X \subseteq S$ relative to $(f(\cdot), g(\cdot))$ to be $g(X)/f(X)$, taking the value to be $+\infty$ when $f(X)$ is zero. For instance if $f(\cdot)$ is the rank function of a graph and $g(X) \equiv\mid X \mid$, the sets of the highest density correspond to subgraphs where we can pack the largest (fractional) number of disjoint forests. As we see below, the sets of the highest density will be the sets in $\mathcal{B}_{\lambda_1}$, where λ_1 is the highest critical value.

The problem of finding a subset T of S of highest density for a given $g(T)$ value would be NP hard even for very simple submodular functions.

Example: Let $f(\cdot) \equiv$ rank function of a graph, $g(X) \equiv\mid X \mid$. In this case $g(T) =\mid T \mid$ and if we could find a set of branches of given size and highest density we can solve the problem of finding the maximal clique subgraph of a given graph. However, as we show below in Theorem 1.6.4, every set in the principal partition has the highest density for its $g(T)$ value and further is easy to construct. This apparent contradiction is resolved when we note that there may be no set of the given value of $g(T)$ in the principal partition.

Theorem 1.6.4
Let $f(\cdot), g(\cdot)$ be polymatroid rank functions on subsets of S with $g(\cdot)$, a positive weight function. Let T be a set in the principal partition of $(f(\cdot), g(\cdot))$. If $T' \subseteq S$ s.t. $g(T) = g(T')$ and T' not in the principal partition, then the density of $T \subseteq S$ is greater than that of T'.

Proof : Suppose otherwise. Let λ be the density of T. We must have $g(T') - \lambda f(T') \geq 0 = g(T) - \lambda f(T)$. Hence, $g(S-T) + \lambda f(T) \geq g(S-T') + \lambda f(T')$. But $g(S - T) = g(S - T')$. Hence, $f(T) \geq f(T')$, since $\lambda > 0$. Let $T \in \mathcal{B}_\sigma$. Then T minimizes the expression $g(S - X) + \sigma f(X) \quad \forall X \subseteq S$. But since $\sigma > 0$ ($\sigma = 0$ minimizes the expression $g(S - X) + \sigma f(X)$ only at $X = S$), $g(S - T) + \sigma f(T) \geq g(S - T') + \sigma f(T')$, a contradiction, since T' is given to be not a set in the principal partition (and therefore, not in B_σ). ∎

1.6.6 Outline of algorithm for Principal Partition

We now present an informal algorithm for building the principal partition of a polymatroid rank function $f(\cdot)$ on subsets of S with respect to a positive weight function $g(\cdot)$.

We assume that we have a subroutine $\int_\lambda(f, g, P)$ for finding all sets $\hat{X}$ which minimize $\lambda f(X) + g(P - X), X \subseteq P$, where $f(\cdot), g(\cdot)$ are as above, $\lambda \geq 0$

and P is the underlying set. By property PP1 (subsection 1.6.2), such sets are closed under union and intersection. This enables them to be represented through a partial order on a suitable partition of P. (Details may be found in [Narayanan97]).

Step 1. Take $\lambda =$ the 'density' $g(S)/f(S)$ and apply $\int_\lambda(f,g,S)$. We obtain X_λ. Output the family $\mathcal{F}_\lambda$ of sets $\hat{X}-X_\lambda$, where $\hat{X}$ is in the family of sets output by $\int_\lambda(f,g,S)$. By Theorem 1.6.2, the sets in this family minimize $\lambda\hat{f}(X)+g(S-X_\lambda-X), X\subseteq S-X_\lambda$, where $\hat{f}(\cdot)\equiv f\diamond(S-X_\lambda)$. If $X_\lambda=\emptyset$ and $X^\lambda=S$, (i.e., if $\lambda f(S)=g(S)$) we stop.

Step 2. Now work with $f_1 \equiv f/X_\lambda, f_2 \equiv f\diamond(S-X_\lambda)$, $g_1 \equiv g/X_\lambda, g_2 \equiv g/(S-X_\lambda)$. Repeat with $(f_1(\cdot),g_1(\cdot)),(f_2(\cdot),g_2(\cdot))$ using $\int_{\lambda_1}(f_1,g_1,X_\lambda),\int_{\lambda_2}(f_2,g_2,S-X_\lambda)$ respectively where λ_1,λ_2 are the corresponding densities.

The only $\lambda's$ in the above sequence of steps which are critical values are those for which the sets X^λ, X_λ $(f_j(\cdot),g_j(\cdot))$ are the full set (at that stage of the algorithm) and the null set respectively.

At the end of the algorithm, we will have a number of families $\mathcal{F}_\lambda$. In the process, we will have a number of disjoint sets $K_1,K_2,\cdots K_j,\cdots$, and a corresponding sequence of critical values such that $\int_{\lambda_j}(f_j,g_j,K_j)$ yields $K_j,\emptyset$, as the maximal and minimal minimizing sets for λ_j. Let the critical values λ_j be reordered as a decreasing sequence λ^j, and let the corresponding sets be $K^1,K^2,\cdots K^j,\cdots$.

The principal sequence then is $K^1, K^1\cup K^2, K^1\cup K^2\cup K^3,\cdots,S$ and the critical values are $\lambda^1,\lambda^2,\cdots$.

We have X^j as a minimizing set corresponding to λ^j, for $f^j(\cdot),g^j(\cdot),K^j$, where $f^j(\cdot)\equiv f\diamond(S-[\bigcup_{i\leq(j-1)}K^i])/K^j, g^j(\cdot)\equiv g/K^j(\cdot)$,
iff
$X^j\cup[\bigcup_{i\leq(j-1)}K^i]$ is a minimizing set for $f(\cdot),g(\cdot),S$ corresponding to λ^j.

If we could take $\int_\lambda(f,g,P)$ to output only the minimal set minimizing $\lambda f(X)+g(P-X), X\subseteq P$, the above algorithm would construct only the principal sequence instead of the complete principal partition. The algorithm is justified through the use of Theorem 1.6.2.

1.6.7 Notes

An excellent overview of submodular functions (and therefore polymatroid rank functions) is available in [Lovász83]. The principal partition of the polymatroid rank function can be generalized to that of a pair of them. Details might be found in [Fujishige91]. If the principal partitions of two matroids on the same underlying set have common sets then this goes over also to the union of the matroids under simple conditions. Details may be found in [Narayanan97]. For the matroid case, applications of the 'structural solvability' kind may be found in the following representative references: [Ozawa76], [Sugihara+Iri80], [Iri+Tsunekawa+Murota82], [Iri83], [Sugihara83], [Sugihara86], [Murota+Iri85], [Murota87]. An up to date survey of principal partition and related ideas may be found in [Fujishige09].

1.7 Matroid Union

In this section we give a self contained description of the matroid union concept and link it to the principal partition of a matroid. We first give an informal algorithm for building a maximal union of bases, one from a matroid $\mathcal{M}_1$ and the other from the matroid $\mathcal{M}_2$. We show that in the process, we really are constructing the base of another matroid, which could aptly be called the 'union' of the two matroids. The algorithm is due to Edmonds [Edmonds65a]. It can be easily modified to give the maximum size common independent set of two matroids. It also allows us to discuss the structure of various standard 'objects' associated with the matroid union viz. f-circuit, the set of coloops etc.

1.7.1 The Matroid Union Algorithm

In this subsection we give an informal description of the matroid union algorithm and justify it.

Let b_1, b_2 be bases of matroids $\mathcal{M}_1$, $\mathcal{M}_2$ respectively on S. We aim to make $b_1 \cup b_2$ a maximal such union (equivalently make b_1, b_2 'maximally distant'). If $b_1 \cup b_2 = S$ there is nothing to be done. Otherwise let $e \in S - (b_1 \cup b_2)$. Now let $L_1(e, b_1)$, $L_2(e, b_2)$ be the unique fundamental circuits that e forms with b_1, b_2 in the matroid $\mathcal{M}_1$, $\mathcal{M}_2$ respectively. The elements in $L_1(e, b_1)$ which do not intersect b_2 in $\mathcal{M}_2$ form fundamental circuits with b_2 in $\mathcal{M}_2$ and similarly elements in $L_2(e, b_2)$ with b_1 in $\mathcal{M}_1$. Let the set of all elements in the fundamental circuits obtained by repeatedly performing these operations be $R(e, b_1, b_2)$ which let us call $\widehat{R}$ temporarily. It is clear that $b_i \cap \widehat{R}$ spans the set $\widehat{R}$ in $\mathcal{M}_i.\widehat{R}$ since all elements in $\widehat{R} - b_i$ form fundamental circuits with it. If $b_1 \cap b_2 \cap \widehat{R}$ is not null we will show that $b_1 \cup b_2$ can be enlarged.

