Preface

This book has grown out of an attempt to understand the role that the topology of an electrical network plays in its **efficient** analysis. The approach taken is to transform the problem of solving a network with a given topology, to that of solving another with a different topology (and same devices), but with additional inputs and constraints. An instance of this approach is network analysis by multiport decomposition - breaking up a network into multiports, solving these in terms of port variables and finally imposing the port connection conditions and getting the complete solution. The motivation for our approach is that of building more efficient circuit simulators, whether they are to run singly or in parallel. Some of the ideas contained in the book have already been implemented - BITSIM, the general purpose circuit simulator built at the VLSI Design Centre, I.I.T. Bombay, is based on the 'topological hybrid analysis' contained in this book and can further be adapted to use topological decomposition ideas.

Many combinatorial optimization problems arise naturally when one adopts the above approach, particularly the hybrid rank problem and its generalizations. The theory required for the solution of these problems was developed by electrical engineers parallel to, and independent of, developments taking place in the theory of matroids and submodular functions. Consider, for instance, the work of Kishi and Kajitani, Iri, Ohtsuki et al in the late 60's on principal partition and its applications, independent of Edmonds' work on matroid partitions (1965). There is a strong case for electrical network topologists and submodular function theorists being aware of each others' fields. It is hoped that the present book would fill this need.

The topological network analysis that we have considered is to be distinguished from the kind of work exemplified by 'Kirchhoff's Third Law' which has been discussed in many books published in the 60's (eg. the book by Seshu and Reed [Seshu+Reed61]). In the 70's much interesting work in this area was done by Iri, Tomizawa, Recski and others using the 'generality assumption' for linear devices. Details may be found, for instance, in Recski's book [Recski89]. In the present book devices play a very secondary role. Mostly we manipulate only Kirchhoff's Laws.

Submodular functions are presented in this book adopting the 'elementary combinatorial' as opposed to the 'polyhedral' approach. Three things made us decide in favour of the former approach.

- It is hoped that the book would be read by designers of VLSI algorithms. In order to be convincing, the algorithms presented would have to be fast. So very general algorithms based on the polyhedral approach are ruled out.

- The polyhedral approach is not very natural to the material on Dilworth truncation.

- There is an excellent and comprehensive monograph, due to S.Fujishige, on the polyhedral approach to submodular functions; a book on polyhedral combinatorics including submodular functions from A.Schrijver is long awaited.

In order to make the book useful to a wider audience, the material on electrical networks and that on submodular functions are presented independently of each other. A final chapter on the hybrid rank problem displays the link. An area which can benefit by algorithms based on submodular functions is that of CAD for VLSI - particularly for building partitioners. Some space has therefore been devoted to partitioning in the chapter on Dilworth truncation.

The book is intended primarily for self study - hence the large number of problems with solutions. However, most of the material has been tested in the class room. The network theory part has been used for many years for an elective course on 'Advanced Network Analysis' - a third course on networks taken by senior undergraduates at the EE Dept, I.I.T. Bombay. The submodular function part has been used for special topics courses on combinatorics taken by doctoral students in Maths and Computer Science. This material can be covered in a semester if the students have a prior background in elementary graphs and matroids, leaving all the starred sections and relegating details and problems to self study.

It is a pleasure to acknowledge the author's indebtedness to his many colleagues, teachers and friends and to express his heartfelt gratitude.

He was introduced to electrical network theory by Professors R.E.Bedford and K.Shankar of the EE Dept., I.I.T. Bombay, and to graph theory by Professor M.N.Vartak of the Dept. of Maths, I.I.T. Bombay. Professor Masao Iri, formerly of the University of Tokyo, now of the University of Chuo, has kept him abreast of the developments in applied matroid theory during the last two decades and has also generously spared time to comment on the viability of lines of research.

He has benefited through interaction with the following: Professors S.D.Agashe, P.R.Bryant,A.N.Chandorkar,M.Chandramouli,C.A.Desoer,A.Diwan,S.Fujishige, P.L.Hammer,M.V.Hariharan,Y.Kajitani,M.V. Kamath,M.S.Kamath,E.L.Lawler, K.V.V. Murthy,T.Ozawa,S.Patkar,S.K.Pillai,P.G.Poonacha,G.N.Revankar,S.Roy, S.C.Sahasrabudhe,P.C.Sharma,M.Sohoni,V.Subbarao,N.J.Sudarshan,V.K.Tandon, N.Tomizawa, P.P.Varaiya, J.M.Vasi.

The friends mentioned below have critically read parts of the manuscript: S.Batterywala, A.Diwan, N.Jayanthi, S.Patkar, P.G.Poonacha and the '96 batch students of the course 'Advanced Network Analysis'. But for Shabbir Batterywala's assistance (technical, editorial, software consultancy), publication of this book would have been delayed by many months.

Mr Z.A.Shirgaonkar has done the typing in Latex and Mr R.S.Patwardhan has drawn the figures.

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Note to the Reader

This book appears too long because of two reasons:

- it is meant for self study - so contains a large number of exercises and problems with solutions.
- it is aimed at **three** different types of readers:
  - Electrical engineers interested in topological methods of network analysis.
  - Engineers interested in submodular function theory
  - Researchers interested in the link between electrical networks and submodular functions.

To shorten the book for oneself it is not necessary to take recourse to drastic physical measures. During first reading all starred sections, starred exercises and problems may be omitted. If the reader belongs to the first two categories mentioned above, she would already find that only about two hundred pages have to be read.

Sections, exercises and problems have been starred to indicate that they are not necessary for a first reading. Length of the solution is a fair indicator of the level of difficulty of a problem - star does not indicate level of difficulty. There are only a handful of routine (drill type) exercises. Most of the others require some effort. Usually the problems are harder than the exercises.

Many of the results, exercises, problems etc. in this book are well known but cannot easily be credited to any one author. Such results are marked with a '(k)'.

Electrical Engineers interested in topological methods

Such readers should first brush up on linear algebra (say first two chapters of the book by Hoffman and Kunze [Hoffman+Kunze72]), read a bit of graph theory (say the chapter on Kirchhoff's laws in the book by Chua et al [Chua+Desoer+Kuh87] and the first four chapters of the book by Narsingh Deo [Narsingh Deo74]) and then read chapters 2 to 8. The chapter on graphs contains material on contraction and restriction which is not easily available in textbooks on circuit theory, but which is essential for an understanding of subsequent chapters. So this chapter should be read carefully, particularly since it is written tersely. The chapter on matroids is optional. The chapter on electrical networks should be easy reading but scanning it is essential since it fixes some notation used subsequently and also because it contains material motivating subsequent chapters, e.g. multiport decomposition. The next three chapters contain whatever the book has to say on topological network analysis.

Engineers interested in submodular functions

Such readers should read Chapters 2 to 4 and Chapters 10 to 14 and the first four sections of Chapter 15. If the reader is not interested in matroids he may skip material (chapters, sections, exercises, examples) dealing with them without serious loss of continuity. This would mean he would have to be satisfied with bipartite graph based instances of the general theory. The key chapter for such a reader is Chapter 10. This is tersely written-so should be gone through carefully.

Researchers interested in the link between submodular functions and electrical networks

The key chapter for such a reader is Chapter 15. To read the first four sections of this chapter the reader has to be familiar with Chapters 5, 6, 7 from the electrical networks part and the unstarred sections of the chapters on submodular functions. If he has some prior familiarity with submodular functions and electrical networks it is possible to directly begin reading the chapter picking up the required results on submodular functions as and when they are referred to in the text. To read the last section of the chapter, familiarity with Chapter 8 is required.

Comments on Notation

Sometimes, instead of numbering equations, key statements etc., we have marked them with symbols such as $(*), (**), (\surd)$. These marks are used over and over again and have validity only within a local area such as a paragraph, a proof or the solution to a problem.

In some cases, where there is no room for confusion, the same symbol denotes different objects. For instance, usually B denotes a bipartite graph. But in Chapter 4, B denotes a base of a matroid- elsewhere a base is always denoted by b. The symbol E is used for the edge set of a graph, in particular a bipartite graph. But $E(X), X \subseteq V(\mathcal{G})$ denotes the set of edges with both endpoints within X, while $E_L(X), X \subseteq V_L$, in the case of a bipartite graph, denotes the set of all vertices adjacent only to vertices in X.

We have often used brackets to write two statements in one.
Example: We say that set X is **contained** in Y (**properly contained in** Y), if

every element of X is also a member of Y (every element of X is a member of Y and $X \neq Y$) and denote it by $X \subseteq Y (X \subset Y)$.
This is to be read as the following two statements.

i. We say that set X is **contained** in Y, if every element of X is also a member of Y and denote it by $X \subseteq Y$.

ii. We say that set X is **properly contained in** Y, if every element of X is a member of Y and $X \neq Y$ and denote it by $X \subset Y$.

List of Commonly Used Symbols

Sets, Partitions, Partial Orders

$\{e_1, e_2, \ldots, e_n\}$	*set whose elements are* $e_1, e_2, \ldots, e_n$
$\{x_i : i \in I\}$	*set whose members are* $x_i,\ i \in I$
$(x_i : i \in I)$	*a family (used only in Chapters* 2 *and* 11)
$x \in X$	*element* x *belongs to set* X
$x \notin X$	*element* x *does not belong to set* X
$\forall x$ *or* $\forall\ x$	*for all elements* x
$\exists x$	*there exists an element* x
$X \subseteq Y$	*set* X *is contained in set* Y
$X \subset Y$	*set* X *is properly contained in set* Y
$X \cup Y$	*union of sets* X *and* Y
$X \cap Y$	*intersection of sets* X *and* Y
$X \uplus Y$	*disjoint union of sets* X *and* Y
$\bigcup_{i=1}^{n} X_i$	*union of the sets* X_i
$\biguplus_{i=1}^{n} X_i$	*disjoint union of the sets* X_i
$X - Y$	*set of elements in* X *but not in* Y
$\bar{X}$	*complement of* X
$X \times Y$	*cartesian product of sets* X *and* Y
$X \oplus Y$	*direct sum of sets* X *and* Y
2^S	*collection of subsets of* S
$\mid X \mid$	*size of the subset* X
$(P, \preceq)$	*preorder on* P
$(P, \leq)$	*partial order on* P
Π	*partition* Π
Π_N	*partition that has* N *as a block and all blocks except* N *as singletons*
$\mathcal{P}_X$	*collection of all partitions of* X
$\Pi \leq \Pi'$	*partition* Π *is finer than* Π'
$\Pi \vee \Pi'$	*finest partition coarser than both* Π *and* Π'
$\Pi \wedge \Pi'$	*coarsest partition finer than both* Π *and* Π'

Functions,Set Functions and Operations on Functions

$f(\cdot)$	*function* $f(\cdot)$
$f/Z(\cdot), f(\cdot)$ *on* S	*restriction of* $f(\cdot)$ *to* $Z \subseteq S$
$gf(X), g \circ f(X)$	$g(f(X))$
$(f_1 \oplus f_2)(\cdot)$	*direct sum of functions* $f_1(\cdot)$ *and* $f_2(\cdot)$
$f_{fus \cdot \Pi}(\cdot), f(\cdot)$ *on* 2^S,	*fusion of* $f(\cdot)$ *relative to* Π *i.e.*, $f_{fus.\Pi}(X_f) \equiv f(\bigcup_{T \in X_f} T), X_f \subseteq \Pi$
$f/\mathbf{X}(\cdot), f(\cdot)$ *on* 2^S	*restriction of* $f(\cdot)$ *to* $2^X, X \subseteq S$ *(usually called) restriction of* $f(\cdot)$ *to* X
$f \diamond \mathbf{X}(\cdot), f(\cdot)$ *on* 2^S	*contraction of* $f(\cdot)$ *to* X $f \diamond \mathbf{X}(Y) \equiv f((S-X) \cup Y) - f(S-X)$
$f^d(\cdot), f(\cdot)$ *on* 2^S	*contramodular dual of* $f(\cdot)$ $f^d(X) \equiv f(S) - f(S-X)$
$f^*(\cdot), f(\cdot)$ *on* 2^S	*comodular dual of* $f(\cdot)$ *(with respect to weight function* $\alpha(\cdot)$*)* $f^*(X) \equiv \alpha(X) - (f(S) - f(S-X))$
$P_f, f(\cdot)$ *on* 2^S	*polyhedron associated with* $f(\cdot)$ $\mathbf{x} \in P_f$ *iff* $x(X) \leq f(X) \quad \forall X \subseteq S$
$P_f^d, f(\cdot)$ *on* 2^S	*dual polyhedron associated with* $f(\cdot)$ $\mathbf{x} \in P_f^d$ *iff* $x(X) \geq f(X) \quad \forall X \subseteq S$

Vectors and Matrices

$\mathcal{F}, \Re, \mathcal{C}, \Re_+$	*field* $\mathcal{F}$, *real field, complex field, set of nonnegative reals*
$\sum x_i$	*summation of elements* x_i
$\mathbf{f}$	*vector* $\mathbf{f}$
$\mathcal{V}$	*vector space* $\mathcal{V}$
$\mathcal{V}^\perp$	*vector space complementary orthogonal to* $\mathcal{V}$
$\mathbf{x}_1 \oplus \mathbf{x}_2$	*direct sum of* $\mathbf{x}_1$ *and* $\mathbf{x}_2$ *(vector obtained by adjoining components of vectors* $\mathbf{x}_1$ *and* $\mathbf{x}_2$*)*
$\mathcal{V}_S \oplus \mathcal{V}_T, S \cap T = \emptyset$	*direct sum of* $\mathcal{V}_S$ *and* $\mathcal{V}_T$ *(obtained by collecting all possible direct sums of vectors in* $\mathcal{V}_S$ *and* $\mathcal{V}_T$*)*

$dim(\mathcal{V}), r(\mathcal{V})$	*dimension of vector space* $\mathcal{V}$
$d(\mathcal{V}, \mathcal{V}')$	$r(\mathcal{V} + \mathcal{V}') - r(\mathcal{V} \cap \mathcal{V}')$
$\mathbf{A}(i,j)$	i, j^{th} *entry of matrix* $\mathbf{A}$
$\mathbf{A}^T$	*transpose of matrix* $\mathbf{A}$
$\mathbf{A}^{-1}$	*inverse of matrix* $\mathbf{A}$
$< \mathbf{f}, \mathbf{g} >$	*dot product of vectors* $\mathbf{f}, \mathbf{g}$
$\mathcal{R}(\mathbf{A})$	*row space of* $\mathbf{A}$
$\mathcal{C}(\mathbf{A})$	*column space of* $\mathbf{A}$
$det(\mathbf{A})$	*determinant of* $\mathbf{A}$

Graphs and Vector Spaces

$\mathcal{G}$	*graph* $\mathcal{G}$
$V(\mathcal{G})$	*vertex set of* $\mathcal{G}$
$E(\mathcal{G})$	*edge set of* $\mathcal{G}$
t	*a tree*
f	*a forest*
$\bar{t}$	*cotree* $(E(\mathcal{G}) - t)$ *of* $\mathcal{G}$
$\bar{f}$	*coforest* $(E(\mathcal{G}) - f)$ *of* $\mathcal{G}$
$L(e,f)$	$f -$ *circuit of* e *with respect to* f
$B(e,f)$	$f -$ *cutset of* e *with respect to* f
$r(\mathcal{G})$	*rank of* $\mathcal{G}$ (= *number of edges in a forest of* $\mathcal{G}$)
$\nu(\mathcal{G})$	*nullity of* $\mathcal{G}$ (= *number of edges in a coforest of* $\mathcal{G}$)
$\mathcal{G}openT$	*graph obtained from* $\mathcal{G}$ *by opening and removing edges* T
$\mathcal{G}shortT$	*graph obtained from* $\mathcal{G}$ *by shorting and removing edges* T
$\mathcal{G} \cdot T$	*graph obtained from* $\mathcal{G}open(E(\mathcal{G}) - T)$ *by removing isolated vertices, restriction of* $\mathcal{G}$ *to* T
$\mathcal{G} \times T$	*graph obtained from* $\mathcal{G}short(E(\mathcal{G}) - T)$ *by removing isolated vertices, contraction of* $\mathcal{G}$ *to* T

$\mathcal{G}_1 \cong \mathcal{G}_2$	$\mathcal{G}_1$ *is* $2-$*isomorphic to* $\mathcal{G}_2$
$r(T)$	$r(\mathcal{G} \cdot T)$
$\nu(T)$	$\nu(\mathcal{G} \times T)$
$\mathcal{H}$	*hypergraph* $\mathcal{H}$
$B(V_L, V_R, E)$	*bipartite graph with left vertex set* V_L, *right vertex set* V_R *and edge set* E
$\mathbf{A}$	*(usually) incidence matrix*
$\mathbf{A_r}$	*reduced incidence matrix*
$\mathbf{Q}_f$	*fundamental cutset matrix of forest* f
$\mathbf{B}_f$	*fundamental circuit matrix of forest* f
KCE	*Kirchhoff's current equations*
KCL	*Kirchhoff's current Law*
KVE	*Kirchhoff's voltage equations*
KVL	*Kirchhoff's voltage Law*
$\mathcal{V}_i(\mathcal{G})$	*solution space of* KCE *of* $\mathcal{G}$
$\mathcal{V}_v(\mathcal{G})$	*solution space of* KVE *of* $\mathcal{G}$
$\mathcal{V} \cdot T$	*restriction of vector space* $\mathcal{V}$ *to* T
$\mathcal{V} \times T$	*contraction of vector space* $\mathcal{V}$ *to* T
$\xi(T)$ *for* $\mathcal{V}$	$r(\mathcal{V} \cdot T) - r(\mathcal{V} \times T)$

Flow Graphs

$F(\mathcal{G}) \equiv (\mathcal{G}, \mathbf{c}, s, t)$	*flow graph on graph* $\mathcal{G}$ *with capacity function* $\mathbf{c}$, *source* s, *sink* t
(A, B)	$cut(A, B)$
$c(A, B)$	*capacity of* $cut(A, B)$
$f(A, B)$	*flow across* $cut(A, B)$, *from* A *to* B
$\lvert\mathbf{f}\rvert$	*value of flow* $\mathbf{f}$
$F(B, \mathbf{c}_L, \mathbf{c}_R)$	*flowgraph associated with bipartite graph* B *with source to left vertex capacity* $\mathbf{c}_L$, *right vertex to sink capacity* $\mathbf{c}_R$ *and (left to right) bipartite graph edge capacity* ∞

Matroids

$\mathcal{M} \equiv (S, \mathcal{I})$	*matroid* $\mathcal{M}$
$\mathcal{I}$	*collection of independent sets*
$\mathcal{M}^*$	*dual of the matroid* $\mathcal{M}$
B	(*only in Chapter* 4) *base of a matroid*
$L(e, B)$	$f-$ *circuit of* e *with respect to base* B
$B(e, B)$	$f-$ *bond of* e *with respect to base* B
$r(T)$	*rank of the subset* T *in the given matroid*
$r(\mathcal{M})$	*rank of the underlying set of the matroid*
$\nu(T)$	*rank of the subset* T *in the dual of the given matroid*
$\nu(\mathcal{M})$	*rank of the underlying set in the dual matroid*
$\mathcal{M}(\mathcal{G})$	*polygon matroid of the graph* $\mathcal{G}$ (*bases are forests*)
$\mathcal{M}^*(\mathcal{G})$	*bond matroid of the graph* $\mathcal{G}$ (*bases are coforests*)
$\mathcal{M}(\mathcal{V})$	*matroid whose bases are maximal independent columns of a representative matrix of* $\mathcal{V}$
$\mathcal{M}^*(\mathcal{V})$	*dual of* $\mathcal{M}(\mathcal{V})$
$\int(X)$	*span* (*closure*) *of the subset* X *in the matroid*
$\mathcal{M} \cdot T$	*restriction of* $\mathcal{M}$ *to* T
$\mathcal{M} \times T$	*contraction of* $\mathcal{M}$ *to* T
$\mathcal{M}_1 \vee \mathcal{M}_2$	*union of matroids* $\mathcal{M}_1$ *and* $\mathcal{M}_2$

Electrical Networks

$\mathbf{v}$	*voltage vector*
$\mathbf{i}$	*current vector*
$\mathcal{N}$	*electrical network*
$\mathcal{N}_{AP}$	*electrical multiport with port set* P *and remaining edge set* A
$\mathcal{E}$	*set of voltage sources in the network*
$\mathcal{J}$	*set of current sources in the network*
R	*resistance, also collection of resistors or 'current controlled voltage' elements in the network*

G	*conductance, also collection of 'voltage controlled current' elements in the network*
L	*inductance, also collection of inductors in the network*
$\mathcal{L}$	*mutual inductance matrix*
C	*capacitance, also collection of capacitors in the network*
vcvs	*voltage controlled voltage source*
vccs	*voltage controlled current source*
ccvs	*current controlled voltage source*
cccs	*current controlled current source*
$\mathcal{D}$	*device characteristic*
$\mathcal{D}_{AB}$	$(\mathbf{v}/A, \mathbf{i}/B),\ where\ \mathbf{v}, \mathbf{i} \in \mathcal{D}$
$\mathcal{D}_A$	$\mathcal{D}_{AA}$
$\mathcal{D}_{AB} \times \mathcal{D}_{PQ}$	$\{(\mathbf{v}, \mathbf{i}),\ \mathbf{v} = \mathbf{v}_A \oplus \mathbf{v}_P, \mathbf{i} = \mathbf{i}_B \oplus \mathbf{i}_Q$ $where\ (\mathbf{v}_A, \mathbf{i}_B) \in \mathcal{D}_{AB}, (\mathbf{v}_P, \mathbf{i}_Q) \in \mathcal{D}_{PQ}\}$
δ_{AB}	$\{(\mathbf{v}_A, \mathbf{i}_B),\ \mathbf{v}_A\ is\ any\ vector\ on\ A,\ \mathbf{i}_B$ $is\ any\ vector\ on\ B\}$

Implicit Duality

$\mathcal{K}_{SP} \leftrightarrow \mathcal{K}_P$	$\{f_S : f_S = f_{SP}/S,\ f_{SP} \in \mathcal{K}_{SP}\ \ s.t.\ f_{SP}/P \in \mathcal{K}_P\}$
$\mathcal{K}_{S_1} \leftrightarrow \mathcal{K}_{S_2}$	$\{\mathbf{f} : \mathbf{f} = \mathbf{f}_1/(S_1 - S_2) \oplus \mathbf{f}_2/(S_2 - S_1),\ \mathbf{f}_1 \in \mathcal{K}_{S_1},$ $\mathbf{f}_2 \in \mathcal{K}_{S_2}\ and\ \mathbf{f}_1/S_1 \cap S_2 = \mathbf{f}_2/S_1 \cap S_2\}$
$\mathcal{K}_{S_1} \rightleftharpoons \mathcal{K}_{S_2}$	$\{\mathbf{f} : \mathbf{f} = \mathbf{f}_1/(S_1 - S_2) \oplus \mathbf{f}_2/(S_2 - S_1),\ \mathbf{f}_1 \in \mathcal{K}_{S_1},$ $\mathbf{f}_2 \in \mathcal{K}_{S_2}\ and\ \mathbf{f}_1/S_1 \cap S_2 = -\mathbf{f}_2/S_1 \cap S_2\}$
$< \cdot, \cdot >$	$a\ q - bilinear\ operation,\ usually\ dot\ product$
$\mathcal{K}^*$	$collection\ of\ vectors\ q - orthogonal\ to\ those\ in\ \mathcal{K}$
$d(\mathcal{V}, \mathcal{V}')$	$r(\mathcal{V} + \mathcal{V}') - r(\mathcal{V} \cap \mathcal{V}')$
$\mathcal{K}^p$	*the collection of vectors polar to those in* $\mathcal{K}$
$\mathcal{K}^d$	*(only in Chapter 7) the collection of vectors integrally dual to those in* $\mathcal{K}$

Multiport Decomposition

$(\mathcal{V}_{E_1P_1}, \cdots, \mathcal{V}_{E_kP_k}; \mathcal{V}_P)$	$k - multiport\ decomposition\ of \mathcal{V}_E$
	$(i.e., (\bigoplus_i \mathcal{V}_{E_iP_i}) \leftrightarrow \mathcal{V}_P = \mathcal{V}_E)$
$((\mathcal{V}_{E_jP_j})_k; \mathcal{V}_P)$	$(\mathcal{V}_{E_1P_1}, \cdots, \mathcal{V}_{E_kP_k}; \mathcal{V}_P)$
$((\mathcal{V}_{E_jP_j})_{j\in I}; \mathcal{V}_{P_I})$	$(\cdots \mathcal{V}_{E_jP_j} \cdots; \mathcal{V}_{P_I})$
	where $j \in I \subseteq \{1, \cdots, k\}$ *and* $P_I = \cup_{j\in I} P_j$
$(\mathcal{V}_{EP}, P)$	*vector space on* $E \uplus P$ *with ports* P
$(\mathcal{V}_{E_1Q_1}, \cdots, \mathcal{V}_{E_kQ_k}; \mathcal{V}_{QP})$	*matched or skewed decomposition of* $(\mathcal{V}_{EP}, P)$

Functions Associated with Graphs and Bipartite Graphs

$V(X), X \subseteq E(\mathcal{G})$	*set of endpoints of edges* X *in graph* $\mathcal{G}$
$\Gamma(X), X \subseteq V(\mathcal{G})$	*set of vertices adjacent to vertices in vertex subset* X *in graph* $\mathcal{G}$
$\Gamma_L(X), X \subseteq V_L$	*in* $B \equiv (V_L, V_R, E)$, *set of vertices adjacent to vertices in* X
$\Gamma_R(X), X \subseteq V_R$	*in* $B \equiv (V_L, V_R, E)$, *set of vertices adjacent to vertices in* X
$E(X), X \subseteq V(\mathcal{G})$	*set of edges with both endpoints in vertex subset* X *in graph* $\mathcal{G}$
$E_L(X), X \subseteq V_L$	*in* $B \equiv (V_L, V_R, E)$ *set of vertices adjacent* **only** *to vertices in* X
$E_R(X), X \subseteq V_R$	*in* $B \equiv (V_L, V_R, E)$ *set of vertices adjacent* **only** *to vertices in* X
$I(X), X \subseteq V(\mathcal{G})$	*set of edges with atleast one endpoint in vertex subset* X *in graph* $\mathcal{G}$
$cut(X), X \subseteq V(\mathcal{G})$	*set of edges with exactly one endpoint in vertex subset* X *in graph* $\mathcal{G}$
$w(\cdot)$	*usually a weight function*
$w_L(\cdot), w_R(\cdot)$	*weight functions on the left vertex set and on the right vertex set respectively of a bipartite graph*

Convolution and PP

$f * g(X)$	*convolution of* $f(\cdot)$ *and* $g(\cdot)$ $(min_{Y \subseteq X}[f(Y) + g(X - Y)])$
$\mathcal{B}_{\lambda f,g}, f(\cdot), g(\cdot)$ *on* 2^S	*collection of sets which minimize* $\lambda f(X) + g(S - X)$ *over subsets of* S
$\mathcal{B}_\lambda$	$\mathcal{B}_{\lambda f,g}$
X^λ, X_λ	*maximal and minimal members of* $\mathcal{B}_\lambda$
$\Pi(\lambda)$	*the partition of* $X^\lambda - X_\lambda$ *induced by* $\mathcal{B}_\lambda$
Π_{pp}	*the partition of* S *obtained by taking the union of all the* $\Pi(\lambda)$
$(\Pi_{pp}, \geq_\pi)$	*partition partial order pair associated with* $(f(\cdot), g(\cdot))$
$\emptyset, E_1, \cdots, E_t$	*(usually) the principal sequence of* $(f(\cdot), g(\cdot))$
$\lambda_1, \cdots, \lambda_t$	*(usually) decreasing sequence of critical values*
$(\geq_R)$	*refined partial order associated with* $(f(\cdot), g(\cdot))$

Truncation and PLP

$\overline{f}(\Pi)$	$\sum_{N_i \in \Pi} f(N_i)$
$f_t(\cdot)$	$f_t(\emptyset) \equiv 0,$ $f_t(X) \equiv min_{\Pi \in \mathcal{P}_X}(\sum_{X_i \in \Pi} f(X_i))$
$\mathcal{L}_{\lambda f}, f(\cdot)$ *on* 2^S	*collection of all partitions of* S *that minimize* $\overline{f - \lambda}(\cdot)$
$\mathcal{L}_\lambda$	$\mathcal{L}_{\lambda f}$
Π^λ, Π_λ	*maximal and minimal member partitions in* $\mathcal{L}_\lambda$
$\lambda_1, \cdots, \lambda_t$	*(usually) decreasing sequence of critical PLP values of* $f(\cdot)$
$\Pi_{\lambda_1}, \Pi_{\lambda_2}, \cdots, \Pi_{\lambda_t}, \Pi^{\lambda_t}$	*principal sequence of partitions of* $f(\cdot)$
$\Pi'_{fus \cdot \Pi}, \Pi' \geq \Pi$	*partition of* Π *with* N_{fus} *as one of its blocks iff the members of* N_{fus} *are the set of blocks of* Π *contained in a single block of* Π'
$(\Pi_{fus})_{exp.\Pi}$	$(\Pi_{fus}, a\ partition\ of\ \Pi)$ *a partition with* N *as a block, iff* N *is the union of all blocks of* Π *which are members of a single block of* Π_{fus}

Contents

Chapter 1

Introduction

Topological Methods

The methods described in this book could be used to study the properties of electrical networks that are independent of the device characteristic. We use only topological constraints, namely, KCL and KVL. Our methods could, therefore, be called 'network topological'. However, in the literature, 'topological' is used more loosely for all those results which use topological ideas, e.g. Kirchhoff's Third Law, where the admittance of a resistive multiport is obtained in terms of products of admittances present in all the trees and certain special kinds of subtrees of the network. These latter results, though important, are not touched upon in this book. Here our aim has been to

- give a detailed description of 'topological methods in the strict sense' for electrical networks,
- present applications:
  - to circuit simulation and circuit partitioning
  - to establish relations between the optimization problems that arise naturally, while using these methods, to the central problems in the theory of submodular functions.

Applications

There are two kinds of applications possible for the approach taken in this book:

i. To build better (faster, numerically more rugged, parallelizable) circuit simulators. Typically, our methods will permit us to speak as follows.

> *'Solution of a network $\mathcal{N}$ containing arbitrary devices is equivalent to solution of topologically derived networks $\mathcal{N}_1, \cdots, \mathcal{N}_k$ under additional topological conditions.'*

An obvious application would be for the (coarse grained) parallelization of circuit simulation. We could have a number of machines $M_1, \cdots, M_k$ which could run general/special purpose circuit simulation of the derived networks $\mathcal{N}_1, \cdots, \mathcal{N}_k$. The central processor could combine their solutions using the additional topological conditions. Optimization problems would arise naturally, e.g. 'how to minimize the additional topological conditions?'
There are more immediate applications possible. The most popular general purpose simulator now running, SPICE, uses the modified nodal analysis approach. In this approach the devices are divided into two classes, generalized admittance type whose currents can be written in terms of voltages appearing somewhere in the circuit, and the remaining devices whose current variables will figure in the list of unknowns. The final variables in terms of which the solution is carried out would be the set of all nodal voltages and the above mentioned current variables. The resulting coefficient matrix is very sparse but suffers from the following defects:

- the matrix often has diagonal zeros;
- even for pure RLC circuits the coefficient matrix is not positive definite;
- if the subnetwork containing the admittance devices is disconnected, then the corresponding principal submatrix is singular.

These problems are not very severe if we resort to sparse LU methods [Hajj81]. However, it is generally accepted that for large enough networks ($\approx$ 5000 nodes) preconditioned conjugate gradient methods would prove superior to sparse LU techniques. The main advantage of the former is that if the matrix is close to a positive definite matrix, then we can bound the number of iterations. The above defects make MNA ill suited to conjugate gradient methods.

There is a simple way out - viz. to use hybrid analysis (partly loop and partly nodal), where we partition elements into admittance type and impedance type. The structure of the coefficient matrix that is obtained in this latter case is well suited to solution obtained by the conjugate gradient technique but could easily, for wrong choice of variables, be dense. A good way of making the matrices sparse is to use the result that we call the '$\mathcal{N}_{AL} - \mathcal{N}_{BK}$ theorem' (see Section 6.4). Here the network is decomposed into two derived networks whose solution under additional topological (boundary) conditions is always equivalent to the solution of the original network. We select $\mathcal{N}_{AL}$ so that it contains the admittance type elements and $\mathcal{N}_{BK}$ so that it contains the impedance type elements. We then write nodal equations for $\mathcal{N}_{AL}$ and generalized mesh type equations for $\mathcal{N}_{BK}$. The result is a sparse matrix with good structure for using conjugate gradient methods - for instance for RLC networks, after discretization, we would get a positive definite matrix and for most practical networks, a large submatrix would be positive definite. A general purpose simulator BITSIM has been built using these ideas [Roy+Gaitonde+Narayanan90].

The application to circuit partitioning arises as a biproduct when we try to solve a version of the hybrid rank problem using the operation of Dilworth truncation on submodular functions. Many problems in the area of CAD for VLSI need the underlying graph/hypergraph to be partitioned such that the 'interaction' between blocks is minimized. For instance we may have to partition the vertex set of a graph so that the number of lines going between blocks is a minimum. This kind of problem is invariably NP-Hard. But, using the idea of principal lattice of partitions (PLP), we can solve a relaxation of such problems exactly. This solution can then be converted to an approximate solution of the original problem [Narayanan91], [Roy+Narayanan91], [Patkar92], [Roy93],[Roy+Narayanan93a] [Narayanan+Roy+Patkar96].

ii. A second kind of application is to establish strong relationships between electrical networks and combinatorial optimization, in particular, submodular function theory. There are a number of optimization problems which arise when we view electrical networks from a topological point of view. These motivate, and are solved by, important concepts such as convolution and Dilworth truncation of submodular functions. The hybrid rank problem and its generalizations are important instances. Other algorithmic problems (though not entirely topological) include the solvability of electrical networks under 'generality' conditions (see for instance [Recski+Iri80]). It is no longer possible for electrical engineers to directly apply well established mathematical concepts. They themselves often have to work out the required ideas. The principal partition is a good instance of such an idea conceived by electrical engineers. A nice way of developing submodular function theory, it appears to the author, is to look for solutions to problems that electrical networks throw up.

We now present three examples which illustrate the concepts that we will be concerned with in network analysis.

The following informal rule should be kept in mind while reading the examples (see Theorem 6.3.1 and also the remark on page 181).

> *Let $\mathcal{N}$ be an electrical network (not necessarily linear) with the set of independent current sources $E_{\mathcal{J}}$ and the set of independent voltage sources $E_{\mathcal{E}}$. We assume that the independent source values do not affect the device characteristic of the remaining devices. Then, the structure of the constraints of the network, in any method of analysis, (as far as variables other than voltage source currents and current source voltages are concerned) is that corresponding to setting the independent sources to zero, i.e., short circuiting voltage sources and open circuiting current sources. In particular, for linear networks, the structure of the coefficient matrix multiplying the unknown vector is that corresponding to the network obtained by short circuiting the voltage sources and open circuiting the current sources.*

Example 1.0.1 The $\mathcal{N}_{AL} - \mathcal{N}_{BK}$ method:
Consider the electrical network whose graph is given in Figure 1.1. We assume

Figure 1.1: To illustrate the $\mathcal{N}_{AL} - \mathcal{N}_{BK}$ Method

that the devices associated with branches $\{1,2,3,4,5\}$ $(= A)$ *are independent of those associated with branches* $\{6,7,8,9,10,11\}$ $(= B)$. *Then we can show that computing the solution of the network in* $\mathcal{N}$ *in Figure 1.1 is always equivalent to the simultaneous computation of the solutions of the networks* $\mathcal{N}_{AL}, \mathcal{N}_{BK}$, *in the same figure, under the boundary conditions*

$$i_{11} \text{ in } \mathcal{N}_{AL} = i_{11} \text{ in } \mathcal{N}_{BK}.$$

$$v_5 \text{ in } \mathcal{N}_{AL} = v_5 \text{ in } \mathcal{N}_{BK}.$$

Here, in $\mathcal{N}_{AL}$, *the devices in* A *are identical to the corresponding devices in* $\mathcal{N}$. *Similarly in* $\mathcal{N}_{BK}$, *devices in* B *are identical to the corresponding devices in* $\mathcal{N}$. *The subset* $L \subseteq B$ *is a set of branches which, when deleted, breaks all circuits intersecting both* A *and* B. *The subset* $K \subseteq A$ *is a set of branches which, when contracted, destroys all circuits intersecting both* A *and* B. *The graph of* $\mathcal{N}_{AL}$ *is obtained from that of* $\mathcal{N}$ *by short circuiting the branches of* $B - L$. *We denote it by* $\mathcal{G} \times (A \cup L)$. *In this case* $L = \{11\}$. *The graph of* $\mathcal{N}_{BK}$ *is obtained from that of* $\mathcal{N}$ *by open circuiting branches of* $A - K$. *We denote it by* $\mathcal{G} \cdot (B \cup K)$. *In this case* $K = \{5\}$.

If the network is linear and A *and* B *are of conductance and impedance type respectively, then we can, if we choose, solve* $\mathcal{N}_{AL}$ *by nodal analysis and* $\mathcal{N}_{BK}$ *by loop analysis. So this method can be regarded as a topological generalization of 'Hybrid Analysis.'*

Figure 1.2: A Network to be decomposed into Multiports

If we so desire, we could try to choose $\mathcal{N}_{AL}$ *or* $\mathcal{N}_{BK}$ *such that they appear (when* i_L, v_k *are set to zero) in several electrically disconnected pieces. So the method can be regarded as a technique of 'Network Analysis by Decomposition'.*

Now we mention some related combinatorial optimization problems.

i. Given a partition of the edges into A *and* B *how to choose* L, K *minimally - this is easy.*

ii. Suppose the network permits arbitrary partitions into A *and* B *and we choose nodal variables for* $\mathcal{N}_{AL}$ *and loop variables for* $\mathcal{N}_{BK}$*. Which partition would give the coefficient matrix of least size?*

It can be shown that the size of the coefficient matrix is $r(\mathcal{G} \cdot A) + \nu(\mathcal{G} \times B)$*, where* $r(\mathcal{G} \cdot A), \nu(\mathcal{G} \times B)$ *respectively denote the rank of* $\mathcal{G} \cdot A$ *and nullity of graph* $\mathcal{G} \times B$*. Minimization of this expression, over all partitions* $\{A, B\}$ *of the edge set* $E(\mathcal{G})$ *of* $\mathcal{G}$*, is the hybrid rank problem which gave rise to the theory of principal partition.*

Example 1.0.2 Multiport Decomposition:
Let $\mathcal{N}$ *be an electrical network with the graph shown in Figure 1.2. We are given that* $A \equiv \{1, 2 \cdots, 10\}$ *and* $B \equiv \{11, \cdots, 24\}$ *(with devices in* A *and* B *decoupled). The problem is to split* $\mathcal{N}$ *into two multiports* $\mathcal{N}_{AP_1}, \mathcal{N}_{BP_2}$ *and a 'port connection diagram'* $\mathcal{N}_{P_1P_2}$ *and solve* $\mathcal{N}$ *by solving* $\mathcal{N}_{AP_1}, \mathcal{N}_{BP_2}, \mathcal{N}_{P_1P_2}$ *simultaneously. (In*

Figure 1.3: Decomposition into Multiports

general this would be a problem involving n multiports). It is desirable to choose P_1, P_2 minimally. It turns out that

$$\mid P_1 \mid = \mid P_2 \mid = r(\mathcal{G} \;.\; A) - r(\mathcal{G} \times A) = r(\mathcal{G} \;.\; B) - r(\mathcal{G} \times B).$$

(Here $\mathcal{G} \;.\; A$ is obtained by open circuiting edges in B, while $\mathcal{G} \times A$ is obtained by short circuiting edges in B). In this case this number is 1. The multiports are shown in Figure 1.3.

The general solution procedure using multiport decomposition is as follows: Find the voltage-current relationship imposed on P_1 by the rest of the network in $\mathcal{N}_{AP_1}$, and on P_2, by the rest of the network in $\mathcal{N}_{BP_2}$. This involves solution of $\mathcal{N}_{AP_1}, \mathcal{N}_{BP_2}$ in terms of some of the current/voltage port variables of N_{AP_1} and some of the current/voltage port variables of N_{BP_2}. The voltage-current relationships imposed on P_1, P_2 (as described above) are treated as their device characteristics in the network $\mathcal{N}_{P_1P_2}$. When this is solved, we get the currents and voltages of P_1, P_2. Networks $\mathcal{N}_{AP_1}, \mathcal{N}_{BP_2}$ have already been solved in terms of these variables. So this completes the solution of $\mathcal{N}$. Like the $\mathcal{N}_{AL} - \mathcal{N}_{BK}$ method (to which it is related), this is also a general method independent of the type of network. As before, the technique is more useful when the network is linear.

This method again may be used as a network decomposition technique (for parallelizing) at a different level. Suppose $\mathcal{N}_{AP_1}$ (or $\mathcal{N}_{BP_2}$) splits into several subnetworks when some of the branches P_{O1} (P_{O2}) of $P_1(P_2)$ are opened and others

$P_{S1}(P_{S2})$ shorted. Then, by using $i_{P_{O1}}(i_{P_{O2}}), v_{P_{S1}}(v_{P_{S2}})$, as variables in terms of which $\mathcal{N}_{AP_1}(\mathcal{N}_{BP_2})$ are solved, we can make the analysis look like the simultaneous solution of several subnetworks under boundary conditions. There is no restriction on the type of network - we only need the subnetworks to be decoupled in the device characteristic. The optimization problem that arises naturally in this case is the following:

> *Given a partition of the edges of a network $\mathcal{N}$ into $E_1, \cdots, E_k$, find a collection of multiports $\mathcal{N}_{E_1 P_1}, \cdots, \mathcal{N}_{E_k P_k}$, and a port connection diagram $\mathcal{N}_{P_1,\cdots,P_k}$, whose combined KCE and KVE are equivalent to those of $\mathcal{N}$, with the size of $\biguplus P_i$ a minimum under these conditions.*

This problem is solved in Chapter 8.

Remark: At an informal level multiport decomposition is an important technique in classical network theory e.g. Thevenin-Norton Theorem, extracting reactances in synthesis, extracting nonlinear elements in nonlinear circuit theory, etc. However, for the kind of topological theory to be discussed in the succeeding pages we need a formal definition of ports that will carry over to vector spaces from graphs. Otherwise the minimization problems cannot be stated clearly, let alone be solved. In the example described above, it is clear that if we match $\mathcal{N}_{AP_1}$ and $\mathcal{N}_{BP_2}$ along P_1, P_2 we **do not** get back $\mathcal{N}$. A purely graph theoretic definition of multiport decomposition would therefore not permit the decomposition given in this particular example. Such a definition would lead to optimization problems with additional constraints which have no relevance for network analysis. Further, even after optimization according to such a definition, we would end up with more ports than required.

Example 1.0.3 Fusion-Fission method:
Consider the network in Figure 1.4. Six subnetworks have been connected together to make up the network. Assume that the devices in the subnetworks are decoupled. Clearly the networks in Figure 1.4 and Figure 1.5 are equivalent, provided the current through the additional unknown voltage source and the voltage across the additional unknown current source are set equal to zero. But the network in Figure 1.5 is equivalent to that in Figure 1.6 under the additional conditions

$$i_{v1} + i_{v2} + i_{v3} + i = 0$$

$$v_{i1} + v_{i2} + v_{i3} - v = 0$$

As can be seen, the subnetworks of Figure 1.6 are decoupled except for the common variables v and i and the additional conditions.
A natural optimization problem here is the following:

> *Given a partition of the edges of a graph into $E_1, \cdots E_k$, what is the minimum size set of node pair fusions and node fissions by which all circuits passing through more than one E_i are destroyed?*

Figure 1.4: Network $\mathcal{N}$ to Illustrate the Fusion-Fission Method

Figure 1.5: A Network equivalent to $\mathcal{N}$ with Virtual Sources

Figure 1.6: Network $\mathcal{N}$ decomposed by the Fusion-Fission Method

In the present example the optimal set of operations is to fuse nodes a *and* b *and cut node* a *into* a_1, a_2 *as in Figure 1.5. Artificial voltage sources are introduced across the node pairs to be fused and artificial current sources are introduced between two halves of a split node.*

It can be shown that this problem generalizes the hybrid rank problem (see Section 15.4). Its solution involves the use of the Dilworth truncation operation on an appropriate submodular function.

We now speak briefly of the mathematical methods needed to derive the kind of results hinted at in the above examples.

The $\mathcal{N}_{AL} - \mathcal{N}_{BK}$ method needs systematic use of the operations of contraction and restriction both for graphs and vector spaces and the notion of duality of operations on vector spaces. These have been discussed in detail in Chapter 3. The $\mathcal{N}_{AL} - \mathcal{N}_{BK}$ method itself is discussed in Chapter 6.

The multiport decomposition method requires the use of the 'Implicit Duality Theorem'. This result, which should be regarded as a part of network theory folklore, has received too little attention in the literature. We have tried to make amends by devoting a full chapter to it. The optimization problem relevant to multiport decomposition ('port minimization') is discussed in Chapter 8.

The fusion-fission method is a special case of the method of topological transformations discussed in Chapter 7. The solution of the optimization problem that it gives rise to (minimization of the number of fusion and fission operations needed to electrically decouple the blocks of a partition of the edge set) is given in Section 15.4. The solution uses the Dilworth truncation operation on submodular functions.

We next give a chapterwise outline of the book.

Chapter 2 is concerned with mathematical preliminaries such as sets, families, vectors and matrices. Also given is a very brief description of inequality systems.

Chapter 3 contains a very terse description of graphs and their vector space representation. Only material that we need later on in this book is included. Emphasis is placed on complementary orthogonality (Tellegen's Theorem) and the important minor operations linked through duality. The duality described corresponds to complementary orthogonality of vector spaces (and not to the vector space - functional space relation).

Also included is a sketch of the basic algorithms relevant to this book - such as bfs, dfs trees, construction of f-circuits, the shortest path algorithm, algorithms for performing graph minor operations and the basic join and meet operations on partitions. Some space is devoted to the flow maximization problem, particularly certain special ones that are associated with a bipartite graph. (Many of the optimization problems considered in this book reduce ultimately to (perhaps repeated) flow maximization).

Chapter 4 gives a brief account of matroids. Important axiom systems such as the ones in terms of independence, circuit, rank, closure etc. are presented and shown to be equivalent to each other. The minor operations and dual matroids are described. Finally the relation between matroids and the greedy algorithm is presented. This chapter is included for two reasons:

- Some of the notions presented in the previous chapter lead very naturally to their extension to matroids
- matroids are perhaps the most important instance of submodular functions which latter is our main preoccupation in the second half of this book.

Chapter 5 contains a brief introduction to conventional electrical network theory, with the aim of making the book self contained. The intention here is also to indicate the author's point of view to a reader who is an electrical engineer. This chapter contains a rapid sketch of the basic methods of network analysis including a very short description of the procedure followed in general purpose circuit simulators. Also included is an informal account of multiport decomposition and of some elementary results including Thevenin-Norton Theorem.

Chapter 6 contains a description of topological hybrid analysis indicated in Example 1.0.1. This chapter is a formalization of the topological ideas behind

Kron's Diakopotics. The methods used involve vector space minors. The main result is Theorem 6.4.1 which has already been illustrated in the above mentioned example.

Chapter 7 contains a detailed description of the Implicit Duality Theorem, its applications and its extensions to linear inequality and linear integrality systems. The operation of generalized minor is introduced and made use of in this chapter. The implicit duality theorem was originally a theorem on ideal transformers and states that if we connect 2-port transformers arbitrarily and expose k-ports, the result would be a k-port ideal transformer. (An ideal transformer, by definition, has its possible port voltage vectors and possible port current vectors as complementary orthogonal spaces.) We show that its power extends beyond these original boundaries. One of the applications described is for the construction of adjoints, another to topological transformations of electrical networks. The latter are used to solve a given network as though it has the topology of a different network, paying a certain cost in terms of additional variables.

Multiport decomposition, from a topological point of view, is the subject of **Chapter 8**. We make free use of the Implicit Duality Theorem of the previous chapter. We indicate that multiport decomposition is perhaps the most natural tool for network analysis by decomposition. It can be shown that multiport decomposition generalizes topological hybrid analysis (see Problem 8.5). We present a few algorithms for minimizing the number of ports for a multiport decomposition corresponding to a given partition of edges of a graph. Finally, we show that this kind of decomposition can be used to construct reduced networks which mimic some of the properties of the original network. In particular we show that any RLMC network can be reduced to a network without zero eigen values (i.e., without trapped voltages or currents) but with, otherwise, the same 'dynamics' as the original network.

The second half of the book is about submodular functions and the link between them and electrical networks.

Chapter 10 contains a compact description of submodular function theory omitting the important operations of convolution and Dilworth truncation. (The latter are developed in subsequent chapters). We begin with the basic definition and some characteristic properties followed by a number of examples of submodular functions which arise in graphs, hypergraphs (represented by bipartite graphs), matrices etc. Basic operations such as contraction, restriction, fusion, dualization etc. are described next. These are immediately illustrated by examples from graphs and bipartite graphs. Some other operations, slightly peripheral, are described next. A section is devoted to the important cases of polymatroid and matroid rank functions. It is shown that any submodular function is a 'translate' through a modular function of a polymatroid rank function. The idea of connectedness is described next. This corresponds to 2-connectedness of graphs. After this there is a very brief but general description of polyhedra associated with set functions in general and with submodular and supermodular functions in particular. The important result due to Frank, usually called the 'Sandwich Theorem' is described in this section. The recent solution, due to Stoer, Wagner and Frank, of the symmetric submodular function minimization problem is described in the next section.

Chapter 11 is devoted to the operation of (lower) convolution of two submodular functions. We begin with purely formal properties and follow it with a number of examples of results from the literature which the operation of convolution unifies. Next we give the polyhedral interpretation for convolution viz. it corresponds to the intersection of the polyhedra of the interacting submodular functions. This is followed by a section in which the operation of convolution is used to show that every polymatroid rank function can be obtained by the fusion of an appropriate matroid rank function.

In the next section, the principal partition (PP) of a submodular function with respect to a strictly increasing polymatroid rank function is dealt with. We begin with the basic properties of PP which give structural insight into many practical problems. An alternative development of PP from the point of view of density of sets is next presented. Finally the PP of naturally derived submodular functions is related to the PP of the original function.

In the next section, the refined partial order associated with the PP is described. After this we present general algorithms for the construction of the PP of a submodular function with respect to a nonnegative weight function. These use submodular function minimization as a basic subroutine. We consider two important special cases of this algorithm. The first, the weighted left adjacency function of a bipartite graph, is described in this chapter. In this case the submodular function minimization reduces to a flow problem. The second is the PP of a matroid rank function which is taken up in the next chapter. The last (starred) section in this chapter describes a peculiar situation where, performing certain operations on the original submodular function, we get functions whose PP is related in a very simple way to the original PP. This section is developed through problems.

Chapter 12 is on the matroid union operation. In the first section, we give a sketch of submodular functions induced through a bipartite graph and end the section with a proof that 'union' of matroids is a matroid. Next we give Edmond's algorithm for constructing the matroid union. We use this algorithm to study the structure of the union matroid - in particular the natural partition of its underlying set into coloops and the complement, and the manner in which the base of the union is built in terms of the bases of the individual matroids. Finally we use the matroid union algorithm to construct the PP of the rank function of a matroid with respect to the '$|\cdot|$' function.

In **Chapter 13** we study the Dilworth truncation operation on a submodular function. This chapter is written in a manner that emphasizes the structural analogies that exist between convolution and Dilworth truncation. As in the case of convolution, we begin with formal properties and follow it with examples of results from the literature unified by the truncation operation.

In the next section, we describe the principal lattice of partitions (PLP) of a submodular function. This notion is analogous to the PP of a submodular function - whereas in the case of the PP there is a nesting of special sets, in the case of the PLP the special partitions get increasingly finer. We begin with basic properties of the PLP, each of which can be regarded as a 'translation' of a corresponding property of PP. We then present an alternative development of the PLP in terms of cost of partitioning. In the next section we use this idea for building approximation algorithms for optimum cost partitioning (this problem is of great practical relevance, particularly in CAD for large scale integrated circuits). After this we describe the relation between the PLP of a submodular function and that of derived functions. Here again there is a strong analogy between the behaviours of PP and PLP.

In **Chapter 14**, we present algorithms for building the PLP of a general submodular function. These algorithms are also analogous to those of the PP. The core subroutine is one that builds a '(strong) fusion' set which uses minimization of an appropriately derived submodular function. We specialize these algorithms to the important special cases of the weighted adjacency and exclusivity functions associated with a bipartite graph. (The matroid rank function case is handled in Section 15.3). Next we present some useful techniques for improving the complexity of PLP algorithms for functions arising in practice. Lastly, using the fact that the PP of the rank function of a graph can be regarded, equivalently, as the PLP of the $\mid V(\cdot) \mid$ function on the edge set, we have presented fast algorithms for the former.

The **last chapter** is on the hybrid rank problem for electrical networks. In this chapter, four different (nonequivalent) formulations of this problem are given. The second, third and fourth formulations can be regarded as generalizations of the first. Except in the case of the fourth formulation, we have given fast algorithms for the solution of the problems. This chapter is intended as the link between electrical networks and submodular functions. Each of the formulations has been shown to arise naturally in electrical network theory. The first two formulations require convolution and the third requires Dilworth truncation for its solution. The fourth formulation gives rise to an optimization problem over vector spaces which is left as an open problem.

Chapter 2

Mathematical Preliminaries

2.1 Sets

A **set** (or **collection**) is specified by the **elements** (or **members**) that **belong** to it. If element x belongs to the set (does not belong to the set) X, we write $x \in X$ ($x \notin X$). Two sets are equal iff they have the same members. The set with no elements is called the empty set and is denoted by $\emptyset$. A set is **finite** if it has a finite number of elements. Otherwise it is **infinite**. A set is often specified by actually listing its members, e.g. $\{e_1, e_2, e_3\}$ is the set with members e_1, e_2, e_3. More usually it is specified by a property, e.g. the set of even numbers is specified as $\{x : x \text{ is an integer and } x \text{ is even }\}$ or as $\{x, x \text{ is an integer and } x \text{ is even }\}$. The symbols $\forall$ and $\exists$ are used to denote 'forall' and 'there exists'. Thus, '$\forall x$' or '$\forall\ x$' should be read as 'forall x' and '$\exists x$' should be read as 'there exists x'. A singleton set is one that has only one element. The singleton set with the element x as its only member, is denoted by $\{x\}$. In this book, very often, we abuse this notation and write x in place of $\{x\}$, if we feel that the context makes the intended object unambiguous.

We say that set X is **contained** in Y (**properly contained in** Y), if every element of X is also a member of Y (every element of X is a member of Y and $X \neq Y$) and denote it by $X \subseteq Y (X \subset Y)$.

The **union** of two sets X and Y denoted by $X \cup Y$, is the set whose members are either in X or in Y (or in both). The **intersection** of X and Y denoted by $X \cap Y$, is the set whose members belong both to X and to Y. When X and Y do not have common elements, they are said to be **disjoint**. Union of disjoint sets X and Y is often denoted by $X \uplus Y$. Union of sets $X_1, \cdots, X_n$ is denoted by $\bigcup_{i=1}^n X_i$ or simply by $\bigcup X_i$. When the X_i are pairwise disjoint, their union is denoted by $\biguplus_{i=1}^n X_i$ or $\biguplus X_i$.

The **difference** of X relative to Y, denoted by $X - Y$, is the set of all elements in X but not in Y. Let $X \subseteq S$. Then the **complement** of X **relative to** S is the set $S - X$ and is denoted by $\bar{X}$ when the set S is clear from the context.

A **mapping** $f : X \rightarrow Y$, denoted by $f(\cdot)$, associates with each element $x \in X$, the element $f(x)$ in Y. The element $f(x)$ is called the image of x under $f(\cdot)$. We say $f(\cdot)$ maps X into Y. The sets X, Y are called, respectively, the **domain** and **codomain** of $f(\cdot)$. We denote by $f(Z), Z \subseteq X$, the subset of Y which has as members, the images of elements in Z. The set $f(X)$ is called the **range** of $f(\cdot)$. The **restriction** of $f(\cdot)$ to $Z \subseteq X$, denoted by $f/Z(\cdot)$ is the mapping from Z to Y defined by $f/Z(x) \equiv f(x), x \in Z$. A mapping that has distinct images for distinct elements in the domain is said to be **one to one** or **injective**. If the range of $f(\cdot)$ equals its codomain, we say that $f(\cdot)$ is **onto** or **surjective**. If the mapping is one to one onto we say it is **bijective**. Let $f : X \rightarrow Y, g : Y \rightarrow Z$. Then the **composition** of g and f is the map, denoted by $gf(\cdot)$ or $g \circ f(\cdot)$, defined by $gf(x) \equiv g(f(x)) \;\; \forall x \in X$. The **Cartesian product** $X \times Y$ of sets X, Y is the collection of all ordered pairs (x, y), where $x \in X$ and $y \in Y$. The **direct sum** $X \oplus Y$ denotes the union of disjoint sets X, Y. We use '**direct sum**' loosely to indicate that structures on two disjoint sets are 'put together'. We give some examples where we anticipate definitions which would be given later. The **direct sum** of vector spaces $\mathcal{V}_1, \mathcal{V}_2$ on disjoint sets S_1, S_2 is the vector space $\mathcal{V}_1 \oplus \mathcal{V}_2$ on $S_1 \oplus S_2$ whose typical vectors are obtained by taking a vector $\mathbf{x}_1 \equiv (a_1, \cdots, a_k)$ in $\mathcal{V}_1$ and a vector $\mathbf{x}_2 \equiv (b_1, \cdots, b_m)$ in $\mathcal{V}_2$ and putting them together as $\mathbf{x}_1 \oplus \mathbf{x}_2 \equiv (a_1, \cdots, a_k, b_1, \cdots, b_m)$. When we have two graphs $\mathcal{G}_1, \mathcal{G}_2$ on disjoint edge sets E_1, E_2, $\mathcal{G}_1 \oplus \mathcal{G}_2$ would have edge set $E_1 \oplus E_2$ and is obtained by 'putting together' $\mathcal{G}_1$ and $\mathcal{G}_2$. Usually the vertex sets would also be disjoint. However, where the context permits, we may relax the latter assumption and allow 'hinging' of vertices.

We speak of a **family** of subsets as distinct from a **collection** of subsets. The collection $\{\{e_1, e_2\}, \{e_1, e_2\}, \{e_1\}\}$ is identical to $\{\{e_1, e_2\}, \{e_1\}\}$. But often (e.g. in the definition of a hypergraph in Subsection 3.6.6) we have to use copies of the same subset many times and distinguish between copies. This we do by 'indexing' them. A family of subsets of S may be defined to be a mapping from an index set I to the collection of all subsets of S. For the purpose of this book, the index set I can be taken to be $\{1, \cdots, n\}$. So the family $(\{e_1, e_2\}, \{e_1, e_2\}, \{e_1\})$ can be thought of as the mapping $\phi(\cdot)$ with

$$\begin{aligned} \phi(1) &\equiv \{e_1, e_2\} \\ \phi(2) &\equiv \{e_1, e_2\} \\ \phi(3) &\equiv \{e_1\}. \end{aligned}$$

(Note that a family is denoted using ordinary brackets while a set is denoted using curly brackets).

2.2 Vectors and Matrices

In this section we define vectors, matrices and related notions. Most present day books on linear algebra treat vectors as primitive elements in a vector space and leave them undefined. We adopt a more old fashioned approach which is convenient for the applications we have in mind. The reader who wants a more leisurely treatment of the topics in this section is referred to [Hoffman+Kunze72].