Let $e_c \in b_1 \cap b_2 \cap \widehat{R}$. We then must have a sequence $e, e_1, e_2, \cdots, e_k = e_c$ with property that $e_1 \in L_1(e, b_1)$, $e_2 \in L_2(e_1, b_2), \cdots, e_k \in L_i(e_{k-1}, b_i)$ where $i = 1$ or 2 depending on whether k is odd or even. We may, without loss of generality, assume that if e_r is in the sequence it does not occur in a fundamental circuit of e_j, $j < r - 1$. Now we update the bases b_1, b_2 as follows. (Let $e_k \in L_1(e_{k-1}, b_1)$ for notational convenience.)

$$b_1^1 = b_1 - e_k + e_{k-1}$$

$$b_2^1 = b_2 - e_{k-1} + e_{k-2}$$

$$b_1^2 = b_1^1 - e_{k-2} + e_{k-3}$$

$$\vdots$$

The claim now is that each of the sets b_1^j, b_2^j is actually a base of $\mathcal{M}_1$, $\mathcal{M}_2$ respectively for every j.

It is clear that b_1^1, b_2^1 are indeed such bases. Consider b_1^2. This would be the base of $\mathcal{M}_1$ provided $e_{k-2} \in L_1(e_{k-3}, b_1^1)$. But this is so because in $L_1(e_{k-3}, b_1^1)$ we know that e_k does not lie so that $L_1(e_{k-3}, b_1^1) = L_1(e_{k-3}, b^1)$.

Repeating this argument it is clear that b_1^j, b_2^j is actually a base of $\mathcal{M}_1$, $\mathcal{M}_2$ respectively for j.

Suppose finally, $b_1^t = b_1^{t-1} - e_1 + e$ and $b_2^{t-1} = b_2^{t-2} - e_2 + e_1$,

$$b_1^t \cup b_2^{t-1} = b_1 \cup b_2 \cup e.$$

Thus $b_1 \cup b_2$ has been enlarged, by including e and making e_c belong only to one of the bases b_1^t, b_2^{t-1}.

We repeat this procedure, which we shall call 'updating using reachability', with every element e outside $b_1 \cup b_2$ and stop when we can proceed no further, i.e., till a stage is reached where no element in $b_1 \cap b_2$ can be reached from an element outside $b_1 \cup b_2$. (We will call the resulting bases b_1, b_2 'maximally distant'.) This is the matroid union algorithm.

Let b_1, b_2 be maximally distant and let R be the set of all such elements reachable from elements of $S - b_1 \cup b_2$ by using fundamental circuits in the matroids $\mathcal{M}_i, i = 1, 2$ with b_1, b_2 repeatedly as above. We then have the following lemma which also contains a justification for the matroid union algorithm.

Lemma 1.7.1 *1. For the matroids $\mathcal{M}_i.R, i = 1, 2$, $b_i \cap R, i = 1, 2$ respectively are disjoint bases.*

2. If b_1, b_2 are the output of the matroid union algorithm (i.e., are maximally distant), then $|b_1 \cup b_2|$ has the maximum size among all unions of bases from $\mathcal{M}_1, \mathcal{M}_2$ respectively and this number is $r_1(R)+r_2(R)+|S-R|$ which is $min_{X \subseteq S} r_1(X) + r_2(X) + |S - X|$.

3. Given an element e in R, it is possible to find maximally distant bases b_1, b_2 of $\mathcal{M}_i$, $i = 1, 2$ respectively such that $e \notin b_1 \cup b_2$.

Proof : All elements in $R - (b_i \cap R)$ form fundamental circuits with $b_i \cap R$ in the matroid $\mathcal{M}_i.R, i = 1, 2$ (equivalently in $\mathcal{M}_i$, $i = 1, 2$). When the algorithm terminates, the set of all elements reachable from outside $b_1 \cup b_2$, by taking repeated fundamental circuit operations with respect to the two matroids, does not contain any element of $b_1 \cap b_2$. This is so since, otherwise, by using the algorithm, we can enlarge $b_1 \cup b_2$ by adding the external element, from which the common element can be reached, to the union of the bases (making the common element not a part of one of the bases). This proves that $b_i \cap R, i = 1, 2$ are disjoint bases of $\mathcal{M}_i.R, i = 1, 2$ respectively.

We have $|b_1 \cup b_2| = r_1(R) + r_2(R) + |S - R|$. If b_1', b_2' are bases of $\mathcal{M}_i$, $i = 1, 2$, then $b_i' \cap X, i = 1, 2, X \subseteq S$, are contained in bases of $\mathcal{M}_i.X, i = 1, 2$ and $(b_1' \cup b_2') \cap (S - X) \subseteq (S - X)$. So $|b_1' \cup b_2'| \leq r_1(X) + r_2(X) + |S - X|$, for every subset X of S. This proves that $|b_1 \cup b_2|$ is the maximum possible and $r_1(R) + r_2(R) + |S - R| = min_{X \subseteq S} r_1(X) + r_2(X) + |S - X|$.

Next, every element e_{in} in $b_i \cap R, i = 1, 2$ is reachable from some element e_{out} outside $b_1 \cup b_2$. If we use the updating using reachability procedure, e_{out} would move into the union of the updated bases and e_{in} would move out. This proves that any element in R lies outside some maximally distant pair of bases. ∎

Example

In Figure 1.1, consider the trees t_1, t_2 of the graph $\mathcal{G}$. We illustrate the algorithm using two trees of $\mathcal{G}$ and obtaining a maximally distant pair by using the matroid union algorithm.

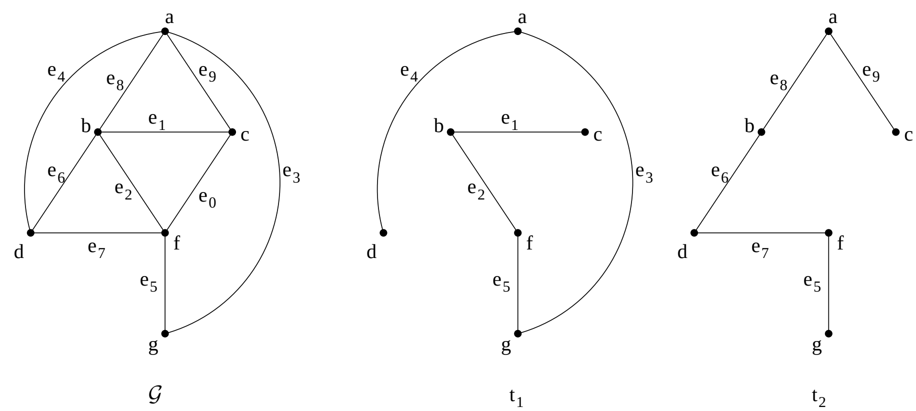

Figure 1.1: Example for matroid union algorithm

In the present case both the matroids are the same being the polygon matroids of $\mathcal{G}$ (i.e., independent set $\equiv$ circuit free set). We have,

$$L(e_0, t_1) = \{e_0, e_1, e_2\}$$

$$L(e_1, t_2) = \{e_1, e_8, e_9\}$$

$$L(e_8, t_1) = \{e_8, e_3, e_5, e_2\}.$$

Note that e_5 belongs to both the trees.

So, we update the trees by,

$$t_1^1 = t_1 - e_5 + e_8$$

$$t_2^1 = t_2 - e_8 + e_1$$

$$t_1^2 = t_1^1 - e_1 + e_0.$$

Observe that $t_1^2 \cup t_2^1 = t_1 \cup t_2 \cup e_0$. Further $e_5 \in t_2^1$ and $e_5 \notin t_1^2$.

For completeness, we give a formal description of the matroid union algorithm below. We make use of a directed graph, $G(b_1, b_2)$, associated with bases

b_1, b_2 of matroids $\mathcal{M}_1, \mathcal{M}_2$ respectively defined on S. The *graph* $G(b_1, b_2)$ *is built as follows:* S is the vertex set of the directed graph. Let v_1, v_2 be vertices. Then there is an edge (v_1, v_2, i) directed from v_1 to v_2 iff $v_2 \in L_i(v_1, b_i)$, i.e., iff v_2 lies in the fundamental circuit of v_1 with respect to b_i in the matroid $\mathcal{M}_i$. If $v_1 \in b_i$ there is no edge of the kind (v_1, v_2, i). The notational difference between the informal algorithm and the present description is that elements of S are here denoted by v and the edges of the graph $G(b_1, b_2)$ by e.

ALGORITHM 1.1 Algorithm Matroid Union

INPUT *Matroids* $\mathcal{M}_1, \mathcal{M}_2$ *on* S. *Bases* b_1, b_2 *of* $\mathcal{M}_1, \mathcal{M}_2$ *respectively.*

OUTPUT
1. *Bases* b_1^f, b_2^f *of* $\mathcal{M}_1, \mathcal{M}_2$ *respectively such that* $b_1^f \cup b_2^f$ *has maximum size (i.e. is a base of* $\mathcal{M}_1 \vee \mathcal{M}_2$*).*
2. *The set* R *of all element reachable from* $S - b_1^f \cup b_2^f$ *in* $G(b_1^f, b_2^f)$.