Let S be a finite set $\{e_1, e_2, \ldots, e_n\}$ and let $\mathcal{F}$ be a field. We will confine ourselves to the field $\Re$ of real numbers, the field $\mathcal{C}$ of complex numbers and the $GF2$ field on elements $0, 1$ $(0+0=0, 0+1=1, 1+0=1, 1+1=0, 1.1=1, 1.0=0, 0.1=0, 0.0=0)$. For a general definition of a field see for instance [Jacobson74]. By a vector on S over $\mathcal{F}$ we mean a mapping $\mathbf{f}$ of S into $\mathcal{F}$.The field $\mathcal{F}$ is called the **scalar field** and its elements are called **scalars**. The **support** of $\mathbf{f}$ is the subset of S over which it takes nonzero values. The **sum** of two vectors $\mathbf{f}, \mathbf{g}$ on S over $\mathcal{F}$ is defined by $(\mathbf{f}+\mathbf{g})(e_i) \equiv \mathbf{f}(e_i) + \mathbf{g}(e_i)\ \forall e_i \in S$. (For convenience the **sum** of two vectors $\mathbf{f}$ on S,$\mathbf{g}$ on T over $\mathcal{F}$ is defined by $(\mathbf{f}+\mathbf{g})(e_i) \equiv \mathbf{f}(e_i)+\mathbf{g}(e_i)\ \forall e_i \in S \cap T$, as agreeing with $\mathbf{f}$ on $S - T$, and as agreeing with $\mathbf{g}$ on $T - S$). The **scalar product** of $\mathbf{f}$ by a scalar λ is a vector $\lambda\mathbf{f}$ defined by $(\lambda\mathbf{f})(e_i) \equiv \lambda(\mathbf{f}(e_i))\ \forall e_i \in S$. A collection $\mathcal{V}$ of vectors on S over $\mathcal{F}$ is a vector space iff it is closed under addition and scalar product. We henceforth omit mention of underlying set and field unless required.

A set of vectors $\{\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_n\}$ is **linearly dependent** iff there exist scalars $\lambda_1, \ldots, \lambda_n$ not all zero such that $\lambda_1\mathbf{f}_1 + \ldots + \lambda_n\mathbf{f}_n = \mathbf{0}$. (Here the $\mathbf{0}$ vector is one which takes value 0 on all elements of S). Vector $\mathbf{f}_n$ is a linear combination of $\mathbf{f}_1, \ldots, \mathbf{f}_{n-1}$ iff $\mathbf{f}_n = \lambda_1\mathbf{f}_1 + \ldots + \lambda_{n-1}\mathbf{f}_{n-1}$ for some $\lambda_1, \ldots, \lambda_{n-1}$.

The set of all vectors linearly dependent on a collection $\mathcal{C}$ of vectors can be shown to form a vector space which is said to be **generated** by or **spanned** by $\mathcal{C}$. Clearly if $\mathcal{V}$ is a vector space and $\mathcal{C} \subseteq \mathcal{V}$, the subset of vectors generated by $\mathcal{C}$ is contained in $\mathcal{V}$. A maximal linearly independent set of vectors of $\mathcal{V}$ is called a **basis** of $\mathcal{V}$.
In general maximal and minimal members of a collection of sets may not be largest and smallest in terms of size.
Example: Consider the collection of sets $\{\{1,2,3\},\{4\},\{5,6\},\{1,2,3,5,6\}\}$. The minimal members of this collection are $\{1,2,3\},\{4\},\{5,6\}$, i.e., these do not contain proper subsets which are members of this collection. The maximal members of this collection are $\{4\},\{1,2,3,5,6\}$, i.e., these are not proper subsets of other sets which are members of this collection.

The following theorem is therefore remarkable.

Theorem 2.2.1 *All bases of a vector space on a finite set have the same cardinality.*

The number of elements in a basis of $\mathcal{V}$ is called the **dimension** of $\mathcal{V}$, denoted by $dim(\mathcal{V})$, or the **rank** of $\mathcal{V}$, denoted by $r(\mathcal{V})$. Using Theorem 2.2.1 one can show that the size of a maximal independent subset contained in a given set $\mathcal{C}$ of vectors is unique. This number is called the **rank** of $\mathcal{C}$. Equivalently, the rank of $\mathcal{C}$ is the dimension of the vector space spanned by $\mathcal{C}$. If $\mathcal{V}_1, \mathcal{V}_2$ are vector spaces and $\mathcal{V}_1 \subseteq \mathcal{V}_2$, we say $\mathcal{V}_1$ is a **subspace** of $\mathcal{V}_2$.

A mapping $\mathbf{A} : \{1, 2, \ldots, m\} \times \{1, 2, \ldots, n\} \longrightarrow \mathcal{F}$ is called a $m \times n$ matrix. It may be thought of as an $m \times n$ array with entries from $\mathcal{F}$. We denote $\mathbf{A}(i,j)$ often by the lower case a_{ij} with i as the row index and j as the column index. We speak of the array $(a_{i1}, \ldots, a_{in})$ as the **ith row** of $\mathbf{A}$ and of the array $(a_{1j}, \ldots, a_{nj})$ as the **jth column** of $\mathbf{A}$. Thus we may think of $\mathbf{A}$ as made up of m row vectors or of n column vectors. Linear dependence, independence and linear combination for row and column vectors are defined the same way as for vectors. We say two matrices are **row equivalent** if the rows of each can be obtained by linearly combining the rows of the other. **Column equivalence** is defined similarly. The vector space spanned by the rows (columns) of $\mathbf{A}$ is called its **row space (column space)** and denoted by $\mathcal{R}(\mathbf{A})(\mathcal{C}(\mathbf{A}))$. The dimension of $\mathcal{R}(\mathbf{A})(\mathcal{C}(\mathbf{A}))$ is called the **row rank (column rank)** of $\mathbf{A}$.

If $\mathbf{A}$ is an $m \times n$ matrix then the **transpose** of $\mathbf{A}$ denoted by $\mathbf{A}^T$ is an $n \times m$ matrix defined by $\mathbf{A}^T(i,j) \equiv \mathbf{A}(j,i)$. Clearly the i^{th} row of $\mathbf{A}$ becomes the i^{th} column of $\mathbf{A}^T$. If $\mathbf{B}$ is also an $m \times n$ matrix the **sum** $\mathbf{A} + \mathbf{B}$ is an $m \times n$ matrix defined by $(\mathbf{A}+\mathbf{B})(i,j) \equiv \mathbf{A}(i,j)+\mathbf{B}(i,j)$. If $\mathbf{D}$ is an $n \times p$ matrix, the product $\mathbf{AD}$ is an $m \times p$ matrix defined by $\mathbf{AD}(i,j) \equiv \sum_{k=1}^{n} a_{ik}d_{kj}$. Clearly if $\mathbf{AD}$ is defined it does not follow that $\mathbf{DA}$ is defined. Even when it is defined, in general $\mathbf{AD} \neq \mathbf{DA}$. The most basic property of this notion of product is that it is **associative** i.e. $\mathbf{A}(\mathbf{DF}) = (\mathbf{AD})\mathbf{F}$.

Matrix operations are often specified by **partitioning**. Here we write a matrix in terms of **submatrices** (i.e., matrices obtained by deleting some rows and columns of the original matrix). A matrix may be partitioned along rows:

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} \\ \cdots \\ \mathbf{A}_{21} \end{bmatrix}$$

or along columns:

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} | \mathbf{A}_{12} \end{bmatrix}$$

or both:

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & \cdots & \mathbf{A}_{1k} \\ \vdots & \vdots & \vdots \\ \mathbf{A}_{p1} & \cdots & \mathbf{A}_{pk} \end{bmatrix}.$$

When two partitioned matrices are multiplied we assume that the partitioning is **compatible**, i.e., for each triple (i,j,k) the number of columns of $\mathbf{A}_{ik}$ equals the number of rows of $\mathbf{B}_{kj}$. Clearly this is achieved if the original matrices $\mathbf{A}, \mathbf{B}$ are compatible for product and each block of the column partition of $\mathbf{A}$ has the same size as the corresponding row partition of $\mathbf{B}$. The following **partitioning rules** can then be verified.

i. $\begin{bmatrix} \mathbf{A}_{11} \\ \cdots \\ \mathbf{A}_{21} \end{bmatrix} \mathbf{C} = \begin{bmatrix} \mathbf{A}_{11}\mathbf{C} \\ \cdots \\ \mathbf{A}_{21}\mathbf{C} \end{bmatrix}$

ii. $\mathbf{C} \begin{bmatrix} \mathbf{A}_{11} | \mathbf{A}_{12} \end{bmatrix} = \begin{bmatrix} \mathbf{CA}_{11} | \mathbf{CA}_{12} \end{bmatrix}$

iii. $\left[\begin{array}{c} \mathbf{A}_{11}|\mathbf{A}_{12} \end{array}\right] \left[\begin{array}{c} \mathbf{C}_{11} \\ \ldots \\ \mathbf{C}_{12} \end{array}\right] = \left[\begin{array}{c} \mathbf{A}_{11}\mathbf{C}_{11} + \mathbf{A}_{12}\mathbf{C}_{12} \end{array}\right].$

In general if $\mathbf{A}$ is partitioned into submatrices $\mathbf{A}_{ik}, \mathbf{B}$ into submatrices $\mathbf{B}_{kj}$ then the product $\mathbf{C} = \mathbf{AB}$ would be naturally partitioned into $\mathbf{C}_{ij} \equiv \sum_k \mathbf{A}_{ik}\mathbf{B}_{kj}$.

Matrices arise most naturally in linear equations such as $\mathbf{Ax} = \mathbf{b}$, where $\mathbf{A}$ and $\mathbf{b}$ are known and $\mathbf{x}$ is an unknown vector. When $\mathbf{b} = \mathbf{0}$ it is easily verified that the set of all solutions of $\mathbf{Ax} = \mathbf{b}$, i.e.,of $\mathbf{Ax} = \mathbf{0}$, form a vector space. This space will be called the **solution space** of $\mathbf{Ax} = \mathbf{0}$, or the **null space** of $\mathbf{A}$. The **nullity** of $\mathbf{A}$ is the dimension of the null space of $\mathbf{A}$. We have the following theorem.

Theorem 2.2.2 *If two matrices are row equivalent then their null spaces are identical.*

Corollary 2.2.1 *If* $\mathbf{A}, \mathbf{B}$ *are row equivalent matrices then a set of columns of* $\mathbf{A}$ *are independent iff the corresponding set of columns of* $\mathbf{B}$ *are independent.*

The following are **elementary row operations** that can be performed on the rows of a matrix:

i. interchanging rows,

ii. adding a multiple of one row to another,

iii. multiplying a row by a nonzero number.

Each of these operations corresponds to premultiplication by a matrix. Such matrices are called **elementary matrices**. It can be seen that these are the matrices we obtain by performing the corresponding elementary row operations on the unit matrix of the same number of rows as the given matrix. We can define elementary column operations similarly. These would correspond to post multiplication by elementary column matrices.

A matrix is said to be in **Row Reduced Echelon** form (RRE) iff it satisfies the following:
Let r be the largest row index for which $a_{ij} \neq 0$ for some j. Then the columns of the $r \times r$ unit matrix (the matrix with $1s$ along the diagonal and zero elsewhere) $\mathbf{e_1}, \ldots, \mathbf{e_r}$ appear as columns, say $\mathbf{C}_{i_1}, \ldots, \mathbf{C}_{i_r}$ of $\mathbf{A}$ with $i_1 < \ldots < i_r$. Further if $p < i_k$ then $a_{kp} = 0$. We have the following theorem.

Theorem 2.2.3 *Every matrix can be reduced to a matrix in the RRE form by a sequence of elementary row transformations and is therefore row equivalent to such a matrix.*

It is easily verified that for an RRE matrix row rank equals column rank. Hence using Theorem 2.2.3 and Corollary 2.2.1 we have

Theorem 2.2.4 *For any matrix, row rank equals column rank.*

The **rank** of a matrix $\mathbf{A}$, denoted by $r(\mathbf{A})$, is its row rank (= column rank).
Let the elements of S be ordered as $(e_1, \ldots, e_n)$. Then for any vector $\mathbf{f}$ on S we define $\mathbf{R}_f$, the **representative vector** of $\mathbf{f}$, as the one rowed matrix $(\mathbf{f}(e_1), \ldots, \mathbf{f}(e_n))$. We will not usually distinguish between a vector and its representative vector. When the rows of a matrix $\mathbf{R}$ are representative vectors of some basis of a vector space $\mathcal{V}$ we say that $\mathbf{R}$ is a **representative matrix** of $\mathcal{V}$. When $\mathbf{R}, \mathbf{R}_1$ both represent $\mathcal{V}$ they can be obtained from each other by row operations. Hence by Corollary 2.2.1 their column independence structure is identical. An $r \times n$ representative matrix $\mathbf{R}$, $r \leq n$, is a **standard representative matrix** iff $\mathbf{R}$ has an $r \times r$ submatrix which can be obtained by permutations of the columns of the $r \times r$ unit matrix. For convenience we will write a standard representative matrix in the form $[\mathbf{I}|\mathbf{R}_{12}]$ or $[\mathbf{R}_{11}|\mathbf{I}]$. (Here $\mathbf{I}$ denotes the unit matrix of appropriate size).

The **dot product** of two vectors $\mathbf{f}, \mathbf{g}$ on S denoted by $< \mathbf{f}, \mathbf{g} >$ over $\mathcal{F}$ is defined by $< \mathbf{f}, \mathbf{g} > \equiv \sum_{e \in S} \mathbf{f}(e).\mathbf{g}(e)$. We say $\mathbf{f}, \mathbf{g}$ are **orthogonal** if their dot product is zero. If $\mathcal{C}$ is a collection of vectors on S then $\mathcal{C}^\perp \equiv$ set of all vectors orthogonal to every vector in $\mathcal{C}$. It can be verified that $\mathcal{C}^\perp$ is a vector space. Let $\mathcal{V}$ be a vector space on S with basis $\mathcal{B}$. Since vectors orthogonal to each vector in $\mathcal{B}$ are also orthogonal to linear combinations of these vectors we have $\mathcal{B}^\perp = \mathcal{V}^\perp$. If $\mathbf{R}$ is a representative matrix of $\mathcal{V}$, it is clear that $\mathcal{V}^\perp$ is its null space. Equivalently $\mathcal{V}^\perp$ is the solution space of $\mathbf{Rx} = \mathbf{0}$. If $\mathbf{R}$ is a standard representative matrix with $\mathbf{R} = [\mathbf{I}_{r \times r}|\mathbf{R}_{12}]$, then the solution space of $\mathbf{Rx} = \mathbf{0}$ can be shown to be the vector space generated by the columns of $\begin{bmatrix} -\mathbf{R}_{12} \\ \cdots \\ \mathbf{I}_{n-r \times n-r} \end{bmatrix}$, where $n = |S|$.(Here $\mathbf{I}_{k \times k}$ denotes the unit matrix with k rows). Equivalently $\mathcal{V}^\perp$ has the representative matrix $[-\mathbf{R}_{12}^T|\mathbf{I}_{n-r \times n-r}]$. The representative matrix of $(\mathcal{V}^\perp)^\perp$ will then be $\mathbf{R}$. We then have the following

Theorem 2.2.5 *i. if* $[\mathbf{I}_{r \times r}|\mathbf{R}_{12}]$ *is a representative matrix of vector space* $\mathcal{V}$ *on* S *then* $[-\mathbf{R}_{12}^T|\mathbf{I}_{n-r \times n-r}]$ *is a representative matrix of* $\mathcal{V}^\perp$.

ii. $r(\mathcal{V}^\perp) = |S| - r(\mathcal{V})$

iii. $(\mathcal{V}^\perp)^\perp = \mathcal{V}$. *Hence two matrices are row equivalent iff their null spaces are identical.*

Consider the collection of $n \times n$ matrices over $\mathcal{F}$. We say that $\mathbf{I}$ is an identity for this collection iff for every $n \times n$ matrix $\mathbf{B}$ we have $\mathbf{IB} = \mathbf{BI} = \mathbf{B}$. If $\mathbf{I}_1, \mathbf{I}_2$ are identity matrices we must have $\mathbf{I}_1 = \mathbf{I}_2 = \mathbf{I}$. The unit matrix (with $1s$ along the diagonal and $0s$ elsewhere) is clearly an identity matrix. It is therefore the only identity matrix. Two $n \times n$ matrices $\mathbf{A}, \mathbf{B}$ are **inverses** of each other iff $\mathbf{AB} = \mathbf{BA} = \mathbf{I}$. We say $\mathbf{A}, \mathbf{B}$ are **invertible** or **nonsingular**. If $\mathbf{A}$ has inverses $\mathbf{B}, \mathbf{C}$ we must have $\mathbf{C} = \mathbf{C}(\mathbf{AB}) = (\mathbf{CA})\mathbf{B} = \mathbf{IB} = \mathbf{B}$. Thus the inverse of a matrix $\mathbf{A}$, if it exists, is unique and is denoted by $\mathbf{A}^{-1}$. We then have the following

Theorem 2.2.6 *i.* $(\mathbf{A}^T)^{-1} = (\mathbf{A}^{-1})^T$

ii. If $\mathbf{A}, \mathbf{D}$ *are* $n \times n$ *invertible matrices, then* $(\mathbf{AD})^{-1} = (\mathbf{D}^{-1}\mathbf{A}^{-1})$.

With a square matrix we associate an important number called its determinant. Its definition requires some preparation.

A bijection of a finite set to itself is also called a **permutation**. A permutation that interchanges two elements (i.e. maps each of them to the other) but leaves all others unchanged is a **transposition**. Every permutation can be obtained by repeated application of transpositions. We then have the following

Theorem 2.2.7 *If a permutation σ can be obtained by composition of an even number of transpositions then every decomposition of σ into transpositions will contain an even number of them.*

By Theorem 2.2.7 we can define a permutation to be **even (odd)** iff it can be decomposed into an even (odd) number of transpositions. The **sign** of a permutation σ denoted by $sgn(\sigma)$ is $+1$ if σ is even and -1 if σ is odd. It is easily seen, since the identity $(= \sigma\sigma^{-1})$ permutation is even, that $sgn(\sigma) = sgn(\sigma^{-1})$. The **determinant** of an $n \times n$ matrix is defined by

$$det(\mathbf{A}) \equiv \sum_{\sigma} sgn(\sigma) a_{1\sigma(1)} \dots a_{n\sigma(n)},$$

where the summation is taken over all possible permutations of $\{1, 2, \dots, n\}$. It is easily seen that determinant of the unit matrix is $+1$. We collect some of the important properties of the determinant in the following

Theorem 2.2.8 *i.* $det(\mathbf{A}) = det(\mathbf{A}^T)$

ii. Let

$$\mathbf{A} = \begin{bmatrix} \mathbf{a}_1 \\ \mathbf{A}_2 \end{bmatrix}, \mathbf{A}' = \begin{bmatrix} \mathbf{a}_1' \\ \mathbf{A}_2 \end{bmatrix}, \mathbf{A}" = \begin{bmatrix} \mathbf{a}_1 + \mathbf{a}_1' \\ \mathbf{A}_2 \end{bmatrix}.$$

Then $det(\mathbf{A}") = det(\mathbf{A}) + det(\mathbf{A}')$.

iii. If $\mathbf{A}$ *has two identical rows, or has two identical columns then* $det(\mathbf{A}) = 0$.

iv. If $\mathbf{E}$ *is an elementary matrix* $det(\mathbf{EA}) = det(\mathbf{E})det(\mathbf{A})$. *Since every invertible matrix can be factored into elementary matrices, it follows that* $det(\mathbf{AB}) = det(\mathbf{A})det(\mathbf{B})$, *for every pair of* $n \times n$ *matrices* $\mathbf{A}, \mathbf{B}$.

v. $det(\mathbf{A}) \neq 0$ *iff* $\mathbf{A}$ *is invertible.*

Problem 2.1 Size of a basis: *Prove*

i. Theorem 2.2.1

ii. If $\mathcal{V}_1$ *is a subspace of vector space* $\mathcal{V}_2$, $\dim \mathcal{V}_1 \leq \dim \mathcal{V}_2$.

iii. If $\mathcal{V}_1 \subseteq \mathcal{V}_2$ *and* $\dim \mathcal{V}_1 = \dim \mathcal{V}_2$ *then* $\mathcal{V}_1 = \mathcal{V}_2$.

iv. an $m \times n$ *matrix with* $m > n$ *cannot have linearly independent rows.*

v. any vector in a vector space $\mathcal{V}$ *can be written uniquely as a linear combination of the vectors in a basis of* $\mathcal{V}$.

Problem 2.2 Ways of interpreting the matrix product: *Define product of matrices in the usual way i.e.* $\mathbf{C} = \mathbf{AB}$ *is equivalent to* $\mathbf{C}_{ij} = \sum_k a_{ik}b_{kj}$. *Now show that it can be thought of as follows*

i. *Columns of* $\mathbf{C}$ *are linear combinations of columns of* $\mathbf{A}$ *using entries of columns of* $\mathbf{B}$ *as coefficients.*

ii. *rows of* $\mathbf{C}$ *are linear combinations of rows of* $\mathbf{B}$ *using entries of rows of* $\mathbf{A}$ *as coefficients.*

Problem 2.3 Properties of matrix product: *Prove, when* $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ *are matrices and the products are defined*

i. $(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$

ii. $(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$

Problem 2.4 Partitioning rules: *Prove*

i. *the partitioning rules.*

ii.

$$\begin{bmatrix} \mathbf{A}_{11} \cdots & \mathbf{A}_{1k} \\ \vdots & \vdots \\ \mathbf{A}_{r1} \cdots & \mathbf{A}_{rk} \end{bmatrix}^T = \begin{bmatrix} \mathbf{A}_{11}^T \cdots & \mathbf{A}_{r1}^T \\ \vdots & \vdots \\ \mathbf{A}_{1k}^T \cdots & \mathbf{A}_{rk}^T \end{bmatrix}$$

Problem 2.5 Solution space and column dependence structure: *Prove theorem 2.2.2 and Corollary 2.2.1.*

Problem 2.6 Algorithm for computing RRE: *Give an algorithm for converting any rectangular matrix into the RRE form. Give an upper bound for the number of arithmetical steps in your algorithm.*

Problem 2.7 Uniqueness of the RRE matrix: *Show that no RRE matrix is row equivalent to a distinct RRE matrix. Hence prove that every matrix is row equivalent to a unique matrix in the RRE form.*

Problem 2.8 RRE of special matrices:

i. *If* $\mathbf{A}$ *is a matrix with linearly independent columns what is its RRE form? If in addition* $\mathbf{A}$ *is square what is its RRE form?*

ii. *If* $\mathbf{A}, \mathbf{B}$ *are square such that* $\mathbf{AB} = \mathbf{I}$ *show that* $\mathbf{BA} = \mathbf{I}$.

iii. *Prove Theorem 2.2.6*

Problem 2.9 Existence and nature of solution for linear equations: *Consider the equation* $\mathbf{Ax} = \mathbf{b}$.

i. *Show that it has a solution*

(a) *iff* $r(\mathbf{A}) = r(\mathbf{A}|\mathbf{b})$.

(b) *iff whenever* $\lambda^T\mathbf{A} = \mathbf{0}, \lambda^T\mathbf{b}$ *is also zero.*

ii. *Show that a vector is a solution of the above equation iff it can be written in the form* $\mathbf{x}_o + \mathbf{x}_p$ *where* $\mathbf{x}_p$ *is a particular solution of the equation while* $\mathbf{x}_o$ *is a vector in the null space of* $\mathbf{A}$ *(i.e. a solution to the linear equation with* $\mathbf{b}$ *set equal to zero).*

iii. **Motivation for the matrix product:** *Why is the matrix product defined as in Problem 2.2? (In the above equation suppose we make the substitution* $\mathbf{x} = \mathbf{B}\mathbf{y}$. *What would the linear equation in terms of* $\mathbf{y}$ *be?)*

iv. **Linear dependence and logical consequence:** *The above equation may be regarded as a set of linear equations (one for each row of* $\mathbf{A}$*) each of which in turn could be thought of as a statement. Show that a linear equation is a logical consequence of others iff it is* **linearly** *dependent on the others.*

Problem 2.10 Positive definite matrices:

i. *Construct an example where* $\mathbf{A}$, $\mathbf{B}$ *are invertible but their sum is not.*

ii. *A matrix* $\mathbf{K}$ *is* **positive semidefinite (positive definite)** *iff* $\mathbf{x}^T\mathbf{K}\mathbf{x} \geq 0 \;\; \forall \mathbf{x} \neq 0$ *(*$\mathbf{x}^T\mathbf{K}\mathbf{x} > 0 \;\; \forall \mathbf{x} \neq 0$*). Show that*

(a) *a matrix is invertible if it is positive definite;*

(b) *sum of two positive semidefinite matrices (positive definite matrices) is positive semidefinite (positive definite);*

(c) *if* $\mathbf{K}$ *is a positive definite matrix,then* $\mathbf{AKA}^T$ *is positive semidefinite and if, further, rows of* $\mathbf{A}$ *are linearly independent, then* $\mathbf{AKA}^T$ *is positive definite;*

(d) *inverse of a symmetric positive definite matrix is also symmetric positive definite.*

Problem 2.11 Projection of a vector on a vector space: *Let* $\mathbf{x}$ *be a vector on* S *and let* $\mathcal{V}$ *be a vector space on* S. *Show that* $\mathbf{x}$ *can be uniquely decomposed as* $\mathbf{x} = \mathbf{x}_1 + \mathbf{x}_2$, *where* $\mathbf{x}_1 \in \mathcal{V}$ *and* $\mathbf{x}_2 \in \mathcal{V}^\perp$. *The vector* $\mathbf{x}_1$ *is called the* **projection** *of* $\mathbf{x}$ *on* $\mathcal{V}$ **along** $\mathcal{V}^\perp$.

Problem 2.12 Parity of a Permutation: *Show that if a permutation can be obtained by composing an odd number of transpositions it cannot also be obtained by composing an even number of transpositions.*

Problem 2.13 Graph of a permutation: *Define the graph* $\mathcal{G}_\sigma$ *of a permutation* σ *on* $\{1, 2, \cdots n\}$ *as follows:* $V(\mathcal{G}_\sigma) \equiv \{1, 2, \cdots, n\}$; *draw an edge with an arrow from* i *to* j *iff* $\sigma(i) = j$.

i. *Show that every vertex in this graph has precisely one arrow coming in and one going out. Hence, conclude that each connected component is a directed circuit.*

ii. Show that if $\mathcal{G}_\sigma$ has an odd (even) number of even length circuits then σ is odd (even).

Problem 2.14 Properties of the determinant: *Prove Theorem 2.2.8.*

Problem 2.15 Equivalence of definitions of a determinant: *Show that the usual definition of a determinant by expanding along a row or column is equivalent to the definition using permutations.*

Problem 2.16 Laplace expansion of the determinant: *Let* $\mathbf{A}$ *be an* $n \times n$ *matrix. Show that*

$$det(\mathbf{A}) = \sum sgn(\sigma)\ det\left(\mathbf{A}\begin{pmatrix} r_1, & \cdots & , r_k \\ i_1, & \cdots & , i_k \end{pmatrix}\right) det\left(\mathbf{A}\begin{pmatrix} r_{k+1}, & \cdots & , r_m \\ i_{k+1}, & \cdots & , i_m \end{pmatrix}\right),$$

($\mathbf{A}\begin{pmatrix} d_1, & \cdots & , d_p \\ i_1, & \cdots & , i_p \end{pmatrix}$ *is the* $p \times p$ *matrix whose* (s,t) *entry is the* (d_s, i_t) *entry of* $\mathbf{A}$*), where the summation is over all subsets* $\{i_1, \cdots, i_k\}$ *of* $\{1, \cdots, n\}$ *and* $\sigma \equiv \begin{pmatrix} i_1, \cdots, i_k, i_{k+1} \cdots i_n \\ r_1, \cdots, r_k, r_{k+1} \cdots r_n \end{pmatrix}$ *i.e.,* $\sigma(i_j) \equiv r_j, j = 1, \cdots, n$.

Problem 2.17 Binet Cauchy Theorem: *Let* $\mathbf{A}$ *be an* $m \times n$ *and* $\mathbf{B}$ *an* $n \times m$ *matrix with* $m \leq n$*. If an* $m \times m$ *submatrix of* $\mathbf{A}$ *is composed of columns* $i_1, \cdots, i_m$*, the* **corresponding** $m \times m$ *submatrix of* $\mathbf{B}$ *is the one with rows* $i_1, \cdots, i_m$*. Prove the Binet Cauchy Theorem:* $det(\mathbf{AB}) = \sum$ *product of determinants of corresponding* $m \times m$ *submatrices of* $\mathbf{A}$ *and* $\mathbf{B}$.

2.3 Linear Inequality Systems

2.3.1 The Kuhn-Fourier Theorem

In this section we summarize basic results on inequality systems which we need later on in the book. Proofs are mostly omitted. They may be found in standard references such as [Stoer+Witzgall70] and [Schrijver86]. This section follows the former reference.

A **linear inequality system** is a set of constraints of the following kind on the vector $\mathbf{x} \in \Re^n$.

$$\left.\begin{array}{lcl} \mathbf{Ax} & = & \mathbf{a}_o \\ \mathbf{Bx} & > & \mathbf{b}_o \\ \mathbf{Cx} & \geq & \mathbf{c}_o \end{array}\right\} \qquad (I)$$

Here, $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are matrices, $\mathbf{a}_o, \mathbf{b}_o, \mathbf{c}_o$ are column vectors with appropriate number of rows. We say $\mathbf{x}_1 > \mathbf{x}_2 (\mathbf{x}_1 \geq \mathbf{x}_2)$ iff each component of $\mathbf{x}_1$ is greater than (greater than or equal to) the corresponding component of $\mathbf{x}_2$.

A **solution** of an inequality system is a vector which satisfies all the inequality constraints of the system. A constraint which is satisfied by every solution of an inequality system is said to be a **consequence** of the system. In particular, we are concerned with constraints of the kind $\mathbf{d}^T\mathbf{x} = \mathbf{d}_o$ or $> \mathbf{d}_o$ or $\geq \mathbf{d}_o$. A **legal linear combination** of the system (I) is obtained by linearly combining the equations and inequalities with real coefficients - α_i for the linear equations, and non-negative real coefficients β_j, γ_k for the '$>$' linear inequalities and '$\geq$' linear inequalities respectively. The resulting constraint would be a linear equation iff β_j, γ_k are all zero. It would be a '$>$' inequality iff at least one of the β_j's is nonzero. It would be a '$\geq$' inequality iff all of β_j are zero but at least one of the γ_k is nonzero. A legal linear combination is thus a consequence of the system. A legal linear combination, with at least one of the $\alpha_i, \beta_j, \gamma_k$ nonzero, that results in the LHS becoming zero is called a **legal linear dependence** of the system. Another important way of deriving consequence relations is by **weakening**. This means to weaken '$=$' to '$\geq$' and '$>$' to '$\geq$' and also in the case of '$>$' and '$\geq$' to lower the right side value.

Example 2.3.1 *Consider the system of linear inequalities:*

$$\begin{aligned} x_1 + 2x_2 &= 3 \\ 2x_1 + x_2 &= 4 \\ x_1 + x_2 &> 1 \\ 2x_1 + 3x_2 &> 2 \\ x_1 + 5x_2 &\geq 2 \\ -x_1 - 2x_2 &\geq 4. \end{aligned}$$

The legal linear combination corresponding to $\alpha_1 = 1, \alpha_2 = 1, \beta_1 = 0, \beta_2 = 0, \gamma_1 = 0, \gamma_2 = 0$ is
$3x_1 + 3x_2 = 7$;
that corresponding to $\alpha_1 = 1, \alpha_2 = 0, \beta_1 = 1, \beta_2 = 0, \gamma_1 = 1, \gamma_2 = 0$ is
$3x_1 + 8x_2 > 6$;
that corresponding to $\alpha_1 = 1, \alpha_2 = 0, \beta_1 = 0, \beta_2 = 0, \gamma_1 = 1, \gamma_2 = 0$ is
$2x_1 + x_2 \geq 5$.
The legal linear combination corresponding to $\alpha_1 = 1, \alpha_2 = 0, \beta_1 = 0, \beta_2 = 0, \gamma_1 = 0, \gamma_2 = 1$ is the zero relation
$0x_1 + 0x_2 \geq 7$.
Thus in this case, the system has a **legal linear dependence** that is a **contradiction**.

We can now state the fundamental theorem of Kuhn and Fourier [Fourier1826], [Kuhn56].

Theorem 2.3.1 (Kuhn-Fourier Theorem) *A linear inequality system has a solution iff no legal linear dependence is a contradiction.*

Sketch of the Proof of Theorem 2.3.1: First reduce the linear equations to the RRE form and express some of the variables in terms of the others. This substitution is now carried out also in the inequalities. So henceforth, without loss of generality, we may assume that we have only inequalities. If we prove the theorem for such a reduced system, it can be extended to one which has equalities also.

Suppose all variables have either zero coefficient or the same sign in all the inequalities. In this case it is easy to see that the system has a solution whether the coefficients are all zero or otherwise. If all the coefficients are zero we are done - the theorem is clearly true. If not, it is not possible to get a legal linear dependence without using zero coefficents. So the theorem is again true in this case.

We now present an elimination procedure which terminates at one of the above mentioned cases.
Let the inequalities be numbered $(1), \cdots, (r), (r+1), \cdots, (k)$. Let x_n be present with coefficient +1 in the inequalities $(1), \cdots, (r)$ and with coefficient -1 in the inequalities $(r+1), \cdots, (k)$. We create $r(k-r)$ inequalities without the variables x_n by adding each of the first r inequalities to each of the last $(k-r)$ inequalities. Note that if both the inequalities are of the $(\geq)$ kind, the addition would result in another of the $(\geq)$ kind and if one of them is of the $(>)$ kind, the addition would result in another of the $(>)$ kind.

If the original system has a solution, it is clear that the reduced system also has one. On the other hand, if the reduced system has a solution $(x_1', \cdots, x_{n-1}')$ it is possible to find a value x_n' of x_n such that $(x_1', \cdots, x_{n-1}', x_n')$ is a solution of the original system. We indicate how, below.
Let the inequalities added be

$$a_{i1}x_1 + \cdots + x_n \geq b_i$$

$$a_{j1}x_1 + \cdots - x_n > b_j$$

(The cases where both are $(\geq)$, both are $(>)$ or first inequality $(>)$ and second $(\geq)$ are similar.) The pair of inequalities can be written equivalently as

$$a_{j1}x_1 + \cdots + a_{j(n-1)}x_{n-1} - b_j > x_n \geq b_i - a_{i1}x_1 - \cdots - a_{i(n-1)}x_{n-1} \qquad (*)$$

The extreme left of the above inequality $(*)$ is always derived from the inequalities $(r+1)$ to (k) while the extreme right is always derived from the (1) to (r) inequalities. When $x_1', \cdots, x_{n-1}'$ is substituted in the above inequality, it would be satisfied for every pair of inequalities, from $(j+1)$ to (k) on the extreme left and (1) to (j) on the extreme right. After substitution, let the least of the extreme left term be reached for inequality (p) and let the highest of the extreme right term be reached for inequality (q). Since

$$a_{p1}x_1' + \cdots + a_{p(n-1)}x_{n-1}' - b_p > b_q - a_{q1}x_1' - \cdots - a_{q(n-1)}x_{n-1}'$$

(this inequality results when (p) and (q) are added), we can find a value x_n' of x_n which lies between left and right sides of the above inequality. Clearly $(x_1', \cdots, x_n')$ is a solution of the original system.

If this procedure were repeated, we would reach a system where either all the coefficients have zero value or where the signs of the coefficients of a variable are all the same in all the inequalities. As mentioned before, for these cases the theorem is true.

□

As an immediate consequence we can prove the celebrated 'Farkas Lemma'.

Theorem 2.3.2 (Farkas Lemma) *The homogeneous system*

$$\mathbf{A}\ \mathbf{x} \leq 0$$

has the consequence

$$\mathbf{d}^T\mathbf{x} \leq 0$$

iff the row vector $\mathbf{d}^T$ *is a nonnegative linear combination of the rows of* $\mathbf{A}$.

Proof : By Kuhn-Fourier Theorem (Theorem 2.3.1), the system

$$\mathbf{A}^T\mathbf{y} = \mathbf{d}$$

$$\mathbf{y} \geq \mathbf{0}$$

has a solution iff

'$\mathbf{x}^T\mathbf{A}^T + \beta^T\mathbf{I} = \mathbf{0}, \beta^T \geq \mathbf{0}$' implies '$\mathbf{x}^T\mathbf{d} \leq \mathbf{0}$';
i.e., iff '$\mathbf{Ax} \leq \mathbf{0}$' implies '$\mathbf{d}^T\mathbf{x} \leq \mathbf{0}$.'

□

The analogue of 'vector spaces' for inequality systems is '**cones**'. A **cone** is a collection of vectors closed under addition and non-negative linear combination. It is easily verified that the solution set of $\mathbf{Ax} \geq \mathbf{0}$ is a cone. Such cones are called **polyhedral**. We say vectors $\mathbf{x}$, $\mathbf{y}$ (on the same set S) are **polar** iff $< \mathbf{x}, \mathbf{y} >$ (i.e., the dot product) is nonpositive. If $\mathcal{K}$ is a collection of vectors, the **polar of** $\mathcal{K}$, denoted by $\mathcal{K}^p$ is the collection of vectors polar to every vector in $\mathcal{K}$. Thus, Farkas Lemma states:

'Let $\mathcal{C}$ be the polyhedral cone defined by $\mathbf{Ax} \leq \mathbf{0}$. A vector $\mathbf{d}$ belongs to $\mathcal{C}^p$ iff $\mathbf{d}^T$ is a nonnegative linear combination of the rows of $\mathbf{A}$.'

2.3.2 Linear Programming

Let $\mathcal{S}$ be a linear inequality system with '$\leq$' and '$=$' constraints ('$\geq$' and '$=$' constraints). The **linear programming problem** or **linear program** is that of finding a solution $\mathbf{x}$ of $\mathcal{S}$ which maximizes a given linear function $\mathbf{c}^T\mathbf{x}$ (minimizes a given linear function $\mathbf{c}^T\mathbf{x}$). The linear function to be optimized is called the **objective function**. A solution of $\mathcal{S}$ is called a **feasible solution**, while a solution which optimizes $\mathbf{c}^T\mathbf{x}$ is called an **optimal solution**, of the linear programming problem. The **value** of a feasible solution is the value of the objective function on it.

The following linear programming problems are said to be **duals** of each other

Primal program

$$\text{Maximize} \qquad \mathbf{c}_1^T\mathbf{x}_1 + \mathbf{c}_2^T\mathbf{x}_2$$

$$\begin{pmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \end{pmatrix} \begin{matrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{matrix} = \mathbf{b}_1$$

$$\begin{pmatrix} \mathbf{A}_{21} & \mathbf{A}_{22} \end{pmatrix} \begin{matrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{matrix} \le \mathbf{b}_2$$

$$\mathbf{x}_2 \ge 0$$

Dual program

$$\text{Minimize} \qquad \mathbf{b}_1^T \mathbf{y}_1 + \mathbf{b}_2^T \mathbf{y}_2$$

$$\begin{pmatrix} \mathbf{A}_{11}^T & \mathbf{A}_{21}^T \end{pmatrix} \begin{matrix} \mathbf{y}_1 \\ \mathbf{y}_2 \end{matrix} = \mathbf{c}_1$$

$$\begin{pmatrix} \mathbf{A}_{12}^T & \mathbf{A}_{22}^T \end{pmatrix} \begin{matrix} \mathbf{y}_1 \\ \mathbf{y}_2 \end{matrix} \ge \mathbf{c}_2$$

$$\mathbf{y}_2 \ge 0.$$

We now present the duality theorem of linear programming [von Neumann47], [Gale+Kuhn+Tucker51].

Theorem 2.3.3 *For dual pairs of linear programs the following statements hold:*

i. The value of each feasible solution of the minimization program is greater than or equal to the value of each feasible solution of the maximization program;

ii. if both programs have feasible solutions then both have optimal solutions and the optimal values are equal;

iii. if one program has an optimal solution then so does the other.

The usual proof uses Farkas Lemma, or more conveniently, Kuhn-Fourier Theorem. We only sketch it.
Sketch of Proof: Part (i) follows by the solution of Exercise 2.1.
Now we write down the inequalities of the primal and dual programs and another '$\le$' inequality which is the **opposite** of the inequality in part (i). Part (ii) would be proved if this system of inequalities has a solution. We assume it has no solution and derive a contradiction by using Kuhn-Fourier Theorem.

□

Exercise 2.1 *Prove part (i) of Theorem 2.3.3.*

A very useful corollary of Theorem 2.3.3 is the following:

Corollary 2.3.1 *(***Complementary Slackness***)*
Let

$$\left\{\begin{array}{c}\max \mathbf{c}^T\mathbf{x}\\ \mathbf{A}\mathbf{x}=\mathbf{b}\\ \mathbf{x}\geq \mathbf{0}\end{array}\right\} \text{ and } \left\{\begin{array}{c}\min \mathbf{b}^T\mathbf{y}\\ \mathbf{A}^T\mathbf{y}\geq \mathbf{c}\end{array}\right\}$$

be dual linear programs. Let $\hat{\mathbf{x}}, \hat{\mathbf{y}}$ *be optimal solutions to the respective programs. Then,*

i. $\hat{x}_i > 0$ *implies* $(\mathbf{A}^T)_i\hat{\mathbf{y}} = c_i$,

ii. $(\mathbf{A}^T)_i\hat{\mathbf{y}} > c_i$ *implies* $\hat{\mathbf{x}}_i = 0$.

Proof : We have by part (ii) of Theorem 2.3.3 $\mathbf{c}^T\hat{\mathbf{x}} = \hat{\mathbf{y}}^T\mathbf{b}$, equivalently

$$\mathbf{c}^T\hat{\mathbf{x}} - \hat{\mathbf{y}}^T\mathbf{A}\hat{\mathbf{x}} = (\mathbf{c}^T - \hat{\mathbf{y}}^T\mathbf{A})\hat{\mathbf{x}} = 0.$$

The result now follows since $(\mathbf{c}^T - \hat{\mathbf{y}}^T\mathbf{A}) \geq 0$ and $\hat{\mathbf{x}} \geq \mathbf{0}$.

□

2.4 Solutions of Exercises

E 2.1: We use the linear programs given in the definition of dual linear programs. We have

$$\begin{aligned}\begin{pmatrix} \mathbf{b}_1^T & \mathbf{b}_2^T \end{pmatrix} \begin{matrix}\mathbf{y}_1\\ \mathbf{y}_2\end{matrix} &\geq \left(\begin{pmatrix} \mathbf{x}_1^T & \mathbf{x}_2^T \end{pmatrix}\begin{bmatrix}\mathbf{A}_{11} & \mathbf{A}_{12}\\ \mathbf{A}_{21} & \mathbf{A}_{22}\end{bmatrix}^T\right) \begin{matrix}\mathbf{y}_1\\ \mathbf{y}_2\end{matrix}\\ &\geq \begin{pmatrix} \mathbf{x}_1^T & \mathbf{x}_2^T \end{pmatrix}\begin{bmatrix}\mathbf{c}_1\\ \mathbf{c}_2\end{bmatrix}.\end{aligned}$$

2.5 Solutions of Problems

Most of these problems can be found as standard results in undergraduate texts on linear algebra (see for instance [Hoffman+Kunze72]). We only give the solution to the last two problems. Here we follow [MacDuffee33], [Gantmacher59] respectively.

P 2.16: We state the following simple lemma without proof.

Lemma 2.5.1 *If* $\alpha_1, \cdots, \alpha_t$ *are permutations of* $\{1, \cdots, n\}$ *then* $sgn(\alpha_1\alpha_2\cdots\alpha_t) = (sgn(\alpha_1))(sgn(\alpha_2))\cdots(sgn(\alpha_t))$ *(where* $\alpha_i\alpha_j$ *denotes composition of permutations* α_i, α_j*).*

We have

$$\sum sgn(\sigma)\ det\left(\mathbf{A}\begin{pmatrix} r_1, & \cdots & ,r_k\\ i_1, & \cdots & ,i_k\end{pmatrix}\right) det\left(\mathbf{A}\begin{pmatrix} r_{k+1}, & \cdots & ,r_m\\ i_{k+1}, & \cdots & ,i_m\end{pmatrix}\right) =$$

$$\sum sgn(\sigma)(\sum sgn(\alpha)(a_{r_1\alpha(r_1)}\cdots a_{r_k\alpha(r_k)}))(\sum sgn(\beta)(a_{r_{k+1}\beta(r_{k+1})}\cdots a_{n\beta(n)})),$$

where α, β are permutations on the sets $\{r_1, \cdots, r_k\}$, $\{r_{k+1}, \cdots, r_n\}$ respectively. Let α' agree with α over $\{r_1, \cdots, r_k\}$ and over $\{r_{k+1}, \cdots, r_n\}$, with the identity permutation. Let β' agree with β over $\{r_{k+1}, \cdots, r_n\}$ and with the identity permutation over $\{r_1, \cdots, r_k\}$. So

$$\begin{aligned} LHS &= \sum sgn(\sigma)sgn(\alpha')sgn(\beta')(a_{r_1\alpha(r_1)}\cdots a_{r_k\alpha(r_k)}a_{r_{k+1}\beta(r_{k+1})}\cdots a_{n\beta(n)}) \\ &= \sum sgn(\beta'\alpha'\sigma)(a_{i_1\alpha\sigma(i_1)}\cdots a_{i_k\alpha\sigma(i_k)}a_{i_{k+1}\beta\sigma(i_{k+1})}\cdots a_{i_n\beta\sigma(i_n)}) \\ &= \sum sgn(\mu)(a_{i_1\mu(i_1)}\cdots a_{i_k\mu(i_k)}a_{i_{k+1}\mu(i_{k+1})}\cdots a_{i_n\mu(i_n)}), \end{aligned}$$

where $\mu \equiv \beta'\alpha'\sigma$. Since the RHS is the usual definition of the determinant of $\mathbf{A}$, the proof is complete.

P 2.17: Let a_{ij}, b_{ij} denote respectively the $(i,j)^{th}$ entry of $\mathbf{A}, \mathbf{B}$. Then the matrix

$$\mathbf{AB} = \begin{bmatrix} \sum_{i_1=1}^{n} a_{1i_1}b_{i_11} & \cdots & \sum_{i_m=1}^{n} a_{1i_m}b_{i_mm} \\ \vdots & & \vdots \\ \sum_{i_1=1}^{n} a_{mi_1}b_{i_11} & \cdots & \sum_{i_m=1}^{n} a_{mi_m}b_{i_mm} \end{bmatrix}.$$

Now each column of $\mathbf{AB}$ can be thought of as the sum of n appropriate columns - for instance the transpose of the first column is made up of rows - a typical one being $(a_{1i_1}b_{i_11}, \cdots, a_{mi_1}b_{i_11})$. Using Theorem 2.2.8

$$\begin{aligned} det(\mathbf{AB}) &= \sum_{i_1,\cdots,i_m} det\left(\begin{bmatrix} a_{1i_1}b_{i_11} & \cdots & a_{1i_m}b_{i_mm} \\ \vdots & & \vdots \\ a_{mi_1}b_{i_11} & \cdots & a_{mi_m}b_{i_mm} \end{bmatrix}\right) \\ &= \sum (b_{i_11}\cdots b_{i_mm})\ det\left(\mathbf{A}\begin{pmatrix} 1, & \cdots & ,m \\ i_1, & \cdots & ,i_m \end{pmatrix}\right), \end{aligned}$$

where $\mathbf{A}\begin{pmatrix} 1, & \cdots & ,m \\ i_1, & \cdots & ,i_m \end{pmatrix}$ is the $m \times m$ matrix which has the first m rows of $\mathbf{A}$ in the same order as in $\mathbf{A}$ but whose j^{th} column is the i_j^{th} column of $\mathbf{A}$. So, again by Theorem 2.2.8,

$$\det(\mathbf{AB}) = \sum_{k_1,\cdots,k_m} det\left(\mathbf{A}\begin{pmatrix} 1, & \cdots & ,m \\ k_1, & \cdots & ,k_m \end{pmatrix}\right)(sgn(\sigma))\ b_{\sigma(k_1)1}\cdots b_{\sigma(k_m)m},$$

where $k_1 < \cdots < k_m, \{k_1, \cdots, k_m\} = \{i_1, \cdots, i_m\}$ and σ is the permutation $\begin{pmatrix} k_1, & \cdots & ,k_n \\ i_1, & \cdots & ,i_n \end{pmatrix}$, i.e.,

$$\sigma(k_j) = i_j.$$

So,

$$det(\mathbf{AB}) =$$

$$\sum_{\substack{k_1,\cdots,k_m \\ k_1<\cdots<k_m}} det\left(\mathbf{A}\begin{pmatrix} 1, & \cdots & ,m \\ k_1, & \cdots & ,k_m \end{pmatrix}\right) det\left(\mathbf{B}\begin{pmatrix} k_1, & \cdots & ,k_m \\ 1, & \cdots & ,m \end{pmatrix}\right).$$

Chapter 3

Graphs

3.1 Introduction

We give definitions of graphs and related notions below. Graphs should be visualized as points joined by lines with or without arrows rather than be thought of as formal objects. We would not hesitate to use informal language in proofs.

3.2 Graphs: Basic Notions

3.2.1 Graphs and Subgraphs

A graph $\mathcal{G}$ is a triple $(V(\mathcal{G}), E(\mathcal{G}), i_{\mathcal{G}})$ where $V(\mathcal{G})$ is a finite set of **vertices**, $E(\mathcal{G})$ is a finite set of **edges** and $i_{\mathcal{G}}$ is an **incidence function** which associates with each edge a pair of vertices, not necessarily distinct, called its **end points** or **end vertices** (i.e., $i_{\mathcal{G}} : E(\mathcal{G}) \rightarrow$ collection of subsets of $V(\mathcal{G})$ of cardinality 2 or 1).
Vertices are also referred to as **nodes** or **junctions** while edges are referred to also as **arcs** or **branches**.
We note

i. an edge may have a single end point - such edges are called **selfloops**.

ii. a vertex may have no edges incident on it - such vertices are said to be **isolated**.

iii. the graph may be in several pieces.

Figure 3.1 shows a typical graph $\mathcal{G}_u$.

A **directed** graph $\mathcal{G}$ is a triple $(V(\mathcal{G}), E(\mathcal{G}), a_{\mathcal{G}})$ where $V(\mathcal{G}), E(\mathcal{G})$ are the vertex set and the edge set respectively and $a_{\mathcal{G}}$ associates with each edge an ordered

Figure 3.1: Undirected and Directed Graphs

pair of vertices not necessarily distinct (i.e., $a_{\mathcal{G}} : E(\mathcal{G}) \rightarrow V(\mathcal{G}) \times V(\mathcal{G})$). The first element of the ordered pair is the **positive end point** or **tail** of the arrow and the second element is the **negative end point** or **head** of the arrow. For selfloops, positive and negative endpoints are the same. Directed graphs are usually drawn as graphs with arrows in the edges. In Figure 3.1, $\mathcal{G}_d$ is a directed graph.
We say a vertex v and an edge e are **incident** on each other iff v is an end point of e. If e has end points u, v we say that u, v are **adjacent** to each other. Two edges e_1, e_2 are **adjacent** if they have a common end point. The **degree** of a vertex is the number of edges incident on it with selfloops counted twice.
A graph $\mathcal{G}_s$ is a **subgraph** of $\mathcal{G}$ iff $\mathcal{G}_s$ is a graph, $V(\mathcal{G}_s) \subseteq V(\mathcal{G}), E(\mathcal{G}_s) \subseteq E(\mathcal{G})$, and the endpoints of an edge in $\mathcal{G}_s$ are the same as its end points in $\mathcal{G}$.
Subgraph $\mathcal{G}_s$ is a **proper subgraph** of $\mathcal{G}$ iff it is a subgraph of $\mathcal{G}$ but not identical to it. The **subgraph** of $\mathcal{G}$ on V_1 is that subgraph of $\mathcal{G}$ which has V_1 as its vertex set and the set of edges of $\mathcal{G}$ with both end points in V_1 as the edge set. The **subgraph** of $\mathcal{G}$ on E_1 has $E_1 \subseteq E(\mathcal{G})$ as the edge set and the endpoints of edges in E_1 as the vertex set. If $\mathcal{G}$ is a directed graph the edges of a subgraph would retain the directions they had in $\mathcal{G}$ (i.e., they would have positive and negative end points as in $\mathcal{G}$).

Exercise 3.1 *(k) In any graph with no parallel edges (edges with the same end points) or selfloops show that the degree of some two vertices must be equal.*

Exercise 3.2 *(k) Show that*

i. the sum of the degrees of vertices of any graph is equal to twice the number of edges of the graph;

ii. the number of odd degree vertices in any graph must be even.

3.2.2 Connectedness

A **vertex edge alternating sequence** (**alternating sequence** for short) of a graph $\mathcal{G}$ is a sequence in which

i. vertices and edges of $\mathcal{G}$ alternate,

ii. the first and last elements are vertices and

iii. whenever a vertex and an edge occur as adjacent elements they are incident on each other in the graph.

Example: For the graph $\mathcal{G}_u$ in Figure 3.1, $(a, e_1, b, e_3, c, e_6, q, e_6, c, e_4, d)$ is an alternating sequence.
A **path** is a graph all of whose edges and vertices can be arranged in an alternating sequence without repetitions.
It can be seen that the degree of precisely two of the vertices of the path is one and the degree of all other vertices (if any) is two. The two vertices of degree one must appear at either end of any alternating sequence containing all nodes and edges of the path without repetition. They are called **terminal nodes**. The path is said to be **between** its terminal nodes. It is clear that there are only two such alternating sequences that we can associate with a path. Each is the reverse of the other. The two alternating sequences associated with the path in Figure 3.2 are (v_1, e_1, v_2, e_2, v_3, e_3, v_4) and (v_4, e_3, v_3, e_2, v_2, e_1, v_1).
We say 'go along the path from v_i to v_j' instead of 'construct the alternating sequence without repetitions having v_i as the first element and v_j as the last element'. Such sequences are constructed by considering the alternating sequence associated with the path in which v_i precedes v_j and taking the subsequence starting with v_i and ending with v_j.
A directed graph may be a path if it satisfies the above conditions. However, the term **strongly directed path** is used if the edges can be arranged in a sequence so that the negative end point of each edge, except the last is the positive end point of the succeeding edge.
A graph is said to be **connected** iff for any given pair of distinct vertices there exists a path subgraph between them. The path graph in Figure 3.2 is connected while the graph $\mathcal{G}_u$ in Figure 3.1 is disconnected.
A **connected component** of a graph $\mathcal{G}$ is a connected subgraph of $\mathcal{G}$ that is not

Figure 3.2: A Path Graph

Figure 3.3: A Circuit Graph and a Strongly Directed Circuit Graph

a proper subgraph of any connected subgraph of $\mathcal{G}$ (i.e., it is a maximal connected subgraph). Connected components correspond to 'pieces' of a disconnected graph.

Exercise 3.3 *(k) Let $\mathcal{G}$ be a connected graph. Show that there is a vertex such that if the vertex and all edges incident on it are removed the remaining graph is still connected.*

3.2.3 Circuits and Cutsets

A connected graph with each vertex having degree two is called a **circuit graph** or a **polygon graph**. ($\mathcal{G}_L$ in Figure 3.3 is a circuit graph). If $\mathcal{G}'$ is a circuit subgraph of $\mathcal{G}$ then $E(\mathcal{G}')$ is a **circuit** of $\mathcal{G}$. A single edged circuit is called a **selfloop**. Each of the following is a characteristic property of circuit graphs (i.e., each can be used to define the notion).
We omit the routine proofs.

i. A circuit graph has precisely two paths between any two of its vertices.

ii. If we start from any vertex v of a circuit graph and follow any path (i.e., follow an edge, reach an adjacent vertex, go along a new edge incident on that vertex and so on) the first vertex to be repeated would be v. Also during the traversal we would have encountered all vertices and edges of the circuit graph.

iii. Deletion of any edge (leaving the end points in place) of a circuit graph reduces it to a path.

Exercise 3.4 *Construct*

i. a graph with all vertices of degree 2 *that is not a circuit graph,*

ii. a non circuit graph which is made up of a path and an additional edge,

iii. a graph which has no circuits,

iv. a graph which has every edge as a circuit.

Exercise 3.5 *Prove*

Lemma 3.2.1 *(k) Deletion of an edge (leaving end points in place) of a circuit subgraph does not increase the number of connected components in the graph.*

Exercise 3.6 *Prove*

Theorem 3.2.1 *(k) A graph contains a circuit if it contains two distinct paths between some two of its vertices.*

Exercise 3.7 *Prove*

Theorem 3.2.2 *(k) A graph contains a circuit if every one of its vertices has degree* ≥ 2.

A set $T \subseteq E(\mathcal{G})$ is a **crossing edge set** of $\mathcal{G}$ if $V(\mathcal{G})$ can be partitioned into sets V_1, V_2 such that $T = \{e : e$ has an end point in V_1 and in $V_2\}$. (In Figure 3.4, C is a crossing edge set). We will call V_1, V_2 the **end vertex sets** of T. Observe that while end vertex sets uniquely determine a crossing edge set there may be more than one pair of end vertex sets consistent with a given crossing edge set. A crossing edge set that is minimal (i.e., does not properly contain another crossing edge set) is called a **cutset** or a **bond**. A single edged cutset is a **coloop**.

Exercise 3.8 *Construct a graph which has (a) no cutsets (b) every edge as a cutset.*

Exercise 3.9 *Construct a crossing edge set that is not a cutset (see Figure 3.4).*

Exercise 3.10 *(k) Show that a cutset is a minimal set of edges with the property that when it is deleted leaving endpoints in place the number of components of the graph increases.*

Figure 3.4: A Crossing Edge Set and a Strongly Directed Crossing Edge Set

Exercise 3.11 *Short (i.e., fuse end points of an edge and remove the edge) all branches of a graph except a cutset. How does the resulting graph look?*

Exercise 3.12 *Prove*

Theorem 3.2.3 *(k) A crossing edge set T is a cutset iff it satisfies the following:*

i. If the graph has more than one component then T must meet the edges of only one component.

ii. If the end vertex sets of T are V_1, V_2 in that component, then the subgraphs on V_1 and V_2 must be connected.

3.2.4 Trees and Forests

A graph that contains no circuits is called a **forest graph** (see graphs $\mathcal{G}_t$ and $\mathcal{G}_f$ in Figure 3.5). A connected forest graph is also called a **tree graph** (see graph $\mathcal{G}_t$ in Figure 3.5).

A **forest** of a graph $\mathcal{G}$ is the set of edges of a forest subgraph of $\mathcal{G}$ that has $V(\mathcal{G})$ as its vertex set and has as many connected components as $\mathcal{G}$ has. A forest of a connected graph $\mathcal{G}$ is also called a **tree** of $\mathcal{G}$. The complement relative to $E(\mathcal{G})$ of a forest (tree) is a **coforest (cotree)** of $\mathcal{G}$. The number of edges in a forest (coforest) of $\mathcal{G}$ is its **rank (nullity)**. Theorem 3.2.4 assures us that this notion is well defined.

Exercise 3.13 *(k) Show that a tree graph on two or more nodes has*

i. precisely one path between any two of its vertices

ii. at least two vertices of degree one.

Figure 3.5: A Tree Graph $\mathcal{G}_t$ and a Forest Graph $\mathcal{G}_f$

Exercise 3.14 *Prove*

Theorem 3.2.4 *(k) A tree graph on n nodes has $(n-1)$ branches. Any connected graph on n nodes with $(n-1)$ edges is a tree graph.*

Corollary 3.2.1 *The forest subgraph on n nodes and p components has $(n-p)$ edges.*

Exercise 3.15 *Prove*

Theorem 3.2.5 *(k) A subset of edges of a graph is a forest (coforest) iff it is a maximal subset not containing any circuit (cutset).*

Exercise 3.16 *(k) Show that every forest (coforest) of a graph $\mathcal{G}$ intersects every cutset (circuit) of $\mathcal{G}$.*

Exercise 3.17 *Prove*

Lemma 3.2.2 *(k) A tree graph splits into two tree graphs if an edge is opened (deleted leaving its end points in place).*

Exercise 3.18 *(k) Show that a tree graph yields another tree graph if an edge is shorted (removed after fusing its end points).*

Exercise 3.19 *Prove*

Theorem 3.2.6 *(k) Let f be a forest of a graph $\mathcal{G}$ and let e be an edge of $\mathcal{G}$ outside f. Then $e \cup f$ contains only one circuit of $\mathcal{G}$.*

Exercise 3.20 *Prove*

Theorem 3.2.7 *(k) Let $\overline{f}$ be a coforest of a graph $\mathcal{G}$ and let e be an edge of $\mathcal{G}$ outside $\overline{f}$ (i.e., $e \in f$). Then $e \cup \overline{f}$ contains only one cutset of $\mathcal{G}$ (i.e., only one cutset of $\mathcal{G}$ intersects f in e).*

Exercise 3.21 *(k) Show that every circuit is an f-circuit with respect to some forest (i.e., intersects some coforest in a single edge).*

Exercise 3.22 *(k) Show that every cutset is an f-cutset with respect to some forest (i.e., intersects some forest in a single edge).*

Exercise 3.23 *(k) Show that shorting an edge in a cutset of a graph does not reduce the nullity of the graph.*

3.2.5 Strongly Directedness

All the above definitions hold also in the case of directed graphs. The subgraphs in each case retain the original orientation for the edges. However, the prefix 'strongly directed' in each case implies a stronger condition. We have already spoken of the strongly directed path. A strongly directed circuit graph has its edges arranged in a sequence so that the negative end point of each edge is the positive end point of the succeeding edge and the positive end point of the last edge is the negative end point of the first (see $\mathcal{G}_{L_d}$ in Figure 3.3). The set of edges of such a graph would be a **strongly directed circuit**.

A **strongly directed crossing edge set** would have the positive end points of all its edges set in the same end vertex set (see C_d in Figure 3.4).

In this book we will invariably assume that the graph is directed but our circuit subgraphs, paths etc. although they are directed graphs, will, unless otherwise stated, not be strongly directed. When it is clear from the context the prefix 'directed' will be omitted when we speak of a graph. For simplicity we would write directed path, directed circuit, directed crossing edge set instead of strongly directed path etc.

Exercise 3.24 *Prove:*
(Minty) Any edge of a directed graph is either in a directed circuit or in a directed cutset but not both.

(For solution see Theorem 3.4.7).

3.2.6 Fundamental Circuits and Cutsets

Let f be a forest of $\mathcal{G}$ and let $e \notin f$. It can be shown (Theorem 3.2.6) that there is a unique circuit contained in $e \cup f$. This circuit is called the **fundamental circuit (f - circuit) of e with respect to f** and is denoted by $L(e, f)$. Let $e_t \in f$. It can be shown (Theorem 3.2.7) that there is a unique cutset contained in $e_t \cup \bar{f}$. This cutset is called the **fundamental cutset of e_t with respect to f** and is denoted by $B(e_t, f)$.

Remark: The f-circuit $L(e, f)$ is obtained by adding e to the unique path in the forest subgraph on f between the end points of e. For the subgraph on f, the edge e_t is a crossing edge set with end vertex sets say V_1, V_2. Then the f-cutset $B(e_t, f)$ is the crossing edge set of $\mathcal{G}$ with end vertex sets V_1, V_2.

Figure 3.6: Circuit subgraph and Crossing Edge Set with Orientations

3.2.7 Orientation

Let $\mathcal{G}$ be a directed graph. We associate **orientations** with circuit subgraphs and crossing edge sets as follows:

An **orientation** of a circuit subgraph is an alternating sequence of its vertices and edges, without repetitions except for the first vertex being also the last (note that each edge is incident on the preceding and succeeding vertices). Two orientations are **equivalent** if one can be obtained by a **cyclic** shift of the other. Diagrammatically an orientation may be represented by a circular arrow. It is easily seen that there can be at most two orientations for a circuit graph. (A single edge circuit subgraph has only one). These are obtained from each other by reversing the sequence. When there are two non equivalent orientations we call them **opposite** to each other. We say that an edge of the circuit subgraph **agrees** with the **orientation** if its positive end point immediately precedes itself in the orientation (or in an equivalent orientation). Otherwise it is opposite to the orientation. The orientation associated with a circuit subgraph would also be called the **orientation** of the **circuit**.

Example: For the circuit subgraph of Figure 3.6 the orientations $(n_1, e,\ n_6,\ e_6,\ n_5,\ e_5,\ n_4,\ e_4,\ n_3,\ e_3,\ n_2,\ e_2,\ n_1)$, and $(n_6, e_6, n_5, e_5, n_4, e_4, n_3, e_3, n_2, e_2, n_1, e, n_6)$ are equivalent. This is the orientation shown in the figure. It is opposite to the orientation $(n_1, e_2, n_2, e_3, n_3, e_4, n_4, e_5, n_5, e_6, n_6, e, n_1)$. The edge e **agrees** with this latter orientation and is **opposite** to the former orientation.

An **orientation** of a crossing edge set is an ordering of its end vertex sets V_1, V_2 as (V_1, V_2) or as (V_2, V_1). An edge e in the crossing edge set with positive end point in V_1 and negative end point in V_2 **agrees** with the orientation (V_1, V_2) and is **opposite** to the orientation (V_2, V_1). In Figure 3.6 the orientation of the crossing edge set is (V_1, V_2).

Theorem 3.2.8 *(k) Let f be a forest of a directed graph $\mathcal{G}$. Let $e_t \in f$ and let $e_c \in \overline{f}$. Let the orientation of $L(e_c, f)$ and $B(e_t, f)$ agree with e_c, e_t, respectively.*

Figure 3.7: Relation between f-circuit and f-cutset

Then $L(e_c, f) \cap B(e_t, f) = \emptyset$ *or* $\{e_c, e_t\}$.
Further when the intersection is nonvoid e_t *agrees with (opposes) the orientation of* $L(e_c, f)$ *iff* e_c *opposes (agrees with) the orientation of* $B(e_t, f)$.

Proof : We confine ourselves to the case where $\mathcal{G}$ is connected since even if it is disconnected we could concentrate on the component where e_t is present.
If $B(e_t, f)$ is deleted from $\mathcal{G}$, two connected subgraphs $\mathcal{G}_1, \mathcal{G}_2$ result whose vertex sets are the end vertex sets V_1, V_2, respectively of $B(e_t, f)$.
Now e_c could have both end points in V_1 both end points in V_2 or one end point in V_1 and another in V_2. In the former two cases $L(e_c, f) \cap B(e_t, f) = \emptyset$. In the last case $L(e_c, f)$ must contain e_t. For, the path in the tree subgraph on f between the endpoints of e_c must use e_t since that is the only edge in f with one endpoint in V_1 and the other in V_2. Now $L(e_c, f)$ contains only one edge namely e_c from $\overline{f}$ and $B(e_t, f)$ contains only one edge namely e_t from f. Hence in the third case

$$L(e_c, f) \cap B(e_t, f) = \{e_c, e_t\}.$$

Let us next assume that the intersection is nonvoid. Suppose that e_c has its positive end point a in V_1 and negative end point b in V_2. Let $(b, \cdots, e_t, \cdots, a, e_c, b)$ be an orientation of the circuit. It is clear that e_t would agree with this orientation if V_2 contains its positive end point and V_1 its negative end point (see Figure 3.7). But in that case e_c would oppose the orientation of $B(e_t, f)$ (which is (V_2, V_1), taken to agree with the orientation of e_t). The other cases can be handled similarly.

3.2.8 Isomorphism

Let $\mathcal{G}_1 \equiv (V_1, E_1, i_1)$, $\mathcal{G}_2 \equiv (V_2, E_2, i_2)$, be two graphs. We say $\mathcal{G}_1$, $\mathcal{G}_2$ are **identical** iff $V_1 = V_2, E_1 = E_2$ and $i_1 = i_2$. However, graphs could be treated as essentially the same even if they satisfy weaker conditions. We say $\mathcal{G}_1$, $\mathcal{G}_2$ are **isomorphic** to each other and denote it by (abusing notation) $\mathcal{G}_1 = \mathcal{G}_2$ iff there exist bijections

Figure 3.8: Isomorphic Directed Graphs

Figure 3.9: Cyclically Connected and Cyclically Disconnected Graphs

$\eta : V_1 \rightarrow V_2$ and $\epsilon : E_1 \rightarrow E_2$ s.t. any edge e has end points a, b in $\mathcal{G}_1$ iff $\epsilon(e)$ has endpoints $\eta(a), \eta(b)$. If $\mathcal{G}_1, \mathcal{G}_2$ are directed graphs then we would further require that an end point a of e, in $\mathcal{G}_1$, is positive (negative) iff $\eta(a)$ is the positive (negative) endpoint of $\epsilon(e)$. When we write $\mathcal{G}_1 = \mathcal{G}_2$ usually the bijections would be clear from the context. However, when two graphs are isomorphic there would be many **isomorphisms** ((η, ϵ) pairs) between them. The graphs $\mathcal{G}, \mathcal{G}'$ in Figure 3.8 are isomorphic. The node and edge bijections are specified by the ('). Clearly there is at least one other (η, ϵ) pair between the graphs.

3.2.9 Cyclically connectedness

A graph $\mathcal{G}$ is said to be **cyclically connected** iff given any pair of vertices there is a circuit subgraph containing them.

Example: The graph $\mathcal{G}_1$ in Figure 3.9 is cyclically connected while $\mathcal{G}_2$ of the same

figure is not cyclically connected since no circuit subgraph contains both nodes a and b.
Whenever a connected graph is not cyclically connected there would be two vertices a, b through which no circuit subgraph passes. If a, b are not joined by an edge there would be a vertex c such that every path between a and b passes through c. We then say c is a **cut vertex** or **hinge**. The graph $\mathcal{G}_2$ of Figure 3.9 has c as a cut vertex.
It can be shown that a graph is cyclically connected iff any pair of edges can be included in the same circuit.

In any graph it can be shown that if edges e_1, e_2 and e_2, e_3 belong to circuits C_{12}, C_{23}, then there exists a circuit $C_{13} \subseteq C_{12} \cup C_{23}$ s.t. $e_1, e_3 \in C_{13}$. It follows that the edges of a graph can be partitioned into blocks such that within each block every pair of distinct edges can be included in some circuit and edges belonging to different blocks cannot be included in the same circuit (each coloop would form a block by itself). We will call such a block an **elementary separator** of the graph. Unions of such blocks will be called **separators**. The subgraphs on elementary separators will be called **2-connected components**.(Note that a coloop is a 2-connected component by itself). If two 2-connected components intersect they would do so at a single vertex which would be a cut vertex. If two graphs have a single common vertex, we would say that they are put together by **hinging**.

3.3 Graphs and Vector Spaces

There are several natural 'electrical' vectors that one may associate with the vertex and edge sets of a directed graph $\mathcal{G}$.

e.g.
- i. potential vectors on the vertex set,
- ii. current vectors on the edge set,
- iii. voltage (potential difference) vectors on the edge set.

Our concern will be with the latter two examples. We need a few preliminary definitions. Henceforth, unless otherwise specified, by **graph** we mean **directed graph**.

The Incidence Matrix

The **incidence matrix A** of a graph $\mathcal{G}$ is defined as follows:
A has one row for each node and one column for each edge.

$\mathbf{A}(i, j) \quad = \quad +1(-1)$ if edge j has its arrow leaving (entering) node i.
0 if edge j is not incident on node i
or if edge j is a selfloop.

Example: The incidence matrix of the directed graph $\mathcal{G}_d$ in Figure 3.1 is

$$
\mathbf{A} = \begin{array}{c} \\ a \\ b \\ c \\ d \\ \hline f \\ \hline g \\ \hline h \end{array}
\begin{array}{c}
\begin{array}{cccccc|cc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \end{array} \\
\left[\begin{array}{cccccc|cc}
+1 & +1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & +1 & +1 & 0 & 0 & 0 & 0 \\
0 & -1 & -1 & 0 & +1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 \\ \hline
0 & 0 & 0 & 0 & 0 & 0 & +1 & +1 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 \\ \hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right]
\end{array}
\tag{3.1}
$$

Note that the selfloop e_6 is represented by a zero column. This is essential for mathematical convenience. The resulting loss of information (as to which node it is incident at) is electrically unimportant. The isolated node h corresponds to a zero row. Since the graph is disconnected the columns and rows can be ordered so that the block diagonal nature of the incidence matrix is evident.

Exercise 3.25 *(k) Prove:*
A matrix $\mathbf{K}$ *is the incidence matrix of some graph* $\mathcal{G}$ *iff it is a* $0, \pm 1$ *matrix and has either zero columns or columns with one* $+1$ *and one* -1 *and remaining entries* 0.

Exercise 3.26 *(k) Prove:*
The sum of the rows of $\mathbf{A}$ *is* $\mathbf{0}$. *Hence the rank of* $\mathbf{A}$ *is less than or equal to the number of its rows minus* 1.

Exercise 3.27 *(k) Prove:*
If the graph is disconnected the sum of the rows of $\mathbf{A}$ *corresponding to any component would add up to* $\mathbf{0}$. *Hence, the rank of* $\mathbf{A}$ *is less than or equal to the number of its rows less the number of components* $(= r(\mathcal{G}))$.

Exercise 3.28 *(k) Prove:*
If $\mathbf{f} = \lambda^{\mathbf{T}}\mathbf{A}$, *then* $\mathbf{f}(e_i) = \lambda(a) - \lambda(b)$ *where* a *is the positive end point of* e_i *and* b, *its negative end point. Thus if* λ *represents a potential vector with* $\lambda(n)$ *denoting the potential at* n *then* $\mathbf{f}$ *represents the corresponding potential difference vector.*

Exercise 3.29 *Construct incidence matrices of various types of graphs e.g. connected, disconnected, tree, circuit, complete graph* K_n *(every pair of* n *vertices joined by an edge), path.*

Exercise 3.30 *Show that the transpose of the incidence matrix of a circuit graph, in which all edges are directed along the orientation of the circuit, is a matrix of the same kind.*

Exercise 3.31 *(k) Show that an incidence matrix remains an incidence matrix under the following operations:*

i. deletion of a subset of the columns,

ii. replacing some rows by their sum.

3.3.1 The Circuit and Crossing Edge Vectors

A **circuit vector** of a graph $\mathcal{G}$ is a vector $\mathbf{f}$ on $E(\mathcal{G})$ corresponding to a circuit of $\mathcal{G}$ with a specified orientation:

$$\begin{aligned} f(e_i) &= +1(-1) \text{ if } e_i \text{ is in the circuit and agrees} \\ &\quad\ \text{with (opposes) the orientation of the circuit.} \\ &= 0 \text{ if } e_i \text{ is not in the circuit.} \end{aligned}$$

Example: The circuit vector associated with the circuit subgraph in Figure 3.6

$$\begin{array}{cccccccccc} & e & e_2 & e_3 & e_4 & e_5 & e_6 & & & \\ f = [& -1 & +1 & -1 & +1 & -1 & +1 & 0 & \ldots & 0\ ] \end{array} \tag{3.2}$$

Exercise 3.32 *(k) Compare a circuit vector with a row of the incidence matrix. Prove:*
A row of the incidence matrix and a circuit vector will

i. *have no nonzero entries common if the corresponding node is not present in the circuit subgraph, or*

ii. *have exactly two nonzero entries common if the node is present in the circuit subgraph. These entries would be ± 1. One of these entries would have opposite sign in the incidence matrix row and the circuit vector and the other entry would be the same in both.*

Exercise 3.33 *Prove*

Theorem 3.3.1 *(k) Every circuit vector of a graph $\mathcal{G}$ is orthogonal to any row of the incidence matrix of $\mathcal{G}$.*

(This follows immediately from the statement of the previous exercise).
A **crossing edge vector** of a graph $\mathcal{G}$ is a vector $\mathbf{f}$ on $E(\mathcal{G})$ corresponding to a crossing edge set with a specified orientation (V_1, V_2):

$$\begin{aligned} \mathbf{f}(e_i) &= +1(-1) \text{ if } e_i \text{ is in the crossing edge set and agrees} \\ &\quad\ \text{with (opposes) the orientation } (V_1, V_2). \\ &= 0 \text{ if } e_i \text{ is not in the crossing edge set.} \end{aligned}$$

If the crossing edge set is a cutset then the corresponding vector is a **cutset vector**.

Example: The crossing edge vector associated with the crossing edge set of Figure 3.6 is

$$\begin{array}{ccccccccc} & e & e_2 & e_3 & e_4 & & & \\ f = [& +1 & -1 & +1 & +1 & 0 & \cdots & 0\ ]. \end{array} \tag{3.3}$$

Exercise 3.34 *Prove*

Theorem 3.3.2 *(k) The crossing edge vector corresponding to the crossing edge set of orientation (V_1, V_2) is obtained by summing the rows of the incidence matrix corresponding to the nodes in V_1.*

Hence, a crossing edge vector of $\mathcal{G}$ is a voltage vector and is orthogonal to every circuit vector of $\mathcal{G}$. (This can also be proved directly).

Exercise 3.35 *(k) When is a row of the incidence matrix also a cutset vector? Can a cutset be a circuit? Can a cutset vector be a circuit vector?*

Exercise 3.36 *(k)* **RRE of an Incidence Matrix:**
Give a simple rule for finding the RRE of an incidence matrix.

3.3.2 Voltage and Current Vectors

For a graph $\mathcal{G}$ a **current vector i** is a vector on $E(\mathcal{G})$ that is orthogonal to the rows of the incidence matrix of $\mathcal{G}$, **equivalently**, that satisfies Kirchhoff's current equations (KCE): $\mathbf{Ax} = \mathbf{0}$ [Kirchhoff1847].
A **voltage vector v** of $\mathcal{G}$ is a vector on $E(\mathcal{G})$ that is linearly dependent on the rows of the incidence matrix of $\mathcal{G}$ i.e.
$\mathbf{v^T} = \lambda^\mathbf{T}\mathbf{A}$ for some vector λ.
The vector λ assigns a value to each node of $\mathcal{G}$ and is called a **potential vector**. We say v is **derived** from the node potential vector λ.
Voltage vectors and current vectors form vector spaces denoted by $\mathcal{V}_v(\mathcal{G}), \mathcal{V}_i(\mathcal{G})$, and called voltage space of $\mathcal{G}$ and current space of $\mathcal{G}$ respectively.

Exercise 3.37 *Prove*
*(***Tellegen's Theorem (weak form)***) Any voltage vector of $\mathcal{G}$ is orthogonal to every current vector of $\mathcal{G}$.*

Remark: When the graph is disconnected with components $\mathcal{G}_1 \ldots \mathcal{G}_p$ it is clear that both the current and voltage space can be written in the form $\oplus_{i=1}^{p}\mathcal{V}(\mathcal{G}_i)$. However, in order to write the space in this decomposed form it is not necessary that the $\mathcal{G}_i$ be disconnected. All that is required is that there be no circuit containing edges from different $\mathcal{G}_i$ (see the discussion on separators). We say that graphs $\mathcal{G}_1, \mathcal{G}_2$ are **2-isomorphic** and denote it by $\mathcal{G}_1 \cong \mathcal{G}_2$ iff there exists a bijection $\in$: $E(\mathcal{G}_1) \to E(\mathcal{G}_2)$ through which an edge in $\mathcal{G}_1$ can be identified with an edge in $\mathcal{G}_2$ so that $\mathcal{V}_v(\mathcal{G}_1) = \mathcal{V}_v(\mathcal{G}_2)$.
Whitney [Whitney33c] has shown that two 2-isomorphic graphs can be made isomorphic through the repeated use of the following operations:

i. Decompose the graphs into their 2-connected components.

ii. Divide one of the graphs into two subgraphs $\mathcal{G}'$ and $\mathcal{G}$" which have precisely two vertices, say a and b, in common. Split the nodes into a_1, a_2 and b_1, b_2 so that the two subgraphs are now disconnected with a_1, b_1, belonging to $\mathcal{G}'$ and a_2, b_2 to $\mathcal{G}$". Let $\mathcal{G}'_e$ be the graph obtained from $\mathcal{G}'$ by adding an edge e between a_1, b_1. Now reverse all arrows of edges of $\mathcal{G}'$ which lie in the 2-connected component containing e in $\mathcal{G}'_e$ and attach a_1 to b_2 and a_2 to b_1.

If $\mathbf{c}$ is a circuit vector corresponding to the circuit C with an orientation then the **Kirchhoff's Voltage Equation** (KVE) [Kirchhoff1847] corresponding to C is

$$\mathbf{c}^T\mathbf{x} = 0$$

We have the following basic characterization of voltage vectors:

Theorem 3.3.3 *(k) A vector* $\mathbf{v}$ *on* $E(\mathcal{G})$ *is a voltage vector iff it satisfies KVE corresponding to each circuit with an orientation.*

Proof : By Theorem 3.3.1 we know that a circuit vector is orthogonal to every row of the incidence matrix. Hence, a circuit vector is orthogonal to any vector that is linearly dependent on the rows of the incidence matrix i.e. orthogonal to a voltage vector. Hence, every voltage vector satisfies KVE corresponding to any circuit with orientation.
Now let $\mathbf{v}$ be a vector that satisfies KVE corresponding to every circuit with an orientation. We will construct a potential vector λ s.t. $\lambda^{\mathbf{T}}\mathbf{A} = \mathbf{v}^{\mathbf{T}}$. Take any node d as the datum node, i.e., $\lambda(d) \equiv 0$. Suppose $\lambda(a)$ is already defined and edge e has a as the positive (negative) end and b as the opposite end. Then we take $\lambda(b) \equiv \lambda(a) - v(e)(\lambda(b) \equiv \lambda(a) + v(e))$. In this manner every node in the same connected component is assigned a λ value. A node that is reachable from d by two different paths will not be assigned two different values as otherwise we can find a circuit with orientation for which KVE is violated. Repeating this procedure for each component yields a λ vector s.t. $\lambda^T\mathbf{A} = \mathbf{v}^{\mathbf{T}}$.

□

3.3.3 Voltage and Current Vector Spaces and Tellegen's Theorem

In this subsection, we compute the rank of $\mathcal{V}_v(\mathcal{G})$ and $\mathcal{V}_i(\mathcal{G})$ and prove that the spaces are complementary orthogonal (**Tellegen's Theorem (strong form)**).

Theorem 3.3.4 *(k) Let G be a graph on n nodes with p connected components. Then*

i. $r(\mathcal{V}_v(\mathcal{G})) = n - p$

ii. Any set of $(n-p)$ *rows of* $\mathbf{A}$ *which omits one row per component of* $\mathcal{G}$ *is a basis of* $\mathcal{V}_v(\mathcal{G})$.

Proof :
i. If $\mathcal{G}$ is made up of p connected components, by (if necessary) rearranging the rows and columns of $\mathbf{A}$ it can be put in the block diagonal form with p blocks.

Hence,

$$r(\mathbf{A}) = \sum_{i=1}^{p} r(\mathbf{A}_i)$$

where $\mathbf{A}_i$ is the incidence matrix of the i^{th} component (Equivalently $\mathcal{V}_v(\mathcal{G}) = \oplus_{i=1}^{p}\mathcal{V}_v(\mathcal{G}_i)$, where $\mathcal{G}_i$ is the i^{th} connected component). If $r(A_i) = n_i - 1, i = 1, \ldots, p$, where n_i is the number of nodes of $\mathcal{G}_i$, we would be done. So let us without loss of generality assume that $\mathcal{G}$ is connected and attempt to show that $r(\mathbf{A}) = n-1$. We select any $(n-1)$ rows of $\mathbf{A}$. Suppose these are linearly dependent. Then there is a non trivial linear combination of these rows which is a zero vector. From this set of rows we omit all the rows which are being multiplied by zeros. The remaining set of rows is nonvoid. Consider the corresponding set of vertices say V_1. This set does not contain all vertices of the graph. Since the graph is connected there must be an edge e with one end point in V_1 and the other outside. The submatrix of $\mathbf{A}$ with rows V_1 has only one nonzero entry in the column e. Hence, by multiplying these rows by nonzero scalars and adding we cannot get a zero row. This contradiction shows that any $(n-1)$ rows of $\mathbf{A}$ must be linearly independent. Since the sum of rows of $\mathbf{A}$ is a zero vector we know already that $r(\mathbf{A}) \leq n-1$. Hence $r(\mathbf{A}) = n-1$.

ii. This is immediate from the above proof noting that the rows of $\mathbf{A}$ span $\mathcal{V}_v(\mathcal{G})$.

□

A **reduced incidence matrix** $\mathbf{A}_r$ of a graph $\mathcal{G}$ is obtained by omitting one row belonging to each component of $\mathcal{G}$.
We know by Theorem 3.3.4 that the reduced incidence matrix is a representative matrix for $\mathcal{V}_v(\mathcal{G})$. A standard representative matrix for $\mathcal{V}_v(\mathcal{G})$ may be built as described below.

3.3.4 Fundamental cutset matrix of a forest f

We know by Theorem 3.2.7 that there is a unique cutset of a graph $\mathcal{G}$ that intersects a forest f in an edge e. This we have called the fundamental cutset of e with respect to f and denoted it by $B(e,f)$. We assign this cutset an orientation agreeing with that of e. Let $e_1, e_2, \ldots, e_r$ be the edges in the forest f and let $\mathbf{v}_1, \ldots, \mathbf{v}_r$ be the corresponding cutset vectors. A matrix which has $\mathbf{v}_1, \ldots, \mathbf{v}_r$ as rows is called the **fundamental cutset matrix $\mathbf{Q}_f$** of f. This matrix is unique within permutation of rows and columns. By reordering rows and columns, if required, this matrix can be cast in the form

$$\mathbf{Q}_f \equiv \begin{array}{c} \begin{array}{cc} \bar{f} & f \end{array} \\ \left[\begin{array}{cc} \mathbf{Q}_{11} & \mathbf{I} \end{array}\right] \end{array} \tag{3.4}$$

It is clear that $\mathbf{Q}_f$ has $\mid f \mid (= (n-p))$ rows which are linearly independent. Since a cutset vector is linearly dependent on the rows of the incidence matrix $\mathbf{A}$ (Theorem 3.3.2) and $r(\mathbf{A}) = n-p$, it follows that $\mathbf{Q}_f$ is a standard representative matrix for $\mathcal{V}_v(\mathcal{G})$.

Example: Consider the graph of Figure 3.10.
Let $f \equiv \{e_3\ e_4\ e_5\ e_6\ e_7\}$ and let $\bar{f} = \{e_1 \quad e_2\}$.

Figure 3.10: f-cutsets and f-circuits

$$
\mathbf{Q}_f = \begin{array}{c} \begin{array}{ccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \end{array} \\ \left[\begin{array}{ccccccc} 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \end{array} \tag{3.5}
$$

3.3.5 Fundamental circuit matrix of a forest f

We have already seen that addition of an edge e to a forest f creates a unique circuit which we have called the fundamental circuit of e with respect to f denoted by $L(e, f)$. As before we assign this circuit an orientation agreeing with that of e. Let $e_1, \cdots, e_\nu$ be edges in the coforest $\bar{f}$. Let $\mathbf{c}_1, \cdots, \mathbf{c}_\nu$ be the corresponding circuit vectors. A matrix with these vectors as rows is called the **fundamental circuit matrix** $\mathbf{B}_f$ of f. This matrix is unique within permutation of rows and columns. By reordering rows and columns, if required, this matrix can be cast in the form

$$
\begin{array}{ccc} & & \begin{array}{cc} \bar{f} & f \end{array} \\ \mathbf{B}_f & \equiv & [\mathbf{I} \quad \mathbf{B}_{12}] \end{array}
$$

It is clear that $\mathbf{B}_f$ has $\mid \bar{f} \mid$ rows which are linearly independent. Since a circuit vector is orthogonal to all the rows of the incidence matrix, it must be a current vector. Thus rows of $\mathbf{B}_f$ are current vectors.

Example: Consider the graph in Figure 3.10. Here $f \equiv \{e_3,\ e_4,\ e_5,\ e_6,\ e_7\}$ and $\bar{f} \equiv \{e_1,\ e_2\}$.

$$\begin{array}{cc} & \begin{array}{ccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \end{array} \\ \mathbf{B}_f = & \left[\begin{array}{ccccccc} 1 & 0 & 0 & -1 & +1 & 0 & -1 \\ 0 & 1 & -1 & 0 & +1 & -1 & 0 \end{array}\right]. \end{array} \tag{3.6}$$

Theorem 3.3.5 *(k) Let $\mathcal{G}$ be a graph on e edges, n nodes and p connected components. Then*
(a) $r(\mathcal{V}_i(\mathcal{G})) = e - n + p$
(b) (**Tellegen's Theorem (strong form)***) $(\mathcal{V}_v(\mathcal{G}))^{\perp} = \mathcal{V}_i(\mathcal{G})$.*

Proof : The rows of a fundamental circuit matrix are current vectors and $e-n+p$ in number. Hence, $r(\mathcal{V}_i(\mathcal{G})) \geq e - n + p$.
On the other hand every voltage vector is orthogonal to every current vector since a voltage vector is linearly dependent on the rows of $\mathbf{A}$ while a current vector is orthogonal to these rows. Thus, $(\mathcal{V}_v(\mathcal{G}))^{\perp} \supseteq \mathcal{V}_i(\mathcal{G})$.
By Theorem 2.2.5, $r(\mathcal{V}_v(\mathcal{G})) + r(\mathcal{V}_v(\mathcal{G}))^{\perp} = e$
We have already seen that $r(\mathcal{V}_v(\mathcal{G})) = n - p$. Hence $r(\mathcal{V}_v(\mathcal{G}))^{\perp} = e - n + p$ and $r(\mathcal{V}_i(\mathcal{G})) \leq e - n + p$ We conclude that $r(\mathcal{V}_i(\mathcal{G})) = e - n + p$ and thus $\mathcal{V}_i(\mathcal{G}) = (\mathcal{V}_v(\mathcal{G}))^{\perp}$.

□

Corollary 3.3.1 *(k) The rows of an f-circuit matrix of a graph $\mathcal{G}$ form a basis for the current space of $\mathcal{G}$.*

Exercise 3.38 *(k) Examine which potential vectors correspond to a zero voltage vector.*

Exercise 3.39 *Consider the column space $\mathcal{C}(\mathbf{A})$ of $\mathbf{A}$. Show that $(\mathcal{C}(\mathbf{A}))^{\perp}$ is one dimensional if the graph is connected. Hence show that $r(\mathbf{A}) = n - 1$.*

Exercise 3.40 *(k) The following is another proof for '$r(\mathbf{A}) = n - 1$ if the graph is connected'. If the graph is connected $r(\mathbf{A}) \leq n - 1$ since the sum of the rows is zero. But $\mathbf{Q}_f$ has $n - 1$ independent rows which are linear combinations of rows of $\mathbf{A}$. Hence $r(\mathbf{A}) = n - 1$.*

Exercise 3.41 *An* **elementary vector** *of a vector space is a nonzero vector with minimal support (subset on which it takes nonzero values).*
Prove

Theorem 3.3.6 *(k) The circuit vector (cutset vector) is an elementary current vector (elementary voltage vector) and every elementary current vector (elementary voltage vector) is a scalar multiple of a circuit vector (cutset vector).*

Exercise 3.42 *Prove*

Theorem 3.3.7 *(k) A set of columns of $\mathbf{A}$ are linearly independent iff the corresponding edges of the graph do not contain a circuit. A set of columns of $\mathbf{B}_f$ are linearly independent iff the corresponding edges of the graph do not contain a cutset.*

Exercise 3.43 *(k) Every standard representative matrix of $\mathcal{V}_v(\mathcal{G})$ (standard representative matrix of $\mathcal{V}_i(\mathcal{G})$) is a fundamental cutset (fundamental circuit) matrix of $\mathcal{G}$.*

Exercise 3.44 An alternative proof of the strong form of Tellegen's Theorem:
(k) Let $\mathbf{B}_f$, $\mathbf{Q}_f$ be the f-circuit and f-cutset matrix with respect to the same forest. Prove:

i. $\mathbf{B}_f^T \mathbf{Q}_f = \mathbf{0}$

ii. If $\mathbf{B}_f = [\mathbf{I}\ \mathbf{B_{12}}]$ *then* $\mathbf{Q}_f = [-\mathbf{B_{12}^T}\ \mathbf{I}]$. *(Note that this implies Theorem 3.2.8).*

iii. Rows of $\mathbf{B}_f$, $\mathbf{Q}_f$ are current vectors (voltage vectors). Their ranks add upto $e(=\mid E(\mathcal{G}) \mid)$. Hence, $(\mathcal{V}_i(\mathcal{G}))^{\perp} = \mathcal{V}_v(\mathcal{G})$.

Exercise 3.45 *(k) Prove*

Theorem 3.3.8 *(k) The maximum number of independent KVE for a graph is $r(\mathcal{V}_i(\mathcal{G}))$ $(= e - n + p)$.*

3.4 Basic Operations on Graphs and Vector Spaces

In this section, we discuss basic operations on graphs (directed and undirected) which correspond to open circuiting some edges and short circuiting some others. These operations are related to two vector space operations: restriction and contraction. Since real vector spaces are associated primarily with directed graphs, henceforth we deal only with such graphs, but, omit the adjective 'directed'.

3.4.1 Restriction and Contraction of Graphs

Let $\mathcal{G}$ be a graph on the set of edges E and let $T \subseteq E$.

Definition 3.4.1 *The graph* $\mathcal{G}$**open**$(\mathbf{E}-\mathbf{T})$ *is the subgraph of $\mathcal{G}$ with T as the edge set and $V(\mathcal{G})$ as the vertex set. Thus $\mathcal{G}open(E-T)$ is obtained by removing (deleting) edges in $E-T$ leaving their end points in place.*
The **restriction** *of $\mathcal{G}$ to T, denoted by* $\mathcal{G} \cdot \mathbf{T}$, *is the subgraph of $\mathcal{G}$ obtained by deleting isolated vertices from $\mathcal{G}open(E-T)$. Thus, $\mathcal{G} \cdot T$ is the subgraph of $\mathcal{G}$ on T.*
If $\mathcal{G}$ is directed, $\mathcal{G}open(E-T), \mathcal{G} \cdot T$, would be directed with edges retaining the original directions they had in $\mathcal{G}$.

Definition 3.4.2 *The graph* $\mathcal{G}$**short** $(\mathbf{E}-\mathbf{T})$ *is built by first building $\mathcal{G}openT$. Let $V_1, \cdots, V_k$ be the vertex sets of the connected components of $\mathcal{G}openT$. The set $\{V_1, \cdots, V_k\}$ is the vertex set and T is the edge set of $\mathcal{G}short$ $(E-T)$. An edge $e \in T$ would have V_i, V_j as its end points in $\mathcal{G}short$ $(E-T)$ iff the end points of e in $\mathcal{G}$ lie in V_i, V_j. If $\mathcal{G}$ is directed, V_i, V_j would be the positive and negative endpoints*

of e in $\mathcal{G}short\ (E-T)$ *provided the positive and negative end points of e in* $\mathcal{G}$ *lie in* V_i, V_j *respectively.*
(Thus, $\mathcal{G}short\ (E-T)$ *is obtained from* $\mathcal{G}$ *by short circuiting the edges in* $(E-T)$ *(fusing their end points) and removing them).*
The **contraction** *of* $\mathcal{G}$ *to* T*, denoted by* $\mathcal{G} \times T$*, is obtained from* $\mathcal{G}short\ (E-T)$ *by deleting the isolated vertices of the latter.*

Example: Consider the graph $\mathcal{G}$ of Figure 3.11.
Let $T = \{e_1, e_6, e_{11}\}$. The graph $\mathcal{G}openT$ is shown in the figure. Graph $\mathcal{G} \cdot (E-T)$ is obtained by omitting isolated vertex v_1 from $\mathcal{G}openT$. Graph $\mathcal{G}short\ (E-T)$ is also shown in the same figure. Graph $\mathcal{G} \times T$ is obtained by omitting the isolated vertex $\{\ v_8, v_9\ \}$ from $\mathcal{G}short\ (E-T)$.

We denote $(\mathcal{G} \times T_1) \cdot T_2$, $T_2 \subseteq T_1 \subseteq E(\mathcal{G})$ by $\mathcal{G} \times T_1 \cdot T_2$ and $(\mathcal{G} \cdot T_1) \times T_2$. $T_2 \subseteq T_1 \subseteq E(\mathcal{G})$ by $\mathcal{G} \cdot T_1 \times T_2$. Graphs denoted by such expressions are called **minors** of $\mathcal{G}$. It can be seen that when a set $A \subseteq E(\mathcal{G})$ is being shorted and a disjoint set $B \subseteq E(\mathcal{G})$, is being opened then the final graph does not depend on the order in which these operations are carried out but only on the sets A and B. Now $\mathcal{G} \times T(\mathcal{G} \cdot T)$ differs from $\mathcal{G}short\ (E-T)$ $(\mathcal{G}open(E-T))$ only in that the isolated vertices are omitted. We thus have the following theorem where equality refers to isomorphism.

Theorem 3.4.1 *(k) Let* $\mathcal{G}$ *be a graph with* $T_2 \subseteq T_1 \subseteq E(G)$*. Then*

i. $\mathcal{G} \times T_1 \times T_2 = \mathcal{G} \times T_2$,

ii. $\mathcal{G} \cdot T_1 \cdot T_2 = \mathcal{G} \cdot T_2$,

iii. $\mathcal{G} \times T_1 \cdot T_2 = \mathcal{G} \cdot (E - (T_1 - T_2)) \times T_2$.

Proof : The theorem is immediate when we note that both graphs are obtained by shorting and opening the same sets. In (i) $E - T_2$ is shorted while in (ii) $E - T_2$ is opened. In (iii) $E - T_1$ is shorted and $T_1 - T_2$ is opened.

□

Exercise 3.46 *(k)* **Simplification of Expression for minors:**
Show that any minor of the form $\mathcal{G} \times T_1 \cdot T_2 \times T_3 \ldots T_n, T_1 \supseteq T_2 \supseteq \ldots \supseteq T_n$ *(the graph being obtained by starting from* $\mathcal{G}$ *and performing the operations from left to right in succession), can be simplified to a minor of the form* $\mathcal{G} \cdot T^{'} \times T_n$ *or* $\mathcal{G} \times T^{'} \cdot T_n$.

Exercise 3.47 *Train yourself to visualize* $\mathcal{G}_1 \equiv \mathcal{G}short\ (E-T)$ *(Put components of* $\mathcal{G}openT$ *inside surfaces which then become nodes of* $\mathcal{G}_1$*). How many components does it have? When would a branch of* $\mathcal{G}$ *become a selfloop of* $\mathcal{G}_1$*? When would a circuit free set of branches of* $\mathcal{G}$ *become dependent in* $\mathcal{G}_1$*?*

Exercise 3.48 Circuits of minors: *Prove*

Lemma 3.4.1 *(k)*

Figure 3.11: Minors of a Graph

i. *A subset C of T is a circuit of $\mathcal{G} \cdot T$ iff C is a circuit of $\mathcal{G}$.*

ii. *A subset C of T is circuit of $\mathcal{G} \times T$ iff C is a minimal intersection of circuits of $\mathcal{G}$ with T (equivalently,iff C is an intersection of a circuit of $\mathcal{G}$ with T but no proper subset of C is such an intersection).*

Exercise 3.49 *(k)* **Cutsets of minors:** *Prove*

Lemma 3.4.2 *(k)*

i. *A subset B of T is a cutset of $\mathcal{G} \cdot T$ iff it is a minimal intersection of cutsets of $\mathcal{G}$ with T.*

ii. *A subset B of T is a cutset of $\mathcal{G} \times T$ iff it is a cutset of $\mathcal{G}$.*

3.4.2 Restriction and Contraction of Vector Spaces

We now describe operations on vector spaces which are analogous to the operations of opening and shorting edges in a graph.
Let $\mathcal{V}$ be a vector space on S and let $T \subseteq S$.

Definition 3.4.3 *The* **restriction** *of $\mathcal{V}$ to T, denoted by $\mathcal{V}.T$, is the collection of vectors $\mathbf{f}_T$ where $\mathbf{f}_T$ is the restriction of some vector $\mathbf{f}$ of $\mathcal{V}$ to T.*
The **contraction** *of $\mathcal{V}$ to T, denoted by $\mathcal{V} \times T$, is the collection of vectors $\mathbf{f}'_T$ where $\mathbf{f}'_T$ is the restriction to T of some vector $\mathbf{f}$ of $\mathcal{V}$ such that $\mathbf{f}/(S-T) = \mathbf{0}$.*

It is easily seen that $\mathcal{V} \cdot T$, $\mathcal{V} \times T$ are vector spaces.

As in the case of graphs we denote $(\mathcal{V} \times T_1) \cdot T_2$ by $\mathcal{V} \times T_1 \cdot T_2$. Such expressions denote vector spaces which are called **minors** of $\mathcal{V}$. To bring out the analogy between graph minor and vector space minor operations we say we 'open' T when we restrict $\mathcal{V}$ to $(S-T)$ and say we 'short' T when we contract $\mathcal{V}$ to $(S-T)$.
It turns out that the order in which we open and short disjoint sets of elements is unimportant. More formally we have

Theorem 3.4.2 *(k) Let $T_2 \subseteq T_1 \subseteq S$. Then*

i. $\mathcal{V} \cdot T_1 \cdot T_2 = \mathcal{V} \cdot T_2$,

ii. $\mathcal{V} \times T_1 \times T_2 = \mathcal{V} \times T_2$,

iii. $\mathcal{V} \times T_1 \cdot T_2 = \mathcal{V} \cdot (S - (T_1 - T_2)) \times T_2$.

Proof of (iii): We show that a vector in the LHS belongs to a vector in the RHS.
Let $\mathbf{f}_{T_2} \in \mathcal{V} \times T_1 \cdot T_2$.
Then there exists a vector $\mathbf{f}_{T_1} \in \mathcal{V} \times T_1$ such that $\mathbf{f}_{T_1}/T_2 = \mathbf{f}_{T_2}$ and a vector $\mathbf{f} \in \mathcal{V}$ with $\mathbf{f}/(S-T_1) = \mathbf{0}$ such that $\mathbf{f}/T_1 = \mathbf{f}_{T_1}$.
Now let $\mathbf{f}'$ denote $\mathbf{f}/(S-(T_1-T_2))$.

Clearly $\mathbf{f}' \in \mathcal{V} \cdot (S-(T_1-T_2))$. Now $\mathbf{f}'/(S-T_1) = \mathbf{0}$.
Hence, $\mathbf{f}'/T_2 \in \mathcal{V} \cdot (S-(T_1-T_2)) \times T_2$.
Thus, $\mathcal{V} \times T_1 \cdot T_2 \subseteq \mathcal{V} \cdot (S-(T_1-T_2)) \times T_2$.
The reverse containment is similarly proved.

□

Remark: To see the proof of the above theorem quickly, observe that a typical vector of both LHS and RHS is obtained by restricting a vector of $\mathcal{V}$, that takes zero value on $S-T_1$, to T_2.

Exercise 3.50 *(k) Prove:*
Any minor of the form $\mathcal{V} \times T_1 \cdot T_2 \times T_3 \ldots T_n, T_1 \supseteq T_2 \supseteq \ldots \supseteq T_n$, *can be simplified to a minor of the form*

$$\mathcal{V} \cdot T' \times T_n \text{ or } \mathcal{V} \times T' \cdot T_n.$$

3.4.3 Vector Space Duality

We now relate the minors of $\mathcal{V}$ to the minors of $\mathcal{V}^{\perp}$. We remind the reader that $\hat{\mathcal{V}}^{\perp}$, the complementary orthogonal space of $\hat{\mathcal{V}}$ is defined to be on the same set as $\hat{\mathcal{V}}$. In the following results we see that the contraction (restriction) of a vector space corresponds to the restriction (contraction) of the orthogonal complement. We say that contraction and restriction are **(orthogonal) duals** of each other.

Theorem 3.4.3 *(k) Let $\mathcal{V}$ be a vector space on S and let $T \subseteq S$. Then,*

i. $(\mathcal{V} \cdot T)^{\perp} = \mathcal{V}^{\perp} \times T$.

ii. $(\mathcal{V} \times T)^{\perp} = \mathcal{V}^{\perp} \cdot T$.

Proof :
i. Let $\mathbf{g}_T \in (\mathcal{V} \cdot T)^{\perp}$. For any $\mathbf{f}$ on S let $\mathbf{f}_T$ denote $\mathbf{f}/T$. Now if $\mathbf{f} \in \mathcal{V}$, then $\mathbf{f}_T \in \mathcal{V} \cdot T$ and $< \mathbf{g}_T, \mathbf{f}_T > = 0$.
Let $\mathbf{g}$ on S be defined by $\mathbf{g}/T \equiv \mathbf{g}_T$, $\mathbf{g}/S-T \equiv \mathbf{0}$. If $\mathbf{f} \in \mathcal{V}$ we have

$$\begin{aligned} < \mathbf{f}, \mathbf{g} > &= < \mathbf{f}_T, \mathbf{g}_T > + < \mathbf{f}_{S-T}, \mathbf{g}_{S-T} > \\ &= 0 + < \mathbf{f}_{S-T}, \mathbf{0}_{S-T} > \\ &= 0. \end{aligned}$$

Thus $\mathbf{g} \in \mathcal{V}^{\perp}$ and therefore, $\mathbf{g}_T \in \mathcal{V}^{\perp} \times T$.Hence, $(\mathcal{V} \,.\, T)^{\perp} \subseteq \mathcal{V}^{\perp} \times T$.
Next let $\mathbf{g}_T \in \mathcal{V}^{\perp} \times T$.
Then there exists $\mathbf{g} \in \mathcal{V}^{\perp}$ s.t. $\mathbf{g}/S-T = \mathbf{0}$ and $\mathbf{g}/T = \mathbf{g}_T$.
Let $\mathbf{f}_T \in \mathcal{V} \cdot T$. There exists $\mathbf{f} \in \mathcal{V}$ s.t. $\mathbf{f}/T = \mathbf{f}_T$.
Now $0 = < \mathbf{f}, \mathbf{g} > = < \mathbf{f}_T, \mathbf{g}_T > + < \mathbf{f}_{S-T}, \mathbf{0}_{S-T} > = < \mathbf{f}_T, \mathbf{g}_T >$.
Hence, $\mathbf{g}_T \in (\mathcal{V} \cdot T)^{\perp}$.
We conclude that
$\mathcal{V}^{\perp} \times T \subseteq (\mathcal{V} \cdot T)^{\perp}$. This proves that $(\mathcal{V} \cdot T)^{\perp} = \mathcal{V}^{\perp} \times T$.

ii. We have $(\mathcal{V}^{\perp} \cdot T)^{\perp} = (\mathcal{V}^{\perp})^{\perp} \times T$.
By Theorem 2.2.5
$((\mathcal{V}^{\perp} \cdot T)^{\perp})^{\perp} = \mathcal{V}^{\perp} \cdot T$ and $(\mathcal{V}^{\perp})^{\perp} = \mathcal{V}$. Hence, $\mathcal{V}^{\perp} \cdot T = (\mathcal{V} \times T)^{\perp}$.

□

The following corollary is immediate.

Corollary 3.4.1 *(k)* $(\mathcal{V} \times P \cdot T)^{\perp} = \mathcal{V}^{\perp} \cdot P \times T, T \subseteq P \subseteq S$.

3.4.4 Relation between Graph Minors and Vector Space Minors

We now show that the analogy between vector space minors and graph minors is more substantial than hitherto indicated - in fact the minors of voltage and current spaces of a graph correspond to appropriate graph minors.

Theorem 3.4.4 *(k) Let $\mathcal{G}$ be a graph with edge set E. Let $T \subseteq E$. Then*

i. $\mathcal{V}_v(\mathcal{G} \cdot T) = (\mathcal{V}_v(\mathcal{G})) \cdot T$

ii. $\mathcal{V}_v(\mathcal{G} \times T) = (\mathcal{V}_v(\mathcal{G})) \times T$

Proof : We remind the reader that by definition a voltage vector $\mathbf{v}$ is a linear combination of the rows of the incidence matrix, the coefficients of the linear combination being given by the entries in a potential vector λ. We say $\mathbf{v}$ is derived from λ.
i. Let $\mathbf{v}_T \in \mathcal{V}_v(\mathcal{G} \cdot T)$
Now $\mathcal{V}_v(\mathcal{G} \cdot T) = \mathcal{V}_v(\mathcal{G}open(E - T))$.
Thus, $\mathbf{v}_T \in \mathcal{V}_v(\mathcal{G}open(E - T))$. The graph $\mathcal{G}open(E - T)$ has the same vertex set as $\mathcal{G}$ but the edges of $(E - T)$ have been removed.
Let $\mathbf{v}_T$ be derived from the potential vector λ of $\mathcal{G}open(E - T)$. Now for any edge $e \in T$, $\mathbf{v}_T(e) = \lambda(a) - \lambda(b)$, where a, b are the positive and negative end points of e. However, λ is also a potential vector of $\mathcal{G}$. Let the voltage vector $\mathbf{v}$ of $\mathcal{G}$ be derived from λ. For the edge $e \in T$, we have, as before, $\mathbf{v}(e) = \lambda(a) - \lambda(b)$. Thus, $\mathbf{v}_T = \mathbf{v}/T$ and therefore, $\mathbf{v}_T \in (\mathcal{V}_v(\mathcal{G})) \cdot T$. Hence $\mathcal{V}_v(\mathcal{G} \cdot T) \subseteq (\mathcal{V}_v(\mathcal{G})) \cdot T$.
The reverse containment is proved similarly.

ii. Let $\mathbf{v}_T \in \mathcal{V}_v(\mathcal{G} \times T)$. Now $\mathcal{V}_v(\mathcal{G} \times T) = \mathcal{V}_v(\mathcal{G}short\ (E - T))$.
Thus, $\mathbf{v}_T \in \mathcal{V}_v(\mathcal{G}short\ (E - T))$.
The vertex set of $\mathcal{G}short\ (E - T)$ is the set $\{V_1, V_2, \ldots V_n\}$ where V_i is the vertex set of the i^{th} component of $\mathcal{G}openT$. Let $\mathbf{v}_T$ be derived from the potential vector $\hat{\lambda}$ in $\mathcal{G}short\ (E - T)$. The vector $\hat{\lambda}$ assigns to each of the V_i the value $\hat{\lambda}(V_i)$. Now define a potential vector λ on the nodes of $\mathcal{G}$ as follows: $\lambda(n) \equiv \hat{\lambda}(V_i), n \in V_i$. Since $\{V_1, \ldots V_k\}$ is a partition of $V(\mathcal{G})$, it is clear that λ is well defined. Let $\mathbf{v}$ be the voltage vector derived from λ in $\mathcal{G}$. Whenever $e \in E - T$ we must have $\mathbf{v}(e) = 0$ since both end points must belong to the same V_i.

Next, whenever $e \in T$ we have $\mathbf{v}(e) = \lambda(a) - \lambda(b)$ where a is the positive end point of e and b, the negative endpoint. Let $a \in V_a$, $b \in V_b$,where $V_a, V_b \in V(\mathcal{G}short\ (E - T))$. Then the positive endpoint of e in $\mathcal{G}short\ (E-T)$ is V_a and the negative end point, V_b.
By definition $\lambda(a) - \lambda(b) = \hat{\lambda}(V_a) - \hat{\lambda}(V_b)$.Thus $\mathbf{v}/T = \mathbf{v}_T$. Hence, $\mathbf{v}_T \in (\mathcal{V}_v(\mathcal{G})) \times T$. Thus, $\mathcal{V}_v(\mathcal{G} \times T) \subseteq (\mathcal{V}_v(\mathcal{G})) \times T$. The reverse containment is proved similarly.

$\square$

Using duality we can now prove

Theorem 3.4.5 *(k) Let $\mathcal{G}$ be a directed graph on edge set E. Let $T \subseteq E$. Then,*

i. $\mathcal{V}_i(\mathcal{G}\ .\ T) = (\mathcal{V}_i(\mathcal{G})) \times T$.

ii. $\mathcal{V}_i(\mathcal{G} \times T) = (\mathcal{V}_i(\mathcal{G})) \cdot T$.

Proof :
i. $\mathcal{V}_i(\mathcal{G}\ .\ T) = (\mathcal{V}_v(\mathcal{G}\ .\ T))^{\perp}$ by the strong form of Tellegen's Theorem.
By Theorem 3.4.4, $\mathcal{V}_v(\mathcal{G}\ .\ T) = (\mathcal{V}_v(\mathcal{G})) \cdot T$.
Hence,

$$\begin{aligned} \mathcal{V}_i(\mathcal{G}\ .\ T) &= ((\mathcal{V}_v(\mathcal{G})) \cdot T)^{\perp} \\ &= (\mathcal{V}_v(\mathcal{G}))^{\perp} \times T \\ &= \mathcal{V}_i(\mathcal{G}) \times T. \end{aligned}$$

ii. The proof is similar.

$\square$

Exercise 3.51 *(k) For a connected directed graph $\mathcal{G}$ on node set $\{v_1, \ldots, v_k\}$ if currents $J_1, J_2 \ldots, J_k$ enter nodes $v_1, v_2, \ldots, v_k$ show that there exists a vector $\mathbf{i}$ on $E(\mathcal{G})$, s.t. $\mathbf{Ai} = \mathbf{J}$ iff $\Sigma J_i = 0$.*

Exercise 3.52 *Prove Theorem 3.4.5 directly. (Hint: the result of the preceding exercise would be useful in extending a current vector of $\mathcal{G} \times T$ to a current vector of $\mathcal{G}$).*

3.4.5 Representative Matrices of Minors

As defined earlier, the **representative matrix R** of a vector space $\mathcal{V}$ on S has the vectors of a basis of $\mathcal{V}$ as its rows. Often the choice of a suitable representative matrix would give us special advantages. We describe how to construct a representative matrix which contains representative matrices of $\mathcal{V}\ .\ T$ and $\mathcal{V} \times (S-T)$ as its submatrices. We say in such a case that $\mathcal{V}\ .\ T$ and $\mathcal{V} \times (S-T)$ become 'visible' in **R**.

Theorem 3.4.6 *(k) Let $\mathcal{V}$ be a vector space on S. Let $T \subseteq S$. Let $\mathbf{R}$ be a representative matrix as shown below*

$$\begin{array}{cc} & \begin{array}{cc} T & S-T \end{array} \\ \mathbf{R} = & \left[\begin{array}{cc} \mathbf{R}_{TT} & \mathbf{R}_{T2} \\ \mathbf{0} & \mathbf{R}_{22} \end{array}\right] \end{array} \tag{3.7}$$

where the rows of $\mathbf{R}_{TT}$ are linearly independent. Then $\mathbf{R}_{TT}$ is a representative matrix for $\mathcal{V} \cdot T$ and $\mathbf{R}_{22}$, a representative matrix for $\mathcal{V} \times (S-T)$.

Proof : The rows of $\mathbf{R}_{TT}$ are restrictions of vectors on S to T. Hence, any linear combination of these rows will yield a vector of $\mathcal{V} \cdot T$. If $\mathbf{f}_T$ is any vector in $\mathcal{V} \cdot T$ there exists a vector $\mathbf{f}$ in $\mathcal{V}$ s.t. $\mathbf{f}/T = \mathbf{f}_T$. Now $\mathbf{f}$ is a linear combination of the rows of $\mathbf{R}$. Hence, $\mathbf{f}/T (= \mathbf{f}_T)$ is a linear combination of the rows of $\mathbf{R}_{TT}$. Further it is given that the rows of $\mathbf{R}_{TT}$ are linearly independent. It follows that $\mathbf{R}_{TT}$ is a representative matrix of $\mathcal{V} \cdot T$.
It is clear from the structure of $\mathbf{R}$ (the zero in the second set of rows) that any linear combination of the rows of $\mathbf{R}_{22}$ belongs to $\mathcal{V} \times (S-T)$. Further if $\mathbf{f}$ is any vector in $\mathcal{V}$ s.t. $\mathbf{f}/T = \mathbf{0}$ then $\mathbf{f}$ must be a linear combination only of the second set of rows of $\mathbf{R}$. For, if the first set of rows are involved in the linear combination, since rows of $\mathbf{R}_{TT}$ are linearly independent, $\mathbf{f}/T$ cannot be zero. We conclude that if $\mathbf{f}/(S-T)$ is a vector in $\mathcal{V} \times (S-T)$, it is linearly dependent on the rows of $\mathbf{R}_{22}$. Now rows of $\mathbf{R}$ are linearly independent. We conclude that $\mathbf{R}_{22}$ is a representative matrix of $\mathcal{V} \times T$.

□

Remark: To build a representative matrix of $\mathcal{V}$ with the form as in Theorem 3.4.6, we start from any representative matrix of $\mathcal{V}$ and perform row operations on it so that under the columns T we have a matrix in the RRE form.

The following corollary is immediate

Corollary 3.4.2 *(k)*

$$r(\mathcal{V}) = r(\mathcal{V} \cdot T) + r(\mathcal{V} \times (S-T)) \ , T \subseteq S$$

Corollary 3.4.3 *(k) Let $\mathcal{G}$ be a graph on E. Then*

$$r(\mathcal{G}) = r(\mathcal{G} \cdot T) + r(\mathcal{G} \times (E-T)) \ , \forall \ T \subseteq E$$

Proof : We observe that $r(\mathcal{G})$ = number of edges in a forest of $\mathcal{G} = r(\mathcal{V}_v(\mathcal{G}))$. The result follows by Theorem 3.4.4.

□

In the representative matrix of Theorem 3.4.6 the submatrix $\mathbf{R}_{T2}$ contains information about how $T, S-T$ are linked by $\mathcal{V}$. If $\mathbf{R}_{T2}$ is a zero matrix then it is clear that $\mathcal{V} = \mathcal{V}_T \oplus \mathcal{V}_{S-T}$ where $\mathcal{V}_T, \mathcal{V}_{S-T}$ are vector spaces on $T, S-T$.

Definition 3.4.4 *A subset T of S is a* **separator** *of $\mathcal{V}$ iff $\mathcal{V} \times T = \mathcal{V} \cdot T$.*

It is immediate that if T is a separator so is $(S-T)$. Thus, we might say that $T, (S-T)$ are decoupled in this case. Now by definition $\mathcal{V} \,.\, T \supseteq \mathcal{V} \times T$. Hence, equality of the spaces follows if their dimensions are the same. Hence, T is a separator iff $r(\mathcal{V} \times T) = r(\mathcal{V} \,.\, T)$.
The connectivity of $\mathcal{V}$ at T is denoted by $\xi(T)$ and defined as follows:

$$\xi(T) \equiv r(\mathcal{V} \,.\, T) - r(\mathcal{V} \times T)$$

It is easily seen that $\xi(T) = \xi(S-T)$. Further, this number is zero if T is a separator.

Exercise 3.53 *(k)*

i. *Let*

$$\begin{array}{cc} & \begin{array}{ccc} T_1 & T_2 & T_3 \end{array} \\ \mathbf{R} = & \left[\begin{array}{ccc} \mathbf{R}_{11} & \mathbf{R}_{12} & \mathbf{R}_{13} \\ \mathbf{R}_{21} & \mathbf{0} & \mathbf{R}_{23} \\ \mathbf{0} & \mathbf{0} & \mathbf{R}_{33} \end{array}\right] \end{array} \tag{3.8}$$

Rows of $\mathbf{R}_{12}$ *and* $\left[\begin{array}{c} \mathbf{R}_{11} \\ \mathbf{R}_{21} \end{array}\right]$ *are given to be linearly independent. Show that* $\mathbf{R}_{33}$ *is a representative matrix of* $\mathcal{V} \times T_3$, $\mathbf{R}_{12}$ *of* $\mathcal{V} \,.\, T_2$, $\mathbf{R}_{21}$ *of* $\mathcal{V} \,.\, (T_1 \cup T_2) \times T_1$ *as well as* $\mathcal{V} \times (T_1 \cup T_3) \,.\, T_1$ *(and hence these spaces must be the same).*

ii. *How would* $\mathbf{R}$ *look if* $\mathcal{V} \,.\, (T_1 \cup T_2)$ *has* T_1, T_2 *as separators?*

Exercise 3.54 *(k) Let*

$$\begin{array}{cc} & \begin{array}{cc} T_1 & T_2 \end{array} \\ \mathbf{R} = & \left[\begin{array}{cc} \mathbf{R}_{11} & \mathbf{0} \\ \mathbf{R}_{21} & \mathbf{R}_{22} \\ \mathbf{0} & \mathbf{R}_{33} \end{array}\right]. \end{array} \tag{3.9}$$

Suppose rows of
$\left(\begin{array}{c} \mathbf{R}_{11} \\ \mathbf{R}_{21} \end{array}\right), \left(\begin{array}{c} \mathbf{R}_{22} \\ \mathbf{R}_{33} \end{array}\right)$, *are linearly independent. Show that the number of rows of* $\mathbf{R}_{22} = r(\mathcal{V} \,.\, T_2) - r(\mathcal{V} \times T_2)$ $(= r(\mathcal{V} \,.\, T_1) - r(\mathcal{V} \times T_1))$.

Exercise 3.55 *(k) Prove:*
Let $\xi'(\cdot)$ *be the* $\xi(\cdot)$ *function for* $\mathcal{V}^{\perp}$. *Then* $\xi'(T) = \xi(T)$, $\forall\, T \subseteq S$.

Exercise 3.56 *(k) Show that the union of a forest of* $\mathcal{G} \times T$ *and a forest of* $\mathcal{G} \,.\, (E - T)$ *is a forest of* $\mathcal{G}$. *Hence, (Corollary 3.4.3)* $r(\mathcal{G} \times T) + r(\mathcal{G} \,.\, (E-T)) = r(\mathcal{G})$.