Initialize $j \leftarrow 0$
(COMMENT: j describes the current index of the base set.)
$b_1^j \leftarrow b_1, b_2^j \leftarrow b_2$.

STEP 1 *Construct* $G(b_1^j, b_2^j)$. *If* $S = b_1^j \cup b_2^j$, *GOTO STEP 7.*

STEP 2 *Mark all vertices which belong to both of the* b_i^j.
For each $v \in S - b_1^j \cup b_2^j$ *in* $G(b_1^j, b_2^j)$, *do*
Starting from v *do a* bfs *(breadth first search) and find the set of all vertices reachable through directed paths from* v.
(COMMENT: The directed edges (v_a, v_b, p), (v_c, v_d, q), $p \neq q$ *may be in the same directed path.)*
If no marked vertex is reachable from v *in* $G(b_1^j, b_2^j)$ *call* v *good. Otherwise* v *is bad.*

STEP 3 *If all* $v \in S - b_1^j \cup b_2^j$ *are good, GOTO STEP 7.*

STEP 4 *Let* v *be a bad vertex of* $G(b_1^j, b_2^j)$ *and let* v_m *be a marked vertex reachable from* v. *Let* $v = v_o, e_1, v_1, \cdots, e_m, v_m$ *be the shortest directed path from* v *to* v_m *(where* e_i *is the directed edge from* v_{i-1} *to* v_i*).*
For $i = 0$ *to* $m - 1$, *do*
If $e_{m-i} \equiv (v_{m-i-1}, v_{m-i}, q)$
$b_q^j \leftarrow (b_q^j \cup v_{m-i-1}) - v_{m-i}$
(COMMENT: The union of the updated bases has size one more than the union of the original bases since v_o *has moved into the union by pushing* v_m *out of one of the bases to which it belonged.)*

STEP 5 *For* $i = 1, 2$, *do*
$b_i^{j+1} \leftarrow b_i^j$

STEP 6 $j \leftarrow j + 1$. *GOTO STEP 1*

STEP 7 *Declare:* $b_1^f = b_1^j, b_2^f = b_2^j$
and R *to be the set of all vertices reachable in* $G(b_1^f, b_2^f)$ *from* $S - b_1^f \cup b_2^f$.

STOP

1.7.2 Complexity of the Matroid Union Algorithm

It is convenient to discuss the complexity of the algorithm in terms of the directed graph, $G(b_1, b_2)$, associated with bases b_1, b_2 of matroids $\mathcal{M}_1, \mathcal{M}_2$ respectively defined on S.

Let us suppose that the matroids are available through the 'independence oracle' which would declare, once per call, whether a particular subset of S is independent in the specified matroid $\mathcal{M}_i, i = 1, 2$.

How many calls do we require to build $G(b_1, b_2)$? This requires the knowledge of the f-circuits of an element outside b_i with respect to it in the matroid $\mathcal{M}_i$. To build $L_i(v, b_i)$ we check for each $v' \in b_i$ whether $v \cup b_i - v'$ is independent in the matroid $\mathcal{M}_i$. This requires atmost $r(\mathcal{M}_i)$ calls to the independence oracle. Thus the total number of calls to the independence oracle to build $G(b_1, b_2)$ is atmost $\mid S - b_1 \mid\mid b_1 \mid + \mid S - b_2 \mid\mid b_2 \mid$.

Finding the reachable set from $S - (b_1 \cup b_2)$ requires $O \mid E(G(b_1, b_2)) \mid$ elementary steps, where $\mid E(G(b_1, b_2)) \mid$ is the number of edges in $G(b_1, b_2)$ treating parallel edges as a single edge.

We may have started with (in the worst case) $b_1 = b_2$ and end with a base of the union of size atmost $\mid b_1 \mid + \mid b_2 \mid$. The graph $G(b_1, b_2)$ has to be rebuilt after each update. Let us call such graphs $\mathcal{G}_j$. So the overall complexity is $O|b_1|(\mid b_1 \mid\mid S - b_1 \mid + \mid b_2 \mid\mid S - b_2 \mid)$ calls to the independence oracle. If r, r' are the maximum and minimum of the ranks of the matroids this simplifies to $O(r^2(\mid S \mid - r'))$ calls to the independence oracle.

There are $O(r \mid E(\mathcal{G}_j) \mid)$ elementary steps involved in building the reachable set for all the $\mathcal{G}_j$. Let us simplify $\mid E(\mathcal{G}_j) \mid$ to $O(\mid S \mid^2)$.

So the *time complexity of the Matroid Union Algorithm* is
$O(r^2(\mid S \mid - r'))$ calls to the independence oracle $+O(r \mid S \mid^2)$ elementary operations.

The space requirement is that of storing the updated version of the graph $G(b_1, b_2)$. This has atmost $\mid S \mid^2$ edges. So the *space complexity of the Matroid Union Algorithm* is $O(\mid S \mid^2)$.

1.7.3 Matroid Union Theorem

We now state and prove the matroid union theorem. Note that a less illuminating but brief proof has been given in Theorem 1.2.3.

Theorem 1.7.1 *Let* $\mathcal{M}_1 \equiv (S, \mathcal{I}_1)$, $\mathcal{M}_2 \equiv (S, \mathcal{I}_2)$ *be matroids with rank functions* $r_1(.)$, $r_2(.)$ *respectively and let* $\mathcal{I}_1 \vee \mathcal{I}_2$ *be the collection of all sets* X *such that* $X = X_1 \cup X_2$,*where* X_1, X_2 *are independent sets respectively in* $\mathcal{M}_1$, $\mathcal{M}_2$.

Then $\mathcal{M}_1 \vee \mathcal{M}_2 \equiv (S, \mathcal{I}_1 \vee \mathcal{I}_2)$ *is a matroid with rank function*

$$r_v(K) = min_{X \subseteq K}(r_1 + r_2)(X) + |K - X|, i.e., r_v(\cdot) = (r_1 + r_2) * |\cdot|.$$

Proof: It is clear that subsets of a set $X \in \mathcal{I}_1 \vee \mathcal{I}_2$ also belong to $\mathcal{I}_1 \vee \mathcal{I}_2$. We will verify that maximal subsets belonging to $\mathcal{I}_1 \vee \mathcal{I}_2$ which are contained in a given subset $K \subseteq S$ have the same size.

We first observe that if b_1, b_2 are bases of $\mathcal{M}_1.K$, $\mathcal{M}_2.K$ respectively then for any set $X \subseteq K$,

$$|(b_1 \cup b_2) \cap X| \leq r_1(X) + r_2(X)$$

$$|(b_1 \cup b_2) \cap (K - X)| \leq |K - X|.$$

Hence,

$$|b_1 \cup b_2| \leq min_{X \subseteq K}[(r_1 + r_2)(X) + |K - X|].$$

We will now construct a subset R of K where we have equality.

If we use the matroid union algorithm on bases of $\mathcal{M}_1.K$ and $\mathcal{M}_2.K$ we will finally reach bases b_1, b_2 respectively of these matroids which are maximally distant. At this stage one of the following two situations will occur.

1. $b_1 \cup b_2 = K$
   In this case $(r_1 + r_2)(\phi) + |K| = |b_1 \cup b_2|$
2. $K - (b_1 \cup b_2) = T \neq \phi$.

By using the repeated fundamental circuit operation in $\mathcal{M}_1$, $\mathcal{M}_2$ starting from each element $e \in T$, it should be impossible to reach any element $e' \in b_1 \cap b_2$ (i.e., there is no directed path in $G(b_1, b_2)$ from the 'vertex' e to the vertex e'), since otherwise we can enlarge $b_1 \cup b_2$.

Let R be the set of all such elements reachable from elements of T by using b_1, b_2. By Lemma 1.7.1, $b_i \cap R$ are disjoint bases in the matroids $\mathcal{M}_i.R$, $i = 1, 2$ respectively. (Note that $\mathcal{M}_i.K.R = \mathcal{M}_i.R$ and the rank function of the matroids $\mathcal{M}_i.R$ and $\mathcal{M}_i$ coincide on subsets of R.) Thus

$$|(b_1 \cup b_2) \cap R| = r_1(R) + r_2(R).$$

The elements in $K - R$ are covered by $(b_1 \cup b_2)$. Hence

$$|(b_1 \cup b_2) \cap (S - R)| = |K - R|$$

Thus the size of the maximal union of independent sets is

$$= (r_1 + r_2)(X) + |K - X| \qquad for\ X = R.$$

Since we have already seen that it is less than or equal to $min_{X \subseteq K}(r_1 + r_2)(X) + |K - X|$, it follows that the size of any maximal union of independent sets of $\mathcal{M}_1$, $\mathcal{M}_2$ contained in K equals $min_{X \subseteq K}(r_1 + r_2)(X) + |K - X|$ and is therefore always the same as required. Further it is clear that $r_v(K) = min_{X \subseteq K}(r_1 + r_2)(X) + |K - X|$. ∎

1.7.4 Fundamental circuits and coloops of $\mathcal{M}_1 \vee \mathcal{M}_2$

Let us now understand details of the matroid union algorithm in the context of the fact that $\mathcal{M}_1 \vee \mathcal{M}_2$ is a matroid. In particular we obtain a picture of fundamental circuits and coloops (i.e., elements not in any circuit) of the matroid $\mathcal{M}_1 \vee \mathcal{M}_2$.