Exercise 3.57 *(k) Prove:*
$\nu(\mathcal{G} \,.\, T) + \nu(\mathcal{G} \times (S-T) = \nu(\mathcal{G})$.

Exercise 3.58 *(k) Prove:*
Let $\mathcal{G}$ *be a graph on* E. *Then* $T \subseteq E$ *is a* **separator** *of* $\mathcal{G}$ *(i.e., no circuit intersects both* T *and* $E - T$ *(Subsection 3.2.9) iff* T *is a separator of* $\mathcal{V}_v(\mathcal{G})$. *Hence,* T *is a separator of* $\mathcal{G}$ *iff* $r(\mathcal{G} \,.\, T) = r(\mathcal{G} \times T)$.

Exercise 3.59 *Let T be a separator of $\mathcal{G}$. Let $\mathcal{G} \cdot T, \mathcal{G} \cdot (E-T)$ have α_1, α_2 forests respectively, β_1, β_2 circuits respectively and γ_1, γ_2 cutsets respectively. How many forests, coforests, circuits and cutsets does $\mathcal{G}$ have?*

3.4.6 Minty's Theorem

Tellegen's Theorem is generally regarded as the most fundamental result in Electrical Network Theory. There is however, another fundamental result which can be proved to be formally equivalent to Tellegen's Theorem [Narayanan85c] and whose utility is comparable to the latter. This is Minty's Theorem (strong form) [Minty60], which we state and prove below.

Theorem 3.4.7 *(***Minty's Theorem (strong form)***) Let $\mathcal{G}$ be a directed graph. Let $E(\mathcal{G})$ be partitioned into red,blue and green edges. Let e be a green edge. Then e* **either** *belongs to a circuit containing only blue and green edges with all green edges of the same direction with respect to the orientation of the circuit* **or** *e belongs to a cutset containing only red and green edges with all green edges of the same direction with respect to the orientation of the cutset but* **not both**.

Proof: We first prove the weak form:

> 'in a graph each edge is present in a directed circuit or in a directed cutset but not both'

Proof of weak form: We claim that a directed circuit and a directed cutset of the same graph cannot intersect. For, suppose otherwise. Let the directed cutset have the orientation (V_1, V_2). The directed circuit subgraph must necessarily have vertices in V_1 as well as in V_2 in order that the intersection be nonvoid. But if we traverse the circuit subgraph starting from the node in V_1 we would at some stage crossover into V_2 by an edge e_{12} and later return to V_1 by an edge e_{21}. Now e_{12}, e_{21} have the same orientation with respect to the circuit which means that if one of them has positive end point in V_1 and negative end point in V_2 the other must have the positive and negative end points in V_2, V_1, respectively. But this contradicts the fact that they both belong to the same directed cutset with orientation (V_1, V_2).

Next we show that any edge e must belong either to a directed circuit or to a directed cutset. To see this, start from the negative end point n_2 of the edge and reach as many nodes of the graph as possible through directed paths. If through one of these paths we reach the positive end point n_1 of e we can complete the directed circuit using e. Suppose n_1 is not reachable through directed paths from n_2. Let the set of all nodes reachable by directed paths from n_2 be enclosed in a surface. This surface cannot contain n_1 and has at least one edge namely e with one end inside the surface and one outside. It is clear that all such edges must be directed into the surface as otherwise the surface can be enlarged by including more reachable nodes. This collection of edges is a directed crossing edge set and contains a directed cutset which has e as a member (see Exercise 3.60). This completes the proof of the weak form.

Proof of strong form: We open the red edges r and short the blue edges b to obtain from $\mathcal{G}$, the graph $\mathcal{G}_g$ on the green edge set g ,i.e., $\mathcal{G}_g = \mathcal{G} \times (E(\mathcal{G}) - b) \cdot g$. In this graph the weak form holds. Suppose the edge e is part of a directed cutset in $\mathcal{G}_g$. Then this is still a directed cutset containing only green edges in $\mathcal{G} \cdot (E(\mathcal{G}) - r)$. (By Lemma 3.4.2, a set $C \subseteq T \subseteq E(\mathcal{G})$ is a cutset of $\mathcal{G} \times T$ iff it is a cutset of $\mathcal{G}$). It would be a part of a red and green cutset in $\mathcal{G}$ when red edges are introduced between existing nodes. On the other hand, suppose the edge e is part of a directed circuit in $\mathcal{G}_g$. Then this is still a directed circuit containing only green edges in $\mathcal{G} \times (E(\mathcal{G}) - b)$. (By Lemma 3.4.1, a set $C \subseteq T \subseteq E(\mathcal{G})$ is a circuit of $\mathcal{G} \cdot T$ iff it is a circuit of $\mathcal{G}$). It would be a part of a blue and green circuit in $\mathcal{G}$ when blue edges are introduced by splitting existing nodes.
Thus, the strong form is proved.

□

Exercise 3.60 *(k) Let e be a member of a directed crossing edge set C. Show that there exists a directed cutset C_1 s.t. $e \in C_1 \subseteq C$.*

Exercise 3.61 *(k)* **A Generalization:** *Prove:*
Let $\mathcal{V}$ be a vector space on S over the real field and let $e \in S$. Then e is in the support of a nonzero nonnegative vector $\mathbf{f}$ in $\mathcal{V}$ or in the support of a nonzero nonnegative vector $\mathbf{g}$ in $\mathcal{V}^{\perp}$ but not in both.

Exercise 3.62 *(k)* **Partition into strongly connected components:** *Prove: The edges of a directed graph can be partitioned into two sets - those that can be included in directed circuits and those which can be included in directed cutsets.*

i. *Hence show that the vertex set of a directed graph can be partitioned into blocks so that any pair of vertices in each block are reachable from each other; partial order can be imposed on the blocks s.t. $B_i \geq B_j$ iff a vertex of B_j can be reached from a vertex of B_i.*

ii. *Give a good algorithm for building the partition as well as the partial order.*

3.5 Problems

Problems on Graphs

Problem 3.1 *(k) If a graph has no odd degree vertices,then it is possible to start from any vertex and travel along all edges without repeating any edge and to return to the starting vertex. (Repetition of nodes is allowed).*

Problem 3.2 *(k) Any graph on 6 nodes has either 3 nodes which are pairwise adjacent or 3 nodes which are pairwise non-adjacent.*

Problem 3.3 *(k) A graph is made up of parallel but oppositely directed edges only. Let $T, E - T$ be a partition of the edges of $\mathcal{G}$ such that* *i. if $e \in T$ then the parallel oppositely directed edge $e' \in T$.*

ii. it is possible to remove from each parallel pair of edges in $T(E-T)$ one of the edges so that the graph is still strongly connected.

Show that it is possible to remove one edge from each parallel pair of edges in $\mathcal{G}$ so that the graph remains strongly connected.

Problem 3.4 *(k) We denote by $\mathcal{K}_n$ the graph on n nodes with a single edge between every pair of nodes and by $\mathcal{K}_{m,n}$ the bipartite graph (i.e.,no edges between left vertices and no edges between right vertices) on m left vertices and n right vertices, with edges between every pair of right and left vertices.*

i. How many edges do $\mathcal{K}_n, \mathcal{K}_{m,n}$ have?

ii. Show that every circuit of $\mathcal{K}_{m,n}$ has an even number of edges.

iii. Show that $\mathcal{K}_n$ has n^{n-2} trees.

iv. A vertex colouring is an assignment of colours to vertices of the graph so that no two of them which have the same colour are adjacent. What is the minimum number of colours required for $\mathcal{K}_n, \mathcal{K}_{m,n}$?

Problems on Circuits

Problem 3.5 *[Whitney35]* **Circuit Matroid:** *Show that the collection $\mathcal{C}$ of circuits of a graph satisfy the* **matroid circuit axioms***:*

i. If $C_1, C_2 \in \mathcal{C}$ then C_1 cannot properly contain C_2.

ii. If $e_c \in C_1 \cap C_2, e_d \in C_1 - C_2$, then there exists $C_3 \in \mathcal{C}$ and $C_3 \subseteq C_1 \cup C_2$ s.t. $e_c \notin C_3$ but e_d does.

Problem 3.6 *(k)* **Circuit Characterization:**

i. A subset of edges C is a circuit of a graph iff it is a minimal set of edges not intersecting any cutset in a single branch.

ii. Same as (i) except 'single branch' is replaced by 'odd number of branches'.

iii. C is a circuit of a graph iff it is a minimal set of branches not contained in any forest (intersecting every coforest).

Problem 3.7 *(k)* **Cyclically Connected in terms of Edges:** *A graph in which any two vertices can be included in a circuit subgraph is said to be cyclically connected. In such a graph any two edges can also be so included.*

Problem 3.8 *(k)* **Cut Vertex:** *A graph with no coloops is cyclically connected iff it has no cut vertex (a vertex whose removal along with its incident edges disconnects the graph).*

Problems on Cutsets

Problem 3.9 *(k)* **Cutset Matroid:** *Show that the collection of cutsets of a graph satisfies the circuit axioms of a matroid.*

Problem 3.10 *(k)* **Cutset Characterization:**

i. A subset of edges C is a cutset of a graph iff it is a minimal set of edges not intersecting any circuit in a single edge (in an odd number of edges).

ii. C is a cutset of a graph iff it is a minimal set of branches not contained in any coforest (intersecting every forest).

Problem 3.11 *(k) Show that every crossing edge set is a disjoint union of cutsets.*

Problem 3.12 *(k)* **Cyclically Connected in terms of Edges in Cutsets:** *In a cyclically connected graph any two edges can be included in a cutset.*

Problems on Graphs and Vector Spaces

Problem 3.13 *(k) Show directly that KCE of a tree graph has only the trivial solution. What is the structure for which KVE has only the trivial solution?*

Problem 3.14 Rank of Incidence Matrix of a Tree Graph:
Give three proofs for 'rank of incidence matrix of a tree graph = number of edges of the graph' using

(a) the determinant of a reduced incidence matrix

(b) current injection

(c) by assuming branches to be voltage sources and evaluating node potentials.

Problem 3.15 *(k)* **Nontrivial KCE Solution and Coforest:**
Prove directly that the support of every nonzero solution to KCE meets every coforest. Hence, the rows of an f-circuit matrix of $\mathcal{G}$ span $\mathcal{V}_i(\mathcal{G})$. Hence, $r(\mathcal{V}_i(\mathcal{G})) = e - (v - p)$.

Problem 3.16 *(k)* **Nontivial KVE Solution and Forest:**
Prove directly that the support of every nonzero solution to KVE meets every forest. Hence, the rows of an f-cutset matrix of $\mathcal{G}$ span $\mathcal{V}_v(\mathcal{G})$. Hence, $r(\mathcal{V}_v(\mathcal{G})) = (v - p)$.

Problem 3.17 *(k)* **Determinants of Submatrices of Incidence Matrix:**
The determinant of every submatrix of the incidence matrix $\mathbf{A}$ is $0, \pm 1$. Hence, this property also holds for every $\mathbf{Q}_f$ and $\mathbf{B}_f$.

Problem 3.18 Interpreting Current Equations:
Let $\mathbf{A}$ be an incidence matrix.

i. Find one solution to $\mathbf{Ax} = \mathbf{b}$, if it exists, by inspection (giving a current injection interpretation).

ii. Find one solution to $\mathbf{A^Ty} = \mathbf{v}$ *by inspection (using voltage sources as branches).*

Problem 3.19 *i. Let* $\mathbf{A}$ *be the incidence matrix of* $\mathcal{G}$. *If* $\mathbf{Ax} = \mathbf{b}$ *is equivalent to* $\mathbf{Q}_f\mathbf{x} = \hat{\mathbf{b}}$, *relate* $\hat{\mathbf{b}}$ *to* $\mathbf{b}$. *Using current injection give a simple rule for obtaining* $\hat{\mathbf{b}}$ *from* $\mathbf{b}$.

ii. If $\mathbf{Q}_{f_1}\mathbf{x} = \mathbf{b}_1$, *and* $\mathbf{Q}_{f_2}\mathbf{x} = \mathbf{b}_2$ *are equivalent give a simple rule for obtaining* $\mathbf{b}_1$ *from* $\mathbf{b}_2$.

iii. If $\mathbf{B}_{f_1}\mathbf{y} = \mathbf{d}_1$, *and* $\mathbf{B}_{f_2}\mathbf{y} = \mathbf{d}_2$ *are equivalent give a simple rule for obtaining* $\mathbf{d}_1$ *from* $\mathbf{d}_2$.

Problem 3.20 *If two circuit (cutset) vectors figure in the same f-circuit (f-cutset) matrix show that the signs of the overlapping portion fully agree or fully oppose. So overlapping f-circuits (f-cutsets) fully agree or fully oppose in their orientations.*

Problem 3.21 *(k) Give simple rules for computing* $\mathbf{AA}^T, \mathbf{B}_f\mathbf{B}_f^T, \mathbf{Q}_f\mathbf{Q}_f^T$. *Show that the number of nonzero entries of* $\mathbf{AA}^T$ *is* $2e$. *Show that* $\mathbf{B}_f\mathbf{B}_f^T, \mathbf{Q}_f\mathbf{Q}_f^T$ *may not have any zero entries. Hence observe that nodal analysis is preferable to fundamental loop analysis and fundamental cutset analysis from the point of view of using Gaussian elimination.*
(Consider the case where a single edge lies in every circuit (cutset) corresponding to rows of $\mathbf{B}_f(\mathbf{Q}_f)$*).*

Problem 3.22 *Under what conditions can two circuit (cutset) vectors of a given graph be a part of the same f-circuit (f-cutset) matrix?*

Problem 3.23 *(k) Construct good algorithms for building f-circuit and f-cutset vectors for a given forest (use dfs or bfs described in Subsections 3.6.1, 3.6.2). Compute the complexity.*

Problem 3.24 Special Technique for Building a Representative Matrix of $\mathcal{V}_i(\mathcal{G})$:
Prove that the following algorithm works for building a representative matrix of $\mathcal{V}_i(\mathcal{G})$:
Let $\mathcal{G}_1$ *be a subgraph of* $\mathcal{G}$,
$\mathcal{G}_2$ *be a subgraph of* $\mathcal{G}$ *s.t.* $E(\mathcal{G}_1) \cap E(\mathcal{G}_2)$ *is a forest of* $\mathcal{G}_1$,
$\vdots$
$\mathcal{G}_k$ *be a subgraph of* $\mathcal{G}$ *s.t.* $E(\mathcal{G}_k) \cap \left[\bigcup_{i=1}^{k-1} E(\mathcal{G}_i)\right]$ *is a forest of the subgraph* $\mathcal{G} \cdot \left(\bigcup_{i=1}^{k-1} E(\mathcal{G}_i)\right)$ *and let* $\bigcup E(\mathcal{G}_i) = E(\mathcal{G})$.
Build representative matrices $\mathbf{R}_j$ *for* $\mathcal{V}_i(\mathcal{G}_j), j = 1, 2, \cdots k$. *Extend the rows of* $\mathbf{R}_j$ *to size* $E(\mathcal{G})$ *by padding with 0s. Call the resulting matrix* $\hat{\mathbf{R}}_j$. *Then* $\mathbf{R}$ *is a representative matrix for* $\mathcal{V}_i(\mathcal{G})$, *where*

$$\mathbf{R} = \begin{bmatrix} \hat{\mathbf{R}}_1 \\ \vdots \\ \hat{\mathbf{R}}_k \end{bmatrix}.$$

Problem 3.25 Equivalence of Minty's and Tellegen's Theorems: *Prove that Minty's Theorem (strong form) and Tellegen's Theorem (strong form) are formally equivalent.*

Problems on Basic Operations of Graphs

Problem 3.26 *(k) Let $\mathcal{G}$ be graph. Let $K \subseteq E(\mathcal{G})$. Then*

i. K is a forest of $\mathcal{G} \cdot T$ iff it is a maximal intersection of forests of $\mathcal{G}$ with T.

ii. K is a forest of $\mathcal{G} \times T$ iff it is a minimal intersection of forests of $\mathcal{G}$ with T.

iii. K is a forest of $\mathcal{G} \times T$ iff $K\cup$ (a forest of $\mathcal{G} \cdot (S - T)$) is a forest of $\mathcal{G}$.

iv. K is a coforest of $\mathcal{G} \cdot T$ iff $K\cup$ (a coforest of $\mathcal{G} \times (S - T)$) is a coforest of $\mathcal{G}$.

Problem 3.27 Relation between Forests Built According to Priority and Graph Minors: *Let $A_1, \cdots A_n$ be pairwise disjoint subsets of $\mathcal{G}$.*

i. A forest f of $\mathcal{G}$ contains edges from these sets in the same priority iff it is the union of forests from $\mathcal{G} \cdot A_1$, $\mathcal{G} \cdot (A_1 \cup A_2) \times A_2$, $\mathcal{G} \cdot (A_1 \cup A_2 \cup A_3) \times A_3, \cdots \mathcal{G} \times A_n$.

ii. Suppose the graph has only such forests what can you conclude?

iii. What can you conclude if the priority sequence $A_i, i = 1, \cdots n$ and $A_{\sigma(i)} i = 1, \cdots n$ for every permutation σ of $1, \cdots n$ yield the same forests?

Problem 3.28 *(k) Show how to build an f-circuit (f-cutset) matrix of $\mathcal{G}$ in which f-circuit (f-cutset) matrices of $\mathcal{G} \cdot T$ and $\mathcal{G} \times (E - T)$ become 'visible' (appear as submatrices). Let $T_2 \subseteq T_1 \subseteq E(\mathcal{G})$. Repeat the above so that the corresponding matrix of $\mathcal{G} \times T_1 \cdot T_2$ is 'visible'.*

Problem 3.29 *(k) Suppose in an electrical network on graph $\mathcal{G}$ the subset T is composed of current (voltage) sources. How will you check that they do not violate KCL (KVL)?*

3.6 Graph Algorithms

In this section we sketch some of the basic graph algorithms which we take for granted in the remaining part of the book. The algorithms we consider are

- construction of trees and forests of various kinds for the graph (bfs, dfs, minimum spanning)
- finding the connected components of the graph
- construction of the shortest path between two vertices of the graph

- construction of restrictions and contractions of the graph
- bipartite graph based algorithms such as for dealing with partitions
- flow maximization in networks

The account in this section is very brief and informal. For more details the readers are referred to [Aho+Hopcroft+Ullman74] [Kozen92] [Cormen+Leiserson+Rivest90].

For each of the above algorithms we compute or mention the 'asymptotic worst case complexity' of the algorithm. Our interest is primarily in computing an upper bound for the worst case running time of the algorithm and sometimes also for the worst case storage space required for the algorithm. A memory unit, for us, contains a single elementary symbol (a number - integer or floating point, or an alphabet). Accessing or modifying such a location would be assumed to cost unit time. Operations such as comparison, addition, multiplication and division are all assumed to cost unit time. Here as well as in the rest of the book we use the 'big Oh' notation:
Let $f, g : \mathcal{N}^p \rightarrow \mathcal{N}$ where $\mathcal{N}$ denotes the set of nonnegative integers and p is a positive integer. We say f **is** $O(g)$ iff there exists a positive integer k s.t. $f(\mathbf{n}) \leq kg(\mathbf{n})$ for all $\mathbf{n}$ outside a finite subset of $\mathcal{N}^p$.

The time and space complexity of an algorithm to solve the problem (the number of elementary steps it takes and the number of bits of memory it requires) would be computed in terms of the **size of the problem instance**. The size normally refers to the number of bits (within independently specified multiplying constant) required to represent the instance of the problem in a computer. It could be specified in terms of several parameters. For example, in the case of a directed graph with capacitated edges the size would be in terms of number of vertices, number of edges and the maximum number of bits required to represent the capacity of an edge. In general, the **size of a set** would be its cardinality while the **size of a number** would be the number of bits required to represent it. Thus, if n is a positive integer, $\log n$ would be its size – the base being any convenient positive integer. All the algorithms we study in this book are polynomial time (and space) algorithms, i.e., their worst case complexity can be written in the form $O(f(n_1, \cdots, n_p))$ where $f(\cdot)$ is a polynomial in the n_i. Further, in almost all cases, the polynomials would have low degree (≤ 5).

Very rarely we have used words such as NP-complete and NP-Hard. Informally, a problem is in P if the 'answer to it' (i.e., the answer to every one of its instances) can be **computed** in polynomial time (i.e., in time polynomial in the size of the instance) and is in NP if the correctness of the candidate answer to every instance of it can be **verified** in polynomial time. It is clear that P $\subseteq$ NP. However, although it is widely believed that P $\neq$ NP, a proof for this statement has not been obtained so far. An **NP-Hard** problem is one which has the property that if its answer can be **computed** in polynomial time, then we can infer that the answer to every problem in NP can be computed in polynomial time. An NP-Hard problem need not necessarily be in NP. If it is in NP, then it is said to be **NP-complete**. The reader interested in formal definitions as well as in additional details is referred to [Garey+Johnson79], [Van Leeuwen90].

Exercise 3.63 *A decision problem is one for which the answer is (***yes** *or* **no***). Convert the problem 'find the shortest path between* v_1 *and* v_2 *in a graph' into a 'short' sequence of decision problems.*

For most of our algorithms elementary **data structures** such as arrays, stacks, queues are adequate. Where more sophisticated data structures (such as Fibonacci Heaps) are used, we mention them by name and their specific property (such as time for retrieval, time for insertion etc.) that is needed in the context. Details are skipped and may be found in [Kozen92].

Storing a graph: A graph can be stored in the form of a sequence whose i^{th} (composite) element contains the information about the i^{th} edge (names of end points; if edge is directed the names of positive and negative end points). This sequence can be converted into another whose i^{th} element contains the information about the i^{th} node (names of incident edges, their other end points; if the graph is directed, the names of out-directed and in-directed edges and their other end points.) We will assume that we can retrieve incidence information about the i^{th} edge in $O(1)$ time and about the (i^{th} node) in $O(degree\ of\ node\ i)$ time. The conversion from one kind of representation to the other can clearly be done in $O(m+n)$ time where m is the number of edges and n is the number of vertices.

Sorting and Searching: For sorting a set of indexed elements in order of increasing indices, there are available, algorithms of complexity $O(n \log n)$, where n is the number of elements [Aho+Hopcroft+Ullman74]. We use such algorithms without naming them. In such a sorted list of elements to search for a given indexed element takes $O(\log n)$ steps by using **binary search**.

3.6.1 Breadth First Search

A **breadth first search** (**bfs**) **tree or forest** for the given graph $\mathcal{G}$ is built as follows:
Start from any vertex v_o and **scan** edges incident on it.
Select these edges and put the vertices $v_1, v_2 \cdots v_{ko}$ which are adjacent to v_o in a **queue** in the order in which the edges between them and v_o were scanned.
Mark v_o as belonging to component 1 and level 0. Mark $v_1, \cdots, v_{ko}$, as belonging to component 1 and level 1 and as **children** of v_o. Mark the vertex v_o additionally as a **parent** of its children (against each of its children).
Suppose at any stage we have the queue $v_{i1}, \cdots, v_{ik}$ and a set M_i of marked vertices. Start from the left end (first) of the queue, scan the edges incident on it and select those edges whose other ends are unmarked. If a selected edge is between v_{ij} and the unmarked vertex v_{um} then the former (latter) is the **parent (child)** of the latter (former).
Put the children of v_{i1} in the queue after v_{ik} and delete v_{i1} from the queue.

Mark these vertices as belonging to the level next to that of v_{i1} and to the same component as v_{i1} and as children of v_{i1} (against v_{i1}). Mark the vertex v_{i1} as a parent of its children (against its children).
Continue.
When the graph is disconnected it can happen that the queue is empty but all vertices have not yet been marked. In this case continue the algorithm by picking an unmarked vertex.
Mark it as of level 0 but as of component number one more than that of the previous vertex. Continue.
STOP when all vertices of the graph have been marked.

At the end of the above algorithm we have a breadth first search forest made up of the selected edges and a partition of the vertex set of the graph whose blocks are the vertex sets of the components of the graph. The starting vertices in each component are called **roots**. The level number of each vertex gives its **distance** from the root (taking the length of each edge to be one). The path in the forest from a given vertex in a component to the root in the component is obtained by travelling from the vertex to its parent and so on back to the root.

In a directed graph a bfs starting from any vertex would yield all vertices reachable from it through directed paths. In this case, while processing a vertex, one selects only the outward directed edges.

The **complexity of the bfs algorithm** is $O(m+n)$ where m is the number of edges and n is the number of vertices. (Each edge is 'touched' atmost twice. Each vertex other than the root is touched when an edge incident on it is touched or when it is a new root. Except where the root formation is involved the labour involved in touching a vertex can always be absorbed in that of touching an edge. Each touching involves a fixed number of operations).

The **complexity** of computing **all the reachable vertices** from a given vertex or a set of vertices of a directed graph through bfs is clearly also $O(m+n)$.

3.6.2 Depth First Search

A **depth first search (dfs) tree or forest** for the given graph $\mathcal{G}$ is built as follows:
Start from any vertex v_o and **scan** the edges incident on it.
Select the first nonselfloop edge. Let v_1 be its other end point. Put v_o, v_1 in a **stack**. (A **stack** is a sequence of data elements in which the last (i.e., latest) element would be processed first). Mark v_o as belonging to component 1 and as having dfs number 0, v_1 as belonging to component 1 and as having dfs number 1. Mark v_o as the parent of v_1 (against v_1) and v_1 as a child of v_o (against v_o).
Suppose at any stage, we have the stack $v_{i1}, \cdots, v_{ik}$ and a set M_i of marked vertices. Start from the top of the stack, i.e., from v_{ik} and scan the edges incident on it. Let e be the first edge whose other end point v_{i+1} is unmarked. Select e. Mark

v_{i+1} as of dfs number one more than that of the highest dfs number of a vertex in M_i and of component number same as that of v_{ik}. Mark (against v_{i+1}) v_{ik} as its parent and (against v_{ik}) v_{i+1} as one of its children. Add v_{i+1} to the top of the stack and repeat the process.
Suppose v_{ik} has no edges incident whose other end points are unmarked. Then delete v_{ik} from the stack (so that $v_{i(k-1)}$ goes to the top of the stack).
Continue.
STOP when all vertices in the graph have been marked.
When the graph is disconnected it can happen that the stack is empty but all vertices have not yet been marked. In this case continue the algorithm by picking an unmarked vertex. Give it a dfs number 0 but component number one more than that of the previous vertex.

At the end of the above algorithm we have a depth first search forest made up of the selected edges and a partition of the vertex set of the graph whose blocks are the vertex sets of the components of the graph. The starting vertices in each component are called roots. The path in the forest from a given vertex in a component to the root in the component is obtained by travelling from the vertex to its parent and so on back to the root. The **time complexity of the dfs algorithm** can be seen to be $O(m+n)$ where m is the number of edges and n, the number of vertices in the graph.

Exercise 3.64 *(k) Let e be an edge outside a dfs tree of the graph. Let v_1, v_2 be the end points of e with dfs numbering a, b respectively. If $b > a$ show that v_1 is necessarily an ancestor of v_2 (ancestor $\equiv$ parent's parent's ... parent).*

The dfs tree can be used to detect 2-connected components of the graph in $O(m+n)$ time [Aho+Hopcroft+Ullman74]. It can be used to construct the planar embedding of a planar graph in $O(n)$ time [Hopcroft+Tarjan74], [Kozen92]. There is a directed version of the dfs tree using which a directed graph can be decomposed into strongly connected components (maximal subsets of vertices which are mutually reachable by directed paths). Using the directed dfs tree this can be done in $O(m+n)$ time [Aho+Hopcroft+Ullman74].

Fundamental circuits: Let t be a forest of graph $\mathcal{G}$ and let $e \in (E(\mathcal{G}) - t)$. To construct $L(e,t)$ we may proceed as follows: Do a dfs of $\mathcal{G} \cdot t$ starting from any of its vertices. This would give a dfs number to every vertex in $\mathcal{G} \cdot t$.
Let v_1, v_2 be the end points of e. From v_1, v_2 proceed towards the root by moving from child to parent until you meet the first common ancestor v_3 of v_1 and v_2. This can be done as follows: Suppose v_1 has a higher dfs number than v_2. Move from v_1 to root until you reach the first v_1' whose dfs number is less or equal to that of v_2. Now repeat the procedure with v_2, v_1' and so on alternately until the first common vertex is reached. This would be v_3. (It can be shown that $v_1' = v_3$).
Then $L(e,t) \equiv \{e\} \cup \{$ edges in paths from v_1 to v_3 and v_2 to $v_3\}$.
To build the circuit vector corresponding to $L(e,t)$ proceed as follows: Let v_1 be the positive end point and v_2, the negative end point of e. The path from v_2 to v_1 in the tree is the path from v_2 to v_3 followed by the path from v_3 to v_1. The circuit vector has value $+1$ at e, 0 outside $L(e,t)$ and $+1$ (-1) at e_j, if it is along

(against) the path from v_2 to v_1 in the tree. Complexity of building the $L(e,t)$ is $O(\mid L(e,t) \mid)$ and that of building all the $L(e_i,t)$ is $O(\sum \mid L(e,t) \mid)$.

Exercise 3.65 *How would you build the f-circuit for a bfs tree?*

3.6.3 Minimum Spanning Tree

We are given a connected undirected graph $\mathcal{G}$ with real weights $(w(\cdot))$ on its edges. The problem is to find a spanning tree of least total weight (total weight = sum of weights of edges in the tree). We give **Prim's algorithm** for this purpose:
Choose an arbitrary vertex v_o. Among the edges incident on v_o select one of least weight.
Suppose at some stage, X is the set of edges selected and $V(X)$, the set of their end points. If $V(X) \neq V(\mathcal{G})$, select an edge e of least weight among those which have only one end point in $V(X)$.
Now replace X by $X \cup e$ and repeat.
Stop when $V(X) = V(\mathcal{G})$.
The selected edges constitute a minimum spanning tree.

Exercise 3.66 *Justify Prim's algorithm for minimum spanning tree.*

Complexity: Let n be the number of vertices and m, the number of edges of the graph. The algorithm has n stages. At each stage we have to find the minimum weight edge among the set of edges with one end point in $V(X)$. Such edges cannot be more than m in number. So finding the minimum is $O(m)$ and the overall complexity is $O(mn)$. However, this complexity can be drastically improved if we store the vertices in $(V(\mathcal{G}) - V(X))$ in a Fibonacci Heap. This data structure permits the extraction of the minimum valued element in $O(\log n)$ amortized time (where n is the number of elements in the heap), changing the value of an element in $O(1)$ amortized time and deleting the minimum element in $O(\log n)$ amortized time. (Loosely, an operation being of amortized time $O(f(n))$ implies that, if the entire running of the algorithm involves performing the operation k times, then the time for performing these operations is $O(kf(n))$.

For each vertex v in $(V(\mathcal{G}) - V(X))$ the **value** is the minimum of the weights of the edges connecting it to $V(X)$. To pick a vertex of least value we have to use $O(\log n)$ amortized time. Suppose v has been added to $V(X)$ and X replaced by $X \cup e$, where e has v as one its ends. Now the value of a vertex v' in $(V(\mathcal{G}) - (V(X) \cup e))$ has to be updated only if there is an edge between v and v'. Throughout the algorithm this updating has to be done only once per edge and each such operation takes $O(1)$ amortized time. So overall the updating takes $O(m)$ time. The extraction of the minimum valued element takes $O(n \log n)$ time over all the n stages. At each stage the minimum element has to be deleted from the heap. This takes $O(\log n)$ amortized time and $O(n \log n)$ time overall. Hence, the running time of the algorithm is $O(m + n \log n)$. (Note that the above analysis shows that, without the use of the Heap, the complexity of Prim's algorithm is $O(n^2)$).

3.6.4 Shortest Paths from a Single Vertex

We are given a graph $\mathcal{G}$, without parallel edges, in which each edge e has a non-negative **length** $l(v_1, v_2)$, where v_1, v_2 are the end points of e. If $v_1 = v_2$, then $l(v_1, v_2) \equiv 0$. The **length of a path** is defined to be the sum of the lengths of the edges in the path.
The problem is to find shortest paths from a given vertex (called the source) to every vertex in the same connected component of the graph. We give **Dijkstra's Algorithm** for this problem.
Start from the source vertex v_o and assign to each adjacent vertex v_i, a **current distance** $d_c(v_i) \equiv l(v_o, v_i)$. **Mark**, against each v_i, the vertex v_o as its foster parent. (We will call v_i, the foster children of v_o).
Let v_1 be the adjacent vertex to v_o with the least value of $d_c(v_i)$. Declare the **final distance of** v_1, $d_f(v_1) \equiv d_c(v_1)$. Mark, against v_1, the vertex v_o as its parent. (We will call v_1, a child of v_o).
(At this stage we have **processed** v_o and **marked** its adjacent vertices).
Assign a current distance ∞ to each unmarked vertex.
Suppose $X \subseteq V(\mathcal{G})$ denotes the processed set of vertices at some stage.
For each neighbour $v_j \in (V(\mathcal{G}) - X)$ of last added vertex v_k,
Check if $d_c(v_j) > d_f(v_k) + l(v_k, v_j)$.
If Yes, then
Mark, against v_j, the vertex v_k as its foster parent
(deleting any earlier mark, if present). (We will call v_j, a foster child of v_k).
Set $d_c(v_j) \equiv d_f(v_k) + l(v_k, v_j)$.
Find a vertex $v_q \in (V(\mathcal{G}) - X)$ with the least current distance $d_c(v_q)$. Declare v_q to have been processed and its final distance $d_f(v_q)$ from v_o to be $d_c(v_q)$. Mark, against v_q, its foster parent u_q as its parent (we will call v_q a child of u_q).
Add v_q to X. Repeat the procedure with $X \cup v_q$ in place of X.
STOP when all vertices in the connected component of v_o are processed.

To find a shortest path from a processed vertex v_j to v_o, we travel back from v_j to its parent and so on, from child to parent, until we reach v_o.

Justification: To justify the above algorithm, we need to show that the shortest distance from v_o to v_q (the vertex with the least current distance in $V(\mathcal{G}) - X$) is indeed $d_f(v_q)$. First, we observe that a finite $d_c(v)$, and therefore $d_f(v)$, for any vertex v is the length of some paths from v_o to v. By induction, we may assume that for every vertex v_{in} in X, $d_f(v_{in})$ = length of the shortest path from v_o to v_{in}. Note that this is justified when $X = \{v_o\}$. Suppose $d_f(v_q)$ is greater than the length of a shortest path $P(v_o, v_q)$ from v_o to v_q. Let $P(v_o, v_q)$ leave X for the first time at v_3 and let the next vertex be $v_{out} \in (V(\mathcal{G}) - X)$. If $v_{out} = v_q$, we must have $d_f(v_q) \leq d_f(v_3) + l(v_3, v_q)$ = length of $P(v_o, v_q)$.
This is a contradiction. So $v_{out} \neq v_q$. Now
$d_c(v_{out}) \leq (d_f(v_3) + l(v_3, v_{out})) \leq$ length of $P(v_o, v_q)$.
Hence, $d_c(v_{out}) < d_c(v_q) = d_f(v_q)$, which contradicts the definition of v_q. We conclude that $d_f(v_q)$ must be the length of the shortest path from v_o to v_q.

Complexity: Let n be the number of vertices and m, the number of edges of the graph. This algorithm has n stages. At each stage we have to compute $d_c(v_j)$ for vertices v_j adjacent to the last added vertex. This computation cannot exceed $O(m)$ over all the stages. Further at each stage we have to find the minimum of $d_c(v_i)$ for each v_i in $(V(\mathcal{G}) - X)$. This is $O(n)$. So we have an overall complexity of $O(n^2 + m)$. Now $m \leq n^2$. So the time complexity reduces to $O(n^2)$.
We note that the complexity of this algorithm reduces to $O(m + n \log n)$ if the elements in $V(\mathcal{G}) - X$ are stored in a Fibonacci Heap (see [Kozen92]).

3.6.5 Restrictions and Contractions of Graphs

Let $\mathcal{G}$ be a graph and let $T \subseteq E(\mathcal{G})$. To build $\mathcal{G} \cdot T$, we merely pick out the edge - end point list corresponding to T. This has complexity $O(\mid T \mid)$. (Note that the edges of T still bear their original index as in the sequence of edges of $\mathcal{G}$).
To build $\mathcal{G} \times T$ we first build $\mathcal{G}openT$. The graph $\mathcal{G}openT$ has $\mathcal{G} \cdot (E(\mathcal{G}) - T)$ + remaining vertices of $\mathcal{G}$ as isolated vertices. Next we find the connected components of $\mathcal{G}openT$. Let the vertex sets of the components be $X_1, \cdots, X_k$. For each X_i, whenever $v \in X_i$, mark it as belonging to X_i. Now in the edge - end point list of T, for each edge e, if v_1, v_2 are its (positive and negative) endpoints, and if $v_1 \in X_i$, $v_2 \in X_j$, then replace v_1 by X_i and v_2 by X_j ($\mathcal{G}shortT$ has vertex set $(X_1, \cdots, X_k)$).
The complexity of building $\mathcal{G}openT$ is $O(n + \mid E - T \mid)$, where n is the number of vertices of $\mathcal{G}$, that of finding its components is $O(n + \mid E - T \mid)$ (using dfs say), that of changing the names of vertices is $O(\mid T \mid)$. So the overall complexity is $O(n+m)$ where $m = \mid E(\mathcal{G}) \mid$.

Elsewhere, we describe methods of network analysis (by decomposition) which require the construction of the graphs $\mathcal{G} \cdot E_1, \cdots, \mathcal{G} \cdot E_k$ or $\mathcal{G} \times E_1, \cdots, \mathcal{G} \times E_k$, where $\{E_1, \cdots, E_k\}$ is a partition of $E(\mathcal{G})$. The complexity of building $\oplus_i \mathcal{G} \cdot E_i$ is clearly $O(n + m)$, while that of building $\oplus_i \mathcal{G} \times E_i$ is $O(k(n + m))$.

3.6.6 Hypergraphs represented by Bipartite Graphs

Hypergraphs are becoming increasingly important for modeling many engineering situations. By definition, a **hypergraph** $\mathcal{H}$ is a pair $(V(\mathcal{H}), E(\mathcal{H}))$, where $V(\mathcal{H})$ is the set of **vertices** of $\mathcal{H}$ and $E(\mathcal{H})$, a family of subsets of $V(\mathcal{H})$ called the **hyperedges** of $\mathcal{H}$. (We remind the reader that in a family, the same member subset could be repeated with distinct indices yielding distinct members of the family). The reader would observe that undirected graphs are a special case of hypergraphs (with the hyperedges having cardinality 1 or 2). The most convenient way of representing a hypergraph is through a **bipartite graph** $B \equiv (V_L, V_R, E)$ - a graph which has a **left vertex set** V_L, a (disjoint) **right vertex set** V_R and the set of edges E each having one end in V_L and the other in V_R. We could represent $\mathcal{H}$ by $B_{\mathcal{H}} \equiv (V_L, V_R, E)$ identifying $V(\mathcal{H})$ with $V_L, E(\mathcal{H})$ with V_R with an edge in the bipartite graph between $v \in V_L$ and $e \in V_R$ iff v is a member of the hyperedge

e of $\mathcal{H}$.

We can define **connectedness** for $\mathcal{H}$ in a manner similar to the way the notion is defined for graphs. $\mathcal{H}$ is **connected** iff for any pair of vertices v_1, v_f there exists an **alternating sequence** $v_1, e_1, v_2, e_2, \cdots, e_f, v_f$, where the v_i are vertices and e_i, edges s.t. each edge has both the preceding and succeeding vertices as members. It is easily seen that $\mathcal{H}$ is connected iff $B_{\mathcal{H}}$ is connected. Hence, checking connectedness of $\mathcal{H}$ can be done in $O(\mid V_L \mid + \mid V_R \mid + \mid E \mid)$ time. Since everything about a hypergraph is captured by a bipartite graph we confine our attention to bipartite graphs in this book. The reader interested in 'standard' hypergraph theory is referred to [Berge73].

3.6.7 Preorders and Partial Orders

A preorder is an ordered pair $(P, \preceq)$ where P is a set and '$\preceq$' is a binary relation on P that satisfies the following:
$x \preceq x, \forall\, x \in P$;
$x \preceq y, y \preceq z \Rightarrow x \preceq z\ , \forall\, x, y, z \in P$.
We can take the elements of P to be vertices and join x and y by an edge directed from y to x if $x \preceq y$. Let $\mathcal{G}_p$ be the resulting directed graph on the vertex set P. Then the vertex sets of the strongly connected components of $\mathcal{G}_P$ are the equivalence classes of the preorder (x, y belong to an equivalence class iff $x \preceq y$ and $y \preceq x$).
Let $\mathcal{P}$ be the collection of equivalence classes. If $X_1, X_2 \in \mathcal{P}$, we define $X_1 \leq X_2$ iff in the graph $\mathcal{G}_p$, a vertex in X_1 can be reached from a vertex in X_2. It is easily seen that this defines a **partial order** ($X_i \leq X_i; X_i \leq X_j$ and $X_j \leq X_i$ iff $X_i = X_j$; $X_i \leq X_j, X_j \leq X_k \Rightarrow X_i \leq X_k$). This partial order $(\mathcal{P}, \leq)$ is said to be **induced by** $(P, \preceq)$. By using a directed dfs forest on the graph $\mathcal{G}_P$ representing the preorder $(P, \preceq)$ we can get a graph representation of the induced partial order in time $O(m+n)$ where m is the number of edges and n is the number of vertices in $\mathcal{G}_P$ [Aho+Hopcroft+Ullman74].

A partial order can be represented more economically by using a Hasse Diagram. Here a directed edge goes from a vertex y to a vertex x iff y **covers** x, i.e., $x \leq y, x \neq y$ and there is no z s.t. $z \neq x$ and $z \neq y$ and $x \leq z \leq y$. An **ideal** I of $(\mathcal{P}, \leq)$ is a collection of elements of $\mathcal{P}$ with the property that if $x \in I$ and $y \leq x$ then $y \in I$. The **principal ideal** I_x in $(\mathcal{P}, \leq)$ of an element $x \in \mathcal{P}$ is the collection of all elements $y \in \mathcal{P}$ s.t. $y \leq x$. Clearly an ideal is the union of the principal ideals of its elements. A **dual ideal** I^d is a subset of $\mathcal{P}$ with the property that if $x \in I$ and $x \leq z$ then $z \in I^d$. **Ideals and dual ideals of preorders** are defined similarly. The **dual** of a partial order $(\mathcal{P}, \leq)$ is the partial order $(\mathcal{P}, \geq)$, where $x \geq y$ iff $y \leq x$. We define the dual of a preorder in the same manner. We use $\leq$ and $\geq$ interchangeably (writing $y \leq x$ or $x \geq y$) while speaking of a partial order or a preorder.

Preorders and partial orders are used repeatedly in this book (see for instance Chapter 11).

Lattices

Let $(\mathcal{P}, \leq)$ be a partial order. An **upper bound** of $e_1, e_2 \in \mathcal{P}$ is an element $e_3 \in \mathcal{P}$ s.t. $e_1 \leq e_3$ and $e_2 \leq e_3$.
A **lower bound** of e_1 and e_2 would be an element $e_4 \in \mathcal{P}$ s.t. $e_4 \leq e_1$ and $e_4 \leq e_2$. A **least upper bound** (l.u.b.) of e_1, e_2 would be an upper bound e^u s.t. whenever e_3 is an upper bound of e_1, e_2 we have $e_3 \geq e^u$. A **greatest lower bound** (g.l.b.) of e_1, e_2 would be a lower bound e_l s.t. whenever e_4 is a lower bound of e_1, e_2 we have $e_4 \leq e_l$. It is easy to see that if l.u.b. (g.l.b.) of e_1, e_2 exists, then it must be unique. We denote the l.u.b. of e_1, e_2 by $e_1 \vee e_2$ and call it the **join** of e_1 and e_2. The g.l.b. of e_1, e_2 is denoted by $e_1 \wedge e_2$ and called the **meet** of e_1 and e_2. If every pair of elements in $\mathcal{P}$ has a g.l.b. and an l.u.b. we say that $(\mathcal{P}, \leq)$ is a **lattice**. A **lattice** can be defined independently of a partial order taking two operations '$\vee$' and '$\wedge$' as primitives satisfying the properties given below:
(idempotency) $x \vee x = x \quad \forall\, x \in \mathcal{P}$; $x \wedge x = x \quad \forall\, x \in \mathcal{P}$.
(commutativity) $x \vee y = y \vee x \quad \forall\, x, y \in \mathcal{P}$; $x \wedge y = y \wedge x \quad \forall\, x, y \in \mathcal{P}$.
(associativity) $(x \vee y) \vee z = x \vee (y \vee z) \quad \forall\, x, y, z \in \mathcal{P}$.
$(x \wedge y) \wedge z = x \wedge (y \wedge z) \quad \forall\, x, y, z \in \mathcal{P}$.
(absorption) $x \wedge (x \vee y) = x \vee (x \wedge y) = x \quad \forall\, x, y \in \mathcal{P}$.
The reader may verify that these properties are indeed satisfied by g.l.b. and l.u.b. operations if we start from a partial order.
A lattice that satisfies the following additional property is called a **distributive lattice**.
(distributivity) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \quad \forall\, x, y, z \in \mathcal{P}$;
$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z) \quad \forall\, x, y, z \in \mathcal{P}$.
e.g. The collection of all subsets of a given set with union as the join operation and intersection as the meet operation is a distributive lattice. (For a comprehensive treatment of lattice theory see [Birkhoff67]).

Exercise 3.67 *Show that the collection of ideals of a partial order form a distributive lattice under union and intersection.*

3.6.8 Partitions

Let S be a finite set. A collection $\{S_1, \cdots, S_k\}$ of nonvoid subsets of S is a **partition** of S iff $\bigcup_i S_i = S$ and $S_i \cap S_j = \emptyset$ whenever i, j are distinct. If $\{S_1, \cdots, S_k\}$ is a partition of S then the S_i are referred to as its **blocks**.
Let $\mathcal{P}_S$ denote the collection of all partitions of S. We may define a partial order $(\mathcal{P}, \leq)$ on $\mathcal{P}_S$ as follows: Let $\Pi_1, \Pi_2 \in \mathcal{P}_S$. Then $\Pi_1 \leq \Pi_2$ (equivalently $\Pi_2 \geq \Pi_1$) iff each block of Π_1 is contained in a block of Π_2. We say Π_1 is **finer than** Π_2 or Π_2 is **coarser than** Π_1. If Π_a, Π_b are two partitions of S, the **join** of Π_a and Π_b, denoted by $\Pi_a \vee \Pi_b$, is the finest partition of S that is coarser than both Π_a and Π_b and the **meet** of Π_a and Π_b denoted by $\Pi_a \wedge \Pi_b$, is the coarsest partition of S that is finer than both Π_a and Π_b. It can be seen that these notions are well defined: To obtain the meet of Π_a and Π_b we take the intersection of each block of Π_a with each block of Π_b and throw away the empty intersections. Observe

that any element of S lies in precisely one such intersection. Clearly, the resulting partition Π_{ab} is finer than both Π_a and Π_b. Suppose Π_c is finer than both Π_a and Π_b. Let N_c be a block of Π_c. Then N_c is contained in some block N_a of Π_a and some block N_b of Π_b. So $N_c \subseteq N_a \cap N_b$ and hence N_c is contained in some block of Π_{ab}. This proves that Π_{ab} is the meet of Π_a and Π_b and therefore, that the 'meet' is well defined. Next let Π, Π' be two partitions coarser than Π_a and Π_b. It is then easy to see that $\Pi \wedge \Pi'$ is also coarser than Π_a and Π_b. Hence there is a unique finest partition of S coarser than Π_a and Π_b. Thus, the 'join' is well defined.

Storing partitions: We can store a partition by marking against an element of S, the name of the block to which it belongs.

Building $\Pi_a \wedge \Pi_b$: When Π_a, Π_b are stored, each element of S would have against it two names - a block of Π_a and a block of Π_b; a pair of names of intersecting blocks of Π_a, Π_b can be taken to be the name of a block of $\Pi_a \wedge \Pi_b$. Thus forming $\Pi_a \wedge \Pi_b$ from Π_a, Π_b is $O(\mid S \mid)$.

Building $\Pi_a \vee \Pi_b$: We first build a bipartite graph B with blocks of Π_a as V_L, blocks of Π_b as V_R with an edge between $N_a \in V_L$ and $N_b \in V_R$ iff $N_a \cap N_b \neq \emptyset$. It can be seen that this bipartite graph can be built in $O(\mid S \mid)$ time (For each element of S, check which blocks of Π_a, Π_b it belongs to). We find the connected components of this bipartite graph. This can be done in $O(m+n)$ time where m is the number of edges and n, the number of vertices in the bipartite graph. But both m and n do not exceed $\mid S \mid$. So $O(m+n) = O(\mid S \mid)$. Now we collect blocks of Π_a (or Π_b) belonging to the same connected component of B. Their union would make up a block of $\Pi_a \vee \Pi_b$. (For, this block is a union of some blocks K of Π_a as well as a union of some blocks of Π_b. Union of any proper subset of blocks of K would cut some block of Π_b). This involves changing the name marked against an element $u \in S$ - instead of say N_a, it would be N_v, which is the name of the connected component of B in which N_a is a vertex. Thus, building $\Pi_a \vee \Pi_b$ is $O(\mid S \mid)$.

3.6.9 The Max-Flow Problem

In this subsection we outline the max-flow problem and a simple solution for it. We also indicate the directions in which more sophisticated solutions lie. In subsequent chapters we use max-flow repeatedly to model various minimization problems. Other than the flexibility in modeling that it offers, the practical advantage of using the concept of max-flow lies in the availability of efficient algorithms.

Let $\mathcal{G}$ be a directed graph. The **flow graph** (or flow network) $F(\mathcal{G})$ is the tuple $(\mathcal{G}, \mathbf{c}, \text{s,t})$ where $c : E(\mathcal{G}) \rightarrow \Re_+$ is a real nonnegative **capacity** function on the edges of $\mathcal{G}$ and s and t are two vertices of $\mathcal{G}$ named **source** and **sink**, respectively. A **flow f** associated with $F(\mathcal{G})$ is a vector on $E(\mathcal{G})$ satisfying the following conditions:

i. $\mathbf{f}$ satisfies KCE at all nodes except s and t, i.e., at each vertex v other than

Figure 3.12: A Flow Graph with a max-flow and a min-cut

s, t, the **net outward flow**

$$\sum_i f(e_{outi}) - \sum_j f(e_{inj}) = 0$$

where $e_{outi}(e_{inj})$ are the edges incident at v and directed out of (directed into) v.

ii. the net outward flow at s is nonnegative, and at t, is non-positive.

iii. $0 \leq f(e) \leq c(e) \quad \forall\ e \in E(\mathcal{G})$.

(Often a flow is defined to be a vector satisfying (i) and (ii) above while a **feasible flow** would be one that satisfies all three conditions). An edge e with $f(e) = c(e)$ is said to be **saturated** with respect to **f**. The **value** of the flow **f**, denoted by $\mid \mathbf{f} \mid$, is the net outward flow at s. A flow of maximum value is called a **max-flow**. An **s,t-cut** (**cut** for short) is an ordered pair (A, B), where A, B are disjoint complementary subsets of $V(\mathcal{G})$ s.t. $s \in A$ and $t \in B$. The **capacity** of the cut (A, B), denoted by $c(A, B)$ is the sum of the capacities of edges with positive end in A and negative end in B. A cut of minimum capacity is called a **min-cut**. The **flow across** (A, B) denoted by $f(A, B)$, is the sum of the flows in the 'forward' edges going from A to B minus the sum of the flows in the 'backward' edges going from B to A.

Example: Figure 3.12 shows a flow graph. Alongside each directed edge is an ordered pair with the second component indicating the capacity of the edge. A feasible flow $\mathbf{f}$ is defined on this flow graph with $f(e)$ being the first component of the ordered pair alongside e. The reader may verify that the net flow leaving any node other than the source s and the sink t is zero. At s there is a net positive outward flow $(= 7)$ and at t there is a net negative outward flow $(= -7)$. Let $A \equiv \{s, a, b, c, d\}$ and let $B \equiv \{g, f, t\}$. Then (A, B) is an s, t cut. It can be verified that $f(A, B) = 4 + 3 - 0 = 7$. Observe that the forward edges (c, g) and (d, f) of the cut (A, B) are saturated while the backward edge (g, d) carries zero flow. It is clear that in the present case $f(A, B) = c(A, B)$. From the arguments given below it would follow that the given flow has the maximum value, i.e, is a max-flow and that the cut (A, B) is a min-cut, i.e., has minimum capacity.

Clearly the flow across an s,t-cut (A, B) cannot exceed the capacity of (A, B), i.e., $f(A, B) \leq c(A, B)$. Let (A, B) be an s,t-cut. If we add the outward flows at all nodes inside A we would get the value $f(A, B)$ (flow of each edge with both ends within A is added once with a $(+)$ sign and another time with a $(-)$ sign and hence cancels) as well as $\mid \mathbf{f} \mid$ (at all nodes other than s the net outward flow is zero). We conclude that $\mid \mathbf{f} \mid = f(A, B)$.

Let $\mathbf{f}$ be a flow in $F(\mathcal{G})$. Let P be a path oriented from s to t. Suppose it is possible to change the flow in the edges of P, without violating capacity constraints, as follows: the flow in each edge e of P is increased (decreased) by $\delta > 0$ if e supports (opposes) the orientation of P.
Such a path is called an **augmenting** path for the flow $\mathbf{f}$. Observe that this process does not disturb the KCE at any node except s, t. At s, the net outward flow goes up by δ, while at t, the net inward flow goes up by δ. Thus, if $\mathbf{f}'$ is the modified flow, $\mid \mathbf{f}' \mid = \mid \mathbf{f} \mid + \delta$. This is the essential idea behind flow maximization algorithms.

It is convenient to describe max-flow algorithms and related results in terms of the **residual graph** $\mathcal{G}_f$ associated with the flow $\mathbf{f}$. The graph $\mathcal{G}_f$ has the vertex set $V(\mathcal{G})$. Whenever $e \in E(\mathcal{G})$ and $f(e) < c(e)$, $\mathcal{G}_f$ has an edge e_+ between the same end points and in the same direction as e; and if $0 < f(e)$, $\mathcal{G}_f$ has an edge e_- in the opposite direction as e. Note that both e_+ and e_- may be present in $\mathcal{G}_f$.
The edge e_+ has the **residual capacity** $r_f(e_+) \equiv c(e) - f(e)$
and the edge e_- has the **residual capacity** $r_f(e_-) \equiv f(e)$.
We note that a directed path P from s to t in the residual graph $\mathcal{G}_f$ corresponds to an augmenting path in $F(\mathcal{G})$ with respect to $\mathbf{f}$. Henceforth we would call such a path P in $\mathcal{G}_f$ also, an **augmenting path** of $\mathbf{f}$. The maximum amount by which the flow can be increased using this augmenting path is clearly the minimum of the residual capacities of the edges of P. This value we would call the **bottle neck capacity** of P.

We now present a simple algorithm for flow maximization. This algorithm is due to Edmonds and Karp [Edmonds+Karp72].

ALGORITHM 3.1 Algorithm Max-Flow
INPUT *A flow graph* $F(\mathcal{G}) \equiv (\mathcal{G}, \mathbf{c}, s, t)$.

OUTPUT *(i) A maximum valued flow* $\mathbf{f}_{max}$ *for* $F(\mathcal{G})$.
(ii) A min-cut (A, B) *s.t.* $\mid \mathbf{f}_{max} \mid \equiv c(A, B)$.

Initialize *Let* $\mathbf{f}$ *be any flow of* $F(\mathcal{G})$ *(* $\mathbf{f}$ *could be the zero flow for instance).*

STEP 1 *Draw the residual graph* $\mathcal{G}_f$. *Do a directed* bfs *starting from* s.

STEP 2 **If** t **is reached,** *we also have a shortest augmenting path* P. *Compute the bottle neck capacity* δ *of* P. *Increase the flow along* P *by* δ. *Let* $\mathbf{f}'$ *be the new flow. Set* $\mathbf{f} \equiv \mathbf{f}'$ *and GOTO STEP 1.*

If t **is not reached,** *let* A *be the set of all vertices reached from* s *and let* $B \equiv V(\mathcal{G}) - A$. *Declare* $\mathbf{f}_{max} \equiv \mathbf{f}$, *min-cut to be* (A, B).

STOP.

Justification of Algorithm 3.1

We need the following

Theorem 3.6.1 *(***Max-Flow Min-Cut Theorem** *[Ford+Fulkerson56], [Ford+Fulkerson62])*

i. The flow reaches its maximum value iff there exists no augmenting path.

ii. The maximum value of a flow in $F(\mathcal{G})$ *is the minimum value of the capacity of a cut.*

Proof : If a flow has maximum value it clearly cannot permit the existence of an augmenting path. If there exists no augmenting path the directed bfs from s in the residual graph will not reach t. Let A be the set of all vertices reached from s and let B be the complement of A. All edges with one end in A and the other in B must be directed into A as otherwise the set of reachable vertices can be enlarged. Now consider the corresponding edges of $F(\mathcal{G})$. Each one of these edges, **if forward** (away from A), must have reached full capacity, i.e., be saturated and **if backward** (into A), must have zero flow. But then, for this cut, $f(A, B) = c(A, B)$. Since for any flow $\mathbf{f}$ and any cut (A', B'), we have $\mid f \mid = f(A', B') \leq c(A', B')$, we conclude that $\mathbf{f}$ is a maximum flow. This completes the proof of (i).
Since $\mid f \mid = c(A, B)$ and $\mid f \mid \leq c(A', B')$ for any cut (A', B'), it is clear that $c(A, B)$ is the minimum capacity of a cut of $F(\mathcal{G})$. This proves (ii).

□

The integral capacity case: We can justify the above algorithm for the case where capacities are **integral** quite simply. Let M be the capacity of the cut $(\{s\}, V(\mathcal{G}) - s)$. The bottle neck capacity of any augmenting path is integral. Whenever we find an augmenting path we would increase the flow by an integer and Theorem 3.6.1 assures us that if we are unable to find an augmenting path we have reached max-flow. Thus, in atmost M augmentations we reach maximum

flow.
This justification also proves the following corollary.

Corollary 3.6.1 *If the capacity function of a flow graph is integral, then there exists a max-flow in the flow graph which is integral.*

Complexity

We consider the integral capacity case. Each augmentation involves a directed bfs. This is $O(m)$ in the present case. Hence, the overall complexity of **Algorithm Max-Flow** is $O(Mm)$, where $m \equiv \mid E(\mathcal{G}) \mid$.

It is not obvious that **Algorithm Max-Flow** would terminate for real capacities. However, it can be shown that it does. Since the augmenting path is constructed through a bfs it is clear that it has minimum length. Edmonds and Karp [Edmonds+Karp72] have shown that if the shortest augmenting path is chosen every time, there are atmost mn augmentations. So the overall complexity of **Algorithm Max-Flow** is $O(m^2 n)$.

Exercise 3.68 *[Edmonds+Karp72] In Algorithm Max-Flow, if the shortest augmenting path is chosen every time, show that there are atmost mn augmentations.*

We mention a few other algorithms which are faster. These are based on Dinic's Algorithm [Dinic70]. This algorithm proceeds in **phases**, in each of which, flow is pushed along a maximal set of shortest paths. Each phase takes $O(mn)$ effort. The total number of phases is bounded by the length L of the longest $s-t$ path in $\mathcal{G}$ (Clearly $L \leq n$). So the overall complexity is $O(Lmn)$.
The MPM Algorithm [MPM78] has the same number of phases as Dinic's Algorithm. But each phase is $O(n^2)$. So the overall complexity is $O(Ln^2)$.
The Sleator Algorithm [Sleator80], [Sleator+Tarjan 83] computes each phase in $O(m \log n)$ time and has an overall complexity $O(Lm \log n)$.
(Usually the above complexities are stated with n in place of L). For a comprehensive treatment of flow algorithms the reader is referred to [Ahuja+Magnanti+Orlin93].

The Nearest Source Side and Sink Side Min-Cuts

When combinatorial problems are modelled as max-flow problems, usually the cuts with minimum capacity have physical significance. Of particular interest would be minimum capacity cuts (A, B) where A or B is minimal. Below we show that these cuts are unique. Further, we show that computing them, after a max-flow has been found, is easy. We begin with a simple lemma.