Firstly if b_1, b_2 is a pair of maximally distant bases of $\mathcal{M}_1, \mathcal{M}_2$ (say as output by the matroid union algorithm), then $b_1 \cup b_2$ is a base of $\mathcal{M}_1 \vee \mathcal{M}_2$. Consider the set R of all elements reachable from elements of $S - b_1 \cup b_2$ in the graph $G(b_1, b_2)$ (equivalently by the process of taking repeated fundamental circuits relative to b_1, b_2 in the matroids $\mathcal{M}_1, \mathcal{M}_2$ respectively). By Lemma 1.7.1 we know that for each $e \in R$, there exist some pair of maximally distant bases b_1', b_2' such that $e \notin b_1' \cup b_2'$, i.e., there exists a base of $\mathcal{M}_1 \vee \mathcal{M}_2$ which does not contain e. So R contains no coloops of $\mathcal{M}_1 \vee \mathcal{M}_2$. Lemma1.7.1 assures us that $r_1(R) + r_2(R) + |S - R| = min_{X \subseteq S} r_1(X) + r_2(X) + |S - X|$. By Theorem 1.5.5, this means that $|S - R|$ is a set of coloops of the matroid whose rank function is $(r_1 + r_2) * |\cdot|$, i.e., of the matroid $\mathcal{M}_1 \vee \mathcal{M}_2$,

Thus the set R that we encounter in the matroid union algorithm is the set of all noncoloops of the matroid $\mathcal{M}_1 \vee \mathcal{M}_2$ and is independent of the pair of maximally distant bases b_1, b_2. Further, again by Theorem 1.5.5, R is the unique minimal set that minimizes $r_1(X) + r_2(X) + |S - X|, X \subseteq S$.

Next let $b_1 \cup b_2$ be a base of $\mathcal{M}_1 \vee \mathcal{M}_2$ and let $e \notin b_1 \cup b_2$. Consider the set R_e of all elements reachable from e in $G(b_1, b_2)$. It is clear that all the elements of R_e are spanned by $b_i \cap R_e, i = 1, 2$ in the matroid $\mathcal{M}_i.R_e$ and further that $b_i \cap R_e, i = 1, 2$ are disjoint. So the union of no pair of maximally distant bases can contain R_e. On the other hand, given any e' in $b_i \cap R_e, i = 1, 2$, by using the updating through reachability process in the algorithm, we can build a pair of maximally distant bases b_1', b_2' such that $b_1' \cup b_2' = b_1 \cup b_2 \cup e - e'$. We conclude, using Theorem 1.2.5, that R_e is the fundamental circuit of e with respect to the base $b_1 \cup b_2$ of $\mathcal{M}_1 \vee \mathcal{M}_2$. Note that if $b_1 \cup b_2 = b_1" \cup b_2"$, R_e would be the same using $G(b_1, b_2)$ or $G(b_1", b_2")$.

1.7.5 Union of Matroids and the Union of Dual Matroids

It is natural to examine the relation between $\mathcal{M}_1 \vee \mathcal{M}_2$ and $\mathcal{M}_1^* \vee \mathcal{M}_2^*$. We show in this section that the complements of coloops of these matroids do not intersect and that this gives a natural partition of S relative to $\mathcal{M}_1, \mathcal{M}_2$.

Theorem 1.7.2
Let $\mathcal{M}_1, \mathcal{M}_2$ be matroids on S and let $\mathcal{M}_1^*, \mathcal{M}_2^*$ be their duals. Let $r_1(\cdot), r_2(\cdot), r_1{}^*(\cdot)$,
$r_2{}^*(\cdot)$ be the rank of functions of $\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_1^*, \mathcal{M}_2^*$ respectively. Let R, R^* be the minimal sets that minimize $(r_1 + r_2)(X) + |S - X|, X \subseteq S$ and
$(r_1{}^* + r_2{}^*)(X) + \mid S - X \mid, X \subseteq S$ respectively. Then,

1. *b_1, b_2 are maximally distant bases of $\mathcal{M}_1, \mathcal{M}_2$ respectively iff $S - b_1, S - b_2$,*

are maximally distant cobases of the same matroids (equivalently maximally distant cobases of $\mathcal{M}_1^*, \mathcal{M}_2^*$).

2. *A set $K \subseteq S$ minimizes $(r_1+r_2)(X)+ \mid S-X \mid, X \subseteq S$ iff $S-K$ minimizes $({r_1}^* + {r_2}^*)(X)+ \mid S - X \mid, X \subseteq S$.*

3. *$S - R^*, S - R$ are the maximal sets that minimize*

$$(r_1 + r_2)(X)+ \mid S - X \mid, X \subseteq S,$$

$$({r_1}^* + {r_2}^*)(X)+ \mid S - X \mid, X \subseteq S$$

respectively.

4. *R is the collection of non-coloops of $\mathcal{M}_1 \vee \mathcal{M}_2$ and is disjoint from R^* which is the collection of non-coloops of $\mathcal{M}_1^* \vee \mathcal{M}_2^*$.*

5. *The set $S-(R\cup R^*)$ can be covered by disjoint bases of $\mathcal{M}_i.(S-R^*)\times(S-(R\cup R^*)), i = 1, 2$ (equivalently by those of $\mathcal{M}_i^*.(S-R)\times(S-(R\cup R^*)), i = 1, 2$).*

Proof : i. This follows essentially by noting that $b_1 \cup b_2$ is of maximum size iff $b_1 \cap b_2$ is of minimum size.
ii. We have

$$({r_1}^* + {r_2}^*)(X)+ \mid S - X \mid$$
$$= (2 \mid X \mid -(r_1(S) + r_2(S) - r_1(S - X) - r_2(S - X))+ \mid S - X \mid$$
$$= ((r_1 + r_2)(S - X)+ \mid S - (S - X) \mid) + (\mid S \mid -(r_1 + r_2)(S)).$$

It is thus clear that K minimizes $(r_1 + r_2)(Y) + |S - Y|, Y \subseteq S$ iff $(S - K)$ minimizes $({r_1}^* + r_2)^*(Y)+ \mid S - Y \mid, Y \subseteq S$.
iii. This is an immediate consequence of the above result when we note that R, R^* are the minimal sets which minimize respectively the expressions $(r_1 + r_2)(X)+ \mid S - X \mid, ({r_1}^* + {r_2}^*)(X)+ \mid S - X \mid$.
iv. We saw in subsection 1.7.4 that the collection of non-coloops of $\mathcal{M}_1 \vee \mathcal{M}_2$ $(\mathcal{M}_1^* \vee \mathcal{M}_2^*)$ is the minimal set R (R^*) that minimizes $(r_1 + r_2)(X)+ \mid S - X \mid, X \subseteq S(({r_1}^* + {r_2}^*)(X)+ \mid S - X \mid, X \subseteq S)$.

However, the second part (above) shows that $S - R \supseteq R^*$.

v. From Lemma 1.7.1 we know that maximally distant bases of $\mathcal{M}_1, \mathcal{M}_2$ respectively must intersect any set T which minimizes the expression $(r_1 + r_2)(X)+ \mid S - X \mid, X \subseteq S$ in disjoint bases of $\mathcal{M}_1.T, \mathcal{M}_2.T$ respectively and the corresponding (maximally distant) cobases must cover T. Similarly maximally distant bases of $\mathcal{M}_1^*, \mathcal{M}_2^*$ respectively must intersect any set P which minimizes the expression $({r_1}^* + {r_2}^*)(X)+ \mid S - X \mid, X \subseteq S$ in disjoint bases of $\mathcal{M}_1^*.P, \mathcal{M}_2^*.P$ respectively and the corresponding (maximally distant) cobases must cover P. It follows that set $P \cap T$ is covered by any pair of maximally distant bases of $\mathcal{M}_1, \mathcal{M}_2$ as well as $\mathcal{M}_1^*, \mathcal{M}_2^*$. Since by (i) above $S - P, S - T$ respectively minimize the expressions $(r_1 + r_2)(X)+ \mid S - X \mid, X \subseteq S, ({r_1}^* + {r_2}^*)(X)+ \mid S - X \mid, X \subseteq S$, (since minimizing sets are closed under intersection) so do

$(S-P)\cap T, (S-T)\cap P$ respectively. But this means, whenever b_1, b_2 are maximally distant bases of $\mathcal{M}_1, \mathcal{M}_2$ respectively they intersect $(S-P)\cap T$ in disjoint bases of $\mathcal{M}_1.((S-P)\cap T), \mathcal{M}_2.((S-P)\cap T)$ respectively, intersect T in disjoint bases of $\mathcal{M}_1.T, \mathcal{M}_2.T$ respectively and cover $P\cap T$. It follows that $b_1\cap P\cap T, b_2\cap P\cap T$ are disjoint bases of $\mathcal{M}_1.T\times(P\cap T), \mathcal{M}_2.T\times(P\cap T)$ which cover $P\cap T$. The result now follows substituting $S-R$ for P and $S-R^*$ for T. The dual result follows by working with dual matroids. ∎

Kishi and Kajitani's principal partition for graphs [Kishi+Kajitani69] is essentially the partition $R, S-R\cup R^*, R^*$ where $\mathcal{M}_1=\mathcal{M}_2=\mathcal{M}(\mathcal{G})$, $\mathcal{G}$ being the given graph.