Lemma 3.6.1 *(k) Let $(A_1, B_1), (A_2, B_2)$ be two minimum capacity cuts. Then $(A_1 \cup A_2, B_1 \cap B_2)$ and $(A_1 \cap A_2, B_1 \cup B_2)$ are also minimum capacity cuts.*

Proof : Let $f(A) \equiv$ sum of the capacity of edges with one end in A and directed away from $A, A \subseteq V(\mathcal{G})$.
Later, in Chapter 10 (see Exercise 10.1, Examples 10.2.5,10.2.6) we show that $f(\cdot)$

is submodular, i.e.,

$$f(X) + f(Y) \geq f(X \cup Y) + f(X \cap Y) \; \forall \; X, Y \subseteq V(\mathcal{G}).$$

Now if X, Y minimize $f(\cdot)$, the only way the above inequality can be satisfied is for $f(\cdot)$ to take the minimum value on $X \cup Y, X \cap Y$ also. This proves the lemma.

□

The following corollary is now immediate.

Corollary 3.6.2 *Let* $F(\mathcal{G}) \equiv (\mathcal{G}, \mathbf{c}, s, t)$. *Then* $F(\mathcal{G})$ *has a unique min-cut* (A, B) *in which* A *is minimal (*B *is minimal).*

We will call the min-cut (A, B) **nearest source side (sink side) min-cut** iff A is minimal (B is minimal). To find the nearest source side (sink side) min cut we proceed as follows
Algorithm Source (Sink) Side Min-Cut: First maximize flow and let $\mathbf{f}$ be the max-flow output by the algorithm. Draw the residual graph $\mathcal{G}_f$. Do a directed bfs in $\mathcal{G}_f$ starting from s and proceeding forward. Let A_s be the set of all vertices reachable from s. Then $(A_s, V(\mathcal{G}) - A_s)$ is the desired nearest source side min-cut. Let $\mathcal{G}_f^-$ denote the directed graph obtained from $\mathcal{G}_f$ by reversing all arrows. The nearest sink side min-cut is obtained by doing a directed bfs starting from t in $\mathcal{G}_f^-$. Let B_t be the set of all vertices reachable in $\mathcal{G}_f^-$ from t. Then $(V(\mathcal{G}) - B_t, B_t)$ is the desired nearest sink side min-cut.

In order to justify the above algorithms we first observe that when we maximize flow for each min-cut (A, B) we would have $f(A, B) = c(A, B)$. Thus, if (A, B) is a min-cut, all the forward edges from A to B would be saturated and all the backward edges from A to B would have zero flow. Therefore, in the residual graph $\mathcal{G}_f$ all edges across the cut would be directed into A. Now $s \in A$ and doing a bfs starting from s we cannot go outside A. Hence, if (A, B) is a min-cut $A_s \subseteq A$, where A_s is the set of all vertices reachable from s in $\mathcal{G}_f$. But $(A_s, V(\mathcal{G}) - A_s)$ is a min-cut. Hence, $(A_s, V(\mathcal{G}) - A_s)$ is the nearest source side min-cut. The justification for the sink side min-cut algorithm is similar. (Note that the above justification provides an alternative proof that min-cuts (A, B), where A or B is minimal, are unique).

The complexity of the above algorithms is $O(m)$. So if they are added to the max-flow algorithms the overall complexity would not increase.

3.6.10 Flow Graphs Associated with Bipartite Graphs

Many optimization problems considered in this book are based on bipartite graphs. Usually they reduce to max- flow problems on a flow graph derived from the bipartite graph in a simple manner. We give below a brief account of the situation and standardize notation.

Let $B \equiv (V_L, V_R, E)$ be a bipartite graph. The flow graph $F(B, \mathbf{c}_L, \mathbf{c}_R)$ **associated with** B **with capacity** $c_L(\cdot) \oplus c_R(\cdot)$ is defined as follows:

Figure 3.13: The Flow Graph associated with B

$c_L(\cdot), c_R(\cdot)$ are nonnegative real functions on V_L, V_R respectively. (They may therefore be treated as weight vectors). Each edge $e \in E$ is directed from V_L to V_R and given a capacity ∞. Additional vertices (source) s and (sink) t are introduced. Directed edges $(s, v_L), (v_R, t)$ are added for each $v_L \in V_L$ and each $v_R \in V_R$. The capacity of the edge (s, v_L) is $c_L(v_L), v_L \in V_L$ and the capacity of the edge (v_R, t) is $c_R(v_R), v_R \in V_R$. Figure 3.13 illustrates the construction of this flow graph.

Let $\Gamma(X)$ denote the set of vertices adjacent to the vertex subset $X \subseteq V_L \uplus V_R$ in the bipartite graph B. Let $c_L(Z)$ $(c_R(Z))$ denote the sum of the values of $c_L(\cdot)$ $(c_R(\cdot))$ on elements of Z. The **cut corresponding to** $X \subseteq V_L$ is the cut $(s \uplus X \uplus \Gamma(X), t \uplus (V_L - X) \uplus (V_R - \Gamma(X)))$ (see Figure 3.14). We now have the following simple theorem which brings out the utility of the flow formulation.

Theorem 3.6.2 *(k) Let $B, c_L(\cdot), c_R(\cdot), F(B, \mathbf{c}_L, \mathbf{c}_R)$ be as defined above.*

i. The capacity of the cut corresponding to X is $c_L(V_L - X) + c_R(\Gamma(X))$.

ii. $Z \subseteq V_L$ minimizes the expression $c_L(V_L - X) + c_R(\Gamma(X)), X \subseteq V_L$, iff the cut corresponding to Z is a min-cut of $F(B, \mathbf{c}_L, \mathbf{c}_R)$. Further, the capacity of the cut corresponding to Z equals $c_L(V_L - Z) + c_R(\Gamma(Z))$.

iii. There is a unique maximal subset Z_{max} and a unique minimal set Z_{min} which minimize the above expression. Let $c_R(\cdot)$ be strictly positive. Then the cuts corresponding to Z_{max}, Z_{min} are respectively the nearest sink side and the nearest source side min-cuts of $F(B, \mathbf{c}_L, \mathbf{c}_R)$.

Figure 3.14: The Cut corresponding to X in the Flow Graph associated with B

Proof :
i. This is immediate (see Figure 3.14).

ii. We will first show that there exist min-cuts which are cuts corresponding to some $X_1 \subseteq V_L$.

Let $(s \uplus X_1 \uplus Y, t \uplus (V_L - X_1) \uplus (V_R - Y))$ be a min-cut of $F(B, \mathbf{c}_L, \mathbf{c}_R)$, where $X_1 \subseteq V_L, Y \subseteq V_R$. Since this is a min-cut, no infinite capacity edge must pass from $s \uplus X_1 \uplus Y$ to its complement. This means that any edge leaving X_1 must terminate in Y, i.e., $\Gamma(X_1) \subseteq Y$. The capacity of the cut is $c_L(V_L - X_1) + c_R(Y)$. Now consider the cut $(s \uplus X_1 \uplus \Gamma(X_1), t \uplus (V_L - X_1) \uplus (V_R - \Gamma(X_1)))$. The capacity of this cut is $c_L(V_L - X_1) + c_R(\Gamma(X_1)) \leq c_L(V_L - X_1) + c_R(Y)$, ($\mathbf{c}_L, \mathbf{c}_R$ are nonnegative vectors). Thus the cut corresponding to X_1, is a min cut.
Let Z minimize the expression $c_L(V_L - X) + c_R(\Gamma(X)), X \subseteq V_L$, and let the cut corresponding to $Z' \subseteq V_L$ be a min-cut of $F(B, \mathbf{c}_L, \mathbf{c}_R)$. The capacity of this cut is $c_L(V_L - Z') + c_R(\Gamma(Z'))$. So $c_L(V_L - Z) + c_R(\Gamma(Z)) \leq c_L(V_L - Z') + c_R(\Gamma(Z'))$. However the LHS is the capacity of the cut corresponding to Z. Since the cut corresponding to Z' is a min cut we must have $c_L(V_L - Z) + c_R(\Gamma(Z))$
$\geq c_L(V_L - Z') + c_R(\Gamma(Z'))$. We conclude that the two capacities are equal. Therefore Z' minimizes $c_L(V_L - X) + c_R(\Gamma(X)), X \subseteq V_L$, and the cut corresponding to Z is a min-cut.

iii. The nearest source side min-cut can be seen to be corresponding to some subset X_1 of V_L even if $c_R(\cdot)$ is nonnegative, but, not necessarily, strictly positive.
Now let $c_R(\cdot)$ be strictly positive.

The nearest sink side min-cut is obtained by travelling backward from t through all unsaturated arcs to reach $T_R \subseteq V_R$ and then backwards to $\Gamma(T_R) \subseteq V_L$. The cut that we obtain by this process is $(s \uplus X_2 \uplus (V_R - T_R), t \uplus \Gamma(T_R) \uplus T_R)$ where $X_2 \equiv V_L - \Gamma(T_R)$. It is clear that $\Gamma(X_2) \subseteq V_R - T_R$. Suppose $\Gamma(X_2) \subset V_R - T_R$. Then the capacity of the cut corresponding to $X_2 = c_L(V_L - X_2) + c_R(\Gamma(X_2))$ $< c_L(V_L - X_2) + c_R(V_R - T_R)$, since c_R is strictly positive. The RHS of the above inequality is the capacity of the min-cut $(s \uplus X_2 \uplus (V_R - T_R), t \uplus \Gamma(T_R) \uplus T_R)$ - a contradiction. We conclude that $\Gamma(X_2) = V_R - T_R$, so that the nearest sink side min-cut corresponds to X_2.

We know that X_1, X_2 minimize the expression $c_L(V_L - X) + c_R(\Gamma(X))$, $X \subseteq V_L$. Let $A \subset s \uplus X_1 \uplus \Gamma(X_1)$ and let B be the complement of A with respect to $V_L \uplus V_R \uplus \{s, t\}$. Then (A, B) cannot be a min-cut (using the justification for the Algorithm Source Side Min Cut). Also $s \uplus X_1 \uplus \Gamma(X_1)$ is the unique set with the above property. It follows that X_1 is the unique minimal set s.t. $(s \uplus X_1 \uplus \Gamma(X_1), t \uplus (V_L - X_1) \uplus (V_R - \Gamma(X_1)))$ is a min-cut. Hence, X_1 is the minimal set that minimizes $c_L(V_L - X) + c_R(\Gamma(X))$. The proof that X_2 is the maximal set that minimizes the above expression is similar.

□

Remark: The expression that was minimized in the above proof is a submodular function. We shall see later, in Chapter 10, that such functions always have a unique minimal and a unique maximal set minimizing them.

Complexity of Max-Flow Algorithms for Bipartite Graph Case

Finally we make some observations on the complexity of the max flow algorithms when the flow graph is associated with a bipartite graph. We note that, in this case, the **longest undirected path from s to t** is $O(\min(|V_L|, |V_R|))$, since every path from s to t has to alternate between vertices of V_L, V_R. So the **number of phases** for Dinic's (and related) algorithms would be $O(\min(|V_L|, |V_R|))$. Therefore the overall complexities of the algorithms for this case would be

$$\begin{array}{rcl} Dinic's & - & O(mn(\min(|V_L|, |V_R|))) \\ MPM & - & O(n^2(\min(|V_L|, |V_R|))) \\ Sleator & - & O(m \log n(\min(|V_L|, |V_R|))). \end{array}$$

Here, n, m refer to the total number of vertices and edges respectively in the flow graph. So
$n = |V_L| + |V_R| + 2$ and $m = |E| + |V_L| + |V_R|$.

Exercise 3.69 *[Menger27] In any graph, show that the number of arc disjoint paths, between any pair of vertices s and t, is the number of branches in a min-cut separating s and t.*

3.7 Duality

Duality is a useful concept often met with in mathematics, e.g. duality of vector spaces and spaces of functionals, duality of partial orders, duality of functions and Fourier transforms etc. When we encounter it we need to know **why** it arises and **how** to use it. The duality that one normally deals with in electrical network theory, arises because the voltage and current spaces of graphs are complementary orthogonal. (For other examples of duality that one encounters within electrical network theory see [Iri+Recski80]). In this section we discuss informally **how to dualize** statements about graphs, vector spaces (and therefore, implicitly, electrical networks) and also as to when we may expect the dual of a true statement to be true.

Let $\mathcal{V}$ be a vector space on S. We associate with $\mathcal{V}$

i. a set of operations each of which converts $\mathcal{V}$ to a vector space on a subset of S - a typical operation is $(S-T_1, T_1-T_2)(\cdot), T_2 \subseteq T_1 \subseteq S$, where

$$(S-T_1, T_1-T_2)(\mathcal{V}) \equiv \mathcal{V} \cdot T_1 \times T_2;$$

ii. classes of objects:

- class of forests
- class of coforests
- class of circuits
- class of cutsets
- primal vectors (vectors in $\mathcal{V}$)
- dual vectors (vectors in $\mathcal{V}^\perp$).

Remark: For convenience we generalize the usual definitions of forest, coforest, circuit, cutset etc. to vector spaces. The reader may verify that, if $\mathcal{V}$ were replaced by $\mathcal{V}_v(\mathcal{G})$, these definitions do reduce to the usual definitions in terms of graphs. A **forest** of $\mathcal{V}$ is a maximally independent subset of columns of a representative matrix of $\mathcal{V}$, a **coforest** of $\mathcal{V}$ is the complement, relative to the set of columns of the representative matrix, of a forest, a **circuit** of $\mathcal{V}$ is a minimal set that is not contained in any forest of $\mathcal{V}$, while a **cutset** of $\mathcal{V}$ is a minimal set that is not contained in any coforest of $\mathcal{V}$. The classes of coforests, circuits and cutsets are used for convenience. Actually any one of the four classes can be treated as primitive and the rest expressed in terms of it.

Now we list the results which 'cause' duality.

i. $(\mathcal{V}^\perp)^\perp = \mathcal{V}$, equivalently, $\mathbf{x}$ is a primal vector for $\mathcal{V}$ iff it is a dual vector for $\mathcal{V}^\perp$.

ii. $(\mathcal{V} \,.\, T_1 \times T_2)^{\perp} = \mathcal{V}^{\perp} \times T_1 \cdot T_2 = \mathcal{V}^{\perp} \cdot (S - (T_1 - T_2)) \times T_2$, i.e., the operation $(S - T_1, T_1 - T_2)(\cdot)$ holds the same place relative to $\mathcal{V}$, that the operation $(T_1 - T_2, S - T_1)(\cdot)$ holds, relative to $\mathcal{V}^{\perp}$. We say $(S - T_1, T_1 - T_2)(\cdot)$ is **dual** to $(T_1 - T_2, S - T_1)(\cdot)$.

iii. (later we add one more operation which includes all the above, namely, that of generalized minor)

$$(\mathcal{V}_S \leftrightarrow \mathcal{V}_P)^{\perp} = \mathcal{V}_S^{\perp} \leftrightarrow \mathcal{V}_P^{\perp}, \quad P \subseteq S.$$

iv. T is a forest (coforest) of $\mathcal{V}$ iff T is a coforest (forest) of $\mathcal{V}^{\perp}$.

v. C is a circuit (cutset) of $\mathcal{V}$ iff C is a cutset (circuit) of $\mathcal{V}^{\perp}$.

Let us consider how to '**dualize**' a statement about a vector space and the associated set of operations and classes of objects. Our procedure requires that the statement to be dualized be in terms of the primitive objects and operations, associated with a vector space, that we described above. Consider the statement
i. 'A subset is a circuit of $\mathcal{V} \times T$ iff it is a minimal intersection of a circuit of $\mathcal{V}$ with T '.
The first step is to write the statement in terms of $\mathcal{V}^{\perp}$:
'A subset is a circuit of $\mathcal{V}^{\perp} \times T$ iff it is a minimal intersection of a circuit of $\mathcal{V}^{\perp}$ with T '.
Next we try to express the sets of objects involved in terms of the appropriate complementary orthogonal space. Thus 'circuit of $\mathcal{V}^{\perp} \times T$ ' becomes 'cutset of $(\mathcal{V}^{\perp} \times T)^{\perp}$ ' and 'circuit of $\mathcal{V}^{\perp}$ ' becomes 'cutset of $(\mathcal{V}^{\perp})^{\perp}$ ' we thus obtain the dual of (i):
i^d. 'A subset is a cutset of $\mathcal{V} \,.\, T$ iff it is a minimal intersection of a cutset of $\mathcal{V}$ with T'.

The above procedure will yield a true (false) dual statement if we start with a true (false) statement. However, as we mentioned before, the statement that we start with must involve only the 'primitives' viz. the sets of operations and the classes of objects.

Next let us consider the case of (directed) graphs. We associate with a graph, a vector space, namely, its voltage space. Given a statement about graphs we first see whether it can be written entirely in terms of its voltage space. If so, then we dualize it and interpret the dual statement in terms of graphs. For instance consider the statement
ii. 'A subset is a circuit of $\mathcal{G} \times T$ iff it is a minimal intersection of a circuit of $\mathcal{G}$ with T '.
This statement can be written entirely in terms of $\mathcal{V}_v(\mathcal{G})$. If we substitute $\mathcal{V}$ in place of $\mathcal{V}_v(\mathcal{G})$ in this latter statement, we get the statement (i) above. Its dual is (i^d). Now we resubstitute $\mathcal{V}_v(\mathcal{G})$ in place of $\mathcal{V}$. This gives us
'A subset is a cutset of $\mathcal{V}_v(\mathcal{G}) \cdot T$ iff it is a minimal intersection of a cutset of $\mathcal{V}_v(\mathcal{G})$ with T '.
Interpreting this statement in terms of $\mathcal{G}$ gives us

ii^d. ' A subset is a cutset of $\mathcal{G} \cdot T$ iff it is a minimal intersection of a cutset of $\mathcal{G}$ with T '.
The above procedure could fail in the beginning when we try to write the statement about $\mathcal{G}$ as a statement about $\mathcal{V}_v(\mathcal{G})$ or when we replace $\mathcal{V}_v(\mathcal{G})$ by a general $\mathcal{V}$. It could fail in the end when we replace $\mathcal{V}$ by $\mathcal{V}_v(\mathcal{G})$ in the dual statement.

Here are a couple of examples of statements which cannot be dualized by our procedure.

i. 'Let $\mathcal{G}$ be a connected graph and let f be a forest of $\mathcal{G}$. Then there exists a unique path between any given pair of vertices using the edges of f alone '.
The procedure fails because 'path' and 'vertices' cannot be extended to vector spaces.

ii. 'There exists a graph $\mathcal{G}$ that has the given sets of edges $C_1, \cdots, C_n$ as circuits'. We can extend this to $\mathcal{V}_v(\mathcal{G})$, thence to $\mathcal{V}$, and dualize the statement involving $\mathcal{V}$. This statement would be:
'There exists a vector space $\mathcal{V}$ that has the given sets of edges $C_1, \cdots, C_n$ as cutsets'. The procedure can fail if we replace $\mathcal{V}$ by $\mathcal{V}_v(\mathcal{G})$ since the latter statement may be false.

Exercise 3.70 *What are the duals of the following?*

i. *rank function of a graph*

ii. $r(\cdot)$ *where* $r(T) \equiv dim(\mathcal{V} \cdot T)$

iii. $\xi(\cdot)$, *where* $\xi(T) \equiv dim(\mathcal{V} \cdot T) - dim(\mathcal{V} \times T)$

iv. *Closed sets of a graph (a subset of edges is closed if its rank is less than that of any proper superset)*

v. *selfloops*

vi. *coloops*

vii. *separators of* $\mathcal{V}$

viii. *separators (2 connected components) of a graph.*

Exercise 3.71 *Dualize the following statements. Assuming the truth of the original statement, comment on the truth of the dual.*

i. *A coforest is a minimal set that intersects every circuit.*

ii. *A circuit is a minimal set that intersects every coforest.*

iii. *Every ring sum of circuits of* $\mathcal{G}$ *is a disjoint union of circuits.* ($C_1 +_r \cdots +_r C_n$ *is the set of all elements which occur in an odd number of the* C_i).

iv. *Let* C_1, C_2 *be circuits of* $\mathcal{G}$ *and let* $e_c \in C_1 \cap C_2$ *and* $e_1 \in C_1 - C_2$. *Then there exists a circuit* C_3 *of* $\mathcal{G}$ *s.t.* $e_1 \in C_3 \subseteq C_1 \cup C_2 - e_c$.

v. *Let $\mathcal{G}$ be a graph and let $E(\mathcal{G})$ be partitioned into $E_1, \cdots, E_n$. Let f be a forest of $\mathcal{G}$ which has as many edges as possible of E_1, then as many as possible of $E_2 \cdots$ upto E_n. Then $f \cap E_j$ is a forest of $\mathcal{G} \; . \; (\bigcup_{i=1}^{j} E_i) \times E_j, j = 1, \cdots, k$.*

vi. *Let $\mathcal{G}$ be a graph. Let $E \equiv E(\mathcal{G})$ be partitioned into sets A, B. Then $L \subseteq B$ is a minimal set such that $\mathcal{G} \; . \; (E - L)$ has A as a separator iff*

   (a) $r(\mathcal{G} \times (A \cup L)) = r(\mathcal{G} \; . \; A)$.

   (b) L has no self loops in $\mathcal{G} \times (A \cup L)$.

vii. *Let $\mathcal{V}$ be a vector space on S and let S be partitioned into A, B. Let $K \subseteq A$ be s.t. $\mathcal{V} \times (E - K)$ has B as a separator. If $\mathbf{x}$ on S is s.t. $\mathbf{x}/A \in \mathcal{V} \; . \; A$, $x/B \cup K \in \mathcal{V} \; . \; (B \cup K)$, then $\mathbf{x} \in \mathcal{V}$.*

Remark: i. We have described a 'sufficient' procedure for dualization. If the procedure fails we cannot be sure that the 'dual' statement is necessarily false. The procedure is, however, applicable wherever duality is found - we merely have to use the appropriate dual objects and operations.

ii. If we restrict ourselves to the class of planar graphs we have the interesting result that there exists a graph $\mathcal{G}^*$ s.t. $\mathcal{V}_i(\mathcal{G}) = \mathcal{V}_v(\mathcal{G}^*)$. In this case a wider range of statements can be dualized. In particular there is the notion of 'mesh' or 'window' that is dual to that of a vertex. In this book we do not exploit 'planar duality'.

3.8 Notes

Graph theory means different things to different authors. The kind of graph theory that electrical networks need was developed systematically (for quite different reasons) by Whitney [Whitney32], [Whitney33a], [Whitney33b], [Whitney33c]. In this chapter the emphasis has been on the topics of direct use to us later on in the book. We have followed, in the main, [Seshu+Reed61] and [Tutte65] for the sections dealing with graphs, their representations and operations on graphs and vector spaces. For the section on graph algorithms we have used [Aho+Hopcroft+Ullman74] and [Kozen92]. For a recent survey of graph algorithms, the reader is referred to [Van Leeuwen90].

3.9 Solutions of Exercises

E 3.1: There are $(n-1)$ possible values for the degree of a node and n vertices (if the graph is connected and $n > 1$).

E 3.2:
i. If we add all the degrees (counting self loops twice) each edge is being counted twice.

ii. The sum of all degrees is even. So is the sum of all even degrees. So the sum of odd degrees is even and therefore the number of odd degree vertices is even.

E 3.3: (Sketch) Start from any vertex v_o and go to a farthest vertex. If this vertex is deleted there would still be paths from v_o to remaining vertices.

E 3.4:
i. circuit graphs disconnected from each other;
ii. add the edge between a non-terminal vertex and another vertex of the path;
iii. a single edge with two end points;
iv. a graph with only self loop edges.

E 3.5: Consider the graph obtained after deleting the edge e. If v_1, v_2 are two vertices of this graph, there must have been a path P_1 between them in the original graph. If this path had no common vertex with the circuit subgraph it would be present in the new graph also. So let us assume that it has some common vertices with the circuit subgraph. If we go along the path from v_1 to v_2 we will encounter a vertex of the circuit graph for the first time (say the vertex a) and a vertex of the circuit graph for the last time (say b). In the circuit subgraph there is a path P_2 between a and b which does not have e as an edge. If we replace the segment in P_1 between a and b by P_2, we would get a path P_3 in the new graph between v_1 and v_2.

E 3.6: Let P_1, P_2 be the two paths between nodes a, b. We start from node a and go along P_1, P_2 towards b until we reach a vertex say c after which the two paths have different edges. (Note that the vertex c could be a itself). From c we follow P_1 towards b until we reach a vertex say d which belongs also to P_2. Such a vertex must exist since b belongs both to P_1 and to P_2. From d we travel back towards a along P_2. The segments c to d along P_1 and d to c along P_2 would constitute a circuit subgraph since it would be connected and every vertex would have degree 2.

E 3.7: If the graph has a self loop then that is the desired circuit. Otherwise we start from any vertex a and travel outward without repeating edges. Since every vertex has degree ≥ 2 if we enter a vertex for the first time we can also leave it by a new edge. Since the graph is finite we must meet some vertex again. We stop as soon as this happens for the first time. Let c be this vertex. The segment (composed of edges and vertices) starting from c and ending back at c is a circuit subgraph.

E 3.8: (a) A graph made up of self loops only.
(b) A single edge with two end points.

E 3.10: A cutset is a set of crossing edges. Hence, it contains a minimal set of edges which when deleted increases the number of components of the graph. Consider any edge set C with the given property. It should be possible to partition the vertices of the graph into two subsets so that all edges between the two subsets are in C since deletion of C increases the number of components of the graph. Thus, we have two collections of subsets each member of which contains a member of the other. Hence, minimal members of both collections must be identical.

E 3.11: All edges are parallel. There may be isolated vertices.

E 3.12:

i. Deletion of T must increase the number of components. Minimality implies that only one of the components should be split.

ii. if the subgraphs on V_1, V_2 are not connected deletion of T would increase the number of components by more than one. On the other hand, if subgraphs on V_1, V_2 are connected the set of edges between them must constitute a cutset because their deletion increases the number of components and deletion of a proper subset will leave a connected subgraph with vertex set $V_1 \cup V_2$.

E 3.13:
i. There must be at least one path because of connectedness. More than one path would imply the presence of a circuit by Theorem 3.2.1.

ii. The tree graph cannot have only nodes of degree greater or equal to two as otherwise by Theorem 3.2.2 it will contain a circuit. Hence, it has a node a of degree less than two. Now if it has more than one node, because of connectedness, a has degree one. If we start from a and proceed away from it we must ultimately reach a node b of degree one since the graph is finite and repetition of a node would imply two distinct paths between some two nodes.

E 3.14: Proof of Theorem 3.2.4: The trivial single node graph with no edges is a tree graph. The graph on two nodes with an edge between them is also a tree graph. It is clear that in these cases the statement of the theorem is true. Suppose it is true for all tree graphs on $(n-1)$ nodes. Let t be a tree graph of n nodes. This graph, by Theorem 3.2.2, has a vertex v of degree less than two. If $n > 1$, since t is connected, this vertex has degree 1. If we delete this vertex and the single edge incident on it, it is clear that the remaining graph t' has no circuits. It must also be connected. For, if nodes v_1, v_2 have no path in t', the path between them in t uses v as a nonterminal node which therefore has degree ≥ 2 in t-a contradiction. Thus t' is a tree graph on $(n-1)$ nodes. By induction it has $(n-2)$ edges. So t has $(n-1)$ edges. On the other hand, let $\mathcal{G}$ be a connected graph on n nodes with $(n-1)$ edges. If it contains a circuit, by Lemma 3.2.1 we can delete an edge of the circuit without destroying connectedness of the graph. Repeating this procedure would ultimately give us a graph on n nodes that is connected but has no circuits. But this would be a tree graph with $(n-1)$ edges. We conclude that $\mathcal{G}$ must itself be a tree graph.
Proof of Corollary 3.2.1: The number of edges $= \sum_{i=1}^{p}(n_i - 1)$, where n_i is the number of nodes of the i^{th} component.

E 3.15: We will only show that maximality implies the subset is a forest (coforest). Suppose the set is maximal with respect to not containing a circuit. Then it must intersect each component of the graph in a tree. For, if not, atleast one more edge can be added without the formation of a circuit. This proves the set is a forest.
Next suppose a set L is maximal with respect to not containing a cutset. Removal of such a set from the graph would leave at least a forest of the graph. However, it cannot leave more edges than a forest for in that case the remaining graph contains a circuit. Let e be in this circuit. Deletion of $L \cup e$ cannot disconnect the graph and so $L \cup e$ contains no cutset – this contradicts the maximality. So removal of L leaves precisely a forest.

E 3.16: Deletion of the edges in a cutset increases the number of components in the graph. Hence, every forest must intersect the cutset (otherwise the corresponding forest subgraph would remain when the cutset is deleted and would ensure that the number of components remains the same).
Removal of edges of a coforest must destroy every circuit as otherwise the corresponding forest would contain a circuit. So a coforest intersects every circuit of the graph.

E 3.17: Proof of Lemma 3.2.2: Let a, b be the end points of the edge e being deleted. Let V_a, V_b be the set of all vertices which can be reached from a, b respectively, by paths in the tree graph which do not use e. Suppose node v is not in V_a or V_b. But the connected component containing v cannot meet V_a or V_b (otherwise v can be reached from a or b by a path) and hence, even if e is put back v cannot be connected to $V_a \cup V_b$ by a path. But this would make the tree graph disconnected. We conclude that $V_a \cup V_b$ is the vertex set of the tree graph. The subgraphs on V_a, V_b are connected and contain no circuits and are therefore tree graphs.

E 3.18: Let a, b be the end points of the edge e being contracted. It is clear that the graph after contraction of an edge e is connected. If it contains a circuit graph this latter must contain the fused node $\{a, b\}$. But if so there exists a path in the original tree graph between a, b which does not use e. This is a contradiction.

E 3.19: Proof of Theorem 3.2.6: Let f_G denote the subgraph of $\mathcal{G}$ on f. By the definition of a forest subgraph, between the end points of e say n_1, n_2 there must be a path, say P in f_G. Addition of e to f_G creates precisely two paths between n_1, n_2, namely, P and the subgraph on e. The path P has n_1, n_2 of degree 1 and remaining vertices of degree two. Hence addition of e to P will create a connected subgraph in which every vertex has degree two. Now this must be the only circuit subgraph created when e is added to f. For if there are two such subgraphs, e must be a part of both of them since f contains no circuit. Hence they and therefore, f_G must have distinct paths between n_1, n_2 which do not use e. But then by Theorem 3.2.1 there must be a circuit subgraph in f_G - a contradiction.

E 3.20: Proof of Theorem 3.2.7: We will prove the result for a connected graph first. Deletion of an edge of a tree graph must increase its connected components by one by Lemma 3.2.2. Deletion of $e \cup \bar{f}$ from the given graph $\mathcal{G}$ is equivalent to first deleting $\bar{f}$ and then, in the resulting tree subgraph f_G on f, deleting e. Therefore, the number of connected components must increase precisely by one when $e \cup \bar{f}$ is deleted. Let a, b be the endpoints of e and let V_a, V_b be the vertex sets of the tree subgraphs (which do not however correspond to trees of $\mathcal{G}$) that result when the edge e is deleted from f_G, equivalently, when $e \cup \bar{f}$ is deleted from $\mathcal{G}$. Any crossing edge set that $e \cup \bar{f}$ contains must have V_a, V_b as end vertex sets. There is only one such. We conclude that $e \cup \bar{f}$ contains only one crossing edge set. This must be a cutset since the subgraphs on V_a, V_b are connected.
If the graph were disconnected, when $e \cup \bar{f}$ is deleted, only one component say $\mathcal{G}_e$ which contains e would be split. Since any cutset contained in $e \cup \bar{f}$ is contained in $\mathcal{G}_e$ we could argue with $\mathcal{G}_e$ in place of $\mathcal{G}$ and the subset of f in $\mathcal{G}_e$ in place of f. So the theorem would be true in this case also.

E 3.21: Let C be a circuit. Let $e \in C$. Then $C - e$ does not contain a circuit and can be grown to a forest of $\mathcal{G}$. C is an f-circuit of this forest.

E 3.22: Let B be a cutset with $e \in B$. By minimality, deletion of $B - e$ will not increase the number of components of the graph, i.e., there is a forest remaining when $B - e$ is deleted. So $B - e$ can be included in a coforest and B is an f-cutset of the corresponding forest.

E 3.23: Let $\hat{f}$ be a forest subgraph of the given graph containing edge e of cutset C.This is possible since e is not a self loop. Contraction of this edge would convert $\hat{f}$ to a forest subgraph f of the new graph. The number of edges in the coforest would not have changed.

E 3.25: Given such a matrix associate a vertex with each row and an edge with each column. The edge has an arrow leaving the vertex (row) where its column has a $+1$ and entering the vertex (row) where its column has a -1. If the column has only zeros the corresponding edge is a self loop incident on any of the vertices.

E 3.31: The matrix retains the property given in Exercise 3.25 when these operations are performed.

E 3.35:
i. When the vertex v is not a cutvertex (i.e., a vertex which lies in every path between some two vertices a, b of the graph which are distinct from itself). In this case deletion of the edges incident at the vertex would break up the graph into atleast three components viz. v alone, component containing a and component containing b.

ii. Consider a graph with two parallel edges.

iii. No. It then has to be orthogonal to itself. Over the real field this would imply that it has null support.

E 3.36: Scan the columns from left. Pick the first column corresponding to a non-selfloop edge. If k columns (edges) have been picked, pick the next column to be corresponding to the first edge which does not form a circuit with previously picked edges. Continue until all columns are exhausted. This gives us a forest of the graph. The f-cutset matrix of this forest with columns in the same order as before and rows such that an identity matrix appears corresponding to the forest would constitute the first set of rows of the RRE matrix. The second set of rows would be zero rows equal in number to the number of components.

E 3.38: If the graph is connected all nodes must have the same potential in order that the voltages of all branches are zero. (Otherwise we can collect all nodes of a particular voltage inside a surface. At least one branch has only one endpoint within this surface. This branch would be assigned a nonzero voltage by the voltage vector). If the graph is disconnected all nodes of the same component must have the same potential by the above argument.

E 3.39: We use the above solution. If the graph is connected we see that $\lambda^T \mathbf{A} = \mathbf{0}$ iff λ has all entries the same, i.e., iff λ belongs to the one dimensional vector space spanned by $(1\ 1 \cdots 1)$. But this means $(\mathcal{C}(A))^{\perp}$ has dimension one. Hence $\dim(\mathcal{C}(A)) = n - 1$, i.e., $r(A) = n - 1$.

E 3.41: Let $\mathbf{i}$ be a nonzero current vector. Let T be the support of $\mathbf{i}$. The subgraph $\mathcal{G} \cdot T$ of $\mathcal{G}$ must have each vertex of degree at least two (otherwise the corresponding row of $\mathbf{A}$ cannot be orthogonal to $\mathbf{i}$). Hence $\mathcal{G} \cdot T$ contains a circuit by Theorem 3.2.2. Thus support of $\mathbf{i}$ contains a circuit. Next every circuit vector is a current vector (Theorem 3.3.1). It follows that its support cannot properly contain the

support of another nonzero current vector since a circuit cannot properly contain another circuit.
Next let $\mathbf{i}$ be an elementary current vector. Clearly its support must be a circuit C. Let $\mathbf{i}_C$ be the corresponding circuit vector. Now by selecting a suitable scalar α, the current vector $\mathbf{i} + \alpha\mathbf{i}_C$ can be made to have a support properly contained in C. But this implies that the support of $\mathbf{i} + \alpha\mathbf{i}_C$ is void, i.e., $\mathbf{i} = -\alpha\mathbf{i}_C$ as needed.
Now regarding the cutset vector. Let $\mathbf{v}$ be a voltage vector. We know that it must be derived from a potential vector. Let V_1 be the set of all nodes having some fixed potential (among the values taken by the potential vector). Then the crossing edge set corresponding to $(V_1, E - V_1)$ must be a subset of the support of $\mathbf{v}$. Thus, the support of $\mathbf{v}$ must contain a cutset. Now every cutset vector is a voltage vector (Theorem 3.3.2). It follows that its support cannot properly contain the support of another nonzero voltage vector since a cutset cannot properly contain another cutset.
Next let $\mathbf{v}$ be an elementary voltage vector. Proceeding analogously to the current vector case we can show that $\mathbf{v}$ must be a scalar multiple of a cutset vector, as required.

E 3.42: A set of columns T of $\mathbf{A}$ are linearly dependent iff there exists a vector $\mathbf{i}$ with support T such that $\mathbf{Ai} = \mathbf{0}$. By definition $\mathbf{i}$ is a current vector. By Theorem 3.3.6 we know that T must contain a circuit of $\mathcal{G}$. Further, if T contains a circuit of $\mathcal{G}$ the corresponding circuit vector of $\mathcal{G}$ is a current vector from which it follows that the set of columns T of $\mathbf{A}$ are linearly dependent.
The rows of $\mathbf{B}_f$ constitute a basis for $\mathcal{V}_i(G)$. By the strong form of Tellegen's Theorem we know that $\mathcal{V}_v(G) = (\mathcal{V}_i(G))^{\perp}$. Hence, $\mathbf{v}$ is a voltage vector iff $\mathbf{B}_f\mathbf{v} = \mathbf{0}$. The rest of the argument parallels that of the linear dependence of columns of $\mathbf{A}$.

E 3.43: An f-cutset matrix $\mathbf{Q}_f$ of $\mathcal{G}$ is a representative matrix of $\mathcal{V}_v(G)$ since by Theorem 3.3.2 its rows are linearly dependent on the rows of the incidence matrix and its rank equals the rank of $\mathbf{A}$. Now we know that (Theorem 3.3.7) the columns of $\mathbf{A}$ are linearly independent iff the corresponding edges do not contain a circuit. This must also be true of any representative matrix $\mathbf{Q}_f$ of $\mathcal{V}_v(\mathcal{G})$ since $\mathbf{A}$ and $\mathbf{Q}_f$ have the same column dependence structure. Let $\mathbf{Q}'$ be any standard representative matrix of $\mathcal{V}_v(G)$. Let us assume without loss of generality that

$$\begin{array}{cc} & \begin{array}{cc} T & E-T \end{array} \\ \mathbf{Q}' = & \left(\begin{array}{cc} \mathbf{I} & \mathbf{Q}'_{12} \end{array} \right). \end{array} \tag{3.10}$$

The columns corresponding to T are linearly independent and $(n-p)$ in number. Hence, T must be a forest of $\mathcal{G}$. Let $\mathbf{Q}_T$ be the f-cutset matrix with respect to T. Then $\mathbf{Q}_T = \left(\mathbf{I} \quad \mathbf{Q}_{12} \right)$ for some $\mathbf{Q}_{12}$. But $\mathbf{Q}'$ and $\mathbf{Q}_T$ are row equivalent to each other. So we conclude that $\mathbf{Q}_{12} = \mathbf{Q}'_{12}$ and $\mathbf{Q}' = \mathbf{Q}_T$. The f-circuit case proof is similar.

E 3.45: Proof of Theorem 3.3.8: Each KVE has the form $\mathbf{c}^T\mathbf{v} = \mathbf{0}$, where $\mathbf{c}$ is a circuit vector.

Now every circuit vector is a current vector. So the size of a maximal independent set of circuit vectors cannot exceed $r(\mathcal{V}_i(G))$. However, the rows of $\mathbf{B}_f$ constitute an independent set of circuit vectors of this size. The result follows.

E 3.48:

i. is immediate.

ii. (Sketch) If we start from any node of a circuit subgraph of $\mathcal{G}$ (that intersects T) and go around it, this would also describe an alternating sequence (without edge repetition) of $\mathcal{G} \times T$ starting and ending at the same vertex. This subgraph of $\mathcal{G} \times T$ has each vertex of degree ≥ 2 and so contains a circuit of $\mathcal{G} \times T$. On the other hand given any circuit subgraph of $\mathcal{G} \times T$ we can trace a closed alternating sequence around it which can be expanded to a closed alternating sequence of $\mathcal{G}$ corresponding to a circuit subgraph. So every circuit of $\mathcal{G} \times T$ is the intersection of some circuit of $\mathcal{G}$ with T.

E 3.49: (Sketch) Assume without loss of generality that $\mathcal{G}$ is connected. Any cutset of $\mathcal{G}$ that intersects T would, when removed, increase the number of components of $\mathcal{G} \,.\, T$. Hence, it contains a cutset B_T of $\mathcal{G} \,.\, T$. Any cutset of $\mathcal{G} \,.\, T$ corresponds to vertex sets V_1, V_2 between which it lies (the subgraphs of $\mathcal{G} \,.\, T$ on V_1, V_2 are connected). Now let V_1 be grown to as large a vertex subset of V_1' of $(V(\mathcal{G}) - V_2)$ as possible using paths that do not intersect B_T. The cutset of $\mathcal{G}$ defined by $V_1', (V(\mathcal{G}) - V_1')$ intersects T in B_T.
Next consider any cutset C_T of $\mathcal{G} \times T$. This corresponds to a partition V_{1T}, V_{2T} of $V(\mathcal{G} \times T)$. Now V_{1T}, V_{2T} are composed of supernodes of $\mathcal{G}$ which are the vertex sets of components of $(\mathcal{G} open T)$. The union of these supernodes yields a partition V_1, V_2 of $V(\mathcal{G})$. Clearly C_T is the set of edges between V_1, V_2. The subgraphs of $\mathcal{G} \times T$ on V_{1T}, V_{2T} are connected. So the subgraphs of $\mathcal{G}$ on V_1, V_2 are also connected. So C_T is a cutset of $\mathcal{G}$. Any cutset of $\mathcal{G}$ made up only of edges in T can similarly be shown to be a cutset of $\mathcal{G} \times T$.

E 3.51: $\mathbf{Ai} = \mathbf{J}$ has a solution iff $\lambda^T \mathbf{A} = \mathbf{0} \Rightarrow \lambda^T \mathbf{J} = \mathbf{0}$. If the graph is connected $\lambda^T \mathbf{A} = \mathbf{0} \Rightarrow$ all components of λ are identical.

E 3.52:

i. A vector satisfies KC Equations of $\mathcal{G} \,.\, T$ iff when padded with $0s$ corresponding to edges in $E(\mathcal{G}) - T$ it satisfies the KC Equations of $\mathcal{G}$. Hence, $\mathcal{V}_i(\mathcal{G} \,.\, T) = (\mathcal{V}_i(\mathcal{G})) \times T$.

ii. Let $\mathbf{i}_T \in \mathcal{V}_i(\mathcal{G} \times T)$. In the graph $\mathcal{G}$ this vector satisfies generalized KCE at supernodes which are vertex sets of components of $\mathcal{G} open T$. The previous exercise implies that we can extend this vector to edges within each of these components. Thus there is a vector $\mathbf{i} \in \mathcal{V}_i(\mathcal{G})$ s.t. $\mathbf{i}/T \in \mathcal{V}_i(\mathcal{G} \times T)$. Thus, $\mathcal{V}_i(\mathcal{G} \times T) \subseteq (\mathcal{V}_i(\mathcal{G})) \cdot T$. Any vector that satisfies KCE of $\mathcal{G}$ would satisfy generalized KCE at supernodes. Hence, if $\mathbf{i} \in \mathcal{V}_i(\mathcal{G})$ then $\mathbf{i}/T \in \mathcal{V}_i(\mathcal{G} \times T)$. Hence, $(\mathcal{V}_i(\mathcal{G})) \cdot T \subseteq \mathcal{V}_i(\mathcal{G} \times T)$.

E 3.53:

i. From Theorem 3.4.6, $\mathbf{R}_{33}$ is a representative matrix of $\mathcal{V} \times T_3$ and

$T_1 \quad T_2$

$$\begin{bmatrix} \mathbf{R}_{11} & \mathbf{R}_{12} \\ \mathbf{R}_{21} & \mathbf{0} \end{bmatrix} \tag{3.11}$$

is a representative matrix of $\mathcal{V} \cdot (T_1 \cup T_2)$.
Now $\mathbf{R}_{21}, \mathbf{R}_{12}$ are given to have linearly independent rows. So $\mathbf{R}_{21}, \mathbf{R}_{12}$ are representive matrices of $\mathcal{V} \cdot (T_1 \cup T_2) \times T_1$ and $\mathcal{V} \cdot (T_1 \cup T_2) \cdot T_2 (= \mathcal{V} \cdot T_2)$ respectively.
Next

$$\begin{array}{cc} T_1 & T_3 \end{array}$$
$$\begin{bmatrix} \mathbf{R}_{21} & \mathbf{R}_{23} \\ \mathbf{0} & \mathbf{R}_{33} \end{bmatrix} \tag{3.12}$$

must be a representive matrix of $\mathcal{V} \times (T_1 \cup T_3)$. So $\mathbf{R}_{21}$ is a representative matrix of $\mathcal{V} \times (T_1 \cup T_3) \cdot T_1$.

ii. If $\mathbf{R}_{11}$ is a zero matrix, then $\mathcal{V} \cdot (T_1 \cup T_2)$ would have T_1, T_2 as separators.

E 3.54: $\mathbf{R}_{33}$ is a representative matrix of $\mathcal{V} \times T_2$ while $\begin{bmatrix} \mathbf{R}_{22} \\ \mathbf{R}_{33} \end{bmatrix}$ is a representative matrix of $\mathcal{V} \,.\, T_2$. The result follows.

E 3.55:

$$\begin{aligned} \xi'(T) &= r(\mathcal{V}^{\perp} \cdot T) - r(\mathcal{V}^{\perp} \times T) \\ &= \mid T \mid - r(\mathcal{V} \times T) - \mid T \mid + r(\mathcal{V} \cdot T) \end{aligned}$$

(by Theorem 3.4.3).

E 3.56: We shall show that the union of a forest f_1 of $\mathcal{G}short(E-T)$ and a forest f_2 of $\mathcal{G}openT$ yields a forest of $= \mathcal{G}$. $\mathcal{G}openT$ has a number of connected components. The forest f_2 intersects each of these components in a tree. The vertex sets (supernodes) V_i of these components $\mathcal{G}_i$ figure as nodes of $\mathcal{G}short(E-T)$. If $f_1 \cup f_2$ contains a circuit of $\mathcal{G}$ it cannot be contained entirely in $\mathcal{G}openT$. The corresponding circuit subgraph can be traced as a closed path starting from some vertex in V_i going through other sets V_j and returning to V_i. When the V_i are fused to single nodes this subgraph would still contain two distinct paths between any pair of its vertices (which are supernodes in the old graph $\mathcal{G}$). Thus, f_1 would contain a circuit of $\mathcal{G}short(E-T)$ which is a contradiction. Hence, $f_1 \cup f_2$ contains no circuit of $\mathcal{G}$. On the other hand we can travel from any vertex v_1 in $\mathcal{G}$ to any other vertex v_f in the same component using only edges of $f_1 \cup f_2$. This is because a connected component of $\mathcal{G}$ would reduce to a connected component of $\mathcal{G}short(E-T)$. So v_1, v_f would be present in supernodes say V_1, V_f which are nodes of $\mathcal{G}short(E-T)$ and which have a path between them using only the edges of f_1. This path P_2 can be exploded into a path P_{12} using only edges of $f_1 \cup f_2$ in $\mathcal{G}$ as follows:

The path P_2 can be thought of as a sequence

$$v_1, v_{11}, e_1, v_2, v_{22}, e_2 \cdots e_f, v'_f, v_f$$

where v_1, v_{11} belong to the same component and in general v_j, v_{jj} belong to the same component of $\mathcal{G}openT$. So would (for notational convenience) v'_f, v_f. Now we can travel from v_1 to v_{11}, v_2 to v_{22}, v_j to v_{jj} etc. using edges of f_2. Addition of these intermediate edges and vertices yields the path P_{12}. Thus, $f_1 \cup f_2$ contains no circuits and contains a tree of each component of $\mathcal{G}$.

E 3.57: Immediate from the above.

E 3.58: Consider the incidence matrix $\mathbf{A}$ of $\mathcal{G}$. A set of columns of $\mathbf{A}$ are linearly independent iff the corresponding edges do not contain a circuit. Thus, T is a separator of $\mathcal{G}$ iff there is no minimal dependent set of columns of $\mathbf{A}$ intersecting both T and $(E - T)$.
Let

$$\begin{array}{cc} T & E-T \end{array}$$
$$\mathbf{R} = \begin{bmatrix} \mathbf{R}_{TT} & \mathbf{R}_{T2} \\ \mathbf{0} & \mathbf{R}_{22} \end{bmatrix} \tag{3.13}$$

be a representative matrix of $\mathcal{V}_v(\mathcal{G})$. This matrix and the incidence matrix are row equivalent and therefore have the same column dependence structure. If the rows of $\mathbf{R_{T2}}$ are linearly dependent on the rows of $\mathbf{R_{22}}$, we can perform reversible row operations using the rows of the latter so that rows of $\mathbf{R_{T2}}$ are made zero.If $\mathbf{R_{T2}}$ is the zero matrix it is clear that no minimal dependent set of columns can intersect both T and $E - T$,where $E \equiv E(\mathcal{G})$. If rows of $\mathbf{R_{T2}}$ are not linearly dependent on those of $\mathbf{R_{22}}$, then $r((\mathcal{V}_v(G)) \cdot T) > r((\mathcal{V}_v(G)) \times T)$. Now, let f_1, f_2 be forests of $\mathcal{G} \,.\, T, \mathcal{G} \,.\, (E-T)$, respectively. The union of these two forests contains more edges than the rank of $\mathcal{G}$ and therefore, contains a circuit. But f_1, f_2 do not individually contain circuits. We conclude that there must exist a circuit that intersects both f_1 and f_2. Thus, we see that T is a separator of $\mathcal{G}$ iff rows of $\mathbf{R_{T2}}$ are linearly dependent on the rows of $\mathbf{R_{22}}$ i.e., iff T is a separator of $\mathcal{V}_v(G)$, i.e., iff $r((\mathcal{V}_v(G)) \cdot T) = r((\mathcal{V}_v(G)) \times T)$. The last statement is equivalent to saying $r(\mathcal{G} \,.\, T) = r(\mathcal{G} \times T)$.

E 3.59: The graph $\mathcal{G}$ has $\alpha_1\alpha_2$ forests as well as coforests, $\beta_1 + \beta_2$ circuits, $\gamma_1 + \gamma_2$ cutsets. This is because every forest of $\mathcal{G}$, when T is a separator, is a union of a forest of $\mathcal{G} \,.\, T$ and a forest of $\mathcal{G} \,.\, (E - T)$. Further, each circuit of $\mathcal{G}$ is either a circuit of $\mathcal{G} \,.\, T$ or a circuit of $\mathcal{G} \,.\, (E - T)$.

E 3.60: Let the directed crossing edge set have the orientation (V_1, V_2). The tail of the edge e lies in a component of the subgraph on V_1.Let V' be the vertex set of this component.Consider the directed crossing edge set defined by $(V', V(\mathcal{G}) - V')$. The head of the edge e lies in a component of the subgraph on $V(\mathcal{G}) - V'$.Let V" be the vertex set of this component.Consider the crossing edge set defined by $(V(\mathcal{G}) - V", V")$.This has e as a member. It can be seen that it is a directed cutset.

E 3.61: We use Kuhn-Fourier Theorem. Let $\mathcal{V}$ be the solution space of $\mathbf{Ax} = \mathbf{0}$. Suppose $\mathcal{V}$ has no nonnegative vector whose support contains e. Then the following system of inequalities has no solution

$$\mathbf{Ax} = \mathbf{0}$$

$$\mathbf{x}(e) > 0$$

$$\mathbf{x} \geq \mathbf{0}.$$

By Kuhn-Fourier Theorem there exists a vector λ, a scalar $\alpha > 0$ and a vector $\sigma \geq \mathbf{0}$ s.t. $\lambda^{\mathbf{T}}\mathbf{A} + \alpha\chi_{\mathbf{e}} + \sigma^{\mathbf{T}} = \mathbf{0}$. Thus, $-\lambda^{\mathbf{T}}\mathbf{A} = (\sigma^{\mathbf{T}} + \alpha\chi_{\mathbf{e}})$. The vector $\sigma^T + \alpha\chi_e$ lies in the space $\mathcal{V}^{\perp}$ and has e in its support.

E 3.62:
ii. From each vertex obtain the set of all reachable vertices (do a bfs). This takes $O(\mid V \mid\mid E \mid)$ time. Sort each of these sets and obtain a list in increasing order of indices. This takes $O(\mid V \mid^2 \log \mid V \mid)$ time. For each pair (v_1, v_2) check if v_2 is reachable from v_1 and if v_1 is reachable from v_2. This takes $O(\mid V \mid^2 \log \mid V \mid)$ time. So overall complexity is $O(\mid V \mid (\max(\mid E \mid, \mid V \mid \log \mid V \mid)))$.

E 3.63: We assume that the length of an edge is an integer. We first find an upper bound u and a lower bound l for the length of this path. The upper bound could be the sum of all the lengths and the lower bound could be the minimum length of an edge. Narrow down to the correct value of the distance between v_1 and v_2 by asking question of the type 'is there a path between v_1 and v_2 of length $\leq d_i$'. The value of d_i in this question could be chosen by binary search between u and l: $d_1 = (l + \frac{u-l}{2})$, if yes $d_2 = (l + \frac{u-l}{4})$, if no $d_2 = (u - \frac{u-l}{4})$ and so on. (Whenever any of these numbers is a fraction we take the nearest integer). Clearly the number of such d_i is $O(\log(u-l))$.
Suppose d is the length of the shortest path. To find the edges of the shortest path we ask, for each edge e between v_1 and v_{11} say, if there is a path of length $d - d(e)$ between v_{11} and v_2. If yes (and it must be yes for one such edge) then e belongs to the path and we now try to find a path of length $d - d(e)$ between v_{11} and v_2. By this process the shortest path can be found by framing $O(\mid E(\mathcal{G}) \mid)$ decision problems. Overall the total number of decision problems is $O(\log(u-l) + \mid E(\mathcal{G}) \mid)$.

E 3.64: Observe that in the stack at any stage the top vertex has the highest dfs numbering and we cannot get below it unless it has been deleted from the stack. Once a vertex has been deleted from the stack it can never reappear. If v_1 is not an ancestor of v_2 then they have a common ancestor v_3 of highest dfs number. Since v_1 has a lower dfs number than v_2 it would have been deleted from the stack before we went back to v_3 and travelled down to v_2. But then the edge e would have been scanned when we were processing v_1 for the last time. At that time the other end of e would have been unmarked and e would then have been included in the dfs tree. This is a contradiction.

E 3.65: The technique described for building f-circuits using dfs would work for any rooted tree (a tree in which each node has a single parent). The advantage in the case of dfs and bfs is that we can stop as soon as we reach the earliest common ancestor. In the case of bfs we walk from v_1 and v_2 towards the root by first equalising levels (if v_1 has a higher level number we first reach an ancestor v_1' of the same level as v_2). Thereafter we move alternately one step at a time in the paths v_1 to root and v_2 to root until the first common ancestor is reached.

E 3.66: Suppose t is a minimum spanning tree whose total weight is less than that of the tree t_{alg} generated by the algorithm. Let t be the nearest such tree to

t_{alg} ($i.e.$, $\mid t_{alg} - t \mid$ is minimum). Let $e \in (t_{alg} - t)$. Consider the f-circuit $L(e,t)$. If $w(e) \leq w(e_j)$ for some $e_j \in (L(e,t) - e)$, then we could replace t by the tree $t \cup e - e_j$ without increasing its weight. This would contradict the fact that t is the nearest minimum spanning tree to t_{alg}. Hence $w(e) > w(e_j)$ for each e_j in $(L(e,t) - e)$. However, e was selected, during some stage of the algorithm, as the edge of least weight with one end in the current set of vertices V_e. The edges of $(L(e,t) - e)$ constitute a path between the end points of e. At least one of them, therefore, has only one end point in V_e and, therefore, has weight not less than that of e. This contradiction proves that $t_{alg} - t$ is void. Since both t_{alg} and t have the same number of edges we conclude that $t_{alg} = t$.

E 3.68: Construct a 'level graph' containing all the edges of a bfs tree in the residual graph from s to t and any other edge of that graph that travels from a lower to a higher level. (The level of a node is the bfs number of the node). Clearly only such edges can figure in a shortest path from s to t. Whenever we augment the flow using a shortest path upto its bottleneck capacity, atleast one of the edges, say e, of the residual graph will drop out of the level graph.
In the residual graph an oppositely directed edge to e would remain. But this edge cannot figure in the level graph unless the length of the shortest path changes (increases), since it would be travelling from a higher to a lower level. An edge that has dropped out cannot return until the length of the shortest path changes. It follows that there can be at most m augmentations at a particular length of the shortest path from s to t. The length of the shortest path cannot decrease and also cannot exceed the number of nodes in the graph. Hence the total number of augmentations cannot exceed mn.

E 3.69: (Sketch) Replace each edge by two oppositely directed edges of capacity 1. Treat s as source and t as sink. Maximize flow from source to sink. Each unit of flow travels along a path whose edges (since their capacity is 1) cannot be used by another unit of flow. Hence, the maximum flow $\leq$ maximum number of arc disjoint paths. The reverse inequality is obvious. In any cut of the flow graph the forward arcs (each of capacity 1) would correspond to arcs in the corresponding cut of the original graph. The result follows.

E 3.70:
i. $r(T)$ = size of the maximum circuit free set contained in T. So the dual function at T would give the size of the maximum cutset free set contained in T, i.e., the dual is $\nu(\cdot)$, the nullity function ($\nu(T) = \mid T \mid - r(\mathcal{G} \times T)$).
ii. Let $r^*(\cdot)$ be the dual. Then

$$r^*(T) \equiv \dim(\mathcal{V}^{\perp} \cdot T) = \mid T \mid - \dim(\mathcal{V} \times T)$$

iii. Let $\xi^*(\cdot)$ be the dual. Then

$$\begin{aligned}\xi^*(T) \equiv \dim(\mathcal{V}^{\perp} \cdot T) - dim(\mathcal{V}^{\perp} \times T) &= \mid T \mid - dim(\mathcal{V} \times T) - \mid T \mid + dim(\mathcal{V} \,.\, T) \\ &= \xi(T).\end{aligned}$$

Thus, $\xi(\cdot)$ is self dual.
iv. Closed sets are complements of unions of cutsets. So the duals are complements

of unions of circuits.
v. Selfloop is a single edged circuit. The dual is a single edged cutset, i.e., a coloop.
vi. the dual is the selfloop.
vii. A separator T satisfies $r(\mathcal{V}\ .\ T) - r(\mathcal{V} \times T) = 0$, i.e., $\xi(T) = 0$. Its dual would satisfy $\xi^*(T) = 0$. But we saw that $\xi(T) = \xi^*(T)$. So the dual of 'separator' is 'separator'.
viii. A separator T satisfies $r(\mathcal{G}\ .\ T) - r(\mathcal{G} \times T) = 0$, i.e.,

$$r((\mathcal{V}_v(\mathcal{G})) \cdot T) - r((\mathcal{V}_v(\mathcal{G})) \times T = 0.$$

If we go through the procedure of dualization we must replace $\mathcal{V}_v(\mathcal{G})$ by $\mathcal{V}, \mathcal{V}$ by $\mathcal{V}^{\perp}$. This would yield

$$r(\mathcal{V}^{\perp} \cdot T) - r(\mathcal{V}^{\perp} \times T) = 0.$$

As we have seen before this is equivalent to

$$r(\mathcal{V}\ .\ T) - r(\mathcal{V} \times T) = 0.$$

Substituting $\mathcal{V}_v(\mathcal{G})$ in place of $\mathcal{V}$ and interpreting in terms of $\mathcal{G}$ we get

$$r(\mathcal{G}\ .\ T) - r(\mathcal{G} \times T) = 0$$

Thus separator of a graph is self dual.

E 3.71:

i. replace 'coforest' by 'forest', 'circuit' by 'cutset'.

ii. as above.

iii. replace 'circuits' by 'cutsets'.

iv. replace 'circuit' by 'cutset'.

v. replace 'forest' by 'coforest', interchange 'dot' and 'cross' operations.

vi. replace '$r(\cdot)$' by '$\nu(\cdot)$', interchange 'dot' and 'cross' operations, replace 'self loops' by 'coloops'

vii. interchange 'dot' and'cross', $\mathcal{V}$ and $\mathcal{V}^{\perp}$.

The dual is true if the original is true (and the original is in fact true) in each of the above cases.