1.7.6 Matroid Union and Matroid Intersection

The problem of matroid intersection (find the maximum size common independent set of two given matroids) and its solution has received more attention in the literature than matroid union. This is probably because Lawler based his well known book [Lawler76] on matroid intersection. In this subsection we consider the relation between the two problems. These results are due essentially to Edmonds [Edmonds70].

Theorem 1.7.3

Let $\mathcal{M}_1, \mathcal{M}_2$ be matroids on S. Let b_{12} be the largest set independent in $\mathcal{M}_1$ as well as in $\mathcal{M}_2$. Then

1. *b_{12} can be represented as $b_{12*}-b_2^*$ where b_{12*} is a base of $\mathcal{M}_1\vee\mathcal{M}_2^*$ which is the union of a base b_1 of $\mathcal{M}_1$ and a base b_2^* of $\mathcal{M}_2^*$,*

2. *every set of the form $b_{12*}-b_2^*$ is a common independent set of $\mathcal{M}_1, \mathcal{M}_2$ of maximum size.*

3. *$\mid b_{12}\mid = r(\mathcal{M}_1\vee\mathcal{M}_2^*) - r(\mathcal{M}_2^*) = min_{X\subseteq S} r(\mathcal{M}_1.X) + r(\mathcal{M}_2.(S-X))$,*

Proof : i. Let b_1, b_2 be bases of $\mathcal{M}_1, \mathcal{M}_2$ s.t. $b_1\cap b_2 = b_{12}$. Let ${b_2}^*\equiv S-b_2$. Then $b_{12} = b_1\cup {b_2}^* - {b_2}^*$. Next $b_1\cup {b_2}^*$ is independent in $\mathcal{M}_1\vee\mathcal{M}_2^*$, since ${b_2}^*$ is a base of $\mathcal{M}_2^*$. Let this be contained in the base $b_{1n}\cup b_{2n}^*$ of $\mathcal{M}_1\vee\mathcal{M}_2^*$. Now $b_{1n}\cup b_{2n}^* - b_{2n}^*$ is independent in $\mathcal{M}_1$ and $\mathcal{M}_2$. Further,

$$\mid b_{1n}\cup b_{2n}^* - b_{2n}^* \mid \geq \mid b_1\cup {b_2}^* - {b_2}^* \mid = \mid b_{12}\mid .$$

But b_{12} is the largest common independent set of $\mathcal{M}_1$ and $\mathcal{M}_2$. We conclude that $|b_{12}| = |b_{1n}\cup b_{2n}^* - b_{2n}^*|$, and $\mid b_1\cup {b_2}^*\mid = \mid b_{1n}\cup b_{2n}^*\mid$. Therefore $b_1\cup {b_2}^*$ is a base of $\mathcal{M}_1\vee\mathcal{M}_2^*$ and the result follows.

ii. If b_{12*} is a base of $\mathcal{M}_1\vee\mathcal{M}_2^*$ with $b_{12*} = b_1\cup {b_2}^*$, where $b_1, {b_2}^*$ are bases of $\mathcal{M}_1, \mathcal{M}_2^*$ respectively, then $b_{12*} - {b_2}^*$ is independent in $\mathcal{M}_1$ as well as in $\mathcal{M}_2$ and further its size equals $r(\mathcal{M}_1\vee\mathcal{M}_2^*) - r(\mathcal{M}_2^*)$.

iii. It is clear from the above that $\mid b_{12}\mid = r(\mathcal{M}_1\vee\mathcal{M}_2^*) - r(\mathcal{M}_2^*) = min_{X\subseteq S}(r(\mathcal{M}_1.X) + r(\mathcal{M}_2^*.X) + |S-X|) - r(\mathcal{M}_2^*)$.

Now $r(\mathcal{M}_2^*.X) - r(\mathcal{M}_2^*) = r(\mathcal{M}_2.(S-X)) - |S-X|$ and the result follows. ■

Note that the collection of maximal common independent sets of two matroids do not form the bases of a matroid since they do not always have the maximum size.

We next show how to convert Algorithm Matroid Union to an algorithm for finding the maximum size common independent set of two matroids $\mathcal{M}_1, \mathcal{M}_2$.

We begin with two bases b_1, b_2 of matroids $\mathcal{M}_1, \mathcal{M}_2$ respectively on S. Let $b_2 = S - b_2^*$, where b_2^* is a base of $\mathcal{M}_2^*$. We now try to push updated versions of b_1, b_2^* apart. However, we would like to work with f-circuits of $\mathcal{M}_2$ rather than with f-circuits of $\mathcal{M}_2^*$. For this it suffices to observe that $v_p \in L_2^*(v_q, b_2^*)$ iff $v_q \in L_2(v_p, b_2)$, where $L_2^*(\cdot,\cdot), L_2(\cdot,\cdot)$ denote f-circuits of $\mathcal{M}_2^*, \mathcal{M}_2$ respectively. So while constructing $G(b_1, b_2^*)$ it is convenient to build edges of the type $(v_p, v_q, 2)$ at the node v_q directed into v_q (rather than at v_p directed away from v_p). If b_1, b_2^* are maximally distant bases of $\mathcal{M}_1, \mathcal{M}_2^*$, then $b_1 \cap b_2$ is a common independent set of $\mathcal{M}_1, \mathcal{M}_2$ of maximum size as we saw in Theorem 1.7.3 above.

1.7.7 Applications of Matroid Union and Matroid Intersection

Representability of matroids (A.Horn [Horn55])
Horn showed that k independent sets of columns can cover the set of all columns of a matrix iff there exists no subset A of columns such that $|A| > kr(A)$. He conjectured that this might be correct only for representable matroids (i.e., for matroids which are associated with column sets of matrices over fields). If the conjecture had been true then there would have been a nice characterization of representability. It is clear that the problem for matroids is to check if S is independent in the matroid $\mathcal{M}_1 \vee \cdots \vee \mathcal{M}_k$, where all the $\mathcal{M}_i$ are the same matroid $\mathcal{M}$. From Theorem 1.7.1, this happens iff $min_{A \subseteq S} kr(A) + |S-A| = |S|$, i.e., iff $|A| \leq kr(A), \forall A \subseteq S$. So the result is true for arbitrary matroids.

Decomposition of a graph into minimum number of subforests [Tutte61], [Nash-Williams61]).
Tutte and Nash-Williams characterized graphs which can be decomposed into k disjoint subforests as those which satisfy $kr(X) \geq |X|, \forall X \subseteq E(\mathcal{G})$. This condition again fits into the matroid union framework as described above.

We need some preliminary definitions to describe the following results. Let $B \equiv (V_L, V_R, E)$ be a bipartite graph, which has all edges (members of E) with one endpoint in ('left vertex set') V_L and another in ('right vertex set') V_R. If $X \subseteq V_L, (X \subseteq V_R)$, then $\Gamma_L(X)(\Gamma_R(X))$ denotes the set of vertices adjacent to X. A matching is a subset of edges with no two incident on the same vertex. A cover is a subset of vertices containing atleast one end point of every edge in E. We give below some fundamental results about bipartite graph matching and derive them using matroid union or matroid intersection. Given a family $S_1, \cdots, S_k$ of S, a transversal of the family is a set $\{v_1, \cdots v_k\}$ of k elements such that $v_i \in S_i$. (Note that the definition of a family permits $S_i = S_j$ even if $i \notin j$). In a bipartite graph $V_L(V_R)$ can be regarded as a family of subsets of $V_R(V_L)$, by identifying a vertex in $V_L(V_R)$ with the subset of vertices of $V_R(V_L)$

it is adjacent to. Thus we could say V_R has a transversal whenever there is a subset T of V_L such that a matching has T as its left end points and V_R as its right end points.