3.10 Solutions of Problems

P 3.1: (Sketch) Break the graph up into disjoint union of circuit subgraphs. This is possible since when a circuit is deleted the remaining graph still has only even degree vertices. Within each circuit subgraph we can start from any vertex, go through all vertices and come back to it. By induction, when one circuit is deleted, in the remaining graph within each component we can start from any vertex go through all vertices and come back to it. Now start from a vertex of the (deleted) circuit subgraph, go around it until a vertex of one of the components of the remaining graph is met. Complete a closed traversal of the component, come back to the vertex of the circuit subgraph and proceed along the circuit subgraph until the next vertex of a component is met. Continue until you come back to the starting vertex of the circuit subgraph.

P 3.5:
i. is easy to see.

ii. Start from the two end points of $e_d \in C_1 - C_2$, proceed outward until you first reach vertices a, b of the subgraph on C_2 (a, b could even be the end points of e_c). Vertices a, b must be distinct as otherwise $C_1 \cap C_2 = \emptyset$. Now in the subgraph on C_2 there are precisely two distinct paths between a, b. Only one of them contains e_c. If we follow the other path we would have constructed the desired circuit subgraph corresponding to C_3.

P 3.9: We use the notation of the Circuit Axioms in Problem 3.5. It is easy to see that axiom (i) is satisfied.
Axiom (ii): If (V_1, V_2) defines C_1 and (V_1', V_2') defines C_2, then e_c lies between $V_1 \cap V_1'$ and $V_2 \cap V_2'$ while e_d lies entirely within V_1' or entirely within V_2'. Delete $C_1 \cup C_2$. The graph is broken up into atleast three pieces (atleast three of the sets $V_1 \cap V_1'$, $V_1 \cap V_2'$, $V_2 \cap V_1'$, $V_2 \cap V_2'$ must be nonvoid). Now add back e_c . The graph would still have atleast two pieces. Consider the crossing edge set corresponding to $((V_1 \cap V_1') \cup (V_2 \cap V_2'), (V_1 \cap V_2') \cup (V_2 \cap V_1'))$. This crossing edge set contains e_d and is itself contained in $C_1 \cup C_2$. Now every crossing edge set is a disjoint union of cutsets (Problem 3.11). So there exists a cutset C_3 s.t. $e_d \in C_3 \subseteq C_1 \cup C_2 - e_c$.

P 3.11: Let $\mathcal{G}$ be connected. Let (V_1, V_2) define the crossing edge set. Let the subgraph on V_1 have components whose vertex sets are $V_{11}, \cdots, V_{1k}$ and V_2 be similarly partitioned into $V_{21}, \cdots, V_{2t}$. When $k = t = 1$ the result is clear since the crossing edge set is a cutset.Otherwise we can break up the crossing edge set into crossing edge sets corresponding to $(V_{11}, (V_2 \cup V_1) - V_{11}), \cdots, (V_{1k}, (V_2 \cup V_1) - V_{1k})$. So without loss of generality we assume $k = 1$. In this case the crossing edge set can be broken up into cutsets corresponding to $(V_{21}, (V_2 \cup V_1) - V_{21}), \cdots, (V_{2t}, (V_2 \cup V_1) - V_{2t})$. (The subgraph on V_{21} is connected and the subgraph on $(V_2 \cup V_1 - V_{21})$ is connected because the graph $\mathcal{G}$ is connected and the subgraph on V_{11} is connected).

P 3.13:
i. By KCE at a node of degree 1, the branch incident at it must carry zero current. So the vertex and the branch can be deleted without affecting KCE at any other node. What is left is a tree graph on a smaller set of nodes so the argument can be repeated.

ii. All selfloops. This is the structure that results when a tree of the (connected) graph is contracted. (Observe that the tree graph results when the cotree is deleted (opened)).

P 3.14: (a) At least one of the non datum nodes has degree 1. This node and the corresponding terminal branch would give us a row and a column which contain only one nonzero entry (± 1, where they meet). Deletion of this node and edge would give us a tree graph on nodes whose size is one less than before. Its determinant could be taken, by induction, to be ± 1. So the original determinant is also ± 1.
(b) For every injecting current vector corresponding to all the non datum nodes, if one can fix the currents in the branches **uniquely** we are done. The current at a tree branch that is terminal (incident at a vertex of degree 1) could be taken as a part of the injecting current source. We are now left with a new tree graph on less number of nodes for which (by induction) we may assume that branch currents are fixed uniquely by injecting currents.
(c) We have $\lambda^T(\mathbf{A}_r) = \mathbf{v}^T$. So if λ^T is uniquely fixed for a given $\mathbf{v}^T$ we are done. Starting from the datum node we travel to a given node along voltage sources (This is possible since the graph is connected). Their algebraic sum gives the **unique** node voltage.

P 3.15: Suppose the support is contained in a forest. Then there is a nontrivial solution to the KCE of a forest graph which is impossible by Problem 3.13 (one can also argue in terms of f-cutsets of this forest).

P 3.16: Suppose the support of the voltage vector meets only the coforest. So we have all the forest voltages zero. But each coforest edge forms an f-circuit with the forest. So its voltage is the algebraic sum of the voltages of the circuit branches in the forest. This would give the coforest edge voltage to be zero. So the voltage vector would be a zero vector.

P 3.17: See solution of Problem 3.14.

P 3.18: See solution of Problem 3.14.

P 3.19: (a) The vector $\mathbf{b}$ gives the injected currents at the nodes. Let $\mathbf{Ax} = \mathbf{b}$ be equivalent to $\mathbf{Q}_f\mathbf{x} \equiv \begin{pmatrix} \mathbf{I} & \mathbf{Q}_{12} \end{pmatrix} \begin{matrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{matrix} = \hat{\mathbf{b}}$. One possible solution of the latter equations has all entries in $\mathbf{x}_2$ zero and $\mathbf{x}_1 = \hat{\mathbf{b}}$. This is also a solution of the former equations. So $\hat{\mathbf{b}}$ is the vector of forest branch currents (the forest being the one corresponding to $\mathbf{Q}_f$) when the node injection currents are given by $\mathbf{b}$. To compute $\hat{\mathbf{b}}$ by inspection proceed as follows: (a) select some node as datum node in each component. (b) for each node v draw a path P_v in the forest graph to the datum node of the component. Associate with P_v the value of $\mathbf{b}$ at v. (c) if e is a branch of the forest the value of $\hat{\mathbf{b}}$ at e is the algebraic sum of the $\mathbf{b}(v)$ where e lies in P_v. (If e agrees with P_v add $\mathbf{b}(v)$, if it opposes subtract $\mathbf{b}(v)$ and if e does not lie in P_v ignore $\mathbf{b}(v)$).

ii. Start with the graph $\mathcal{G} \cdot f_1$. To this add a copy $f_2{}'$ of f_2. Now associate each branch e' of $f_2{}'$ with a current source of value equal to $\mathbf{b}_2(e')$. Find the currents in branches of f_1 by constructing f-circuits of branches in $f_2{}'$ and taking the value of $\mathbf{b}_1(e)$ to be the algebraic sum of $\mathbf{b}_2(e')$ where e lies in the f-circuit of e'.

P 3.20: Consequence of total unimodularity of $\mathbf{Q}_f, \mathbf{B}_f$, i.e., every subdeterminant has value $0, \pm 1$ (see Problem 3.17).

P 3.21: $\mathbf{A}\mathbf{A}^T$: The (i,j) entry is the negative of the number of edges between i, j, if $i \neq j$, and equal to the number of edges incident at i, if $i = j$.
$\mathbf{B}_f\ \mathbf{B}_f{}^T$: The (i,j) entry is (number of edges which lie in i^{th} and j^{th} f-circuits with same orientation relative to f-circuit orientation) - (number of edges which lie in i^{th} and j^{th} f-circuits with opposite orientation).
$\mathbf{Q}_f\ \mathbf{Q}_f{}^T$: similar to $\mathbf{B}_f\ \mathbf{B}_f{}^T$ case.

P 3.22: (sketch) Let $\mathbf{C}_1, \mathbf{C}_2$ be the circuits (cutsets). The corresponding circuit (cutset) vectors can be a part of the same f-circuit (f-cutset) matrix iff $\mathcal{G}\ .\ (\mathbf{C}_1 \cup \mathbf{C}_2)$ $(\mathcal{G} \times (\mathbf{C}_1 \cup \mathbf{C}_2))$ has nullity 2 (rank 2). These ideas follow by noting that if $\mathbf{C}_1, \mathbf{C}_2$ correspond to f-circuit vectors of some forest then the submatrix of the f-circuit matrix of that forest composed of these two vectors and columns $\mathbf{C}_1 \cup \mathbf{C}_2$ must be a representative matrix of $\mathcal{V}_i(\mathcal{G}\ .\ (\mathbf{C}_1 \cup \mathbf{C}_2))$ (using Theorem 3.4.6 and the fact that $\mathcal{V}_i(\mathcal{G}\ .\ T) = (\mathcal{V}_i(\mathcal{G})) \times T)$. The cutset case arguments are dual to the above.

P 3.23: See Subsection 3.6.2 for a good algorithm for building the f-circuit.Building all f-circuits of a tree has been shown there to be $O(\sum \mid L(e,t) \mid)$. For building the f-cutset, with respect to a tree t, of a branch e_t, find the sets of all nodes reachable from either of the end points of e_t in the graph $\mathcal{G} \cdot t$ by doing a bfs. If these sets are V_1, V_2 respectively then the desired f-cutset is defined by (V_1, V_2). The complexity of this algorithm is $O(|V(\mathcal{G})|)$. However building all f-cutsets of a tree is clearly equivalent to building all f-circuits of the same tree (see Exercise 3.44).

P 3.24: (Sketch) If $\mathcal{G}'$ is a subgraph of $\mathcal{G}$ then let $\hat{\mathcal{V}}_i(\mathcal{G}')$ denote the vectors obtained from those of $\mathcal{V}_i(\mathcal{G}')$ by adjoining zeros corresponding to edges outside $\mathcal{G}'$. Clearly $\hat{\mathcal{V}}_i(\mathcal{G}_j) \subseteq \mathcal{V}_i(\mathcal{G})$. Now, by construction, there exists a coforest of $\mathcal{G}\ .\ (\bigcup_{t=1}^{j-1} E(\mathcal{G}_t))$ that does not intersect $E(\mathcal{G}_j), j = 2, \cdots, k$. Assume by induction that $\hat{\mathcal{V}}_i(\mathcal{G}\ .\ (\bigcup_{t=1}^{j-1} E(\mathcal{G}_t)))$ has $\mathbf{R}'_{j-1} \equiv \begin{bmatrix} \hat{\mathbf{R}}_1 \\ \vdots \\ \hat{\mathbf{R}}_{j-1} \end{bmatrix}$ as a representative matrix. The rows of $\hat{\mathbf{R}}_j$ are linearly independent of these rows since the columns corresponding to the above coforest are independent in $\mathbf{R}'_{j-1}$ and have zero entries in $\hat{\mathbf{R}}_j$. So if we show that $\mathbf{R}'_j$ has the correct number of rows $(= \nu(\mathcal{G}\ .\ (\bigcup_{t=1}^{j} E(\mathcal{G}_t))))$ we are done.We have,

$$\nu(\mathcal{G}\ .\ (\bigcup_{t=1}^{j} E(\mathcal{G}_t))) = \nu(\mathcal{G}\ .\ (\bigcup_{t=1}^{j-1} E(\mathcal{G}_t))) + \nu(\mathcal{G}\ .\ (\bigcup_{t=1}^{j} E(\mathcal{G}_t)) \times (E(\mathcal{G}_j) - (\bigcup_{t=1}^{j-1} E(\mathcal{G}_t))))$$

(using Corollary 3.4.3).
Now

$$\nu(\mathcal{G}\ .\ (\bigcup_{t=1}^{j} E(\mathcal{G}_t)) \times (E(\mathcal{G}_j) - (\bigcup_{t=1}^{j-1} E(\mathcal{G}_t)))) = \nu(\mathcal{G}_j \times (E(\mathcal{G}_j) - (\bigcup_{t=1}^{j-1} E(\mathcal{G}_t))))$$

(since,as far as $E(\mathcal{G}_j)$ is concerned, contracting all of $(\bigcup_{t=1}^{j-1} E(\mathcal{G}_t))$ is the same as contracting the forest $E(\mathcal{G}_j) \cap \left[\bigcup_{i=1}^{j-1} E(\mathcal{G}_i)\right]$ of $\mathcal{G} \, . \, (\bigcup_{t=1}^{j-1} E(\mathcal{G}_t))$). But

$$\nu(\mathcal{G}_j \times (E(\mathcal{G}_j) - (\bigcup_{t=1}^{j-1} E(\mathcal{G}_t)))) = \nu(\mathcal{G}_j) - \nu(\mathcal{G}_j \cdot (E(\mathcal{G}_j) \cap (\bigcup_{t=1}^{j-1} E(\mathcal{G}_t)))) = \nu(\mathcal{G}_j)$$

(since $E(\mathcal{G}_j) \cap \left[\bigcup_{i=1}^{j-1} E(\mathcal{G}_i)\right]$ is a subforest of $\mathcal{G}_j$). This proves the required result.

P 3.25: See [Narayanan85c].

P 3.26: (Sketch)
i. Every forest of $\mathcal{G}$ intersects T in a subforest of $\mathcal{G} \, . \, T$ and every forest of $\mathcal{G} \, . \, T$ can be grown to a forest of $\mathcal{G}$.

ii. Union of a forest of $\mathcal{G} \, . \, (E-T)$ and a forest of $\mathcal{G} \times T$ is a forest of $\mathcal{G}$. The result now follows from the previous part.

iii. 'Only if' is clear. Suppose $K \cup$ (a forest f_{E-T} of $\mathcal{G} \, . \, (E-T)$) is a forest of $\mathcal{G}$. When edges of f_{E-T} are contracted K would not contain a circuit. The remaining edges of $(E-T)$ would have become selfloops by then. So K must also be a subforest of $\mathcal{G} \times T$. But $\mid K \mid = r(\mathcal{G} \times T)$. So K is a forest of $\mathcal{G} \times T$.

iv. Arguments similar to the previous part.

P 3.27: (Sketch)
i. We know that union of a forest of $\mathcal{G} \, . \, A_1$ and a forest of $\mathcal{G} \times (E - A_1)$ is a forest of $\mathcal{G}$ and if a forest of $\mathcal{G}$ contains a forest of $\mathcal{G} \, . \, A_1$ then its intersection with $(E - A_1)$ is a forest of $\mathcal{G} \times (E - A_1)$. So the given statement is true for $n = 2$. If it is true for $n = k - 1$ then by working with $\mathcal{G} \, . \, (A_1 \cup \cdots \cup A_{k-1})$ and $\mathcal{G} \times A_k$ we see that it must be true for $n = k$ also.

ii. If the graph has only such forests $A_1, \cdots, A_n$ become separators. Proof by induction.

iii. If this is true for each σ then $A_1, \cdots, A_n$ become separators.

P 3.28: (Sketch for the second part) Select a forest with priority $E-T_1, T_2, T_1-T_2$. Now use ideas of Exercise 3.53.

P 3.29: We need to check that there is no violation of KCL (KVL) in any cutset (circuit) contained in T. So check if there is violation of KCL in $\mathcal{G} \times T$ and violation of KVL in $\mathcal{G} \, . \, T$.

Chapter 4

Matroids

4.1 Introduction

Matroids are important combinatorial structures both from the point of view of theory and from that of applications. One of the subjects to which applications were found early was electrical network theory [Seshu+Reed61]. In this chapter we give a brief sketch of the theory with electrical networks in mind. Additional material on the matroid union theorem and related results is presented in Chapter 12.

4.2 Axiom Systems for Matroids

A **matroid** can be defined in several equivalent ways. Each of these is based on an axiom system. The primitive objects of each axiom system can be identified with either the primitive or some derived objects of every other axiom system. We restrict ourselves to finite underlying sets. The words maximal and minimal are used in the statements of the axioms below. The reader might like to look at the example in page 17.

4.2.1 Independence and Base Axioms

Independent sets of a matroid correspond to subtrees (or, dually, subcotrees) of graphs and to independent sets of columns of matrices.

I. **Independence Axioms:** A **matroid** $\mathcal{M}$ on S is a pair $(S, \mathcal{I})$, where S is a finite set and $\mathcal{I}$ is a family of subsets of S called **independent sets**, satisfying the following:
i. if $I_1 \in \mathcal{I}$ and $I_2 \subseteq I_1$, then $I_2 \in \mathcal{I}$.
ii. maximally independent sets contained in a subset of S have the same cardinality.

A **base** of the matroid $\mathcal{M} \equiv (S, \mathcal{I})$, is a maximally independent subset of $\mathcal{M}$ contained in S. The complement of a base relative to S is called a **cobase** of $\mathcal{M}$.

Remark: In this chapter a base is usually denoted by the symbol B. In subsequent chapters, however, in order to avoid confusion with 'bipartite graph' we denote a base by b.

Example 4.2.1 *(k) Let $\mathcal{G}$ be a graph. Let $\mathcal{I}_t$ be the collection of subforests of $\mathcal{G}$ and let $\mathcal{I}_c$ be the collection of subcoforests of $\mathcal{G}$. Then $(E(\mathcal{G}), \mathcal{I}_t), (E(\mathcal{G}), \mathcal{I}_c)$ are matroids. Further, the bases of each matroid are cobases of the other. The matroid $(E(\mathcal{G}), \mathcal{I}_t)$ is called the* **polygon matroid** *of $\mathcal{G}$, denoted by $\mathcal{M}(\mathcal{G})$ and the matroid $(E(\mathcal{G}), \mathcal{I}_c)$ is called the* **bond matroid** *of $\mathcal{G}$, denoted by $\mathcal{M}^*(\mathcal{G})$.*

Example 4.2.2 *(k) Let $\mathcal{V}$be a vector space on S and let $\mathbf{R}, \mathbf{R}^*$ be representative matrices of $\mathcal{V}, \mathcal{V}^\perp$, respectively. We note that column dependence structure of all representative matrices of a vector space is the same. Let $\mathcal{I}$ be the collection of independent column sets of $\mathbf{R}$ (identified with corresponding subsets of S) and let $\mathcal{I}^*$ be the collection of independent column sets of $\mathbf{R}^*$. Then $(S, \mathcal{I}), (S, \mathcal{I}^*)$ are matroids. Further, bases of each matroid are cobases of the other. We say that $(S, \mathcal{I})$ $((S, \mathcal{I}^*))$ is the matroid (dual matroid)* **associated** *with $\mathcal{V}$ and denote it by $\mathcal{M}(\mathcal{V})$ $(\mathcal{M}^*(\mathcal{V}))$.*

Example 4.2.3 *[Edmonds68][Nash-Williams67] Let $\mathcal{G}$ be a graph and let k be a positive integer. Let $\mathcal{I}_\cup$ be the collection of unions of k subforests of $\mathcal{G}$. Then $(E(\mathcal{G}), \mathcal{I}_\cup)$ is a matroid.*

Example 4.2.4 *[Edmonds+Fulkerson65] Let $\mathcal{G}$ be a graph. A matching of $\mathcal{G}$ is a subset of edges no two of which have a common end point. We say that a set of vertices and a matching* **meet** *iff the former is a subset of the end points of the edges in the matching. Let $\mathcal{I}_m$ be the collection of subsets of vertices each of which meets a matching. Then $(V(\mathcal{G}), \mathcal{I}_m)$ is a matroid.*

Exercise 4.1 *Show that the structures in the above examples are indeed matroids.*

As we know, a circuit of a graph is not contained in any forest and is the minimal such subset of edges (see Problem 3.6). This motivates us to define a **circuit** of a matroid $(S, \mathcal{I})$ to be a minimal subset of S not contained in any independent set, equivalently, we could say that a circuit is a minimal **dependent** (or non-independent) subset of the matroid.

Let B_1, B_2 be two bases of the matroid $\mathcal{M} \equiv (S, \mathcal{I})$. Let $e \in B_2 - B_1$. Then $B_1 \cup e$ is not independent and, therefore, contains a circuit (a minimal dependent set). We claim that this circuit is unique. For, suppose C_1, C_2 are two circuits contained in $B_1 \cup e$. Clearly $e \in C_1 \cap C_2$. Now $\{e\} \subseteq B_2$ and hence $\{e\}$ is not a circuit. Consider a maximally independent subset of $C_1 \cup C_2$ containing e. If C_1, C_2 are distinct, the cardinality of this set cannot exceed $\mid C_1 \cup C_2 \mid -2$. On the other hand $C_1 \cup C_2 - e$ is independent. Thus maximally independent sets contained in $C_1 \cup C_2$ do not always have the same cardinality. This violates the second characteristic property of independent sets in the Axiom system above for a matroid. We conclude that $C_1 = C_2$. (We refer to the unique circuit contained

in $e \cup B_1$ as the **fundamental circuit (f-circuit) of** e **with respect to** B_1 and denote it by $L(e, B_1)$.)
Now $L(e, B_1)$ has a nonvoid intersection with B_1 since $\{e\}$ is not a circuit. Let $e' \in L(e, B_1) \cap B_1$. Let $B_1{}' \equiv e \cup (B_1 - e')$. Clearly $B_1{}'$ contains no circuit and has the same cardinality as B_1. It follows that $B_1{}'$ is a base of $\mathcal{M}$. On the other hand if $e" \in B_1$ then $(B_1 - e") \cup e$ is independent only if $e" \in L(e, B_1)$. We therefore have the following theorem.

Theorem 4.2.1 *(k) Let B_1, B_2 be bases of a matroid $\mathcal{M}$ on S. Let $e \in B_2 - B_1$. Then,*

i. $e \cup B_1$ contains a unique circuit $L(e, B_1)$. This has a nonvoid intersection with B_1.

ii. If $e' \in B_1$, then $(B_1 - e') \cup e_1$ is a base of $\mathcal{M}$ iff $e' \in L(e, B_1)$.

Next let B_1, B_2 be as above and let $e_1 \in B_1 - B_2$. In the set $B_1 \cup B_2 - e_1$, we have B_2 as a maximally independent set. Since $| B_1 |=| B_2 |$, clearly $B_1 - e_1$ is not maximally independent in this set. Hence, there exists an element $e_2 \in B_2 - B_1$ s.t. $(B_1 - e_1) \cup e_2$ is independent and, therefore, a base of $\mathcal{M}$. (From Theorem 4.2.1 above it is clear that only those elements e_2 would qualify for which $e_1 \in L(e_2, B_1)$). We therefore have the following.

Theorem 4.2.2 *(k) Let B_1, B_2 be bases of $\mathcal{M}$ and let $e_1 \in B_1 - B_2$. Then there exists $e_2 \in B_2 - B_1$, such that $(B_1 - e_1) \cup e_2$ is a base of $\mathcal{M}$.*

Theorems 4.2.1 and 4.2.2 can be used to generate axiom systems for matroids. We state these below:

II. **Base Axioms:** A matroid $\mathcal{M}$ on S is a pair $(S, \mathcal{I})$ where $\mathcal{I}$ is a collection of subsets of S satisfying
i. If $I_1 \in \mathcal{I}$ and $I_2 \subseteq I_1$ then $I_2 \in \mathcal{I}$.
ii. If B_1, B_2 are maximally independent sets (bases) of $\mathcal{M}$ and if $e_2 \in B_2 - B_1$, then there exists $e_1 \in B_1 - B_2$ s.t. $(B_1 - e_1) \cup e_2$ is maximally independent in $\mathcal{M}$.
Condition (ii) above can be replaced by (ii') below.
ii'. If B_1, B_2 are maximally independent sets of $\mathcal{M}$ and if $e_1 \in B_1 - B_2$ then there exists $e_2 \in B_2 - B_1$ s.t. $(B_1 - e_1) \cup e_2$ is maximally independent in $\mathcal{M}$.

Exercise 4.2 *(k) Show that a collection $\mathcal{I}$ of subsets of S that satisfies Base Axioms satisfies the Independence Axioms.*

4.2.2 Rank Axioms

Let $\mathcal{M}$ be a matroid on S. The **rank** of a subset $T \subseteq S$ is defined to be the size of the maximally independent set contained in T. We know that this number is well defined since all maximally independent subsets of T have the same cardinality.

We denote rank of T by $r(T)$. The number $r(S)$ is often called the rank of $\mathcal{M}$ and is also denoted by $r(\mathcal{M})$.

It is clear that $r(\cdot)$ takes value 0 on $\emptyset$ and that it is an integral, increasing function on subsets of S. Also $r(X \cup e) - r(X) \leq 1 \ \forall \ X \subseteq S, e \in S$. We then have the following:

Theorem 4.2.3 *(k) Let $r(\cdot)$ be the rank function of a matroid on S. Let $X \subseteq S$ and let $e_1, e_2 \in S$.*

i. If $r(X \cup e_1) = r(X \cup e_2) = r(X)$, then $r(X \cup e_1 \cup e_2) = r(X)$.

ii. $r(X \cup e) - r(X) \geq r(Y \cup e) - r(Y)$ whenever $X \subseteq Y \subseteq S - e$.

iii. $r(\cdot)$ is submodular, i.e.,

$$r(T_1) + r(T_2) \geq r(T_1 \cup T_2) + r(T_1 \cap T_2) \ \forall \ T_1, T_2 \subseteq S.$$

Proof :
i. Let B_x be a maximally independent subset of X. Clearly B_x is also a maximally independent subset of $X \cup e_1$ as well as of $X \cup e_2$. Suppose it is not a maximally independent subset of $X \cup e_1 \cup e_2$. Then $B_x \cup e_1$ or $B_x \cup e_2$ must be independent. But this would violate the conditions of the theorem.

ii. Select any maximally independent subset B_x of X and grow it into a maximally independent subset B_y of Y. Now $r(X \cup e) - r(X) \geq 0$ and $r(Y \cup e) - r(Y)$ can be 0 or 1. So we need only consider the case where $r(Y \cup e) - r(Y) = 1$. In this case B_y is not maximally independent in $Y \cup e$. So $B_y \cup e$ must be independent. But then so must $B_x \cup e$ be independent (by the first characteristic property of independent sets) and $r(X \cup e) - r(X) = 1$. Thus in all cases we have $r(X \cup e) - r(X) \geq r(Y \cup e) - r(Y)$.

iii. Let $T_2 - T_1 = \{e_1, \cdots, e_k\}$. We then have

$$\begin{aligned}
r(T_2) - r(T_1 \cap T_2) = & \quad r((T_1 \cap T_2) \cup e_1) - r(T_1 \cap T_2) \\
& + \ r((T_1 \cap T_2) \cup e_1 \cup e_2) - r((T_1 \cap T_2) \cup e_1) + \cdots \\
& + \ r((T_1 \cap T_2) \cup e_1 \cdots \cup e_k) - r((T_1 \cap T_2) \cup e_1 \cup \cdots \cup e_{k-1}) \\
\geq & \quad r(T_1 \cup e_1) - r(T_1) + \cdots \\
& + \ r(T_1 \cup e_1 \cup \cdots \cup e_k) - r(T_1 \cup e_1 \cup \cdots \cup e_{k-1}) \\
\geq & \quad r(T_1 \cup T_2) - r(T_1).
\end{aligned}$$

□

Example 4.2.5 *In the case of the polygon matroid $\mathcal{M}(\mathcal{G})$ associated with graph $\mathcal{G}$ (independent set $\equiv$ subforest), the rank function of the matroid is the same as the rank function of the graph. For the bond matroid $\mathcal{M}^*(\mathcal{G})$ associated with $\mathcal{G}$, the rank function of the matroid is the same as the nullity function of the graph.*

Submodular functions play a key role in subsequent developments in this book. Matroid rank functions are an important example of submodular functions. One can choose rank functions as a starting point for the description of matroids.

III. **Rank Axioms:** Let S be a finite set and let $r(\cdot)$ be an integer valued submodular function on subsets of S satisfying in addition

$$r(\emptyset) = 0$$

$$0 \leq r(X \cup e) - r(X) \leq 1 \quad \forall\, X \subseteq S, e \in S.$$

We call $r(\cdot)$ a **matroid rank function.**

Exercise 4.3 *(k) Let $r(\cdot)$ be a matroid rank function on subsets of S. Let $\mathcal{I}$ be the collection of all subsets X of S for which $r(X) =\mid X \mid$. Let members of $\mathcal{I}$ be called* **independent**. *Show that the pair $(S, \mathcal{I})$ is a matroid (satisfying the Independence Axioms).*

4.2.3 Circuit Axioms

Circuits of matroids satisfy the conditions given in the following theorem. These conditions can be used to define an axiom system for matroids using circuits as primitive objects.

Theorem 4.2.4 *(k) Let $\mathcal{M} \equiv (S, \mathcal{I})$ be a matroid and let C_1, C_2 be circuits of $\mathcal{M}$ with $e_c \in C_1 \cap C_2$ and $e_1 \in C_1 - C_2$. Then there exists a circuit $C_3 \subseteq C_1 \cup C_2 - e_c$ s.t. $e_1 \in C_3$.*

We need the following lemma for the proof of the theorem.

Lemma 4.2.1 *(k) Let $\mathcal{M}, C_1, C_2$ be as in Theorem 4.2.4. Then there exists a circuit $C_3' \subseteq C_1 \cup C_2 - e_c$.*

Proof of Lemma 4.2.1: We use the Independence Axioms. Let B_c be a maximally independent set contained in $C_1 \cup C_2$ and containing $C_1 - e_1$. Now $C_1 \not\subseteq B_c$ and $C_2 \not\subseteq B_c$. Hence, $e_1 \notin C_1$ and $C_2 - C_1 \not\subseteq B_c$. Also C_2 has at least one element not belonging to the other circuit. Hence, $\mid B_c \mid \leq \mid C_1 \cup C_2 \mid -2$.
Next, let B be a maximally independent set contained in $C_1 \cup C_2$ and containing $C_2 - e_c$. Clearly $e_c \notin B$. Now $\mid B \mid = \mid B_c \mid = \mid C_1 \cup C_2 \mid -2$. Hence there exists an element $e_1' \in C_1 - B - e_c$. Now $e_1' \cup B$ is dependent and therefore contains a circuit. This circuit does not contain e_c and may be taken as the desired circuit C_3'.

□

Proof of Theorem 4.2.4: The result is clearly true when the union of the two circuits has size 3 or, trivially, when it is 2. Let us suppose that the result is true when the size of the union of the two circuits is less than n.
Now let $\mid C_1 \cup C_2 \mid = n$, $e_c \in C_1 \cap C_2$ and $e_1 \in C_1 - C_2$. There exists a circuit $C_3' \subseteq C_1 \cup C_2 - e_c$, by Lemma 4.2.1. If $e_1 \in C_3'$ we are done. So let us assume that $e_1 \notin C_3'$.
We have, $C_3' \not\subseteq C_1$. So $C_3' \cap C_2 \not\subseteq C_1 \cap C_2$. Let $e_2 \in C_3' \cap C_2 - C_1 \cap C_2$.
Consider $C_2 \cup C_3'$. Now $e_1 \notin C_2 \cup C_3'$ so that $\mid C_2 \cup C_3' \mid < n$. Further $e_2 \in C_3' \cap C_2$ and $e_c \in C_2 - C_3'$. By the induction hypothesis there is a circuit $C_2' \subseteq C_2 \cup C_3' - e_2$ s.t. $e_c \in C_2'$.

Consider $C_1 \cup C_2'$. We have $e_c \in C_2' \cap C_1$ and $e_1 \in C_1 - C_2'$. Further $e_2 \notin C_1 \cup C_2'$ so that $\mid C_1 \cup C_2' \mid < n$. Hence, by the induction hypothesis, there exists a circuit $C_3 \subseteq C_1 \cup C_2' - e_c$ s.t. $e_1 \in C_3$.

□

Example 4.2.6 *(k) In the case of the polygon matroid $\mathcal{M}(\mathcal{G})$ associated with graph $\mathcal{G}$ (independent set $\equiv$ subforest), a circuit of the matroid is the same as a circuit of the graph. For the bond matroid $\mathcal{M}^*(\mathcal{G})$ associated with $\mathcal{G}$ (independent set $\equiv$ subcoforest), a circuit of the matroid is the same as a cutset of the graph. For the matroid associated with the columns of a matrix, a circuit is a minimal dependent set of columns.*

IV. **Circuit Axioms:** Let S be a finite set. A matroid $\mathcal{M}$ is a pair $(S, \mathcal{C})$ where $\mathcal{C}$ is a family of subsets of S called circuits satisfying
i. No member of $\mathcal{C}$ is a proper subset of another.
ii. Let $C_1, C_2 \in \mathcal{C}$ and let $e_c \in C_1 \cap C_2$ and $e_1 \in C_1 - C_2$. Then there exists $C_3 \in \mathcal{C}$ s.t. $C_3 \subseteq C_1 \cup C_2 - e_c$ and $e_1 \in C_3$.

Exercise 4.4 *(k) Let $\mathcal{M} \equiv (S, \mathcal{C})$ satisfy the circuit axioms. Let $\mathcal{I}$ be the class of subsets of S that do not contain a member of $\mathcal{C}$. Show that $(S, \mathcal{I})$ satisfies the independence axioms.*

4.2.4 Closure Axioms

The **span** of a set of vectors X is the collection of all linear combinations of vectors in X. Sets of vectors X and Y are said to **cospan** each other, if each is contained in the span of the other. The span of X is also spoken of as the vector space **generated by** X. Suppose $Y \supseteq X$. Then the **span of** X **relative to** Y is the collection of all linear combinations of vectors in X which lie in Y. We generalize this notion to matroids as follows:
Let $\mathcal{M} \equiv (S, \mathcal{I})$ be a matroid and let $r(\cdot)$ be its rank function (i.e., $r(X)$ is the size of the maximally independent set contained in X). The **span** or **closure** of $X \subseteq S$, denoted $\int(X)$, is the maximal superset of X that has the same rank as X. As in the case of vectors, subsets X and Y are said to **cospan** each other in $\mathcal{M}$ if each is contained in the span of the other. A subset is **closed** if it is equal to its closure. Closed sets of matroids are also called its **flats**.
Suppose $Y_1, Y_2 \supseteq X$ and $r(Y_1) = r(Y_2) = r(X)$. Then it follows by submodularity that $r(Y_1 \cup Y_2) = r(Y_1 \cap Y_2) = r(X)$. Thus, there is a unique maximal superset of X which has rank $r(X)$. Hence, the notion of span is well defined.
Suppose $e \in \int(X) - X$. Then it is clear that $r(e \cup X) = r(X)$. Thus, there is a maximally independent set $B_x \subseteq X$ s.t. $B_x \cup e$ is dependent. The circuit $L(e, B_x)$ has only the element e outside X. On the other hand if C is a circuit s.t. $C - X = \{e'\}$, then $C - e'$ can be grown to a maximally independent set B contained in $e' \cup X$. Clearly $e' \cup B \supseteq C$ and therefore, $e' \notin B$. Hence, $B \subseteq X$ and $r(X) = r(X \cup e')$.
We thus see that

Lemma 4.2.2 *(k) An element $e_i \in (\int(X) - X)$ iff there exists a circuit C_i s.t. $\{e_i\} = C_i - X$.*

Example 4.2.7 *(k) We have already described the closure (span) operator for vectors.*
In the case of the polygon matroid $\mathcal{M}(\mathcal{G})$ associated with graph $\mathcal{G}$ (independent set $\equiv$ subforest), the closure of a set of edges T is obtained by first building the subgraph $\mathcal{G}_T$ on T and then adding all the edges of $\mathcal{G}$ with both end points in the same component of $\mathcal{G} \,.\, T$. The edge set of this new graph is $\int(T)$.
In the case of the bond matroid $\mathcal{M}^*(\mathcal{G})$ associated with $\mathcal{G}$ (independent set $\equiv$ subcoforest), the closure of $T \subseteq E(\mathcal{G})$ is obtained as follows: Delete T from $\mathcal{G}$. Let T_a be the set of coloops of $\mathcal{G} \,.\, (E-T)$. Then in the matroid $\mathcal{M}^*(\mathcal{G}), \int(T) = T \cup T_a$.

It can be verified that the 'closure (span) operator' satisfies the following properties:
(S1) $X \subseteq \int(X) \; \forall \; X \subseteq S$;
(S2) $Y \supseteq X \Rightarrow \int(Y) \supseteq \int(X) \;\; \forall \; X, Y \subseteq S$;
(S3) $\int(X) = \int(\int(X)) \; \forall \; X \subseteq S$;
(S4) if $y \notin \int(X)$, *but* $y \in \int(X \cup x)$, then $x \in \int(X \cup y) \; \forall \; X \subseteq S, \;\; \forall \; x, y \in S$.

Exercise 4.5 *(k) Prove property (S4) for the span operator of a matroid.*

These properties can be used to construct an axiom system for matroids in terms of the notion of span or closure.

V. **Closure Axioms:** Let S be a finite set and let $\int : 2^S \rightarrow 2^S$. Then $\int(\cdot)$ is a **matroid closure operator** iff it satisfies the properties (S1), (S2), (S3), (S4).
We then have the following theorem.

Theorem 4.2.5 *(k) Let $\int(\cdot)$ be a matroid closure operator on subsets of a set S. Let $\mathcal{I}$ be the collection of subsets of S defined by*

$$X \in \mathcal{I} \Rightarrow e \notin \int(X - e) \; \forall \; e \in X.$$

Then $(S, \mathcal{I})$ satisfies the Independence Axioms of a matroid.

The proof of this theorem is contained in the solution to the following exercises.

Exercise 4.6 *(k) Let $\mathcal{I}$ be defined as in Theorem 4.2.5. Show that $Y \in \mathcal{I}$ and $X \subseteq Y \Rightarrow X \in \mathcal{I}$.*

Exercise 4.7 *(k) Let $\mathcal{I}$ be as in Theorem 4.2.5. Show that, if $T \subseteq S$, and X is a maximal subset of T that is also a member of $\mathcal{I}$, then $\int(X) = \int(T)$.*

Exercise 4.8 *(k) Let $\mathcal{I}$ be as in Theorem 4.2.5. Let B_1, B_2 be maximal members of $\mathcal{I}$. Let $e_1 \in B_1$. Show that there exists $e_2 \in B_2$ s.t. $(B_1 - e_1) \cup e_2$ is a maximal member of $\mathcal{I}$.*

4.3 Dual of a Matroid

Matroids occur naturally in pairs. Consider a vector space $\mathcal{V}$ on S. Let $\mathbf{R}$ be a standard representative matrix of $\mathcal{V}$. If

$$\mathbf{R} \equiv \begin{array}{c} \begin{array}{ccc} B & & S-B \end{array} \\ \left[\begin{array}{ccc} \mathbf{I} & \vdots & \mathbf{K} \end{array}\right] \end{array}, \tag{4.1}$$

then we know that

$$\mathbf{R}^* \equiv \begin{array}{c} \begin{array}{ccc} B & & S-B \end{array} \\ \left[\begin{array}{ccc} -\mathbf{K}^T & \vdots & \mathbf{I} \end{array}\right] \end{array}, \tag{4.2}$$

is a representative matrix of $\mathcal{V}^{\perp}$. Let $\mathcal{M}(\mathcal{V})$ denote the matroid on S whose independent sets are independent column sets of a representative matrix of $\mathcal{V}$ (note that column dependence structure of all representative matrices of $\mathcal{V}$ is identical). Then it is clear that the bases of $\mathcal{M}(\mathcal{V}^{\perp})$ are cobases of $\mathcal{M}(\mathcal{V})$ and vice versa. This situation holds also for arbitrary matroids and the pairs of matroids are said to be dual to each other.

Theorem 4.3.1 *(k) Let $\mathcal{M} \equiv (S, \mathcal{I})$ be a matroid. Then, $\mathcal{M}^* \equiv (S, \mathcal{I}^*)$, where $X \in \mathcal{I}^*$ iff $S - X$ contains a base of $\mathcal{M}$, is a matroid.*

Proof : It is clear that if $X \subseteq Y$ and $Y \in \mathcal{I}^*$, then $X \in \mathcal{I}^*$. So we need only verify that $\mathcal{M}^*$ satisfies the Base Axioms with condition (ii'). Let B_1^*, B_2^* be two maximal members of $\mathcal{I}^*$. Let $e_1 \in B_1^* - B_2^*$. Now, by the definition of $\mathcal{M}^*, B_1 \equiv S - B_1^*, B_2 \equiv S - B_2^*$ are bases of $\mathcal{M}$ and $e_1 \in B_2 - B_1$. Hence, by condition (ii) of Base Axioms, there exists $e_2 \in B_1 - B_2$ s.t. $(B_1 - e_2) \cup e_1$ is a base of $\mathcal{M}$. But then $e_2 \in B_2^* - B_1^*$ and $S - ((B_1 - e_2) \cup e_1)$ is a maximal member of $\mathcal{I}^*$, i.e., $(B_1^* - e_1) \cup e_2$ is a maximal member of $\mathcal{I}^*$. Thus $\mathcal{M}^* \equiv (S, \mathcal{I}^*)$ satisfies the Base Axioms with condition (ii') and hence is a matroid.

□

Let $\mathcal{M} \equiv (S, \mathcal{I})$ be a matroid. Then the **dual matroid** of $\mathcal{M}$ denoted by $\mathcal{M}^*$, is the pair $(S, \mathcal{I}^*)$ where $X \in \mathcal{I}^*$ iff $S - X$ contains a base of $\mathcal{M}$.
It is clear that $(\mathcal{M}^*)^* = \mathcal{M}$.

Example 4.3.1 *(k) If $\mathcal{V}$ is a vector space on S then $(\mathcal{M}(\mathcal{V}))^* = \mathcal{M}(\mathcal{V}^{\perp})$. If $\mathcal{M}(\mathcal{G})$ is the polygon matroid associated with graph $\mathcal{G}$, then we know that $\mathcal{M}(\mathcal{G}) = \mathcal{M}(\mathcal{V}_v(\mathcal{G}))$ and $(\mathcal{M}(\mathcal{G}))^* = \mathcal{M}((\mathcal{V}_v(\mathcal{G}))^{\perp}) = \mathcal{M}(\mathcal{V}_i(\mathcal{G}))$. Thus matroid $(\mathcal{M}(\mathcal{G}))^*$ is the bond matroid associated with $\mathcal{G}$, for which the independent subsets are subcoforests. When the graph $\mathcal{G}$ is planar there exists a graph $\mathcal{G}^*$ s.t. $\mathcal{V}_v(\mathcal{G}^*) = \mathcal{V}_i(\mathcal{G})$. It would follow that $\mathcal{M}(\mathcal{G}^*) = (\mathcal{M}(\mathcal{G}))^*$.*

Exercise 4.9 *(k) Give an informal algorithm for the construction of $\mathcal{G}^*$, where $\mathcal{G}$ is a planar graph, whose embedding on a plane is given.*

The circuit of the matroid $\mathcal{M}^*$ is called a **bond** of $\mathcal{M}$. The following theorem gives some characterizations of a bond.

Theorem 4.3.2 *(k) Let $\mathcal{M} \equiv (S, \mathcal{I})$ be a matroid. A subset $K \subseteq S$ is a bond of $\mathcal{M}$ iff any of the following equivalent conditions are satisfied:*

i. K is a circuit of $\mathcal{M}^*$.

ii. K is a minimal set that intersects every base of $\mathcal{M}$.

iii. K is a minimal set that meets no circuit of $\mathcal{M}$ in a single element.

We need the following lemma to prove the theorem.

Lemma 4.3.1 *(k) Let L be independent in $\mathcal{M} \equiv (S, \mathcal{I})$ and let K be independent in $\mathcal{M}^*$. Further let $L \cap K = \emptyset$. Then there exists a base of $\mathcal{M}$ that contains L and does not intersect K.*

Proof of Lemma 4.3.1 By the definition of independence in $\mathcal{M}^*$, $S - K$ contains a base B of $\mathcal{M}$. Now L is independent in $\mathcal{M}$ and is contained in $S - K$. So there exists a subset B', that is maximally independent in $\mathcal{M}$ under the condition that it contains L and is contained in $S - K$. Now $\mid B' \mid = \mid B \mid$ (condition (ii) of Independence axioms). Hence B' is the desired base of $\mathcal{M}$.

□

Proof of Theorem 4.3.2: Condition (i) is the definition of a bond. We will show that (ii) and (iii) are equivalent to (i).

(i) ⇔ (ii): K is a minimal set that is not contained in any base of $\mathcal{M}^*$, equivalently, that intersects every base of $\mathcal{M}$ (since bases of $\mathcal{M}$ are complements of bases of $\mathcal{M}^*$).

(i) ⇔ (iii): Suppose K is a circuit of $\mathcal{M}^*$. Let $T \subset K$. Then T is independent in $\mathcal{M}^*$. Let B_T^* be a base of $\mathcal{M}^*$ that contains T. Let B_T be the complement of B_T^*. Clearly B_T is a base of $\mathcal{M}$ and $B_T \cap T = \emptyset$. Let $e \in T$. Consider the circuit $L(e, B_T)$ of $\mathcal{M}$ (the unique circuit of $\mathcal{M}$ contained in $e \cup B_T$). This circuit intersects T in $\{e\}$. Thus every proper subset of K meets some circuit of $\mathcal{M}$ in a single element. Next, suppose K meets a circuit C of $\mathcal{M}$. Let $e \in C \cap K$. We have, $K - e$ independent in $\mathcal{M}^*$ and $C - e$ independent in $\mathcal{M}$. If $(C - e) \cap (K - e) = \emptyset$, by Lemma 4.3.1, there exists a base B of $\mathcal{M}$ that contains $C - e$ but does not intersect $K - e$. Now $e \in B$ or $e \in (S - B)$. In the former case $C \subseteq B$ and in the latter $K \subseteq (S - B)$ which contradicts the independence of B in $\mathcal{M}$ and $S - B$ in $\mathcal{M}^*$ respectively. We conclude that $C - e$ and $K - e$ must intersect. Thus, $\mid C \cap K \mid > 1$. Thus, K is a minimal set that meets no circuit of $\mathcal{M}$ in a single element.

Next, let K be a subset of S that has the property $\mathcal{P}$ of meeting no circuit of $\mathcal{M}$ in a single element. Then, K cannot be contained in a cobase of $\mathcal{M}$, since, if $e \in S - B$, where B is a base of $\mathcal{M}$, then $L(e, B)$ meets $(S - B)$ in e. Hence, K contains a circuit of $\mathcal{M}^*$. So if K is a minimal subset having the property $\mathcal{P}$, then K contains a circuit K_1 of $\mathcal{M}^*$. We have already seen that the circuit K_1 of $\mathcal{M}^*$ must have the property $\mathcal{P}$. So if $K_1 \subset K$ there would be a contradiction. We conclude that $K_1 = K$, i.e., K is a circuit of $\mathcal{M}^*$.

□

We next relate the rank function of a matroid to that of its dual.

Theorem 4.3.3 *(k) Let $\mathcal{M} \equiv (S, \mathcal{I})$ be a matroid and let $\mathcal{M}^* \equiv (S, \mathcal{I}^*)$ be its dual. Let $r(\cdot), r^*(\cdot)$ be the rank functions of $\mathcal{M}$ and $\mathcal{M}^*$ respectively. Then*

$$r^*(X) = \mid X \mid - (r(S) - r(S - X))$$

Proof : We have, $r^*(X) \equiv$ size of a maximal subset of X that is a member of $\mathcal{I}^*$. Now $T \in \mathcal{I}^*$ iff there exists a base of $\mathcal{M}$ that is contained in $S - T$. Thus,

$r^*(X)$ = size of a maximal subset of X whose complement contains a base of $\mathcal{M}$
= size of the complement relative to X of a minimal subset of X that is the intersection of a base of $\mathcal{M}$ with X.

Now a base of $\mathcal{M}$ has minimal intersection with X iff it has maximal intersection with $(S - X)$. Hence, the size of a minimal intersection of a base of $\mathcal{M}$ with $X = r(S) - r(S - X)$. Therefore, $r^*(X)$ is the size of the complement of a set of size $(r(S) - r(S - X))$ relative to X. It follows that, $r^*(X) = \mid X \mid - (r(S) - r(S - X))$.

□

We know that, if B is a base of $\mathcal{M}$ and $e \notin B$, then $e \cup B$ contains a unique circuit $L(e, B)$ called the fundamental circuit of e with respect to B in the matroid $\mathcal{M}$. Let $e_t \in B$. Now $(S - B)$ is a base of $\mathcal{M}^*$. Consider the fundamental circuit of e_t with respect to $(S - B)$ in the matroid $\mathcal{M}^*$. This is a bond of $\mathcal{M}$ and meets B in e_t. We call this bond, the **fundamental bond of e_t with respect to B in the matroid $\mathcal{M}$ and denote it by** $B(e_t, B)$. We then have the following theorem.

Theorem 4.3.4 *(k) Let B be a base of a matroid $\mathcal{M}$ on S. Let $e_t \in B$ and let $e_c \in S - B$. Then*

i. $B \cup e_c - e_t$ is a base of $\mathcal{M}$ iff $e_c \in B(e_t, B)$ or, equivalently, $e_t \in L(e_c, B)$ and hence

ii. $e_c \in B(e_t, B)$ iff $e_t \in L(e_c, B)$.

Proof :
i. We know that $B \cup e_c$ contains a unique circuit $L(e_c, B)$. If $e_t \in L(e_c, B)$, then $B \cup e_c - e_t$ is independent in $\mathcal{M}$ and has the same size as B and, therefore, is a base of $\mathcal{M}$. If $e_t \notin L(e_c, B)$, then $B \cup e_c - e_t$ contains $L(e_c, B)$ and is therefore not a base of $\mathcal{M}$. Thus, $B \cup e_c - e_t$ is a base of $\mathcal{M}$ iff $e_t \in L(e_c, B)$.
On the other hand, working with the dual matroid, $(S - B) \cup e_t - e_c$ is a base of $\mathcal{M}^*$ iff e_c belongs to the fundamental circuit of e_t with respect to $S - B$ in $\mathcal{M}^*$. Equivalently, $B \cup e_c - e_t$ is a base of $\mathcal{M}$ iff $e_c \in B(e_t, B)$.

ii. We have, from the above, that $B \cup e_c - e_t$ is a base of $\mathcal{M}$ iff $e_t \in L(e_c, B)$ and also iff $e_c \in B(e_t, B)$. The result follows.

□

Let us next consider the closure operators of a matroid and its dual. We have the following

Theorem 4.3.5 *(k) Let $\mathcal{M}$ be a matroid on S. Let $\int(\cdot), \int^*(\cdot)$ be the closure operators of $\mathcal{M}, \mathcal{M}^*$ respectively. Let $T \subseteq S$ and let B, B' be bases of $\mathcal{M}$ that intersect T maximally and minimally, respectively, among all bases of $\mathcal{M}$. Then*

i. $e \in \int(T)$ iff $e \in T$ or $e \cup (B \cap T)$ is dependent in $\mathcal{M}$.

ii. $e \in \int(T)$ iff $e \in T$ or $e \in S - B$ and $(L(e, B)) \cap B \subseteq T$.

iii. $e \in \int^*(T)$ iff $e \in T$ or $e \in B'$ and $(B(e, B')) \cap (S - B') \subseteq T$, where the fundamental circuit and bond are taken with respect to $\mathcal{M}$.

Proof : Let $B_T \equiv B \cap T, B'_T \equiv B' \cap T$. We will throughout consider the case where $e \notin T$.
i. We have, $e \in \int(T)$ iff there exists a circuit C s.t. $e \in C$ and $C - e \subseteq T$ (Lemma 4.2.2). Now, if $e \cup B_T$ is dependent, there exists a circuit contained in it. This circuit has e as a member, since B_T is independent. Hence, $e \in \int(T)$.
Next, if $e \in \int(T)$, we have, $r(B_T) = r(T) = r(T \cup e) \geq r(B_T \cup e)$. We conclude (since $r(\cdot)$ is increasing), that $r(B_T) = r(B_T \cup e)$. Hence $B_T \cup e$ is dependent.

ii. Let $e \in \int(T)$. Then by (i) above, $e \cup B_T$ is dependent in $\mathcal{M}$. So it contains a circuit which has e as a member. This must be the unique circuit $L(e, B)$ contained in $e \cup B$. Hence, $L(e, B) \cap B \subseteq T$.
Next suppose $L(e, B) \cap B \subseteq T$, i.e., $L(e, B) \cap B \subseteq B_T$. But this means that $e \cup B_T$ contains the circuit $L(e, B)$. So $e \cup B_T$ is dependent in $\mathcal{M}$. So by (i) $e \in \int(T)$.

iii. We first observe that B' has a minimal intersection among all bases of $\mathcal{M}$ with T iff $S - B'$ has a maximal intersection among all bases of $\mathcal{M}^*$ with T. So by (ii) above, it follows that $e \in \int^*(T)$ iff $e \in T$ or
$e \in (S - (S - B'))$ and $L^*(e, (S - B')) \cap (S - B') \subseteq T$,
where $L^*(e, (S - B'))$ is the fundamental circuit of e with respect to $(S - B')$ in the matroid $\mathcal{M}^*$.
Now $L^*(e, (S - B')) = B(e, B')$. So $e \in \int^*(T)$ iff $e \in T$ or $e \in B'$ and $(B(e, B')) \cap (S - B') \subseteq T$.

□

Exercise 4.10 *(k) Let $\mathcal{G}$ be a graph and let $\mathbf{v} \in \mathcal{V}_v(\mathcal{G})$ and let $\mathbf{i} \in \mathcal{V}_i(\mathcal{G})$. Let $T \subseteq E \equiv E(\mathcal{G})$. Suppose only $\mathbf{v}/T, \mathbf{i}/T$ are known about $\mathbf{v}$ and $\mathbf{i}$. Let $\int(\cdot), \int^*(\cdot)$ be the closure operators of $\mathcal{M}(\mathcal{G}), \mathcal{M}^*(\mathcal{G})$ respectively. Show that $v(e)$ $(i(e))$ can be uniquely determined iff $e \in \int(T)$ $(e \in \int^*(T))$.*

Exercise 4.11 *(k) Let $f, g : \Re \to \Re$ and let $f(\cdot)$ be increasing. Consider the functions $|\cdot|, r(\cdot)$ on subsets of S where $| X |$ is the size of X and $r(\cdot)$ is the rank function of a matroid $\mathcal{M}$. Show that $f(|\cdot|) - g(r(\cdot))$ reaches a maximum on a closed subset of $\mathcal{M}$.*

4.4 Minors of Matroids

In this section we generalize the notion of minors of graphs and vector spaces to matroids.

Let $\mathcal{M} \equiv (S, \mathcal{I})$ be a matroid. Let $T \subseteq S$. The **restriction** (or **reduction**) of $\mathcal{M}$ to T, denoted by $\mathcal{M} \cdot T$, is the matroid $(T, \mathcal{I}_T)$ where $\mathcal{I}_T$ is the collection of all subsets of T which are members of $\mathcal{I}$. The **contraction** of $\mathcal{M}$ to T, denoted by $\mathcal{M} \times T$, is the pair $(T, \mathcal{I}'_T)$ in which $X \in \mathcal{I}'$ iff $X \cup B_{S-T} \in \mathcal{I}$ whenever B_{S-T} is a base of $\mathcal{M} \cdot (S-T)$. We show below that $\mathcal{M} \times T$ is also a matroid. A **minor** of $\mathcal{M}$ is a matroid of the form $(\mathcal{M} \times T_1) \cdot T_2$ or $(\mathcal{M} \cdot T_1) \times T_2, T_2 \subseteq T_1 \subseteq S$. Since there is no room for confusion we omit the bracket while denoting minors.
It is clear from the definition that $\mathcal{M} \cdot T$ is a matroid. We prove below that $\mathcal{M} \times T$ is also a matroid.

Lemma 4.4.1 *(k) Let $\mathcal{M}$ be a matroid on S and let $X \subseteq T \subseteq S$. Suppose B^1_{S-T}, B^2_{S-T} are two bases of $\mathcal{M} \cdot (S-T)$ and $X \cup B^1_{S-T}$ is independent in $\mathcal{M}$. Then $X \cup B^2_{S-T}$ is also independent in $\mathcal{M}$.*

Proof : Suppose the lemma fails. Then there exist two bases B^1_{S-T}, B^2_{S-T} of $\mathcal{M} \cdot (S-T)$ s.t. $X \cup B^1_{S-T}$ is independent, $X \cup B^2_{S-T}$ is dependent and $\mid B^1_{S-T} - B^2_{S-T} \mid$ is a minimum for these conditions. Let $e \in B^2_{S-T} - B^1_{S-T}$. Then $e \cup B^1_{S-T}$ contains the unique circuit $L(e, B^1_{S-T})$ of $\mathcal{M} \cdot (S-T)$. Now $L(e, B^1_{S-T})$ has some element $e' \in (B^1_{S-T} - B^2_{S-T})$. Hence, $B^3_{S-T} = (B^1_{S-T} - e') \cup e$ is a base of $\mathcal{M} \cdot (S-T)$, using Theorem 4.2.2.
Let B^1 be a base of $\mathcal{M}$ containing $X \cup B^1_{S-T}$. We know that $e \cup B^1$ contains the unique circuit $L(e, B^1)$. Now by the definition of $\mathcal{M} \cdot (S-T)$ it follows that circuits of $\mathcal{M} \cdot (S-T)$ are the same as circuits of $\mathcal{M}$ contained in $(S-T)$. Hence, $L(e, B^1) = L(e, B^1_{S-T})$ and $B^3 \equiv (B^1 - e') \cup e$ is a base of $\mathcal{M}$. Now $X \cup B^3_{S-T} \subseteq B^3$ and therefore, $X \cup B^3_{S-T}$ is independent in $\mathcal{M}$. But $\mid B^3_{S-T} - B^2_{S-T} \mid < \mid B^1_{S-T} - B^2_{S-T} \mid$, a contradiction. We conclude that $X \cup B^2_{S-T}$ is independent in $\mathcal{M}$.

□

To prove that $\mathcal{M} \times T \equiv (S, \mathcal{I}'_T)$ is a matroid, we will verify that it satisfies the Independence Axioms. If $Y \in \mathcal{I}'_T$ and $X \subseteq Y$ it is clear from the definition of $\mathcal{I}'_T$ that $X \in \mathcal{I}'_T$. Let $T_1 \subseteq T$ and let X_1, X_2 be maximal members of $\mathcal{I}'_T$ contained in T_1. Then X_1, X_2 are maximal with respect to the property that $X_1, X_2 \subseteq T_1$ and $X_1 \uplus B_{S-T}$, $X_2 \uplus B_{S-T}$ are independent in $\mathcal{M}$, for each base B_{S-T} of $\mathcal{M} \cdot (S-T)$. Lemma 4.4.1 assures us that it is sufficient that they be independent for some base B_{S-T} of $\mathcal{M} \cdot (S-T)$. Hence, $X_1 \uplus B_{S-T}, X_2 \uplus B_{S-T}$ are maximally independent subsets of $T_1 \uplus (S-T)$ in $\mathcal{M}$. Thus, $\mid X_1 \uplus B_{S-T} \mid = \mid X_2 \uplus B_{S-T} \mid$ and therefore, $\mid X_1 \mid = \mid X_2 \mid$ as required.

Exercise 4.12 *(k) Let $\mathcal{M}$ be a matroid on S and let $T \subseteq S$. Show that*

i. the union of a base of $\mathcal{M} \times T$ and a base of $\mathcal{M} \cdot (S-T)$ is a base of $\mathcal{M}$.

ii. $r(\mathcal{M} \times T) + r(\mathcal{M} \cdot (S-T)) = r(\mathcal{M})$.

We now study the relation between primitive notions (such as bases, circuits, bonds) associated with a matroid and those associated with restrictions and contractions of a matroid. We begin with bases.

Theorem 4.4.1 *(k) Let $\mathcal{M}$ be a matroid on S. Let $T \subseteq S$. Then*

i. B_T is a base of $\mathcal{M} \cdot T$ iff it is a maximal intersection of a base of $\mathcal{M}$ with T.

ii. B'_T is a base of $\mathcal{M} \times T$ iff it is a minimal intersection of a base of $\mathcal{M}$ with T.

Proof :
i. B_T is a maximal intersection of a base of $\mathcal{M}$ with T iff it is a maximal subset of T that is independent in $\mathcal{M}$, i.e., iff B_T is a base of $\mathcal{M} \cdot T$.

ii. By the definition of $\mathcal{M} \times T$ and Lemma 4.4.1 (see Exercise 4.12) B'_T is a base of $\mathcal{M} \times T$ iff $B'_T \cup B_{S-T}$ is a base of $\mathcal{M}$ for some base B_{S-T} of $\mathcal{M} \cdot (S-T)$, i.e., iff a base B of $\mathcal{M}$ intersects T in B'_T and intersects $(S-T)$ maximally among all bases of $\mathcal{M}$, i.e., iff a base of B of $\mathcal{M}$ intersects T in B'_T and this intersection is minimal among all bases of $\mathcal{M}$.

□

We next characterize circuits of minors.

Theorem 4.4.2 *(k) Let $\mathcal{M}$ be a matroid on S and let $T \subseteq S$. Then*

i. C_T is a circuit of $\mathcal{M} \cdot T$ iff it is a circuit of $\mathcal{M}$ contained in T.

ii. C_T is circuit of $\mathcal{M} \times T$ iff it is a minimal nonvoid intersection of a circuit of $\mathcal{M}$ with T.

Proof :
i. Independent sets of $\mathcal{M} \cdot T$ are just the independent sets of $\mathcal{M}$ contained in T. Hence, C_T is a minimal dependent set of $\mathcal{M} \cdot T$ iff it is a minimal dependent set of $\mathcal{M}$ contained in T.

ii. Let C_T be a circuit of $\mathcal{M} \times T$. Let B_{S-T} be a base of $\mathcal{M} \cdot T$. Then $C_T \cup B_{S-T}$ is a dependent set in $\mathcal{M}$, by the definition of $\mathcal{M} \times T$ and Lemma 4.4.1. Hence, $C_T \cup B_{S-T}$ contains a circuit which however cannot be contained in B_{S-T} (B_{S-T} is independent in $\mathcal{M}$). Hence, there exists a circuit C of $\mathcal{M}$ s.t. $C_T \supseteq C \cap T$.

Next consider a circuit C of $\mathcal{M}$ that has nonvoid intersection with T. Now $C - T$ is independent in $\mathcal{M}$ and, therefore, in $\mathcal{M} \cdot (S-T)$. Hence, there exists a base B_{S-T} of $\mathcal{M} \cdot (S-T)$ that contains $C - T$. Now $(C \cap T) \cup (B_{S-T})$ contains C and is therefore dependent in $\mathcal{M}$. Hence, $C \cap T$ is dependent in $\mathcal{M} \times T$, by the definition of independence in $\mathcal{M} \times T$ and using Lemma 4.4.1. Hence, $C \cap T$ contains a circuit of $\mathcal{M} \times T$.

Let ℓ_1 be the collection of circuits of $\mathcal{M} \times T$ and ℓ_2, the collection of minimal nonvoid intersections of circuits of $\mathcal{M}$ with T. If $C \in \ell_1 (C \in \ell_2)$ no proper subset of C belongs to ℓ_1 (belongs to ℓ_2). We have further shown that each member of

$\ell_1(\ell_2)$ contains a member of $\ell_2(\ell_1)$. It follows therefore that $\ell_1 = \ell_2$.

□

Next we consider the closure operators of minors.

Theorem 4.4.3 *(k) Let $\mathcal{M}$ be a matroid on S and let $T \subseteq S$. Let $\int(\cdot), \int_r(\cdot), \int_c(\cdot)$ be the closure operators respectively of $\mathcal{M}, \mathcal{M} \cdot T, \mathcal{M} \times T$. Let $P \subseteq T$. Then*

i. $\int_r(P) = (\int(P)) \cap T$.

ii. $\int_c(P) = (\int(P \cup (S - T))) \cap T$.

Proof :
i. We know by Lemma 4.2.2 that $e \in \int_r(P)$ iff $e \in P$ or there exists a circuit C of $\mathcal{M} \cdot T$ s.t. $e \in C$ and $(C - e) \subseteq P$. Now circuits of $\mathcal{M} \cdot T$ are identical to circuits of $\mathcal{M}$ contained in T. So $e \in \int_r(P)$ iff $e \in P$ or there exists a circuit C of $\mathcal{M}$ s.t. $C \subseteq T, e \in C$ and $(C - e) \subseteq P$. So $e \in \int_r(P)$ iff $e \in T$ and $e \in \int(P)$. This proves the required result.

ii. We have, $e \in \int_c(P)$ iff $e \in P$ or there exists a circuit C_T of $\mathcal{M} \times T$ s.t. $e \in C_T$ and $(C_T - e) \subseteq P$. If C_T is a circuit of $\mathcal{M} \times T$, then there exists a circuit C of $\mathcal{M}$ s.t. $C \cap T = C_T$. Now $e \in C$ and $(C - e) \subseteq P \cup (S - T)$. Hence, $e \in \int(P \cup (S - T))$, i.e., $e \in (\int(P \cup (S - T))) \cap T$. Hence, $e \in \int_c(P)$ implies $e \in (\int(P \cup (S - T))) \cap T$.

Next let $e \in (\int(P \cup (S - T))) \cap T$. Then either $e \in P$ or there exists a circuit C of $\mathcal{M}$ s.t. $e \in C \cap T$, $(C - e) \subseteq P \cup (S - T)$. We consider the latter (nontrivial) situation. Let us suppose that $C \cap T$ is a minimal subset under the conditions that C is a circuit of $\mathcal{M}$ and $e \in C \cap T$. We claim that $C \cap T$ is a circuit of $\mathcal{M} \times T$. Suppose otherwise. Then there exists a circuit C_T of $\mathcal{M} \times T$ that is a proper subset of $C \cap T$. (In fact, by the minimality of $C \cap T$, $e \notin C_T$). Let $C_T = C' \cap T$ where C' is a circuit of $\mathcal{M}$ and let $e' \in C_T$. Now $e \in C - C', e' \in C \cap C'$. Hence, by Theorem 4.2.4, there exists a circuit $C_3 \subseteq C \cup C'$ s.t. $C_3 \subseteq C \cup C' - e'$ and $e \in C_3$. But $C_3 \cap T \subset C \cap T$ and $e \in C_3 \cap T$. This contradicts the minimality of $C \cap T$. We conclude therefore, that $C \cap T$ is a circuit of $\mathcal{M} \times T$. We thus have that e belongs to the circuit $C \cap T$ of $\mathcal{M} \times T$ and $C \cap T - e \subseteq P$. Hence, $e \in \int_c(P)$.

□

The next result speaks of the rank function of minors. We omit the proof since it is discussed in Exercise 10.11.

Theorem 4.4.4 *(k) Let $\mathcal{M}$ be matroid on S. Let $T \subseteq S$. Let $r(\cdot), r_r(\cdot), r_c(\cdot)$ be the rank functions of $\mathcal{M}, \mathcal{M} \cdot T, \mathcal{M} \times T$ respectively. Then*

i. $r_r(X) = r(X), X \subseteq T$,

ii. $r_c(X) = r(X \cup (S - T)) - r(S - T), X \subseteq T$.

Lastly we relate the minors of the dual matroid to the duals of the minors of the original matroid.

Theorem 4.4.5 *(k) Let $\mathcal{M}$ be a matroid on S. Let $T \subseteq S$. Then*

i. $(\mathcal{M} \times T)^* = \mathcal{M}^* \cdot T$,

ii. $(\mathcal{M} \cdot T)^* = \mathcal{M}^* \times T$.

Proof :
i. We observe that B_T is a base of $\mathcal{M} \times T$ iff it is a minimal intersection of a base of $\mathcal{M}$ with T, i.e., iff it is the complement of a maximal intersection of a cobase of $\mathcal{M}$ with T, i.e., iff it is the complement of a maximal intersection of a base of $\mathcal{M}^*$ with T i.e., iff it is the complement of a base of $\mathcal{M}^* \cdot T$.

ii. We have, by the definition of dual, a matroid is the dual of its dual. Hence, by (i) above, $(\mathcal{M}^* \times T)^* = (\mathcal{M}^*)^* \cdot T = \mathcal{M} \cdot T$

$$i.e., \qquad \mathcal{M}^* \times T = (\mathcal{M}^* \times T)^{**} = (\mathcal{M} \cdot T)^*.$$

□

Exercise 4.13 *(k) Let $\mathcal{M}$ be a matroid on S. Let $P \subseteq Q \subseteq S$. Show that*

i. $(\mathcal{M} \cdot Q \cdot P)^* = \mathcal{M}^* \times Q \times P$,

ii. $(\mathcal{M} \times Q \cdot P)^* = \mathcal{M}^* \cdot Q \times P$.

Exercise 4.14 *(k) Let $\mathcal{M}$ be a matroid on S and let $T \subseteq S$. Then,*

i. C_T is a bond of $\mathcal{M} \times T$ iff it is a bond of $\mathcal{M}$ contained in T,

ii. C_T is a bond of $\mathcal{M} \cdot T$ iff it is a minimal nonvoid intersection of a bond of $\mathcal{M}$ with T.

Exercise 4.15 *(k) Let $\mathcal{G}$ be a graph on edge set E. Let $\mathcal{M}(\mathcal{G}), \mathcal{M}^*(\mathcal{G})$ be the polygon and bond matroids associated with $\mathcal{G}$. Let $T \subseteq E$. Then*

i. $\mathcal{M}(\mathcal{G} \,.\, T) = (\mathcal{M}(\mathcal{G})) \cdot T$,

ii. $\mathcal{M}(\mathcal{G} \times T) = (\mathcal{M}(\mathcal{G})) \times T$,

iii. $\mathcal{M}^*(\mathcal{G} \,.\, T) = (\mathcal{M}^*(\mathcal{G})) \times T$,

iv. $\mathcal{M}^*(\mathcal{G} \times T) = (\mathcal{M}^*(\mathcal{G})) \cdot T$.

Exercise 4.16 *(k) Let $\mathcal{V}$ be a vector space on S. Let $\mathcal{M}(\mathcal{V})$, $(\mathcal{M}^*(\mathcal{V}))$ be the matroid (dual matroid) associated with $\mathcal{V}$. Let $T \subseteq E$. Show that*

i. $\mathcal{M}(\mathcal{V} \,.\, T) = (\mathcal{M}(\mathcal{V})) \cdot T$.

ii. $\mathcal{M}(\mathcal{V} \times T) = (\mathcal{M}(\mathcal{V})) \times T$.

iii. $\mathcal{M}^*(\mathcal{V} \,.\, T) = (\mathcal{M}^*(\mathcal{V})) \times T$.

iv. $\mathcal{M}^*(\mathcal{V} \times T) = (\mathcal{M}^*(\mathcal{V})) \cdot T$.

Exercise 4.17 *(k) Let $\mathcal{G}$ be a graph on E and let $T \subseteq E$. Use Exercise 4.15 to show that*

i. C is a circuit of $\mathcal{G} \,.\, T$ (cutset of $\mathcal{G} \,.\, T$) iff it is a circuit of $\mathcal{G}$ contained in T (it is a minimal nonvoid intersection of a cutset of $\mathcal{G}$ with T) and

ii. C is a circuit of $\mathcal{G} \times T$ (cutset of $\mathcal{G} \times T$) iff it is a minimal nonvoid intersection of a circuit of $\mathcal{G}$ with T (it is a cutset of $\mathcal{G}$ contained in T).

A minor of a general form could be obtained from the original matroid by a sequence of restrictions and contractions. As in the case of graphs we can simplify these operations to a single contraction followed by a single restriction or vice versa. The following result is needed for such simplification.

Theorem 4.4.6 *(k) Let $\mathcal{M}$ be a matroid on S and let $P \subseteq Q \subseteq S$. Then,*

i. $\mathcal{M} \cdot Q \cdot P = \mathcal{M} \cdot P$.

ii. $\mathcal{M} \times Q \times P = \mathcal{M} \times P$.

iii. $\mathcal{M} \times Q \cdot P = \mathcal{M} \cdot (S - (Q - P)) \times P$.

Proof :
i. Immediate from the definition of restriction.

ii. We have, by Theorem 4.4.5,
$(\mathcal{M}^* \cdot P)^* = (\mathcal{M}^* \cdot Q \cdot P)^* = (\mathcal{M}^* \cdot Q)^* \times P = (\mathcal{M}^*)^* \times Q \times P = \mathcal{M} \times Q \times P$
But $(\mathcal{M}^* \cdot P)^* = (\mathcal{M}^*)^* \times P = \mathcal{M} \times P$.
We conclude that $\mathcal{M} \times Q \times P = \mathcal{M} \times P$.

iii. Let X be an independent set of $\mathcal{M} \times Q \cdot P$. Then, by the definition of restriction, $X \subseteq P$ and X is independent in $\mathcal{M} \times Q$. Let B_{S-Q} be a base of $\mathcal{M} \cdot (S - Q)$. Then by the definition of contraction, $X \cup B_{S-Q}$ is independent in $\mathcal{M}$. By the definition of restriction $X \cup B_{S-Q}$ must be independent in $\mathcal{M} \cdot (S - (Q - P))$. Now B_{S-Q} is a base of $\mathcal{M} \cdot (S - Q) = \mathcal{M} \cdot (S - (Q - P)) \cdot (S - Q)$. Hence, X is independent in $\mathcal{M} \cdot (S - (Q - P)) \times P$ (note that $(S - Q) \uplus P = S - (Q - P)$ since $P \subseteq Q$). It is easy to see that the above sequence of implications can be reversed. Hence, if X is independent in $\mathcal{M} \cdot (S - (Q - P)) \times P$ then X is also independent in $\mathcal{M} \times Q \cdot P$. Thus,

$$\mathcal{M} \times Q \cdot P = \mathcal{M} \cdot (S - (Q - P)) \times P$$

□

Exercise 4.18 *(k) Let $\mathcal{M}$ be a matroid on S and let $S \supseteq T_1 \supseteq T_2 \supseteq \cdots \supseteq T_n$. Show that $\mathcal{M} \times T_1 \cdot T_2 \times T_3 \cdots \cdot T_n$ can be written in the form $\mathcal{M} \times P \cdot T_n$ for a suitable $P \supseteq T_n$.*

4.5 Connectedness in Matroids

Connectedness for matroids is defined analogous to 2-connectedness for graphs. Let $\mathcal{M}$ be a matroid on S. We say $T \subseteq S$ is a **separator** of $\mathcal{M}$ iff no circuit of $\mathcal{M}$ has a nonvoid intersection with both T and $(S-T)$. A separator that contains no other separator as a proper subset is called an **elementary separator**. The matroid $\mathcal{M}$ is **connected** iff its underlying set is an elementary separator. The structure of a matroid can be studied conveniently by studying the restrictions (equivalently, contractions) on elementary separators. If $\mathcal{V}$ is a vector space and $\mathcal{M}(\mathcal{V})$ is the matroid associated with it, then the separators of $\mathcal{V}$ are the same as the separators of $\mathcal{M}(\mathcal{V})$. In this section we study connectedness in terms of the various primitive notions associated with a matroid.

Theorem 4.5.1 *(k) Let $\mathcal{M}$ be a matroid on S. Let $T \subseteq S$. Then T is a separator of $\mathcal{M}$ iff the following equivalent conditions are satisfied*

i. No circuit of $\mathcal{M}$ intersects both T and $S-T$.

ii. $\mathcal{M} \times T = \mathcal{M} \cdot T$.

iii. $r(\mathcal{M} \times T) = r(\mathcal{M} \cdot T)$.

iv. $r(\mathcal{M}^* \times T) = r(\mathcal{M}^* \cdot T)$.

v. No bond of $\mathcal{M}$ intersects both T and $S-T$.

Proof : By definition, T is a separator of $\mathcal{M}$ iff no circuit of $\mathcal{M}$ intersects both T and $S-T$.
(i) and (ii) are equivalent:
Let no circuit of $\mathcal{M}$ intersect both T and $S-T$.
Let B_1, B_2 be two bases of $\mathcal{M}$ s.t. $B_1 \cap T, B_2 \cap T$ respectively have minimal and maximal intersections with T among all bases of $\mathcal{M}$. Let $e \in (B_2 - B_1) \cap T$. Then $L(e, B_1)$ does not intersect $(S-T)$. Let $e' \in L(e, B_1) - B_2$. Clearly $e' \in T$ and $B_{11} \equiv (B_1 - e') \cup e$ is a base of $\mathcal{M}$. This base has more elements in common with B_2 than B_1. Further $\mid B_{11} \cap T \mid = \mid B_1 \cap T \mid$. Repeating this procedure we would ultimately reach a base B_{1n} s.t. $B_{1n} \cap T \supseteq B_2 \cap T$ and $\mid B_{1n} \cap T \mid = \mid B_1 \cap T \mid$. This shows that $\mid B_1 \cap T \mid = \mid B_2 \cap T \mid$. Thus, bases of $\mathcal{M} \cdot T$ and $\mathcal{M} \times T$ are identical and hence $\mathcal{M} \cdot T = \mathcal{M} \times T$.
Next let $\mathcal{M} \cdot T = \mathcal{M} \times T$. Suppose there exists a circuit C that intersects both T and $S-T$. Grow $C \cap T$ into a base B_T of $\mathcal{M} \cdot T (= \mathcal{M} \times T)$ and $C \cap (S-T)$ into a base B_{S-T} of $\mathcal{M} \cdot (S-T)$. Now since B_T is a base of $\mathcal{M} \times T$ we must have $B_T \uplus B_{S-T}$ as a base of $\mathcal{M}$, by Exercise 4.12. This is a contradiction, since the union contains the circuit C. We conclude that there exists no circuit that intersects both T and $S-T$.

(ii) and (iii) are equivalent:
(Note that $r(\mathcal{M})$ denotes the size of a base of $\mathcal{M}$ and not the rank function of $\mathcal{M}$). We need only show that '$r(\mathcal{M} \cdot T) = r(\mathcal{M} \times T)$' implies '$\mathcal{M} \cdot T = \mathcal{M} \times T$',

since the reverse implication is trivial. Let B'_T be a base of $\mathcal{M} \times T$. Then B'_T is an independent set of $\mathcal{M}$ contained in T. Hence B'_T is independent in $\mathcal{M} \cdot T$. But $r(\mathcal{M} \cdot T) = r(\mathcal{M} \times T) =\mid B'_T \mid$. Hence, B'_T is a base of $\mathcal{M} \cdot T$. Conversely let B_T be a base of $\mathcal{M} \cdot T$. Then B_T is a maximal intersection of a base of $\mathcal{M}$ with T. Hence, it contains a minimal intersection of a base of $\mathcal{M}$ with T, i.e., B_T contains a base B'_T of $\mathcal{M} \times T$. Since $r(\mathcal{M} \cdot T) = r(\mathcal{M} \times T)$ we conclude that $\mid B_T \mid=\mid B'_T \mid$ and $B_T = B'_T$. Hence, B_T is a base of $\mathcal{M} \times T$.
Thus, $\mathcal{M} \cdot T = \mathcal{M} \times T$.

(iii) and (iv) are equivalent:
We have $r(\mathcal{M} \cdot T) = r(\mathcal{M} \times T)$ iff $\mid T \mid -r(\mathcal{M} \cdot T) =\mid T \mid -r(\mathcal{M} \times T)$, i.e., iff $r(\mathcal{M}^* \times T) = r(\mathcal{M}^* \cdot T)$.

(iv) and (v) are equivalent:
Since (ii) and (iii) above are equivalent, we conclude that $r(\mathcal{M}^* \times T) = r(\mathcal{M}^* \cdot T)$ iff T is a separator of $\mathcal{M}^*$. But T is a separator of $\mathcal{M}^*$ iff no circuit of $\mathcal{M}^*$ intersects both T and $(S - T)$, i.e., iff no bond of $\mathcal{M}$ intersects both T and $(S - T)$.

□

Exercise 4.19 *(k) Let $\mathcal{V}$ be a vector space on S. Let $\mathcal{M}(\mathcal{V})$ on S be the matroid associated with $\mathcal{V}$ (independent sets correspond to independent columns of a representative matrix of $\mathcal{V}$). A subset $T \subseteq S$ is a separator of $\mathcal{M}(\mathcal{V})$ iff it is a separator of $\mathcal{V}$.*

4.5.1 Duality for Matroids

The mode of construction of dual statements involving matroids is very similar to the case of vector spaces (Section 3.7). For completeness we describe the basic notions and the procedure of dualization briefly.
Let $\mathcal{M}$ be a matroid on S. We associate with $\mathcal{M}$

i. A set of operations each of which converts $\mathcal{M}$ to a matroid on a subset of S. A typical operation is $(S - T_1, T_1 - T_2)(\cdot)$, $T_2 \subseteq T_1 \subseteq S$, where

$$(S - T_1, T_1 - T_2)(\mathcal{M}) \equiv \mathcal{M} \cdot T_1 \times T_2.$$

ii. Classes of objects:

(a) class of bases

(b) class of cobases

(c) class of circuits

(d) class of bonds (circuits of $\mathcal{M}^*$)

(Actually any one of the four classes, circuits, bonds, bases and cobases can be taken to be primitive and the others expressed in terms of it. We have included all four classes for convenience).

Here is a list of results which 'cause' duality:

i. $(\mathcal{M}^*)^* = \mathcal{M}$.

ii. $(\mathcal{M} \cdot T_1 \times T_2)^* = \mathcal{M}^* \times T_1 \cdot T_2 = \mathcal{M}^* \times (S - (T_1 - T_2)) \cdot T_2$
i.e., the operation $(S - T_1, T_1 - T_2)(\cdot)$ holds the same place relative to $\mathcal{M}$ that the operation $(T_1 - T_2, S - T_1)(\cdot)$ holds relative to $\mathcal{M}^*$. We say $(S - T_1, T_1 - T_2)(\cdot)$ is **dual** to $(T_1 - T_2, S - T_1)(\cdot)$.

iii. (Definition) T is a base of $\mathcal{M}$ iff T is a cobase of $\mathcal{M}^*$.

iv. (Definition) C is a circuit of $\mathcal{M}$ iff C is a bond of $\mathcal{M}^*$.

Below we dualize a statement about a matroid and the associated set of operations and classes of objects. Consider the statement
i. 'A subset is a circuit of $\mathcal{M} \times T$ iff it is a minimal intersection of a circuit of $\mathcal{M}$ with T'.
The first step is to write the statement in terms of $\mathcal{M}^*$
'A subset is a circuit of $\mathcal{M}^* \times T$ iff it is a minimal intersection of a circuit of $\mathcal{M}^*$ with T.'
Next we try to express the sets of objects involved in terms of the appropriate dual matroid. Thus, 'circuit of $\mathcal{M}^* \times T$' becomes 'bond of $(\mathcal{M}^* \times T)^*$' and 'circuit of $\mathcal{M}^*$' becomes 'bond of $(\mathcal{M}^*)^*$'. We thus obtain the dual of (i):
i^d. 'A subset is a bond of $\mathcal{M} \cdot T$ iff it is a minimal intersection of a bond of $\mathcal{M}$ with T.'

4.6 Matroids and the Greedy Algorithm

A common class of optimization problems is characterized by the fact that if one performs 'local optimization' at every step of the algorithm, then, at the end of the algorithm, we have a global optimum. An algorithm which performs local optimization at every step and does no back tracking is called a **greedy algorithm**. Loosely, matroids can be characterized as those structures for which the greedy algorithms works. We make this statement more precise in this section.

Example 4.6.1 *(k) We describe some problems for which the greedy algorithm works.*
i. *Generate a forest of a graph.*

Start with any nonselfloop branch, add additional branches always avoiding circuit formation. When you can proceed no further the final set of branches is a forest.

ii. *Given a weight for each branch, generate the heaviest (lightest) forest of the graph.*

(Heaviest forest case) Start with the heaviest edge. If at any stage T is the constructed set, add to T the heaviest edge e outside it under the condition that $e \cup T$

contains no circuit. The algorithm terminates with the heaviest forest of the graph. (The reader might like to compare the above with Prim's Algorithm given in Subsection 3.6.3).

Example 4.6.2 *Here is a problem for which the greedy strategy fails:*
Let $\mathcal{F} = \{\{1,2\},\{1,2,3\},\{2,3,4,5\},\{6\},\{7,8\}\}$
Find a maximum size subset of $\{1,2,3,4,5,6,7,8\}$ *that is a member of* $\mathcal{F}$.

Suppose we start with $\{6\}$ *and check if there is a member of* $\mathcal{F}$ *properly containing the set, we would find there is no such member. However,* $\{6\}$ *is not the maximum size subset that is a member of* $\mathcal{F}$ *(the required subset is* $\{2,3,4,5\}$*).*

Our main result relates matroids to the greedy algorithm. We need some preliminary definitions.

Let S be a finite set and let $\mathcal{I}$ be a collection of subsets of S such that '$Y \in \mathcal{I}, X \subseteq Y$' implies '$X \in \mathcal{I}$.' Let $w(\cdot)$ be a real weight function on S. Let the **weight** of a subset X of S, denoted by $w(X)$, be defined by $w(X) \equiv \sum_{e \in X} w(e), X \subseteq S$. Let us call a maximal member of $\mathcal{I}$, a **base of** $\mathcal{I}$. Let X_1, X_2 be two bases of $\mathcal{I}$. Let $X_1 \equiv \{a_1, \cdots, a_k\}, X_2 \equiv \{e_1, \cdots, e_m\}$ and further let $w(a_1) \geq \cdots \geq w(a_k)$ and let $w(e_1) \geq \cdots \geq w(e_m)$. We define a preorder on the maximal members of $\mathcal{I}$ as follows:
$X_1 \succeq X_2$ iff $w(a_i) > w(e_i)$ whenever i is the least index s.t. $w(a_i) \neq w(e_i)$. If X_0 is a base of $\mathcal{I}$ s.t. $X_0 \succeq X_j$, whenever X_j is a base of $\mathcal{I}$, we say that X_0 is a **lexicographically optimum base of** $\mathcal{I}$ **relative to** $w(\cdot)$.

Assume that we have an oracle ($\mathcal{I}-$oracle) which tells us whether a given subset of S belongs to $\mathcal{I}$. Then it is clear that with $\mid S \mid$ queries to the $\mathcal{I}-$oracle one can determine a lexicographically optimum base of $\mathcal{I}$ relative to $w(\cdot)$:
Let $S \equiv \{e_1, \cdots, e_n\}$. Without loss of generality let us assume that $w(e_1) \geq \cdots \geq w(e_n)$. Let e_{i1} be the heaviest element s.t. $\{e_{i1}\} \in \mathcal{I}$. Suppose at some stage we have constructed a member $T \in \mathcal{I}$. Let e_k be the lightest element of T. To grow T further we look for the heaviest element e_j in $\{e_{k+1}, \cdots, e_n\}$ for which $T \cup e_j \in \mathcal{I}$. If no such e_j exists T is the lexicographically optimum base of $\mathcal{I}$.

Clearly the above algorithm can be called 'greedy' since the set is grown to its full size by doing only local optimization with no back tracking.

Theorem 4.6.1 *(k) Let S be a finite set and $\mathcal{I}$, a collection of subsets of S s.t. $X \subseteq Y, Y \in \mathcal{I}$ implies $X \in \mathcal{I}$,*

i. [Gale68] Let $w(\cdot)$ be a weight function on S. If $\mathcal{M} \equiv (S, \mathcal{I})$ is a matroid with $\mathcal{I}$ as the collection of independent subsets of $\mathcal{M}$, then a base of $\mathcal{M}$, relative to $w(\cdot)$, is lexicographically optimum iff it is a base of maximum weight.

ii. If, for every weight function, the lexicographically optimum base of $\mathcal{I}$ is also the base with maximum weight, then $(S, \mathcal{I})$ is a matroid.

Proof :
i. Only if: Let X be a lexicographically optimum base of $\mathcal{M}$ relative to $w(\cdot)$ and

let Y be a base of maximum weight s.t. $\mid Y \cap X \mid$ is the maximum possible. Let $X \equiv \{a_1, \cdots, a_k\}, Y \equiv \{e_1, \cdots, e_k\}$ with $w(a_i) \geq w(a_j),\ i \leq j$ and $w(e_i) \geq w(e_j), i \leq j$. Let $X \neq Y$. Let r be the highest index for which $\{a_1, \cdots, a_r\} = \{e_1, \cdots, e_r\}$. (If $a_1 \neq e_1$, we take r to be zero.)
Now $a_{r+1} \neq e_{r+1}$. We have $X \succeq Y$. Hence, $w(a_{r+1}) \geq w(e_{r+1})$. Consider the fundamental circuit $L(a_{r+1}, Y)$. There exists an element $e_i \in (Y - X) \cap L(a_{r+1}, Y)$ s.t. $Y' \equiv Y \cup a_{r+1} - e_i$ is a base of $\mathcal{M}$.
We have $i \geq r+1$. Hence, $w(e_i) \leq w(e_{r+1}) \leq w(a_{r+1})$. Hence, $w(Y') \geq w(Y)$. But this contradicts the fact that Y is the maximum weight base nearest to X. We conclude therefore that $X = Y$.

if: Let $Y \equiv \{e_1, \cdots, e_k\}$,$w(e_i) \geq w(e_j), i \leq j$, be a base of $\mathcal{M}$ of maximum weight and let $X \equiv \{a_1, \cdots, a_k\}, w(a_i) \geq w(a_j), i \leq j$, be a lexicographically optimum base of $\mathcal{M}$. Suppose Y is not lexicographically optimum. Let r be the least index for which $w(a_r) > w(e_r)$. (For $i < r$ we must have $w(a_i) = w(e_i)$). Now $\{e_1, \cdots, e_r\}$ cannot span $\{a_1, \cdots, a_r\}$ as otherwise the base $(Y - \{e_1, \cdots, e_r\}) \cup \{a_1, \cdots, a_r\}$ would have greater weight than Y. Let $a_j, j \leq r$, be an element that does not belong to the span of $\{e_1, \cdots, e_r\}$. Consider $L(a_j, Y)$. This set must intersect $\{e_{r+1}, \cdots, e_k\}$. Let e_m belong to the intersection. Clearly $(a_j \cup Y) - e_m$ is a base of $\mathcal{M}$ of greater weight than Y. This contradiction shows that $w(a_i) = w(e_i), i \leq k$. Hence Y is a lexicographically optimum base of $\mathcal{M}$.

ii. We will show that $(S, \mathcal{I})$ satisfies the Independence Axioms. We already have that $X \subseteq Y, Y \in \mathcal{I}$ implies $X \in \mathcal{I}$. So we need only verify that maximal members of $\mathcal{I}$, contained in any subset T of S, have the same size. Let B_1, B_2 be two such maximal members. Choose $w(\cdot)$ as follows:
$w(e_i) = 0,\ e_i \notin B_1 \cup B_2$
$w(e_i) = 1,\ e_i \in B_1$
$w(e_i) = \alpha,\ e_i \in B_2 - B_1$.
Let $\mid B_1 \mid = n, \mid B_2 - B_1 \mid = p$ and let $\mid B_2 \mid = m$. Suppose $m > n$. Then we can select α so that $\alpha < 1$ but $n < (m-p) + p\alpha$. In this case the lexicographically optimum base clearly contains B_1 (and no element from $B_2 - B_1$). But we have $w(B_2) > w(B_1)$, so that a base that contains B_1 (and does not intersect $B_2 - B_1$), cannot have maximum weight. We can avoid this contradiction only if $n = m$.

□

Exercise 4.20 *Let $\mathcal{M}$ be a matroid on S and let $w(\cdot)$ be a weight function taking the values $1, 2, \cdots, k$. Let $T_j \equiv w^{-1}(k - j + 1)$. Show that a base B of $\mathcal{M}$ has maximum weight iff $B = B_1 \uplus \cdots \uplus B_k$ where B_j is a base of $\mathcal{M} \cdot (\bigcup_{i=1}^{j} T_i) \times T_j$.*

4.7 Notes

Matroids were introduced into combinatorics by H.Whitney in 1935 [Whitney35], when he described several equivalent axiom systems which characterized the 'abstract properties of linear independence'. One such axiom system was also described by Van Der Waerden in his book on Modern Algebra [Van der Waerden37].

Early work on the lattice of flats of a matroid was done by Birkhoff [Birkhoff35]. In the 1940's Rado and Dilworth made important contributions to this theory [Rado42], [Dilworth44]. The subject received a big impetus when Tutte solved the regular matroid and graphic matroid characterization problems in 1959 [Tutte58], [Tutte59]. In mid 60's important applications to combinatorial optimization were discovered by Edmonds and Fulkerson [Edmonds+Fulkerson65], [Edmonds65a], [Edmonds65b]. Since then, research in this area has remained very active, both in theory and in applications. The reader who wishes to pursue the subject further may refer to [Tutte65], [Tutte71], [Crapo+Rota70], [Randow76], [Welsh76], [Aigner79], [White86]. Applications may be found in [Papadimitriou+Steiglitz82], [Lawler76], [Faigle87], [Recski89]. A good way of accessing the basic papers of the subject is through [Kung86].

4.8 Solutions of Exercises

E 4.1:
Example 4.2.1: This follows from the facts that

- maximal intersection of a forest (coforest) of $\mathcal{G}$ with $T, T \subseteq E(\mathcal{G})$, is a forest of $\mathcal{G} \,.\, T$ (coforest of $\mathcal{G} \times T$),
- all forests of $\mathcal{G} \,.\, T$ have the same cardinality and that
- all coforests of $\mathcal{G} \times T$ have the same cardinality.

Since forests and coforests are complements of each other, the bases of either of $(E(\mathcal{G}), \mathcal{I}_t), (E(\mathcal{G}), \mathcal{I}_c)$ are cobases of the other.

Example 4.2.2: This follows from the fact that maximally independent subsets of columns of any submatrix of $\mathbf{R}$ $(\mathbf{R}^*)$ have the same cardinality. Further if $(\mathbf{I} : \mathbf{K})$ is a standard representative matrix of $\mathcal{V}$ then $(-\mathbf{K}^T : \mathbf{I})$ is a standard representative matrix of $\mathcal{V}^\perp$. Since there is a standard representative matrix corresponding to each maximally independent subset of columns, we conclude that bases of either of $(S, \mathcal{I}), (S, \mathcal{I}^*)$ are cobases of the other.

Example 4.2.3: This follows from Example 4.2.1 and Theorem 12.2.6.

Example 4.2.4: (sketch) Let $T \subseteq V(\mathcal{G})$ and let I_1, I_2 be two maximal members of $\mathcal{I}_m$ contained in T. There exist matchings M_1, M_2 which meet I_1, I_2. Consider the subgraph of $\mathcal{G}$ on $M_1 \cup M_2$. (Note that the vertex set of this subgraph may contain vertices outside T). It can be seen that each component of this subgraph is either a circuit graph or a path graph. If $\mid I_2 \mid > \mid I_1 \mid$ we must have the subset ${I_2}'$ of I_2 in one of these components of larger size than the subset ${I_1}'$ of I_1 in the same component. It is then possible to find a matching in this component which meets ${I_1}' \cup v$ for some $v \in {I_2}' - {I_1}'$. Hence, $I_1 \cup v \in \mathcal{I}_m$. This is a contradiction. We conclude that $\mid I_1 \mid = \mid I_2 \mid$.

E 4.2: Let $\mathcal{I}$ satisfy Base Axioms with condition (ii') (the case where condition (ii) is satisfied is similar). In order to show that it satisfies the Independence Axioms we need only show that maximally independent subsets contained in $T \subseteq S$ have the same cardinality.

Case 1: $T = S$.
If B_1, B_2 are bases and $e \in B_1 - B_2$, we can find an $e' \in B_2 - B_1$ s.t. $(B_1 - e) \cup e'$ is a base. If we repeat this procedure we would finally get a base $B_k \subseteq B_2$ s.t. $| B_k |=| B_1 |$. But one base cannot properly contain another. So $B_k = B_2$ and $| B_2 |=| B_1 |$.

Case 2: $T \subset S$.
Suppose $X \equiv \{x_1, \cdots, x_k\}$ and $Y \equiv \{y_1, \cdots, y_m\}$ are maximally independent sets contained in T. Further let $k < m$. First grow X to a base B_x and Y to a base B_y. Let

$$B_x \equiv \{x_1, \cdots, x_k, p_{k+1}, \cdots, p_r\}$$

$$B_y \equiv \{y_1, \cdots, y_m, q_{m+1}, \cdots, q_r\}$$

Since $k < m$, there is an element $p_t \in B_x - B_y$. Hence, there is an element z in $B_y - B_x$ s.t. $(B_x - p_t) \cup z$ is a base. Now z cannot be one of the y_i as otherwise X would not be a maximally independent subset contained in T. So $z = q_s$ say. We thus have a new base $B'_x \equiv (B_x - p_t) \cup q_s$. Observe that $(B_y - B'_x) \cap (S - T) \subset (B_y - B_x) \cap (S - T)$. Repeating this procedure we would reach a base B^f_x s.t. $B^f_x \cap (S - T) \supseteq B_y \cap (S - T)$.
Now if $e \in B^f_x - B_y$, then there must exist $z' \in B_y - B^f_x$ s.t. $(B^f_x - e) \cup z'$ is a base. But then $z' \in Y$ and $X \cup z'$ is independent, which contradicts the fact that X is a maximally independent subset of T.
We conclude therefore that $| X |=| Y |$, i.e., that maximally independent subsets contained in T have the same cardinality.

E 4.3:
i. Let $Y \in \mathcal{I}$ and let $X \subseteq Y$. We need to show that $X \in \mathcal{I}$.
We have $r(\emptyset) = 0$ and $r(A \cup e) \leq r(A) + 1 \ \forall \ A \in S$. Hence, $r(X) \leq| X |$ and $r(Y) - r(X) \leq| Y | - | X |$. Hence, if $r(X) <| X |$, we must have $r(Y) <| Y |$, a contradiction. We conclude that $r(X) =| X |$, i.e., $X \in I$.

ii. Let B_1, B_2 be two maximal members of $\mathcal{I}$ contained in a subset T of S. We need to show that $| B_1 |=| B_2 |$.
For each $e_i \in T - B_1$ we have $r(B_1) \leq r(B_1 \cup e_i) <| B_1 | +1$, since B_1 is a maximal subset of T with the property that size and rank are equal. Hence, $r(B_1 \cup e_i) = r(B_1) \ \forall \ e_i \in T - B_1$. Now, if P, Q are sets such that $r(B_1 \cup P) = r(B_1 \cup Q) = r(B_1)$ then

$$r(B_1 \cup P) + r(B_1 \cup Q) \geq r(B_1 \cup P \cup Q) + r(B_1 \cup (P \cap Q)),$$

by submodularity of $r(\cdot)$. Since LHS equals $2r(B_1)$ and RHS is greater or equal to $2r(B_1)$ ($r(\cdot)$ is an increasing function), It follows that

$$r(B_1 \cup P \cup Q) = r(B_1 \cup (P \cap Q)) = r(B_1).$$

Thus by induction we can prove that

$$r(B_1 \cup e_1 \cup \cdots \cup e_k) = r(B_1),$$

where $\{e_1, \cdots, e_k\} = T - B_1$. Hence, $r(B_1) = r(T)$.
Similarly we must have $r(B_2) = r(T)$.
We conclude that $\mid B_1 \mid = r(B_1) = r(B_2) = \mid B_2 \mid$ as required.

E 4.4: We will show that maximal subsets of $T \subseteq S$ that do not contain a circuit have the same cardinality.
Let B_1, B_2 be two such 'maximally independent' sets contained in T. If $B_1 \neq B_2$, clearly $B_1 \not\supseteq B_2$. Let $e_2 \in B_2 - B_1$. Then $e_2 \cup B_1$ contains a circuit.
We claim this circuit is unique. Otherwise if C_1, C_2 are two such circuits since both have e_2 as a member, by the circuit axioms there exists a circuit $C_3 \subseteq C_1 \cup C_2 - e_2$. This is a contradiction, since $C_3 \subseteq B_1$.
Let $L(e_2, B_1)$ be the unique circuit contained in $e_2 \cup B_1$. Since $\{e_2\}$ is not a circuit (it is contained in B_2) we must have that $L(e_2, B_1) \cap B_1 \neq \emptyset$. Further $L(e_2, B_1)$ cannot be wholly contained in B_2. Hence, $L(e_2, B_1) \cap (B_1 - B_2) \neq \emptyset$. Let e_1 belong to this intersection. Then $B_1' \equiv e_2 \cup B_1 - e_1$ is independent.
We claim that B_1' is also maximally independent. For, let $e' \in T - B_1'$. Clearly, if $e' = e_1$ we have $e' \cup B_1'$ containing $L(e_2, B_1)$. Suppose $e' \neq e_1$. Now $e' \cup B_1$ contains a circuit $L(e', B_1)$. If this circuit does not contain e_1 then $L(e', B_1) \subseteq e' \cup B_1'$. So let $e_1 \in L(e', B_1)$. Now $L(e_2, B_1)$ and $L(e', B_1)$ have a nonvoid intersection which contains e_1. Hence, $L(e_2, B_1) \cup L(e', B_1) - e_1$ contains a circuit. This circuit is contained in $e' \cup B_1'$.
Thus, we see that $e' \cup B_1'$ is dependent for all $e' \in T - B_1'$.
Now we have a maximally independent subset B_1' of T which has the same cardinality as B_1 but satisfies $\mid B_2 - B_1' \mid < \mid B_2 - B_1 \mid$. Repeating this procedure we get a maximally independent subset B_k of T that has the same cardinality as B_1 but contains B_2. But this means $B_k = B_2$ and hence, $\mid B_1 \mid = \mid B_2 \mid$.

E 4.5: We have $r(X) \neq r(X \cup y)$. Hence, $r(X \cup y) = r(X) + 1$. Since $y \in \int(X \cup x)$ it follows that

$$r(X \cup x) = r(X \cup x \cup y) \geq r(X \cup y) = r(X) + 1.$$

Further, $r(X \cup x) \leq r(X) + 1$. Thus, $x \notin \int(X)$ and $r(X \cup x) = r(X) + 1$.
Let B be a maximally independent set contained in X. Then B cannot be maximally independent in $X \cup x$ since $r(X \cup x) = \mid B \mid + 1$. Hence, $B \cup x$ must be a maximally independent subset of $X \cup x$.
We know that $B \cup x \cup y$ must be dependent, since $r(X \cup x \cup y) = \mid B \cup x \mid$. Hence there exists a (unique) circuit $C \subseteq B \cup x \cup y$ that has y as a member. If $x \notin C$ we would have that $r(B \cup y) = r(B)$ and therefore, by submodularity, $r(X \cup y) = r(X)$ which is a contradiction.
We conclude that $x \in C$. Noting that $C - (X \cup y) = \{x\}$ and using Lemma 4.2.2 we see that $x \in \int(X \cup y)$ as required.

E 4.6: [Welsh76] It is clear that $\emptyset \in \mathcal{I}$. Let $X \subseteq Y, Y \in \mathcal{I}$ but $X \notin \mathcal{I}$. Since $X \notin \mathcal{I}$ there exists $e \in X$ s.t. $e \in \int(X - e)$. But then $e \in (Y - e)$ using (S2). Hence, $Y \notin \mathcal{I}$, a contradiction.

E 4.7: [Welsh76] We will first show by contradiction that $T - \int(X) = \emptyset$. Let $e \in T - \int(X)$. We will show that $X \cup e \in \mathcal{I}$, i.e., if $e' \in X \cup e$ then $e' \notin \int((X \cup e) - e')$. This is obviously true when $e' = e$. Let $e' \neq e$. Then $e' \in X$. Suppose $e' \in \int((X - e') \cup e)$. We know that $e' \notin \int(X - e')$ since $X \in \mathcal{I}$. Therefore, by (S4) $e \in \int((X - e') \cup e') = \int(X)$, a contradiction. Hence, we conclude that $e' \notin \int((X \cup e) - e')$ and therefore that $X \cup e \in \mathcal{I}$. But this contradicts the maximality of X. Hence, $T - \int(X) = \emptyset$, i.e., $\int(X) \supseteq T$.
So by (S3) and (S2) we have $\int(X) = \int(\int(X)) \supseteq \int(T)$. But $X \subseteq T$ and by (S2) $\int(X) \subseteq \int(T)$. Therefore, $\int(X) = \int(T)$.

E 4.8: By Exercise 4.7, $\int(B_1) = \int(B_2) = S$. Since $B_1 \in \mathcal{I}, e_1 \notin \int(B_1 - e_1)$. Hence, $\int(B_1 - e_1) \not\supseteq B_2$, as otherwise, using (S3) and (S2), $\int(B_1 - e_1) = \int(\int(B_1 - e_1)) \supseteq \int(B_2) = S$. Hence, there exists $e_2 \in B_2 - \int(B_1 - e_1)$. But $e_2 \in \int((B_1 - e_1) \cup e_1)$. So by (S4), $e_1 \in \int((B_1 - e_1) \cup e_2)$. Hence, by (S1), $\int((B_1 - e_1) \cup e_2) \supseteq (B_1 - e_1) \cup e_1$. Now by (S3) and (S2),

$$\int((B_1 - e_1) \cup e_2) = \int(\int((B_1 - e_1) \cup e_2)) \supseteq \int(B_1) = S.$$

Next, by using arguments of Exercise 4.7, we have $(B_1 - e_1) \cup e_2 \in \mathcal{I}$. Since $\int((B_1 - e_1) \cup e_2) = S, (B_1 - e_1) \cup e_2$ is a maximal member of $\mathcal{I}$ as required.

E 4.9: We have the embedding of $\mathcal{G}$ on a plane, i.e., $\mathcal{G}$ is drawn on a plane so that edges do not cross. This divides the plane into regions or meshes including the outermost region. Give all the internal meshes a clockwise orientation and the outermost region an anticlockwise orientation. Let $\mathbf{B}$ be the matrix whose rows are circuit vectors corresponding to the meshes including the outermost mesh. Now each edge lies in precisely two meshes and its direction agrees with the orientation of one of the meshes and opposes the orientation of the other mesh. From this we infer that in each column of $\mathbf{B}$ there is precisely one +1, one -1 and the rest zero entries. So $\mathbf{B}$ is the incidence matrix of another graph $\mathcal{G}_2$. This is the desired dual graph $\mathcal{G}^*$.

Sketch of justification
Let v be the number of vertices, e be the number of edges, and p the number of components of $\mathcal{G}$. Then, by induction, one can show that $\mathcal{G}$ has $(e - v + p + 1)$ regions (including the outermost). By our construction $\mathcal{G}^*$ is connected and has $(e - v + p + 1)$ vertices. So the rank of $\mathbf{B}$ must be $e - v + p$. Further each row is a circuit vector of $\mathcal{G}$. Hence rows of $\mathbf{B}$ span $\mathcal{V}_i(\mathcal{G})$. Thus $\mathcal{V}_v(\mathcal{G}^*) = \mathcal{V}_i(\mathcal{G})$ as required.

E 4.10: Construct a forest f of $\mathcal{G}$ (i.e., a base of $\mathcal{M}(\mathcal{G})$) that has the maximal intersection among all forests of $\mathcal{G}$ with T (i.e., build a forest of $\mathcal{G} \cdot T$ and extend it to a forest f of $\mathcal{G}$). Let $e \in E - f$. Suppose $L(e, f)$ meets f within $f \cap T$. Let $\mathbf{i}_c$ be the circuit vector corresponding to $L(e, f)$. Now $\mathbf{i}_c^T \mathbf{v} = 0$. Hence, $-\mathbf{i}_c(e)v(e) = (\mathbf{i}_c/f)^T(-\mathbf{v}_c/f) = (\mathbf{i}_c/T)^T(\mathbf{v}_c/T)$. The RHS is known and hence $v(e)$ is uniquely determined. Hence, by Theorem 4.3.5 if $e \in \int(T)$ then $v(e)$ can be uniquely determined.
Next suppose $e \notin \int(T)$. Then we know by the above theorem, that $e \cup (f \cap T)$ is independent. Grow $(f \cap T) \cup e$ into a forest f' of $\mathcal{G}$. Choose a voltage vector $\mathbf{v}'$ as follows: $\mathbf{v}'/(f \cap T) = \mathbf{v}/(f \cap T), v'(e) \neq v(e)$ and $v'(e') = 0, \ \forall \ e' \in f' -$

$(T \cup e)$. Let $[\mathbf{I}|\mathbf{K}]$ be the f-circuit matrix of $\mathcal{G}$ relative to f' with the identity matrix corresponding to $E - f'$. Let $\mathbf{v}'_{E-f'} \equiv \mathbf{v}'/E - f', \mathbf{v}'_{f'} \equiv \mathbf{v}'/f'$. Then, since $[\mathbf{I}|\mathbf{K}]\mathbf{v}' = \mathbf{0}$, $\mathbf{v}'_{E-f'} = -K\mathbf{v}'_{f'}$. Thus, $\mathbf{v}'$ can be assigned values on $E - f'$ consistent with the earlier assignment on f' so that it becomes a voltage vector of $\mathcal{G}$. Further $\mathbf{v}'$ and $\mathbf{v}$ agree over $f \cap T$. So by the earlier argument we have $\mathbf{v}/T = \mathbf{v}'/T$. It is thus clear that if $e \notin \int(T)$, then $v(e)$ cannot be uniquely determined from $\mathbf{v}/T$. The argument for the current case is similar (dual) working with coforests, f-cutsets and f-cutset matrices.

E 4.11: Suppose $f(\mid X \mid) - g(r(X))$ is a maximum among all subsets of S. Let $Y \equiv \int(X)$. Now $Y \supseteq X$ and $r(Y) = r(X)$. Hence, since $f(\cdot)$ is an increasing function

$$f(\mid Y \mid) - g(r(Y)) \geq f(\mid X \mid) - g(r(X)).$$

This proves the required result.

E 4.12:
i. Let B_T be a base of $\mathcal{M} \times T$ and let B_{S-T} be a base of $\mathcal{M} \cdot (S - T)$. By the definition of $\mathcal{M} \times T$, $B_{S-T} \cup B_T$ is independent in $\mathcal{M}$. Hence, $r(\mathcal{M} \times T) \leq r(\mathcal{M}) - r(\mathcal{M} \cdot (S - T))$.
Next, B_{S-T} can be extended to a base B of $\mathcal{M}$. By the definition of $\mathcal{M} \cdot (S - T)$, we must have $B_{S-T} = B \cap (S - T)$. Now $(B \cap T) \cup B_{S-T}$ is independent in $\mathcal{M}$. Hence by the definition of $\mathcal{M} \times T$ and Lemma 4.4.1, $B \cap T$ must be independent in $\mathcal{M} \times T$. Hence, $r(\mathcal{M} \times T) \geq r(\mathcal{M}) - r(\mathcal{M} \cdot (S - T))$. We conclude therefore that $r(\mathcal{M} \times T) = r(\mathcal{M}) - r(\mathcal{M} \cdot (S - T))$ and $B_{S-T} \cup B_T$ is a base of $\mathcal{M}$.

ii. This is immediate from the above.

E 4.13: Using Theorem 4.4.5 we get
i. $(\mathcal{M} \cdot Q \cdot P)^* = (\mathcal{M} \cdot Q)^* \times P = (\mathcal{M}^* \times Q) \times P = \mathcal{M}^* \times Q \times P$

ii. $(\mathcal{M} \times Q \cdot P)^* = (\mathcal{M} \times Q)^* \times P = (\mathcal{M}^* \cdot Q) \times P = \mathcal{M}^* \cdot Q \times P.$

E 4.14:
i. The subset C_T is a bond of $\mathcal{M} \times T$ iff it is a circuit of $(\mathcal{M} \times T)^*$, i.e., of $\mathcal{M}^* \cdot T$. Now a circuit of $\mathcal{M}^* \cdot T$ is simply a circuit of $\mathcal{M}^*$ contained in T, or equivalently, a bond of $\mathcal{M}$ contained in T.

ii. The subset C_T is a bond $\mathcal{M} \cdot T$ iff it is a circuit of $(\mathcal{M} \cdot T)^*$, i.e., of $\mathcal{M}^* \times T$. Now a circuit of $\mathcal{M}^* \times T$ is a minimal nonvoid intersection of a circuit of $\mathcal{M}^*$ with T, i.e., a minimal nonvoid intersection of a bond of $\mathcal{M}$ with T.

E 4.15: From the chapter on graphs we know that forests of $\mathcal{G}\ .\ T$ are maximal intersections of forests of $\mathcal{G}$ with T, forests of $\mathcal{G} \times T$ are minimal intersections of forests of $\mathcal{G}$ with T, coforests of $\mathcal{G}\ .\ T$ are minimal intersections of coforests of $\mathcal{G}$ with T, coforests of $\mathcal{G} \times T$ are maximal intersections of coforests of $\mathcal{G}$ with T. Now bases of the polygon matroid associated with a graph are precisely the forests of the graph and bases of the bond matroid associated with a graph are precisely the coforests of the graph. The results now follow from the definitions of contraction and restriction of a matroid.

E 4.16:

i. Let $\mathbf{R} \equiv (\mathbf{R}_T \vdots \mathbf{R}_{S-T})$ be a representative matrix of $\mathcal{V}$. Then rows of $\mathbf{R}_T$ span rows of $\mathcal{V} \,.\, T$. Hence the column dependence structure of $\mathbf{R}_T$ is the same as the column dependence structure of any representative matrix of $\mathcal{V} \cdot T$. Now a set of columns of $\mathbf{R}_T$ are independent iff they are independent in $\mathbf{R}$. Hence, $\mathcal{M}(\mathcal{V} \,.\, T) = (\mathcal{M}(\mathcal{V})) \cdot T$.

ii. We have $\mathcal{M}(\mathcal{V}^{\perp}) = \mathcal{M}^*(\mathcal{V})$ for any vector space.
So $\mathcal{M}(\mathcal{V} \times T) = \mathcal{M}((\mathcal{V}^{\perp})^{\perp} \times T) = \mathcal{M}((\mathcal{V}^{\perp} \cdot T)^{\perp}) = (\mathcal{M}(\mathcal{V}^{\perp} \cdot T))^*$
Now $\mathcal{M}(\mathcal{V}^{\perp} \cdot T) = (\mathcal{M}(\mathcal{V}^{\perp})) \cdot T$.
Hence, $\mathcal{M}(\mathcal{V} \times T) = ((\mathcal{M}(\mathcal{V}^{\perp})) \cdot T)^* = (\mathcal{M}(\mathcal{V}^{\perp}))^* \times T = (\mathcal{M}(\mathcal{V})) \times T$ as required.

iii. $\mathcal{M}^*(\mathcal{V} \,.\, T) \equiv (\mathcal{M}(\mathcal{V} \,.\, T))^* = ((\mathcal{M}(\mathcal{V})) \cdot T)^*, = (\mathcal{M}(\mathcal{V}))^* \times T, = (\mathcal{M}^*(\mathcal{V})) \times T.$

iv. $\mathcal{M}^*(\mathcal{V} \times T) = (\mathcal{M}(\mathcal{V} \times T))^* = ((\mathcal{M}(\mathcal{V})) \times T)^* = (\mathcal{M}(\mathcal{V}))^* \cdot T = (\mathcal{M}^*(\mathcal{V})) \cdot T.$

E 4.17: Use the facts that
(a) A circuit of a graph is the same as a circuit of the polygon matroid of the graph.
(b) A cutset of a graph is the same as a bond of the polygon matroid of the graph.
Now use Theorem 4.4.2, Exercise 4.14 and Exercise 4.15.

E 4.18: Use Theorem 4.4.6 whenever there is a sequence 'contraction, restriction, contraction' and convert to 'restriction, contraction, contraction'. The latter simplifies to just two operations - restriction followed by contraction. Do this repeatedly.

E 4.19: We have, by Exercise 4.16, $\mathcal{M}(\mathcal{V} \,.\, T) = (\mathcal{M}(\mathcal{V})) \cdot T$ and $\mathcal{M}(\mathcal{V} \times T) = (\mathcal{M}(\mathcal{V})) \times T$. Further T is a separator of $\mathcal{V}$ iff $\mathcal{V} \,.\, T = \mathcal{V} \times T$ and T is a separator of $\mathcal{M}(\mathcal{V})$ iff $(\mathcal{M}(\mathcal{V})) \cdot T = (\mathcal{M}(\mathcal{V})) \times T$.
Hence, $T \subseteq S$ is a separator of $\mathcal{M}(\mathcal{V})$, if it is a separator of $\mathcal{V}$.
On the other hand, since $\mathcal{V} \,.\, T \supseteq \mathcal{V} \times T$, if $\mathcal{V} \,.\, T \neq \mathcal{V} \times T$, we must have $r(\mathcal{V} \,.\, T) \neq r(\mathcal{V} \times T)$. Hence, $\mathcal{M}(\mathcal{V} \,.\, T) \neq \mathcal{M}(\mathcal{V} \times T)$, i.e., $(\mathcal{M}(\mathcal{V})) \cdot T \neq (\mathcal{M}(\mathcal{V})) \times T$. So, if T is not a separator of $\mathcal{V}$, it would not be a separator of $\mathcal{M}(\mathcal{V})$.

E 4.20: A maximum weight base is the same as a lexicographically optimum base relative to $w(\cdot)$. A base B of the latter type would have edges from the subsets of T_i in the priority sequence $T_1, \cdots, T_k$. Let $B = B_1 \uplus \cdots \uplus B_k$ where $B_i \equiv B \cap T_i$. Clearly $\bigcup_{i=1}^{j} B_i$ is a maximal intersection of a base of $\mathcal{M}$ with $\bigcup_{i=1}^{j} T_i$. Hence, by the definition of restriction, $\bigcup_{i=1}^{j} B_i$ is a base of $\mathcal{M} \cdot (\bigcup_{i=1}^{j} T_i)$. If $j = 1$, since $\mathcal{M} \cdot T_1 \times T_1 = \mathcal{M} \cdot T_1$ we have B_1 as a base of $\mathcal{M} \cdot T_1 \times T_1$. Let $j > 1$. In this case $\bigcup_{i=1}^{j} B_i$ is a maximal intersection of a base of $\mathcal{M} \cdot (\bigcup_{i=1}^{j} T_i)$ with $(\bigcup_{i=1}^{j} T_i)$. Therefore, B_j is a minimal intersection of a base of $\mathcal{M} \cdot (\bigcup_{i=1}^{j} T_i)$ with T_j, i.e., B_j is a base of $\mathcal{M} \cdot (\bigcup_{i=1}^{j} T_i) \times T_j$.
Conversely, let $B = B_1 \uplus \cdots \uplus B_k$, where B_j is a base of $\mathcal{M} \cdot (\bigcup_{i=1}^{j} T_i) \times T_j$. Clearly B_1 is a maximal intersection of a base of $\mathcal{M}$ with T_1. Suppose $\bigcup_{i=1}^{j-1} B_i$ is a maximal intersection of a base of $\mathcal{M}$ with $\bigcup_{i=1}^{j-1} T_i$. Then $\bigcup_{i=1}^{j-1} B_i$ is a base of $\mathcal{M} \cdot (\bigcup_{i=1}^{j-1} T_i)$ (by the definition of restriction). Now we are given that B_j is a base of $\mathcal{M} \cdot (\bigcup_{i=1}^{j} T_i) \times T_j$. It follows that (Exercise 4.12) $(\bigcup_{i=1}^{j-1} B_i) \cup B_j$ is a base of $\mathcal{M} \cdot (\bigcup_{i=1}^{j} T_i)$. Thus, $\bigcup_{i=1}^{j} B_i$ is a maximal intersection of a base of $\mathcal{M}$ with

$\bigcup_{i=1}^{j} T_i$. Thus, B contains elements of T_i in the priority sequence $T_1, \cdots, T_k$, i.e., it is a lexicographically optimum base of $\mathcal{M}$ relative to $w(\cdot)$, i.e., it is a maximum weight base of $\mathcal{M}$.

Chapter 5

Electrical Networks

5.1 Introduction

In this chapter we give a brief introduction to electrical network analysis. The aim is to make the book self contained and also to fix notations and conventions. We begin by giving the multiterminal and 2-terminal descriptions of an electrical network. We then give a list of the standard devices that are used in electrical networks. Next we give a short description of the common methods of analysis with, in the case of modified nodal analysis (MNA), a mention of its merits and demerits. After this we give a sketch of the working of a general purpose simulator. This sketch and the description of MNA are given so that one may better appreciate the need for topological hybrid analysis, which is dealt with in the next chapter. This is followed by an informal account of state equations for networks assuming that the initial conditions of all the capacitors and inductors can be assigned independently of each other. (In Chapter 8 we show that using multiport decomposition we can handle more general situations and also reduce the network without losing information of its dynamics). We then give an informal description of multiport decomposition for networks. As an application of this idea we state and prove the 'generalized Thevenin - Norton Theorem'. Finally we discuss two elementary results of network theory - substitution theorem and superposition theorem.

5.2 In Terms of Multiterminal Devices

An electrical network is obtained by the **interconnection** of a collection of **multiterminal devices**. Each of the devices may have one or more terminals. The behaviour of the devices can usually be described independently of each other, but this is not always the case.

The multiterminal device behaviour is specified through a relation of the form

Figure 5.1: A Multiterminal Device

$\mathbf{f}(\int \mathbf{v}, \mathbf{v}, \dot{\mathbf{v}}, \int \mathbf{i}, \mathbf{i}, \dot{\mathbf{i}}) = \mathbf{0}$,

where $\int \mathbf{v} = (\int v_1 dt, \int v_2 dt \ldots, \int v_k dt)$ and the other symbols have similar meanings, $v_1, v_2 \ldots, v_k$ are the terminal potentials of the device and $i_1, i_2 \ldots, i_k$, the current entering these terminals (see Figure 5.1).