Transversal Matroids [Edmonds+Fulkerson65] For each vertex $v \in V_R$, we define a matroid $\mathcal{M}_v$ on the set V_L. In this matroid the set of all vertices, say $v_{l1}, \cdots, v_{lk}$, which are adjacent to v, has rank one and contains no selfloops (rank zero elements). The complementary subset of vertices of V_L are all selfloops. Let its rank function be denoted by r_v. The union of all the matroids $\mathcal{M}_v, v \in V_R$, is a matroid which has, as independent sets, the subsets of V_L which are endpoints of matchings. This matroid is called the transversal matroid $\mathcal{M}_{tr}$. Its rank function, by using Theorem 1.7.1, can be seen to be $r_{tr}(X) \equiv min_{Y \subseteq X}(\sum_{v \in V_R} r_v(Y) + |X - Y|) = min_{Y \subseteq X}(|\Gamma_L(Y)| + |X - Y|), X \subseteq V_L$.

Konig's Theorem [König36]. Let $B \equiv (V_L, V_R, E)$ be a bipartite graph. A cover meets the edges of every matching. No two edges of any matching can meet the same vertex of any cover and therefore the size of a matching can never exceed the size of any cover. The following result is therefore remarkable.

Theorem 1.7.4 *[König36] In a bipartite graph the sizes of a maximum matching and a minimum cover are equal.*

This follows naturally from the matroid intersection result in Theorem 1.7.3 part (ii). We have the following matroids defined on E: $\mathcal{M}_L$ where the independent sets are subsets of E which do not meet any vertex of V_L in more than one edge and $\mathcal{M}_R$ where the independent sets are subsets of E which do not meet any vertex of V_R in more than one edge. ($\mathcal{M}_L$,$\mathcal{M}_R$ are easily seen to be matroids). A subset of E is a matching iff it is independent in both matroids. The size of the maximum matching is therefore $min_{X \subseteq E}(r_L(X) + r_R(E - X))$, where $r_L(\cdot)$, $r_R(\cdot)$ are the rank functions respectively of $\mathcal{M}_L$,$\mathcal{M}_R$. Now $r_L(X)$ ($r_R(E - X)$) is the size of the left vertex subset $v_L(X)$ ($v_R(E - X)$) meeting X ($E - X$). It is clear that $v_L(X) \cup v_R(E - X)$ is a cover. The result now follows.

Rado's Theorem [Rado42]

Theorem 1.7.5 *(Rado's Theorem [Rado42])*
Let $B \equiv (V_L, V_R, E)$ be a bipartite graph. Let $\mathcal{M}$ be a matroid on V_L with rank function $f(\cdot)$. Then V_R has a transversal that is independent in $\mathcal{M}$ iff

$$f(\Gamma_R(Z)) \geq | Z | \quad \forall Z \subseteq V_R.$$

Here we consider the intersection of two matroids on V_L, viz., $\mathcal{M}$ and the above mentioned transversal matroid $\mathcal{M}_t$. By Theorem 1.7.3, part (ii), the maximum size of a common independent set in the two matroids is
$min_{X \subseteq V_L}(r_t(X) + f(V_L - X))$
$= min_{X \subseteq V_L}(min_{Y \subseteq X}(|\Gamma_L(Y)| + |X - Y|) + f(V_L - X)))$
$= min_{X \subseteq V_L}(|\Gamma_L(X)| + f(V_L - X))$, where we have used the fact that $f(K) \leq |K|, K \subseteq V_L$. Clearly the maximum size of common independent set in the two matroids must become $|V_R|$ for V_R to have an independent transversal. This will happen iff $min_{X \subseteq V_L}(|\Gamma_L(X)| + f(V_L - X)) = |V_R|$,

i.e., iff $min_{X \subseteq V_L}(|\Gamma_L(X)| + f(V_L - X)) - |V_R| = 0$,
i.e., iff $f(V_L - X)) \geq |V_R| - |\Gamma_L(X)|, \forall X \subseteq V_L$. (*)
We claim that the condition (*) is equivalent to
$f(\Gamma_R(Z)) \geq |Z|, \forall Z \subseteq V_R$. (**)
To see this, first observe since $\Gamma_R(V_R - \Gamma_L(X)) \subseteq V_L - X$, and $f(\cdot)$ is an increasing function, it follows that (**) implies (*). Next, define $\overline{Z} \equiv V_R - \Gamma_L(V_L - \Gamma_R(Z))$. If (*) is true, taking $X \equiv V_L - \Gamma_R(Z)$, we have $f(\Gamma_R(Z)) \geq |\overline{Z}|, \forall Z \subseteq V_R$. But $\overline{Z} \supseteq Z$ and $\Gamma_R(\overline{Z}) = \Gamma_R(Z)$. So (**) is true.

1.7.8 Algorithm for construction of the Principal Sequence of a Matroid rank function

In this subsection we outline an algorithm for building the principal sequence of a matroid rank function with respect to a positive rational weight function. The main subroutine is the matroid union algorithm. The algorithm for the complete principal partition is along the same lines and may be found, for instance, in [Narayanan97]. This algorithm is elementary and handles the weight function in a naive manner. The case of real weight function may be tackled by using the methods in [Cunningham84],[Narayanan95a].

We need some preliminary ideas about parallel elements in a matroid for describing our algorithm.

For a matroid $\mathcal{M}$ on S, two elements e_1, e_2 are in parallel iff $\{e_1, e_2\}$ is a circuit or e_1, e_2 are both selfloops. It is immediate that if $e_1 \in I$, where I is independent in $\mathcal{M}$ and e_1, e_2 are in parallel, then $(I - e_1) \cup e_2$ is also independent. Given a matroid $\mathcal{M}$ on S, and $e \in S$, we can create a new matroid $\mathcal{M}'$ on $S \cup e', e' \notin S$, by making e, e' parallel. The independent sets of this new matroid are simply all the independent sets of $\mathcal{M}$ and in addition sets of the form $(I - e) \cup e'$, where $e \in I$, I independent in $\mathcal{M}$. This process can be repeated by adding more than one element in parallel with a given element. In particular, we could replace each element e of $\mathcal{M}$ by k parallel elements $\{e^1, \cdots, e^k\}$. The resulting matroid $\mathcal{M}_k$ is on $S^k \equiv \{\bigcup_{e_j \in S} P^k(e_j)\}$, where $P^k(e_j) \equiv \{e_j^1, \cdots, e_j^k\}$. The sets $P^k(e_j)$ constitute a partition of S^k. We denote by $P^k(T), T \subseteq S$, the set $\bigcup_{e_j \in T} P^k(e_j)$. If $r(\cdot), r_k(\cdot)$, are the rank functions of $\mathcal{M}, \mathcal{M}_k$, then $r(T) = r_k(P^k(T)), T \subseteq S$. More generally, given a positive integral weight function $g(\cdot)$ on subsets of S, we can build the g- copy $\mathcal{M}_g$ on S^g, of $\mathcal{M}$ on S, with each element $e \in S$ replaced by the set $P^g(e)$ of $g(e)$ parallel elements in S^g. Here again $r(T) = r_g(P^g(T)), T \subseteq S$, where $P^g(T)$ is defined as $\bigcup_{e_j \in T} P^g(e_j)$ and $r_g(\cdot)$ is the rank function of the matroid $\mathcal{M}^g$.

The principal partition of $r(\cdot), g(\cdot)$ is essentially the same as that of $r_g(\cdot), |\cdot|$. This situation basically does not change even if $g(\cdot)$ were divided by a positive integer. We formalize these ideas in the following theorem.

Theorem 1.7.6 *Let $\mathcal{M}$ be a matroid with rank function $r(\cdot)$. Let q be a positive and λ, a nonnegative number.*

1. *Let $g(\cdot)$ be a positive integral weight function. Then, $min_{X \subseteq S} \lambda r(X) + g(S-X)$ occurs at K iff $min_{Y \subseteq S^g} \lambda r_g(Y) + |S^g - Y|$ occurs at $P^g(K)$.*

2. *Let $qg(\cdot)$ be a positive integral weight function. Then, $min_{X \subseteq S} (\lambda/q) r(X) + g(S-X)$ occurs at K iff $min_{X \subseteq S} \lambda r(X) + qg(S-X)$ occurs at K, equivalently, $min_{Y \subseteq S^g} \lambda r_{qg}(Y) + |S^{qg} - Y|$ occurs at $P^{qg}(K)$.*

Proof : Let e_i, e_j be in parallel and let $e_i \in X, e_j \notin X$. Then $\lambda r(X) + g(S-X) > \lambda r(X \cup e_j) + g(S-(X \cup e_j))$. Thus a minimizing set of $\lambda r(X) + g(S-X)$ must contain all elements parallel to e if it contains e.

Now consider the principal partition of $(r(\cdot), g(\cdot))$. If we replace every element e by $g(e)$ parallel elements $\lambda r(X) + g(S-X) = \lambda r_g(P^g(X)) + |S^g - P^g(X)|$. Next if $\hat{X}$ minimizes $min_{Y \subseteq S^g} \lambda r_g(Y) + |S^g - Y|$, it must be of the form $P^g(X)$ for some $X \subseteq S$.
This proves (i).
(ii) is a routine consequence. ∎

To build the principal sequence of $(r(\cdot), g(\cdot))$, where $r(\cdot)$ is a matroid rank function and $g(\cdot)$, a positive integral weight function, the key step is the construction of $\int_\lambda(r, g, Q)$, which we will take as outputting the minimal minimizing set for $\lambda r(X) + g(Q-X)$. Since $g(\cdot), r(\cdot)$ are integral, the λs for which we need $\int_\lambda(r, g, Q)$, are of the form p/q, where p, q are positive integers. By Theorem 1.7.6, we need to build $\int_p(r_{qg}(\cdot), |\cdot|, P^{qg}(Q))$. The output of this subroutine is simply the set of noncoloops of the matroid $\mathcal{M}_{qg}^p$, which is the union of $\mathcal{M}_{qg}$ with itself p times. As we have seen in Theorem 1.7.6 this set, since it minimizes $pr_{qg}(Y) + |Q^{qg} - Y|$ must have the form $P^{qg}(K)$, for some set $K \subseteq P$. The set K is the minimal minimizing set for $\lambda r(X) + g(Q-X)$ and therefore the desired output of $\int_\lambda(r, g, Q)$.

Algorithmically speaking, parallel elements can be handled by using just one of the elements and simply remembering how many elements are in parallel to it. So the underlying size of set does not go up in an essential way.

1.7.9 Example

Principal sequence of $(r(\cdot), |\cdot|)$ where $r(\cdot)$ is the rank function of a graph

Consider the graph $\mathcal{G}$ in Figure 1.2. We have, $E(\mathcal{G}) \equiv \{1, \cdots, 20\}$. We need to compute the principal sequence of $(r(\cdot), |\cdot|)$. We trace the steps of the algorithm of Subsection 1.6.6 as specialized in Subsection 1.7.8 for this purpose.

First we compute the density $\lambda \equiv E(\mathcal{G})/r(\mathcal{G})$. This is 20/11. So we use the subroutine $\int_\lambda(r, |\cdot|, E(\mathcal{G}))$ which is essentially $\int_{20}(r_{11}, |\cdot|, P^{11}(E(\mathcal{G})))$
$\equiv \int_{20}(r_{11}, |\cdot|, E(\mathcal{G}_{11}))$, where $\mathcal{G}_{11}$ is obtained by putting in place of each edge of $\mathcal{G}$, 11 parallel edges. The subroutine does its job by computing the set of noncoloop elements of the matroid $\mathcal{M}_{11}^{20}$ (where $\mathcal{M}_{11}$ is the polygon matroid of $\mathcal{G}_{11}$ and $\mathcal{M}_{11}^{20}$ is the union of $\mathcal{M}_{11}$ with itself 20 times).
This set is the parallel copy of the subset $\{1, \cdots, 13\}$. So we build the graphs

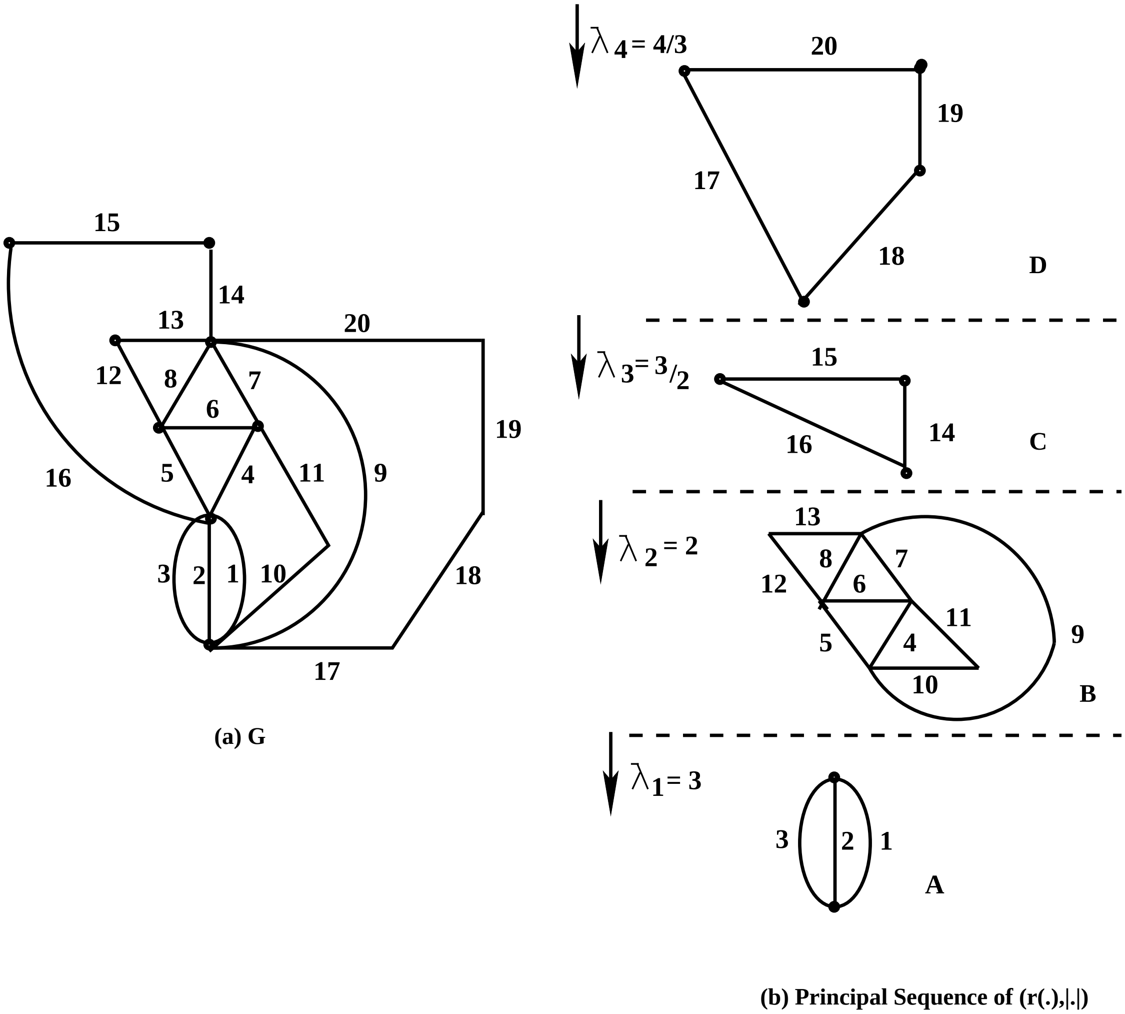

Figure 1.2: Example of Principal Sequence for a graph

$\mathcal{G}_\alpha \equiv \mathcal{G} \cdot \{1, \cdots, 13\}$ and $\mathcal{G}_\beta \equiv \mathcal{G} \times \{14, \cdots, 20\}$ and repeat the algorithm on $\mathcal{G}_\alpha, \mathcal{G}_\beta$.
For $\mathcal{G}_\alpha$ the density is $13/6$ and for $\mathcal{G}_\beta$, it is $7/5$. So we build the parallel $6-$ copy of $\mathcal{G}_\alpha$ and parallel $5-$ copy of $\mathcal{G}_\beta$ and find the set of non coloops of $(\mathcal{M}_\alpha)_6^{13}, (\mathcal{M}_\beta)_5^7$. This yields the appropriate parallel copies of sets $\{1, 2, 3\}, \{14, 15, 16\}$ respectively.
We are now left with the graphs
$\mathcal{G}_{\alpha 1} \equiv \mathcal{G} \cdot \{1, 2, 3\}, \mathcal{G}_{\alpha 2} \equiv \mathcal{G} \cdot \{1, \cdots, 13\} \times \{4, \cdots, 13\}$,
$\mathcal{G}_{\beta 1} \equiv \mathcal{G} \times \{14, \cdots, 20\} \cdot \{14, 15, 16\}, \mathcal{G}_{\beta 2} \equiv \mathcal{G} \times \{17, \cdots, 20\}$.
On these graphs when we apply our subroutine $\int_\lambda(r, |\cdot|, E(\mathcal{G}'))$, ($\mathcal{G}'$ being the

appropriate graph), we find the set of noncoloops is the null set which means that the null set minimizes $\lambda r(X)+|E(\mathcal{G}')-X|$. Further, we find that the full set also minimizes $\lambda r(X)+|E(\mathcal{G}')-X|$ since $\lambda r(\emptyset)+|E(\mathcal{G}')| = \lambda r(E(\mathcal{G}'))+|\emptyset|$. So at this stage we get the sets $A \equiv \{1,2,3\}, B \equiv \{4,\cdots,13\}, C \equiv \{14,15,16\}, D \equiv \{17,\cdots,20\}$ with the corresponding critical values $3,2,3/2,4/3$ (being the densities of the graphs $\mathcal{G}\cdot\{1,2,3\}, \mathcal{G}\cdot\{1,\cdots,13\}\times\{4,\cdots,13\}, \mathcal{G}\times\{14,\cdots,20\}\cdot\{14,15,16\}, \mathcal{G}\times\{17,\cdots,20\}$). Thus the principal sequence is $E_0 \equiv \emptyset = X_{\lambda_1}, E_1 = X_{\lambda_2} = A, E_2 = X_{\lambda_3} = A\cup B, E_3 = X_{\lambda_4} = A\cup B\cup C, E_4 = X^{\lambda_4} = A\cup B\cup C\cup D \equiv E(\mathcal{G})$. The critical values are $\lambda_1 = 3, \lambda_2 = 2, \lambda_3 = 3/2, \lambda_4 = 4/3$.

1.8 Notes

A good way of approaching matroid union is through submodular functions induced through a bipartite graph [Welsh76]. Related material may be found in the survey paper by Brualdi [Brualdi74]. A book that emphasizes the algorithmic uses of matroid intersection is [Lawler76]. An important class of applications of the matroid union, intersection and its generalizations, is in the structural solvability of systems [Murota87], [Recski89].

Bibliography

[Brualdi74] R.A. Brualdi: Matroids induced by directed graphs - a survey. In: *Recent Advances in Graph Theory* (Proceedings of the Symposium, Prague, Academia Praha, June 1974) 115-134.

[Crapo+Rota70] H.H. Crapo and G.C. Rota: *On the Foundations of Combinatorial Theory - Combinatorial Geometry* (MIT Press, Cambridge, MA, 1970).

[Cunningham84] W.H. Cunningham: Testing membership in matroid polyhedra. *Journal of Combinatorial Theory* **B36** (1984) 161-188.

[Edmonds65a] J. Edmonds: Minimum partition of a matroid into independent subsets. *Journal of Research of the National Bureau of Standards* **69B** (1965) 67-72.

[Edmonds68] J. Edmonds: Matroid partition. In: *Mathematics of the Decision Sciences, Part I* (Lectures in Applied Mathematics) **11** (1968) 335-345.

[Edmonds70] J. Edmonds: Submodular functions, matroids, and certain polyhedra. *Proceedings of the Calgary International Conference on Combinatorial Structures and Their Applications* (R.Guy, H.Hanani, N.Sauer and J.Schönheim, eds., Gordon and Breach, New York, 1970), 69-87.

[Edmonds+Fulkerson65] J. Edmonds and D.R. Fulkerson: Transversals and matroid partition. *Journal of Research of the National Bureau of Standards* **69B** (1965) 147-157.

[Fujishige80b] S. Fujishige: Principal structures of submodular systems. *Discrete Applied Mathematics* **2** (1980) 77-79.

[Fujishige91] S.Fujishige: *Submodular functions and optimization* (Annals of Discrete Maths **47**) (North Holland,Amsterdam,New York,Oxford,Tokyo,1991).

[Fujishige09] S. Fujishige: Theory of principal partitions revisited. In: Research Trends in Combinatorial Optimization (W. J. Cook, L. Lovász, and J. Vygen, eds.) (Springer, 2009), pp. 127–162.

[Hall35] P. Hall: On representatives of subsets. *Journal of the London Mathematical Society* **10** (1935) 26-30.

[Horn55] A. Horn: A characterization of unions of linearly independent sets. *Journal of the London Mathematical Society* **30** (1955) 494-496.

[Iri83] M. Iri: Applications of matroid theory. In: *Mathematical Programming – The State of the Art* (A. Bachem, M. Grötschel and B. Korte, eds., Springer, Berlin, 1983) 158-201.

[Iri+Tsunekawa+Murota82] M. Iri, J. Tsunekawa and K. Murota: Graph theoretical approach to large-scale systems – Structural solvability and block-triangularization. *Transactions of Information Processing Society of Japan* **23** (1982) 88-95.

[Kishi+Kajitani69] G. Kishi and Y. Kajitani: Maximally Distant Trees and Principal Partition of a Linear Graph. *IEEE Transactions on Circuit Theory* **CT-16** (1969) 323-329.

[König36] D.König: *Theorie der Endlichen und Unendlichen Graphen* (Leipsig, 1936, Reprinted New York, Chelsea, 1950).

[Kung86] J.P.S. Kung: *A Source Book in Matroid Theory* (Birkhäuser, Boston, 1986).

[Lawler76] E.L. Lawler: *Combinatorial Optimization – Networks and Matroids* (Holt, Rinehart and Winston, New York, 1976).

[Lovász83] L. Lovász: Submodular functions and convexity. In: *Mathematical Programmming – The State of the Art* (A.Bachem. M. Grötschel and B.Korte, eds., Springer, Berlin, 1983), 235-257.

[Mirsky71] L. Mirsky: *Transversal Theory* (Academic Press, London, 1971).

[Murota87] K. Murota: *Systems Analysis by Graphs and Matroids – Structural Solvability and Controllability* (Algorithms and Combinatorics **3**) (Springer, 1987).

[Murota+Iri85] K. Murota and M. Iri: Structural solvability of systems of equations – a mathematical formulation for distinguishing accurate and inaccurate numbers in structural analysis of systems. *Japan Journal of Applied Mathematics* **2** (1985) 247-271.

[Narayanan74] H. Narayanan: *Theory of Matroids and Network Analysis* Ph.D. Thesis, Department of Electrical Engineering, Indian Institute of Technology, Bombay, February 1974.

[Narayanan95a] H. Narayanan: A rounding technique for the polymatroid membership problem. *Linear Algebra and its Applications* **221** (1995) 41-57.

[Narayanan97] H. Narayanan: Submodular Functions and Electrical Networks. *Annals of Discrete Mathematics* **54** North Holland (London, New York, Amsterdam) (1997). (Revised version at http://www.ee.iitb.ac.in/ hn/book/).

[Nash-Williams61] C. St. J.A. Nash-Williams: Edge-disjoint spanning trees of finite graphs. *Journal of the London Mathematical Society* **36** (1961) 445-450.

[Nash-Williams67] C. St. J.A. Nash-Williams: An application of matroids to graph theory. In: *Theory of Graphs* Proceedings of the International Symposium, (Rome, 1966) (P. Rosenstiehl, ed., Gordon & Breach, New York, 1967) 263-265.

[Ozawa76] T. Ozawa: Topological conditions for the solvability of active linear networks. *International Journal of Circuit Theory and its Applications* **4** (1976) 125-136.

[Oxley92] *Matroid Theory* (Oxford University Press, New York, 1992).

[Rado42] R. Rado: A theorem on independence relations. *Quarterly Journal of Mathematics*, Oxford **13** (1942) 83-89.

[Recski89] A. Recski: *Matroid Theory and its Applications in Electric Network Theory and in Statics.* (Springer-Verlag,Berlin,Heidelberg,New York,London,Paris,Tokyo,1989).

[Seshu+Reed61] S. Seshu and M.B. Reed: *Linear Graphs and Electrical Networks* (Addison-Wesley, Reading, Mass., London, 1961).

[Sugihara+Iri80] K. Sugihara and M. Iri: A mathematical approach to the determination of the structure of concepts. *Matrix and Tensor Quarterly* **30** (1980) 62-75.

[Sugihara83] K. Sugihara: A unifying approach to descriptive geometry and mechanisms. *Discrete Applied Mathematics* **5** (1983) 313-328.

[Sugihara86] K. Sugihara: *Machine Interpretion of Line Drawings* (The MIT Press, Cambridge, Massachusetts, 1986).

[Tomizawa76] N. Tomizawa: Strongly irreducible matroids and principal partition of a matroid into irreducible minors (in Japanese). *Transactions of the Institute of Electronics and Communication Engineers of Japan* **59A** (1976) 83-91.

[Tutte61] W.T. Tutte: On the problem of decomposing a graph into $n-$ connected factors. *Journal of the London Mathematical Society* **36** (1961) 221-230.

[Tutte65] W.T. Tutte: Lectures on matroids. *Journal of Research of the National Bureau of Standards* **69B** (1965) 1-48.

[Tutte71] W.T. Tutte: *Introduction to the Theory of Matroids* (American Elsevier, New York, 1971).

[Van der Waerden37] B.L. van der Waerden: *Moderne Algebra* (2nd ed.) (Springer, Berlin, 1937).

[Welsh76] D.J.A. Welsh: *Matroid Theory* (Academic Press, Cambridge, 1976).

[White86] N.L. White: *Theory of Matroids* (Cambridge University Press, Cambridge, 1986).

[White87] N.White: *Combinatorial Geometries* (N. White, ed., Encyclopedia of Mathematics and Its Applications **29**, Cambridge University Press, 1987).

[Whitney35] H. Whitney: On the abstract properties of linear dependence. *American Journal of Mathematics* **57** (1935) 509-533.

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