The interconnection of these devices consists in identifying the terminals of different devices with single nodes. This interconnection may be represented by a bipartite graph $B \equiv (V_L, V_R, E)$. Here,

V_L is the set of nodes of the network
V_R is the set of devices in the network
E is the collection of all terminals of all the devices in the network.

So if terminal t^i_j of device d_i is incident on node n_r of the network we join node $n_r \in V_L$ with node $d_i \in V_R$ by an edge t^i_j directed towards d_i. Figure 5.2 shows an electrical network $\mathcal{N}$ made up of multiterminal devices and the bipartite graph representing the interconnection of the devices in the network.

Multiterminal Network Constraints

The constraints of the network are

i. the **device characteristics:**
These are the relations between the terminal voltages and terminal currents of each of the devices.

ii. the **topological constraints:**

(a) the potentials of all terminals, which are connected to the same node, are to be the same.

Figure 5.2: An Electrical Network $\mathcal{N}$ and its Bipartite Graph Representation

(b) the sum of the currents entering all the terminals of a device is zero and the sum of the currents leaving each node is zero.

5.3 In Terms of 2-Terminal Devices

It is possible to develop network theory using multiterminal devices as the basis. There are advantages to doing this atleast in some special situations. For historical reasons electrical network theory has developed using only 2-terminal devices (which may be coupled in the device characteristic). Since it can be shown that any n-terminal device is equivalent to $(n-1)$ coupled two terminal devices there is no real loss of generality. Thus we have the second formulation:

An **electrical network** is obtained by **interconnecting** two terminal devices which thus become the edges of a directed graph $\mathcal{G}$. Each two terminal device d_j is represented by a directed edge j and is associated with a voltage v_j and a current i_j.

We may call the tail of the arrow of the edge the 'positive terminal' and the head of the arrow the 'negative terminal' of the device. The voltage v_j associated with the j^{th} edge is equal to $(v_{j+} - v_{j-})$ where v_{j+} is the positive terminal potential and v_{j-}, the negative terminal potential. Of course the actual value of v_{j+}, v_{j-} may not necessarily be positive or negative. The current i_j associated with the edge j is the current flowing through the device in the direction of the arrow.

Figure 5.3: KCE and KVE

2-terminal Network Constraints

The network has the following constraints:

i. (**device characteristic**) $\mathbf{f}(\int \mathbf{v}, \mathbf{v}, \dot{\mathbf{v}}, \int \mathbf{i}, \mathbf{i}, \dot{\mathbf{i}}) = \mathbf{0}$ (here $\mathbf{v}$ is the voltage vector composed of the voltages of all the devices, and $\mathbf{i}$ is the current vector composed of the currents of all the devices).

ii. (**KCE**) the net current leaving any node is zero.

iii. (**KVE**) the algebraic sum of the voltages around any loop is zero, equivalently, the voltage vector is derived from a node potential vector (see Subsection 3.3.2).

Example: In Figure 5.3 the net current leaving node n_1 is $-i_1 + i_2 - i_3 + i_4$. KCE sets this expression to zero. In the KVE part of the same figure the algebraic sum of the voltages around the loop in the given orientation is $v_1 + v_2 + v_3 - v_4 - v_5$. KVE sets this expression to zero.

The KCE and KVE constraints can be stated equivalently as $\mathbf{v}(t) \in \mathcal{V}_v(G)$, $\mathbf{i}(t) \in \mathcal{V}_i(G)$, $\forall t \in \Re$ (see Subsection 3.3.2). These are the **topological** constraints.

The problem of **network analysis** is to **solve** the network, i.e., to find the set of all ordered pairs $(\mathbf{v}(\cdot), \mathbf{i}(\cdot))$ which satisfy the above mentioned device characteristic and topological constraints. This latter set is called the **solution set** of the network.

Figure 5.4: The Sources

5.4 Standard Devices

We now list a number of standard devices which are used in electrical networks. Some of these devices are made up of more than one coupled two terminal device. Both the voltage and current associated with a two terminal device would have the same reference arrow with the voltage being the 'tail potential minus head potential' and the current being the value of current flowing in the direction of the arrow.

i. **Voltage Source:** Characteristic $v_e(t) = e(t)$.
(Figure 5.4). A voltage source imposes no constraint on the current through it. We will denote the collection of voltage sources by $\mathcal{E}$.

ii. **Current Source:** Characteristic $i_j(t) = j(t)$.
(Figure 5.4). A current source imposes no constraint on the voltage across it. We will denote the collection of current sources by $\mathcal{J}$.

iii. **Resistor:** Characteristic $v_r = Ri_r$.
R is a positive number (Figure 5.5).
We will denote the collection of resistors also by R. During the discussion of methods of analysis we permit coupled resistors with the characteristic $\mathbf{v}_R = \mathbf{R}\mathbf{i}_R$. Here $\mathbf{R}$ is an arbitrary matrix.

iv. **Inductor:** Characteristic $v_L = L\dot{i}_L$.
L is a positive number (Figure 5.5).
We denote the collection of inductors by L.

v. **Coupled inductors:** Characteristic $\mathbf{v}_L = \mathcal{L}\dot{\mathbf{i}}_L$. Here $\mathcal{L}$ is a symmetric positive definite matrix (Figure 5.5).

Figure 5.5: Passive Linear Devices

We denote the collection of coupled (or 'mutual') inductors by M.

vi. **Capacitor:** Characteristic $i_C = C\dot{v}_C$
C is a positive number (Figure 5.5). We will denote the collection of capacitors by C. During the discussion of methods of analysis we permit coupled capacitors with the characteristic

$$\mathbf{i}_C = \mathcal{C}\dot{\mathbf{v}}_C.$$

vii. **Nonlinear resistors, inductors and capacitors:** A **nonlinear resistor** would have the general characteristic $f_R(v_r, i_r) = 0$ (Figure 5.6). The resistors could be coupled, in which case the characteristic would be

$$\mathbf{f}_R(\mathbf{v}_R, \mathbf{i}_R) = \mathbf{0}.$$

A **nonlinear inductor** would have the general characteristic

$$v_L = \frac{d\psi_L}{dt}$$

$$f_L(i_L, \psi_L) = 0$$

(Figure 5.6).
Coupled nonlinear inductors would have a general characteristic of the

Figure 5.6: Nonlinear Devices

same form as above except that $\mathbf{i}_L, \psi_L, \mathbf{f}_L(\cdot,\cdot)$ would be vectors.
A **nonlinear capacitor** would have the general characteristic

$$i_C = \frac{dq_C}{dt}$$

$$f_C(v_C, q_C) = 0$$

(Figure 5.6).

viii. **Ideal transformer:** An ideal transformer can be thought of as several coupled 2-terminal devices called 'ports of the transformer' with an overall characteristic

$$\mathbf{v} \in \mathcal{V}$$

$$\mathbf{i} \in \mathcal{V}^{\perp}$$

where $\mathcal{V}, \mathcal{V}^{\perp}$ are complementary orthogonal real vector spaces on the set S of the ports (Figure 5.7). In the figure,

$$(v_1, \cdots, v_k) = \mathbf{v}$$

$$(i_1, \cdots, i_k) = \mathbf{i}$$

The reader may observe that the voltage and current constraints imposed by KCE and KVE of a graph make it into a special case of an ideal transformer.
A 2-port $(1:n)$ ideal transformer satisfies

$$nv_1 - v_2 = 0$$

Figure 5.7: Ideal Transformer

Figure 5.8: Gyrator

$$i_1 + n i_2 = 0$$

v_1, i_1 are **primary** voltage and current; v_2, i_2 are **secondary** voltage and current. Clearly, (v_1, v_2) belongs to the space spanned by $(1, n)$ and (i_1, i_2) belongs to the complementary orthogonal space spanned by $(-n, 1)$.

ix. **Gyrator:** Characteristic

$$v_1 = -r i_2$$
$$v_2 = r i_1$$

(Figure 5.8).
r is called the **gyration resistance** and the arrow associated with r is called the **direction of gyration**. The gyration resistance is taken to be positive. However, if the direction of gyration is opposite to that in the figure we will have

$$v_1 = r i_2$$
$$v_2 = -r i_1$$

Figure 5.9: Ideal Diode

x. **Ideal Diode:** Characteristic

$$v \leq 0, i \geq 0$$

$$v = 0, \ if \ i > 0$$
$$i = 0, \ if \ v < 0$$

(Figure 5.9).

xi. **Controlled sources:** These are not independent sources but coupled resistors of special kinds. In each case there are two coupled 2-terminal devices (Figure 5.10).

(a) **vccs (Voltage controlled current source):** Characteristic

$$i_1 = 0$$

$$i_2 = gv_1$$

(b) **vcvs (Voltage controlled voltage source):** Characteristic

$$i_1 = 0$$

$$v_2 = \alpha v_1$$

(c) **ccvs (Current controlled voltage source):** Characteristic

$$v_1 = 0$$

$$v_2 = ri_1$$

(d) **cccs (Current controlled current source):** Characteristic

$$v_1 = 0$$

$$i_2 = \beta i_1.$$

Figure 5.10: Controlled Sources

xii. **Norators and nullators:** These devices are pathological –introduced primarily for notational convenience. A **norator** permits every (v, i) pair to be associated with it, i.e., it imposes no device characteristic constraint. A **nullator** imposes $v = 0$ and $i = 0$. Norator is useful for topological purposes such as to specify ports.

Exercise 5.1 *(k) Our reference convention is that for a two terminal device a single arrow is shown for both $v(\cdot)$ and $i(\cdot)$ associated with the device with $v(\cdot)$ being the 'voltage drop across' the device in the direction of the arrow and $i(\cdot)$ the 'current through' the device in the direction of the arrow. Power absorbed (in watts) by the device at some time t is defined to be the product $v(t)\ i(t)$.*

i. Justify this definition on physical grounds.

ii. Show that the power absorbed by a positive resistor is always nonnegative, that by a negative resistor is always nonpositive and that by sources can be positive, negative or zero.

Exercise 5.2 *i. (k) Suppose a network contains a voltage source circuit. If the network has a solution, then show that the current through the voltage sources in the circuit cannot be determined uniquely.*

ii. (k) If the network contains a current source cutset and is known to have a solution, then show that the voltage across the current sources in the cutset cannot be determined uniquely.

Exercise 5.3 *(k) Let the network contain only voltage sources* $\mathbf{v}_{\mathcal{E}} = \mathbf{e}$*, current sources* $\mathbf{i}_{\mathcal{J}} = \mathbf{j}$ *and* $\mathbf{v}_R = \mathbf{R}\ \mathbf{i}_R$*. Suppose* $(\mathbf{v}^1, \mathbf{i}^1)$ *is a solution of the network when voltage sources value is given by the vector* $\mathbf{e}^1$ *and the current sources value by the vector* $\mathbf{j}^1$*. Let* $(\mathbf{v}^2, \mathbf{i}^2)$ *correspond to* $(\mathbf{e}^2, \mathbf{j}^2)$*. Show that there is a solution of the network* $(\alpha\mathbf{v}^1 + \beta\mathbf{v}^2, \alpha\mathbf{i}^1 + \beta\mathbf{i}^2)$ *corresponding to* $(\alpha\mathbf{e}^1 + \beta\mathbf{e}^2, \alpha\mathbf{j}^1 + \beta\mathbf{j}^2)$*.*

Exercise 5.4 *(k) We have defined four types of controlled sources, cccs, ccvs, vcvs, vccs*

i. *Suppose, in a given physical network, currents and voltages associated with some device, say a resistor, control the current or voltage associated with some other device. How would you model this situation using our controlled sources?*

ii. *Show how to model cccs and vcvs using the other two controlled sources.*

Exercise 5.5 *(k) Let a capacitor* C *carry an initial voltage* v_o *in a certain circuit. Show that*

i. *if the final voltage of the capacitor is* v_f *the capacitor has lost energy* $= \frac{1}{2}C(v_o^2 - v_f^2)$ *joules.*

ii. *the maximum electrical energy that can be extracted from the capacitor equals* $\frac{1}{2}Cv_o^2$ *joules.*

Exercise 5.6 *(k) Let a nonlinear capacitor be defined by* $i = \dfrac{dq}{dt}$ *,* $q = \hat{q}(v)$*. Show that the energy delivered by this capacitor depends only on the initial and final voltage.*

Exercise 5.7 *(k) Let a set of coupled inductors with the characteristic* $\mathbf{v} = \mathcal{L}\ \dot{\mathbf{i}}$ *carry an initial current vector* $\mathbf{i}_o$*. Show that*

i. *if the final current of the set of inductors is* $\mathbf{i}_f$ *show that the coupled system has gained energy* $= \frac{1}{2}[\mathbf{i}_f^T\mathcal{L}\mathbf{i}_f - \mathbf{i}_o^T\mathcal{L}\mathbf{i}_o]$ *joules.*

ii. *the maximum electrical energy that can be extracted from the coupled system equals* $\frac{1}{2}[\mathbf{i}_o^T\mathcal{L}\mathbf{i}_o]$ *joules.*

Exercise 5.8 *(k) Let a nonlinear inductor be defined by* $\mathbf{v} = \dfrac{d\psi}{dt}$*,* $\psi = \hat{\psi}(i)$*. Show that the energy delivered by this inductor depends only on the initial and final current.*

Exercise 5.9 i. *(k) Show that the power absorbed by an ideal transformer is always zero.*

ii. *(k) Let a resistor* R *be connected across the secondary of a* $(1 : n)$ *ideal transformer, i.e.,* $v_2 = v_R$*,* $i_2 = -i_R$*. (Observe that the current through the secondary has to be taken to be the negative of the current through the resistor if the voltages are taken to be the same). Find the relation between the primary voltage* v_1 *and the primary current* i_1*.*

iii. *Repeat the above if the secondary were 'terminated' alternatively by inductor L, capacitor C, current source $j(.)$ and voltage source $e(.)$.*

iv. *Let an ideal transformer be defined by the current equations*

$$\begin{bmatrix} \mathbf{K} & \mathbf{I} \end{bmatrix} \begin{matrix} \mathbf{i}_A \\ \mathbf{i}_B \end{matrix} = \mathbf{0}$$

Suppose we terminate the B ports by resistors with the characteristic $\mathbf{v}_R = \mathbf{R}\mathbf{i}_R$. What would be the relation between $\mathbf{v}_A$ and $\mathbf{i}_A$?

Exercise 5.10 *(k) Let a gyrator have the characteristic*

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 & r \\ -r & 0 \end{bmatrix} \begin{matrix} i_1 \\ i_2 \end{matrix}$$

i. *Show that the power absorbed by the gyrator is always zero.*

ii. *What would be the relation betwen v_1 and i_1 if across the secondary we connect a resistor, a capacitor, inductor, voltage source or current source?*

Exercise 5.11 *(k) Show that the power absorbed by an ideal diode is always zero.*

Exercise 5.12 *The ideal transformer, gyrator and diode all absorb zero power. However, it can be claimed that the ideal transformer satisfies this condition in a way much stronger than the other two. Why?*

Exercise 5.13 *[Belevitch68] Show that any multiport ideal transformer can be realized by a suitable connection of 2-port ideal transformers.*

5.5 Common Methods of Analysis

Next we describe some common methods of analysis. We observe that all methods of analysis must use KCE, KVE and device characteristic constraints. They differ from one another only in the manner in which these constraints are imposed. Although the methods of analysis that we describe are valid for both linear and nonlinear networks, we confine ourselves to the former for two reasons: (i) notational simplicity (ii) most general purpose circuit simulators convert nonlinear network analysis to the analysis of a sequence of linear networks.

5.5.1 Nodal Analysis

This method is directly applicable when the network has only **current sources** and **conductance type devices**, i.e., devices whose currents can be expressed in terms of voltages of the same or other nonsource devices.

STEPS

i. **Write KCE** for the network using a reduced incidence matrix (obtained by omitting one row of the incidence matrix per component of the graph). Shift the current source terms to the right.

ii. Express non source device currents in terms of device voltages, i.e., **impose device characteristic constraints**.

iii. Express device voltages in terms of node potentials, i.e., **impose KVE constraints**.

The final equations have node potentials as unknowns.

Now we derive the nodal equations according to the above steps. We assume there are two types of devices G and $\mathcal{J}$. The reduced incidence matrix is denoted by $\mathbf{A}_r$. We have $\mathbf{A}_r = (\mathbf{A}_{rG} \vdots \mathbf{A}_{rJ})$, where the columns have been partitioned into those corresponding to G and those corresponding to $\mathcal{J}$.

We then have

STEP 1

$$\mathbf{A}_{rG}\mathbf{i}_G = -\mathbf{A}_{rj}\mathbf{i}_{\mathcal{J}} \qquad (KCE)$$

STEP 2 Let

$$\mathbf{i}_G = \mathbf{G}\mathbf{v}_G \qquad (device\ characteristics)$$

Then,

$$\mathbf{A}_{rG}\mathbf{G}\mathbf{v}_G = -\mathbf{A}_{rJ}\mathbf{i}_{\mathcal{J}}$$

STEP 3

$$\begin{bmatrix} \mathbf{v}_G \\ \cdots \\ \mathbf{v}_{\mathcal{J}} \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{rG}^T \\ \cdots \\ \mathbf{A}_{rJ}^T \end{bmatrix} \mathbf{v}_n \qquad (KV\ constraints) \tag{5.1}$$

Hence, $(\mathbf{A}_{rG}\mathbf{G}\mathbf{A}_{rG}^T)\mathbf{v}_n = -\mathbf{A}_{rJ}\mathbf{i}_{\mathcal{J}}$.

The last equation is the **set of nodal equations** for the network.

Necessity and Sufficiency of Nodal Equations

Solving the nodal equations would give us the collection of all node potential vectors. Starting from such a node potential vector $\mathbf{v}_n$, we can obtain $\mathbf{v}_G, \mathbf{v}_{\mathcal{J}}$ by using the KV constraints of STEP 3. We can then obtain $\mathbf{i}$ using the device characteristic constraints of STEP 2. Since $\mathbf{i}_{\mathcal{J}}$ is already specified, we have found a solution $(\mathbf{v}, \mathbf{i})$ of the network consistent with a node potential vector $\mathbf{v}_n$. On the other hand suppose we have a solution $(\mathbf{v}, \mathbf{i})$. It is clear that STEP 1, STEP 2 would be satisfied by it. Further $\mathbf{v}$ would satisfy the KV constraints of STEP 3 for some $\mathbf{v}_n$. This latter vector would satisfy the nodal equations of the network. Thus, the nodal equations are **necessary** and **sufficient** for the solution of **this type** of network. By this we mean (a) every solution of the network after transformation to node potential vector must satisfy the nodal equations, and (b) from each solution of nodal equations we can derive a unique solution of the network. In particular if the nodal equations have no solution, neither will the network have a solution. If they have nonunique solutions, the network will have nonunique solutions.

Effect of Current Source Topology

It can be seen that if $\mathcal{J}$ does not contain a cutset of the graph $\mathcal{G}$ then deletion of $\mathcal{J}$ does not disconnect the graph and $\mathbf{A}_{rG}$ remains the reduced incidence matrix of the resulting graph. So $r(\mathbf{A}_{rG}) = r(\mathbf{A}_{rG}\mathbf{A}_{rJ})$. If the matrix $\mathbf{G}$ is positive definite (in particular positive diagonal) then the coefficient matrix $\mathbf{A}_{rG}\mathbf{G}\mathbf{A}_{rG}^T$ is positive definite. So the network has a unique solution.

If $\mathcal{J}$ contains a cutset there are two possibilities:

i. The current sources violate KCE, i.e, there is a cutset composed only of current sources and the sum of the currents in the cutset from the positive end vertex set to the negative end vertex set is nonzero. In this case the network has no solution.

ii. The current sources do not violate KCE. In this case the nodal equations have nonunique solutions since $r(\mathbf{A}_{rG}) < r(\mathbf{A}_{rG}\mathbf{A}_{rJ})$ = number of rows of $\mathbf{A}_{rG}$.

Sparsity when G is Positive Diagonal

The case where $\mathbf{G}$ is a positive diagonal matrix is particularly important. In this case the entries of $\mathbf{A}_{rG}\mathbf{G}\mathbf{A}_{rG}^T$ can be written down by inspection. The diagonal entries are the sum of the conductances incident on the corresponding nodes while the $(i,j)^{th}$ offdiagonal entry is the negative of the conductance between i and j.

The total number of nonzero entries is equal to 2(number of conductance edges) + number of nodes - (number of conductance edges incident at the node omitted from the reduced incidence matrix + 1). So the matrix is sparse if the number of nodes $\gg$ 2(number of edges).
This method can be modified to take care of the case where voltage sources are also present.
Where the type of network permits it, nodal analysis is currently the most popular method of network analysis because of the sparsity of the coefficient matrix in the method.

5.5.2 Loop Analysis

This method is directly applicable when the network has only voltage sources and resistance type devices, i.e., devices whose voltages can be expressed in terms of currents of the same or other nonsource devices.

STEPS

i. **Write KVE** for the network using a representative matrix $\mathbf{B}$ of $\mathcal{V}_i(G)$, where $\mathcal{G}$ is the graph of the network. Shift the voltage source terms to the right.

ii. Express nonsource device voltages in terms of device currents, i.e., **impose device characteristic constraints**.

iii. Express device currents in terms of 'loop currents', i.e., **impose KCE constraints**.

The final equations have loop currents as unknowns.

Now we derive the loop equations according to the above steps. We assume there are two types of devices R and $\mathcal{E}$. The representative matrix $\mathbf{B}$ of $\mathcal{V}_i(G)$ is partitioned into $(\mathbf{B}_R \vdots \mathbf{B}_{\mathcal{E}})$ according to the branches in R and $\mathcal{E}$.

We then have

STEP 1

$$\mathbf{B}_R \mathbf{v}_R = -\mathbf{B}_{\mathcal{E}} \mathbf{v}_{\mathcal{E}} \qquad (KCE)$$

STEP 2 Let

$$\mathbf{v}_R = \mathbf{R}\ \mathbf{i}_R \qquad (\textit{device characteristic})$$

Then,

$$\mathbf{B}_R \mathbf{R} \mathbf{i}_R = -\mathbf{B}_{\mathcal{E}} \mathbf{v}_{\mathcal{E}}$$

STEP 3

$$\begin{bmatrix} \mathbf{i}_R \\ \cdots \\ \mathbf{i}_{\mathcal{E}} \end{bmatrix} = \begin{bmatrix} \mathbf{B}_R^T \\ \cdots \\ \mathbf{B}_{\mathcal{E}}^T \end{bmatrix} \mathbf{i}_l \qquad (\textit{KV constraint}) \tag{5.2}$$

Hence, $(\mathbf{B}_R \mathbf{R} \mathbf{B}_R^T)\mathbf{i}_l = -\mathbf{B}_{\mathcal{E}} \mathbf{v}_{\mathcal{E}}$.

The last equation is the set of **loop equations for the network**. The vector $\mathbf{i}_l$ is called the vector of loop currents associated with the matrix $\mathbf{B}$. It is the coordinate vector of $\mathbf{i}$ in the basis corresponding to the rows of $\mathbf{B}$.

By an argument parallel to the one that we used for nodal equations we can show that loop equations are **necessary** and **sufficient** for the solution of the above type of network: (a) Every solution of the network after transformation to loop current vector must satisfy the loop equations. (b) from each solution of the loop equations we can derive a unique solution of the network.

Effect of Voltage Source Topology

It can be shown that if $\mathcal{E}$ does not contain a circuit of the graph $\mathcal{G}$ then contraction of $\mathcal{E}$ does not reduce the nullity of $\mathcal{G}$. So $r(\mathbf{B}_R) = r(\mathbf{B}_R \vdots \mathbf{B}_{\mathcal{E}})$. If the matrix $\mathbf{R}$ is positive definite (in particular positive diagonal) then the coefficient matrix $\mathbf{B}_R\mathbf{R}\mathbf{B}_R^T$ is positive definite. So the network has a unique solution.

If $\mathcal{E}$ contains a circuit there are two possibilities:

i. The voltage sources violate KVE, i.e., there is a circuit composed only of voltage sources and the sum of the voltages of the sources along the orientation of the circuit is nonzero. In this case the network has no solution.

ii. The voltage sources do not violate KVE. In this case the loop equations have nonunique solution since

$r(\mathbf{B}_R) < r(\mathbf{B}_R : \mathbf{B}_{\mathcal{E}}) =$ number of rows of $\mathbf{B}_R$.

Sparsity when R is Positive Diagonal

The case where $\mathbf{R}$ is a positive diagonal matrix is particularly important. In this case the entries of $\mathbf{B}_R\mathbf{R}\mathbf{B}_R^T$ can be written down by inspection. The diagonal entry (k, k) is the sum of the resistances corresponding to the support of the k^{th} row of $\mathbf{B}$. The offdiagonal entry (k, m) is equal to the sum of the resistances whose edges have the same sign in row k and row m minus the sum of the resistances whose edges have opposite sign in row k and row m. It is easily seen that the total number of nonzero entries in the matrix can be quite large if the matrix $\mathbf{B}$ is chosen carelessly. For instance it can be shown that if $\mathbf{B}$ is an f-circuit matrix and has a forest edge common to all the f-circuits then $\mathbf{B}_R\mathbf{R}\mathbf{B}_R^T$ has in general no zero entries. When the graph is planar we can choose $\mathbf{B}$ to be the mesh matrix. In this case $\mathbf{B}$ has the properties of a reduced incidence matrix and $\mathbf{B}_R\mathbf{R}\mathbf{B}_R^T$ would be sparse. (Meshes are the windows into which the network divides the plane when drawn on it without crossing of edges. The mesh matrix has one row per mesh except the outer most mesh which is omitted. All the meshes are given the same orientation, either clockwise or anticlockwise).

The Method of Planar Slices

If the graph is not planar one can construct a number of planar subgraphs $\mathcal{G}_1, \mathcal{G}_2 \ldots \mathcal{G}_k$ s.t. (a) $\bigcup E(\mathcal{G}_i) = E(\mathcal{G})$ (b) $E(\mathcal{G}_{j+1}) \cap (\bigcup_{i=1}^{j} E(\mathcal{G}_i))$ is a forest of the graph $\mathcal{G} \cdot (\bigcup_{i=1}^{j} E(\mathcal{G}_i)), j = 1, \cdots, k-1$. It can be shown that $\mathcal{V}_i(\mathcal{G}) = \sum \mathcal{V}_i(\mathcal{G}_i)$ and $\nu(\mathcal{G}) = \sum \nu(\mathcal{G}_i)$. So we can select $\mathbf{B}$ to be the matrix obtained by taking the collection of all the rows of the mesh matrices of $\mathcal{G}_i$. If the intersecting forests are chosen with care the resulting $\mathbf{B}$ matrix yields a sparse $\mathbf{B}_R\mathbf{R}\mathbf{B}_R^T$ (see Problem 3.24 [Ovalekar+Narayanan92]).

Loop analysis can also be modified to handle circuits which have both voltage and current sources. However, where there is a choice between nodal and loop analysis the former is invariably preferred. This is because it is difficult to achieve sparsity for the loop analysis matrix comparable to that of the nodal analysis matrix. Even if the network is planar, since in practice ranks of network graphs are lower than their nullities, the matrix size would come out to be larger than the nodal analysis matrix. However, there is a case for using loop analysis as a supplement to nodal analysis. This **hybrid analysis** is sometimes preferable to **modified nodal analysis** which latter we describe next. Hybrid analysis is described in the next chapter.

5.5.3 Modified Nodal Analysis

This is a very simple technique which is applicable to all kinds of networks. It coincides with nodal analysis where the latter is applicable.

STEP 1 Divide the network into current sources $\mathcal{J}$, conductance type branches G and the remaining branches T. Write nodal analysis equations for the network treating both $\mathcal{J}$ and T as current source branches.

STEP 2 For each device in T write down the device characteristic equations.

STEP 3 In the device characteristic equations of STEP 2 write branch voltages in terms of node potentials.

The unknowns of the resulting MNA equations are node voltages $\mathbf{v}_n$ and the currents $\mathbf{i}_T$.

Now we derive the MNA equations according to the above steps. The reduced incidence matrix $\mathbf{A}_r$ of $\mathcal{G}$ is partitioned as $(\mathbf{A}_{rG} \vdots \mathbf{A}_{rT} \vdots \mathbf{A}_{rJ})$. We then have
STEP 1.1

$$(\mathbf{A}_{rG} \vdots \mathbf{A}_{rT})^{\mathbf{i}_G}_{\mathbf{i}_T} = -\mathbf{A}_{rJ}\mathbf{i}_{\mathcal{J}} \qquad (KCE)$$

STEP 1.2 Let

$$\mathbf{i}_G = \mathbf{G}\mathbf{v}_G \qquad (device\ characteristic)$$

Then

$$(\mathbf{A}_{rG}\mathbf{G} \vdots \mathbf{A}_{rT})^{\mathbf{v}_G}_{\mathbf{i}_T} = -\mathbf{A}_{rJ}\mathbf{i}_{\mathcal{J}}.$$

STEP 1.3

$$\begin{bmatrix} \mathbf{v}_G \\ \cdots \\ \mathbf{v}_T \\ \cdots \\ \mathbf{v}_{\mathcal{J}} \end{bmatrix} = \begin{bmatrix} \mathbf{A}^T_{rG} \\ \cdots \\ \mathbf{A}^T_{rT} \\ \cdots \\ \mathbf{A}^T_{rJ} \end{bmatrix} \mathbf{v}_n \qquad (KV constraint) \qquad (5.3)$$

Hence,

$$(\mathbf{A}_{rG}\mathbf{G}\mathbf{A}_{rG}^T \vdots \mathbf{A}_{rT}) \begin{bmatrix} \mathbf{v}_n \\ \mathbf{i}_T \end{bmatrix} = -\mathbf{A}_{rJ}\mathbf{i}_{\mathcal{J}}.$$

STEP 2 Let the device characteristic of the branches in T be

$$\begin{pmatrix} \mathbf{M} & \mathbf{N} \end{pmatrix} \begin{bmatrix} \mathbf{i}_T \\ \mathbf{v}_T \end{bmatrix} = \mathbf{s}_T.$$

STEP 3 We have

$$\mathbf{v}_T = \mathbf{A}_{rT}^T \mathbf{v}_n.$$

Hence,

$$(\mathbf{M} \vdots (\mathbf{N}\mathbf{A}_{rT}^T)) \begin{bmatrix} \mathbf{i}_T \\ \mathbf{v}_n \end{bmatrix} = \mathbf{s}_T.$$

Thus, the MNA equations for the linear static case are

$$\begin{bmatrix} \mathbf{A}_{rG}\mathbf{G}\mathbf{A}_{rG}^T & \mathbf{A}_{rT} \\ \mathbf{N}\mathbf{A}_{rT}^T & \mathbf{M} \end{bmatrix} \begin{bmatrix} \mathbf{v}_n \\ \mathbf{i}_T \end{bmatrix} = \begin{bmatrix} -\mathbf{A}_{rJ}\mathbf{i}_{\mathcal{J}} \\ \mathbf{s}_T \end{bmatrix}.$$

Static Nonlinear Networks

Let the network be static but nonlinear with the device characteristic equations given below:

$\mathbf{i}_G = \mathbf{g}(\mathbf{v}_G)$

$\mathbf{h}(\mathbf{v}_T, \mathbf{i}_T) = \mathbf{0}.$

Then the MNA equations for static nonlinear case would be

$\mathbf{A}_{rG}\mathbf{g}(\mathbf{A}_{rG}^T \mathbf{v}_n) + \mathbf{A}_{rT}\mathbf{i}_T = -\mathbf{A}_{rJ}\mathbf{i}_{\mathcal{J}}$

$\mathbf{h}(\mathbf{A}_{rT}^T \mathbf{v}_n,\ \mathbf{i}_T) = \mathbf{0}$

Dynamic Nonlinear Networks

Let the network be dynamic and nonlinear with the device characteristic equations given below:

$$\begin{aligned} \mathbf{i}_{G1} &= \mathbf{g}(\mathbf{v}_{G1}) \\ \mathbf{i}_{C2} &= \frac{d}{dt}\mathcal{C}(\mathbf{v}_{C2}) \\ \mathbf{v}_{T1} &= \frac{d}{dt}\psi_{T1} \\ \mathbf{h}_1(\psi_{T1}, \mathbf{i}_{T1}) &= \mathbf{0} \end{aligned}$$

$$\mathbf{i}_{T2} = \frac{d\mathbf{q}_{T2}}{dt}$$
$$\mathbf{h}_2(\mathbf{q}_{T2}, \mathbf{v}_{T2}) = \mathbf{0}$$

$$\begin{pmatrix} \mathbf{M} & \mathbf{N} \end{pmatrix} \begin{matrix} \mathbf{i}_{T3} \\ \mathbf{v}_{T3} \end{matrix} = \mathbf{s}_{T3}$$

Note that $\mathbf{g}(\cdot), \mathcal{C}(\cdot)$ etc. are arbitrary functions.

The MNA equations for this case would be

$$\mathbf{A}_{rG1}\ \mathbf{g}(\mathbf{A}_{rG1}^T \mathbf{v}_n) + \mathbf{A}_{rC2}\frac{d}{dt}(\mathcal{C}(\mathbf{A}_{rC2}^T \mathbf{v}_n)) + \mathbf{A}_{rT1}\ \mathbf{i}_{T1} + \mathbf{A}_{rT2}\ \mathbf{i}_{T2}$$
$$+\mathbf{A}_{rT3}\ \mathbf{i}_{T3} = -\mathbf{A}_{rJ}\ \mathbf{i}_{\mathcal{J}}$$

$$\frac{d}{dt}\psi_{T1} - \mathbf{A}_{rT1}^T\ \mathbf{v}_n = \mathbf{0}$$
$$\mathbf{h}_1(\psi_{T1}, \mathbf{i}_{T1}) = \mathbf{0}$$
$$\frac{d\mathbf{q}_{T2}}{dt} - \mathbf{i}_{T2} = \mathbf{0}$$
$$\mathbf{h}_2(\mathbf{q}_{T2}, \mathbf{A}_{rT2}^T \mathbf{v}_n) = \mathbf{0}$$
$$(\mathbf{M} \vdots (\mathbf{N}\ \mathbf{A}_{rT3}^T))\ \begin{matrix} \mathbf{i}_{T3} \\ \mathbf{v}_n \end{matrix} = \mathbf{s}_{T3}.$$

We have described the MNA equations for nonlinear static and dynamic circuits only for completeness. For the purpose of studying the structure of network equations the **static linear** case is the most important as would be clear by the end of this chapter. We therefore make a few remarks on the MNA equations for this case.

Necessity and Sufficiency of MNA Equations

It is easy to show that the MNA equations are necessary and sufficient. Given any solution $(\mathbf{v}(\cdot), \mathbf{i}(\cdot))$ we can transform it into appropriate $\mathbf{v}_n, \mathbf{i}_T$ which satisify the MNA equations. On the other hand given any solution $\mathbf{v}_n, \mathbf{i}_T$ of the MNA equations we can find a corresponding solution $(\mathbf{v}(\cdot), \mathbf{i}(\cdot))$ of the network.

Sparsity of MNA Equations

During the last thirty years the problem of computing the solution of sparse linear equations has been extensively studied. Although there are hardly any theoretical results in this area, extremely efficient practical algorithms using heuristics are available. The great merit of MNA equations is that they are very sparse. Below we indicate why.

In most practical networks the bulk of the elements would be of the conductance type. So the sparsity of the matrix is controlled by the submatrix $\mathbf{A}_{rG}\mathbf{G}\mathbf{A}_{rG}^T$. The structure of this matrix has already been described in the discussion on nodal analysis. The matrix $(\mathbf{M}\ \mathbf{N})$ usually has about two entries per row. (Consider ordinary resistors, controlled sources etc.). Under this assumption it is clear that the matrix $(\mathbf{N}\mathbf{A}_{rT}^T \vdots \mathbf{M})$ has atmost four entries per row since $\mathbf{A}_{rT}^T$ has at most two entries per row. The coefficient matrix of the MNA equations is therefore very sparse. This in combination with the generality and simplicity of the method has made it very popular. Indeed it is the method used in the general purpose circuit simulator SPICE (see for instance [McCalla88]).

Defects of MNA

We will speak now of some of the defects of this way of writing equations. It is often the case that $T \cup \mathcal{J}$ will contain a cutset of the graph. In such a case $r(\mathbf{A}_{rG}) < r(\mathbf{A}_{rG} \vdots \mathbf{A}_{rT} \vdots \mathbf{A}_{rJ}) =$ number of rows of $\mathbf{A}_{rR}$. So the matrix $\mathbf{A}_{rG}\mathbf{G}\mathbf{A}_{rG}^T$ would be singular. If we convert the coefficient matrix into the upper triangular form using the order of the rows as the order of pivoting (using a nonzero entry of a column to reduce all entries underneath it to zero) we would encounter a zero pivot by the time we reach the end of the rows of $\mathbf{A}_{rG}\mathbf{G}\mathbf{A}_{rG}^T$. Zero diagonal elements are very often found in the matrix $\mathbf{M}$ (for example when the branch corresponds to a voltage source). Thus, it is often the case that the rows have to be reordered (differently from the columns as otherwise the undesirable features would still remain). This disturbs the symmetry of the matrix and reduces the efficiency of the solution technique.

If the set T is relatively large the coefficient matrix is unsuitable for iterative methods of solution of linear equations such as the conjugate gradient technique. These usually perform well when the matrix is positive definite or failing that has a large (relative to the size of the matrix) positive definite principal diagonal submatrix. MNA matrix does not yield a positive definite matrix even where hybrid method (which we shall describe later) yields one. One way of looking at this situation is to observe that MNA sacrifices both structural advantages and reduction of unknowns in order to gain sparsity. The variables $\mathbf{v}_n$ can be reduced when $\mathcal{G} \cdot G$ has more components than $\mathcal{G}$ since for each component of $\mathcal{G} \cdot G$ one node can be taken as the pseudo datum. The variables $\mathbf{i}_T$ can be reduced when T contains cutsets, i.e., when $\mathcal{G} \times T$ is not made up only of selfloops. We will show later how to reduce variables without sacrificing sparsity excessively by the use of hybrid methods.

5.5.4 Sparse Tableau Approach

Here KCE is written in the form $\mathbf{A}_r\mathbf{i} = \mathbf{0}$. The KV constraints are imposed by writing $\mathbf{A}_r^T\mathbf{v}_n - \mathbf{v} = \mathbf{0}$. Device characteristic constraints are put down as they are.

The final sets of unknowns are $\mathbf{i}, \mathbf{v}$ and $\mathbf{v}_n$. The method relies heavily on sparse LU techniques since the number of variables is very large.

5.6 Procedures used in Circuit Simulators

In this book, among other things, we advocate certain techniques for building better circuit simulators. It is therefore necessary that we give a very brief description of how general purpose circuit simulators are built.

5.6.1 Example to Illustrate Working of Circuit Simulators

Consider the following problem:
Solve numerically the differential equation

$$\frac{dx}{dt} + x^2 = u(t) \tag{5.4}$$

over a given interval $[0, T]$ with time steps $0, h, 2h \cdots$ for a given h. The function $u(.)$ is given over the same interval.

STEP 1 Discretization
We first discretize the differential equation to yield a nonlinear algebraic equation. Let us discretize using the simplest practical multistep method, namely the 'Backward Euler'. Here we replace

$$\frac{dx}{dt} \; at \; t = t_n \; by \; \frac{x(t_n) - x(t_{n-1})}{t_n - t_{n-1}}.$$

For the present problem we take $t_n \equiv nh$ and $t_n - t_{n-1} \equiv h$. This converts Equation 5.4 to the equation

$$\frac{x(nh) - x((n-1)h)}{h} + (x(nh))^2 = u(nh) \tag{5.5}$$

At this stage instead of a differential equation we have a difference equation. Note that we are not looking for a closed form solution.

In this equation observe that $x((n-1)h)$ and $u(nh)$ are known – the former by prior computation and the latter since it is given. We now have to solve this nonlinear algebraic equation numerically.

STEP 2 Newton Raphson Procedure
We use the Newton Raphson (NR) procedure on the nonlinear algebraic equation. Suppose we have to solve $f(x) = 0$, where $f(\cdot)$ is a smooth function of x. The NR procedure consists in guessing a solution $x = x^\circ$ and replacing $f(x) = 0$ by

$$f(x^\circ) + (df/dx)_{x=x^\circ}(x^1 - x^\circ) = 0.$$

The next guess would be x^1. The procedure is repeated until the $x^i s$ differ from each other within some bound chosen according to some criterion. To apply this procedure to Equation 5.5 we first simplify the equation to the form

$$(x/h) + (x)^2 - \hat{u} = 0.$$

Here $x(\equiv x(nh))$ is the unknown. All the known terms are concentrated in $\hat{u}$. Using the NR procedure on the simplified equation we get, since $\frac{df}{dx} \equiv \frac{1}{h} + 2x$,

$$\frac{x^\circ}{h} + (x^\circ)^2 - \hat{u} + (\frac{1}{h} + 2x^\circ)(x^1 - x^\circ) = 0. \tag{5.6}$$

Equation 5.6 can be seen to be a linear equation in which the unknown is the next guess.

So the numerical solution of Equation 5.4 *entails the solution of a number (= number of time steps) of nonlinear equations. Each of these nonlinear equations is solved by solving a number of linear equations successively.*

Remark: In the above discussion we have glossed over the usual difficulties that one encounters while using multistep methods (such as instability and inaccuracy) and while using the NR procedure (such as singularity of the Jacobian).

5.6.2 Working of General Purpose Circuit Simulators

A general nonlinear dynamic circuit can be thought of as being made up of a number of nonlinear differential equations. To solve these equations using the above technique we have to solve a number of sets of nonlinear equations (one set per time point) which in turn are solved by solving a number of sets of linear equations (one set per guess).

However, the discretization and NR procedures can be **adapted** to electrical networks. The problem of solving a set of equations of the kind exemplified by Equation 5.5 can be converted into the problem of solving a nonlinear static circuit, while that of solving a set of equations like Equation 5.6 can be converted into the problem of solving a linear static circuit.

Adaptation of Discretization Procedures to Network Analysis

We will now show how to adapt discretization procedures for the analysis of nonlinear dynamic circuits.

Let $\mathcal{N}$ be an electrical network on graph $\mathcal{G}$ with device characteristic equations:

$$\mathbf{f}_R(\mathbf{v}_R, \mathbf{i}_R) = \mathbf{0}$$
$$\mathbf{i}_C - \frac{d\mathbf{q}_C}{dt} = \mathbf{0}$$
$$\mathbf{f}_C(\mathbf{q}_C, \mathbf{v}_C) = \mathbf{0}$$

$$\mathbf{v}_L - \frac{d\psi_L}{dt} = \mathbf{0}$$
$$\mathbf{f}_L(\psi_L, \mathbf{i}_L) = \mathbf{0}$$
$$\mathbf{v}_{\mathcal{E}} - \mathbf{e} = \mathbf{0}$$
$$\mathbf{i}_{\mathcal{J}} - \mathbf{j} = \mathbf{0}$$

The functions $\mathbf{f}_R, \mathbf{f}_C, \mathbf{f}_L$ are nonlinear functions of arguments which are ordered pairs of vectors. Note that the set of R elements includes all non-source, non-dynamic elements. The set of $\mathcal{C}$ elements are nonlinear capacitors which may be coupled, while $\mathcal{L}$ elements are coupled nonlinear inductors. The topological constraints of this network are

$\mathbf{A}_r\mathbf{i} = \mathbf{0}$ (KCE)

$\mathbf{B}\mathbf{v} = \mathbf{0}$ (KVE)

To illustrate the method of adaptation of discretization procedures to network problems we use the Backward Euler discretization.

We assume that the interval of interest is $[0,\ T]$ and this interval has been broken up into time steps each of value h. We replace

$$\frac{d\mathbf{x}}{dt}|_{t=nh} \quad by \quad \frac{\mathbf{x}(nh) - \mathbf{x}((n-1)h)}{h}.$$

Now $\mathbf{v}(nh), \mathbf{i}(nh)$ must any way satisfy KVE, KCE respectively and these vectors along with $\mathbf{q}(nh)$ and $\psi(nh)$ must satisfy the device characteristic under the above replacement of the derivative by an expression linear in the present and immediate past value of the variable. We then have the following constraints on $\mathbf{v}(nh), \mathbf{i}(nh), \mathbf{q}(nh), \psi(nh)$:

$$\mathbf{f}_R(\mathbf{v}_R(nh), \mathbf{i}_R(nh)) = \mathbf{0}$$
$$\mathbf{i}_C(nh) - \frac{\mathbf{q}_C(nh) - \mathbf{q}_C((n-1)h)}{h} = \mathbf{0}$$
$$\mathbf{f}_C(\mathbf{q}_C(nh), \mathbf{v}_C(nh)) = \mathbf{0}$$
$$\mathbf{v}_L(nh) - \frac{\psi_L(nh) - \psi_L((n-1)h)}{h} = \mathbf{0}$$
$$\mathbf{f}_L(\psi_L(nh), \mathbf{i}_L(nh)) = \mathbf{0}$$
$$\mathbf{v}_{\mathcal{E}}(nh) = \mathbf{e}(nh)$$
$$\mathbf{i}_J(nh) = \mathbf{j}(nh)$$

The above is the transformed version of the device characteristic.

In addition we have the KCE and KVE

$\mathbf{A}_r(\mathbf{i}(nh)) = \mathbf{0}$

$\mathbf{B}(\mathbf{v}(nh)) = \mathbf{0}$

Figure 5.11: A dynamic network for discretization

Consider the problem of solving for $\mathbf{v}(nh)$, $\mathbf{i}(nh)$ in terms of variables whose (time) argument is less than nh. This is equivalent to solving an **electrical network** since we have KCE, KVE and device characteristic constraints. The latter are transformed **nonlinear static** versions of the original (nonlinear dynamic) device characteristics. Notice that only the dynamic constraints are transformed into static constraints. Variables such as q_C, ψ_L are not eliminated but are left as part of the statement of the (in general nonlinear) static constraints on $\mathbf{v}_C(nh), \mathbf{i}_C(nh), \mathbf{v}_L(nh), \mathbf{i}_L(nh)$ etc. The static constraints such as $\mathbf{f}_R(\mathbf{v}_R, \mathbf{i}_R) = 0$, $\mathbf{v}_{\mathcal{E}} = \mathbf{e}$, $\mathbf{i}_{\mathcal{J}} = \mathbf{j}$ which do not involve the past are left unchanged. The KCE and KVE constraints remain the same.

So the 'new' network topology is identical to the original one.

Exercise 5.14 *Transform the network in Figure 5.11 to a static network corresponding to the 'Backward Euler' discretization procedure.*

Exercise 5.15 *Discretization procedures based on higher order multistep algorithms are usually preferred for greater accuracy. We give some of them below. Transform the network in Figure 5.11 corresponding to each of them.*

$$\frac{dx}{dt}|_{t=nh} = \frac{2}{h}(x_n - x_{n-1}) - \frac{dx}{dt}|_{t=(n-1)h} \qquad (Trapezoidal)$$

$$\frac{dx}{dt}|_{t=nh} = \frac{12}{5h}(x_n - x_{n-1}) - (\frac{8}{5})\frac{dx}{dt}|_{t=(n-1)h} + (\frac{1}{5})\frac{dx}{dt}|_{t=(n-2)h} \qquad (Third\ Order\ A{-}M)$$

$$\frac{dx}{dt}|_{t=nh} = \frac{3}{2h}(x_n - \frac{4}{3}x_{n-1} + \frac{1}{3}x_{n-2}) \qquad (Second\ Order\ Gear)$$

Adaptation of the NR Procedure to Electrical Networks

Let $\mathcal{N}$ be a static nonlinear network on graph $\mathcal{G}$. Let the device characteristic equations be

$$\mathbf{f}_R(\mathbf{v}_R, \mathbf{i}_R) = \mathbf{0}$$

$$\mathbf{v}_{\mathcal{E}} - \mathbf{e} = \mathbf{0}$$
$$\mathbf{i}_{\mathcal{J}} - \mathbf{j} = \mathbf{0}$$

where $\mathbf{f}_R$ is a nonlinear function of the ordered pair $(\mathbf{v}_R, \mathbf{i}_R)$. The set of R elements includes all the nonsource elements. The topological constraints are:

$$\mathbf{A}_r\mathbf{i} = \mathbf{0} \qquad (KCE)$$
$$\mathbf{B}\mathbf{v} = \mathbf{0} \qquad (KVE)$$

We can view the above set of constraints as a set of nonlinear equations and apply the NR procedure. We exploit the fact that some of the constraints are linear/affine. Thus, we have the constraints:

$$\begin{aligned} \mathbf{f}_R(\mathbf{v}_R, \mathbf{i}_R) &= \mathbf{0} \\ \mathbf{f}_{\mathcal{E}}(\mathbf{v}_{\mathcal{E}}) &= \mathbf{0} \\ \mathbf{f}_{\mathcal{J}}(\mathbf{i}_{\mathcal{J}}) &= \mathbf{0} \\ \mathbf{f}_i(\mathbf{i}) &= \mathbf{0} \\ \mathbf{f}_v(\mathbf{v}) &= \mathbf{0} \end{aligned}$$

(The last two constraints are KCE and KVE).

Let us start with a guess $(\mathbf{v}^\circ, \mathbf{i}^\circ)$.
We then get a set of linear equations in which the unknown is the **next** guess $(\mathbf{v}^1, \mathbf{i}^1)$ by applying the NR procedure as follows:

i. the nonsource devices:

$$\mathbf{f}_R(\mathbf{v}_R^\circ, \mathbf{i}_R^\circ) + \frac{\partial \mathbf{f}_R}{\partial \mathbf{v}_R}|_\circ(\mathbf{v}_R^1 - \mathbf{v}_R^\circ) + \frac{\partial \mathbf{f}_R}{\partial \mathbf{i}_R}|_\circ(\mathbf{i}_R^1 - \mathbf{i}_R^\circ) = \mathbf{0}$$

where, assuming that $(\mathbf{f}_R)^T = (f^1, \cdots, f^k)$

and $(\mathbf{v}_R)^T = (v_{R_1}, \cdots, v_{R_k})$,

$\frac{\partial \mathbf{f}_R}{\partial \mathbf{v}_R}|_\circ$ denotes the Jacobian $\begin{bmatrix} \frac{\partial f^1}{\partial v_{R_1}} & \cdots & \frac{\partial f^1}{\partial v_{R_k}} \\ \vdots & & \\ \frac{\partial f^k}{\partial v_{R_1}} & \cdots & \frac{\partial f^k}{\partial v_{R_k}} \end{bmatrix}$ *at* $\mathbf{v}_R^\circ, \mathbf{i}_R^\circ$ and $\frac{\partial \mathbf{f}_R}{\partial \mathbf{i}_R}|_\circ$ denotes a similar Jacobian with respect to $\mathbf{i}_R$.

Observe that $\mathbf{f}_R(\mathbf{v}_R^\circ, \mathbf{i}_R^\circ) - \frac{\partial \mathbf{f}_R}{\partial \mathbf{v}_R}|_\circ \mathbf{v}_R^\circ - \frac{\partial \mathbf{f}_R}{\partial \mathbf{i}_R}|_\circ(\mathbf{i}_R^\circ)$ is an expression whose value is known and can be shifted to the right.

ii. the voltage sources:
Observe that $\mathbf{f}_{\mathcal{E}}(\mathbf{v}_{\mathcal{E}}) = \mathbf{v}_{\mathcal{E}} - \mathbf{e}$. So the constraint $\mathbf{v}_{\mathcal{E}} - \mathbf{e} = \mathbf{0}$ transforms, in

the variable $\mathbf{v}^1_{\mathcal{E}}$, to the constraint $\mathbf{v}^\circ_{\mathcal{E}} - \mathbf{e} + (\mathbf{I})(\mathbf{v}^1_{\mathcal{E}} - \mathbf{v}^\circ_{\mathcal{E}}) = \mathbf{0}$, i.e., $\mathbf{v}^1_{\mathcal{E}} - \mathbf{e} = \mathbf{0}$. Note that the Jacobian $\frac{\partial \mathbf{f}_{\mathcal{E}}}{\partial \mathbf{v}_{\mathcal{E}}}$ is the identity matrix. This is to be expected since the original expression $\mathbf{v}_{\mathcal{E}} - \mathbf{e}$ is **affine** in $\mathbf{v}_{\mathcal{E}}$, (i.e., sum of a linear function of $\mathbf{v}_{\mathcal{E}}$ and a constant term).

iii. the current sources:
Observe that $\mathbf{f}_{\mathcal{J}}(\mathbf{i}_{\mathcal{J}}) = \mathbf{i}_{\mathcal{J}} - \mathbf{j}$. So as in the case of the voltage sources we get $\mathbf{i}^1_{\mathcal{J}} - \mathbf{j} = \mathbf{0}$. Thus, the next guesses $\mathbf{v}^1_{\mathcal{E}}, \mathbf{i}^1_{\mathcal{J}}$ have to agree with source values even if $\mathbf{v}^\circ_{\mathcal{E}}, \mathbf{i}^\circ_{\mathcal{E}}$ do not.

iv. the KCE:
We have $\mathbf{f}_i(\mathbf{i}) = \mathbf{A}\ \mathbf{i}$. So we get $\mathbf{A}\ \mathbf{i}^\circ + \mathbf{A}(\mathbf{i}^1 - \mathbf{i}^\circ) = \mathbf{0}$,
i.e., $\mathbf{A}\mathbf{i}^1 = \mathbf{0}$.
Thus the next guess currents have to satisfy KCE of the original network even if the first guess currents do not.

v. the KVE:
Here $\mathbf{B}\ \mathbf{v} = \mathbf{0}$ transforms to $\mathbf{B}\ \mathbf{v}^1 = \mathbf{0}$. So the next guess voltages satisfy KVE of the original network even if the first guess voltages do not.

We thus see that the problem of obtaining the next guess $\mathbf{v}^1, \mathbf{i}^1$ from $\mathbf{v}^\circ, \mathbf{i}^\circ$ or of obtaining $(\mathbf{v}^n, \mathbf{i}^n)$ from $(\mathbf{v}^{n-1}, \mathbf{i}^{n-1})$ is one of solving a **linear static network** in which the previous guess voltages and currents appear as sources.

Remark: The adaptations of discretization procedures and the NR procedure to network problems provide good examples of a theme common in network theory, namely, the conversion of a problem arising in the course of the solution of a network into the problem of solving an appropriately defined network with the same graph as that of the original network. Other examples of this theme are the construction of a 'phasor' network for steady state sinusoidal analysis, of an 's-domain' network for obtaining the $\mathcal{L}$- transforms of the solution of a linear time invariant network etc. In all these transformations the *linear algebraic homogeneous equations remain unaltered.* Therefore *KCE and KVE remain unaltered* - which is another way of saying that the *graph of the transformed network remains the same as before.* In subsequent chapters we talk of various network decomposition procedures which are essentially topological, independent of the type of devices present. These procedures remain invariant even if networks are transformed as above.

Exercise 5.16 *Transform the nonlinear network in Figure 5.12 to a linear network corresponding to one iteration of the NR method.*

The device characteristic of the nonlinear elements is as follows:

$v_{R_1} = i^3_{R_1};\ \ v_5 - e^{i_1} = 0$.

Exercise 5.17 *Transform the nonlinear flow graph in Figure 5.13 to a linear flow graph (i.e., a flow graph with linear blocks) corresponding to one iteration of the NR method. Observe that the topological structure of the flow graph does not change.*

Figure 5.12: A Nonlinear Network for Application of the NR Procedure

Figure 5.13: A Flow Graph with Nonlinear Blocks

5.7 State Equations for Dynamic Networks

For theoretical studies on the temporal evolution of the solution of a dynamic network it is often convenient to isolate the constraints on the dynamic variables (the variables which appear under the derivative sign). These isolated constraints would involve the dynamic variables, their derivatives and source terms. The solution as far as these variables are concerned can then be studied independent of the remaining variables. These latter variables are related through other constraints to the dynamic variables and inputs. We discuss the problem of deriving such constraints from network constraints in a relatively informal manner in this section. Our concern is not, however, to discuss methods of solution.

We will attempt to put the constraints in the form

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}) \qquad \text{(state equations)}$$

$$\mathbf{y} = \hat{\mathbf{y}}(\mathbf{x}, \mathbf{u}) \qquad \text{(output equations)}$$

Here $\mathbf{x}$ is composed of some or all of the dynamic variables of the network and is called the **state vector**, the components of $\mathbf{u}$ are the inputs (sources) to the network and $\mathbf{y}$ is composed of the nondynamic variables in the network. The right side of the above constraints would sometimes involve higher derivatives of $\mathbf{u}$.

We consider networks composed of linear capacitors, linear coupled inductors, positive linear resistors and voltage and current sources. In this section, for simplicity, we *assume that there are no circuits composed only of capacitors and voltage sources and no cutsets composed only of inductors and current sources.* The reasons for these assumptions will be clear shortly. In Section 8.5 we relax these assumptions.

The device characteristic of this network has the form:

$$\begin{aligned}
\mathbf{v}_R - \mathbf{R}\,\mathbf{i}_R &= \mathbf{0}\\
\mathbf{i}_C - \mathcal{C}\frac{d\mathbf{v}_C}{dt} &= \mathbf{0}\\
\mathbf{v}_L - \mathcal{L}\frac{d\mathbf{i}_L}{dt} &= \mathbf{0}\\
\mathbf{v}_{\mathcal{E}} - \mathbf{e} &= \mathbf{0}\\
\mathbf{i}_{\mathcal{J}} - \mathbf{j} &= \mathbf{0}
\end{aligned}$$

Observe that $\mathbf{v}_C$ and $\mathbf{i}_L$ appear under the derivative sign. We will make them the state variables. We begin with their device characteristic equations namely,

$$\begin{aligned}
\mathcal{C}\frac{d\mathbf{v}_C}{dt} &= \mathbf{i}_C\\
\mathcal{L}\frac{d\mathbf{i}_L}{dt} &= \mathbf{v}_L
\end{aligned}$$

We will express the right side variables in terms of $\mathbf{v}_C$, $\mathbf{i}_L$ and the source variables.

To do this we replace the capacitors by 'unknown voltage sources' $\mathbf{v}_C$ and inductors by 'unknown current sources' $\mathbf{i}_L$ and solve the resulting static network.

We can state the constraints of this static network as follows:

$$
\begin{aligned}
\mathbf{v}_C &= \mathbf{v}_C \\
\mathbf{i}_L &= \mathbf{i}_L \\
\mathbf{v}_R - R\,\mathbf{i}_R &= \mathbf{0} \\
\mathbf{v}_{\mathcal{E}} &= \mathbf{e} \\
\mathbf{i}_{\mathcal{J}} &= \mathbf{j} \\
\mathbf{A}_r\mathbf{i} &= \mathbf{0} \qquad (KCE) \\
\mathbf{B}\mathbf{v} &= \mathbf{0} \qquad (KVE)
\end{aligned}
$$

We now have a circuit with positive linear resistors, voltage and current sources. By our **assumption** these 'new' voltage sources do not contain circuits and 'new' current sources do not contain cutsets. It can be shown that such a network can be **solved uniquely** given (arbitrary) values for $\mathbf{v}_C$ and $\mathbf{i}_L$. Hence, we can write

$$\mathbf{i}_C = [\mathbf{K}_C \quad \mathbf{K}_L]_{\mathbf{i}_L}^{\mathbf{v}_C} + [\mathbf{K}_{\mathcal{E}}]\mathbf{e} + [\mathbf{K}_{\mathcal{J}}]\mathbf{j}$$

$$\mathbf{v}_L = [\mathbf{M}_C \quad \mathbf{M}_L]_{\mathbf{i}_L}^{\mathbf{v}_C} + [\mathbf{M}_{\mathcal{E}}]\mathbf{e} + [\mathbf{M}_{\mathcal{J}}]\mathbf{j}$$

and

$$\mathbf{y} = [\mathbf{C}_C \quad \mathbf{C}_L]_{\mathbf{i}_L}^{\mathbf{v}_C} + [\mathbf{D}_{\mathcal{E}}]\mathbf{e} + [\mathbf{D}_{\mathcal{J}}]\mathbf{j}$$

where $\mathbf{y}$ denotes other voltages and currents in the network and perhaps their linear combinations.

Substitution of the above expressions for $\mathbf{i}_C$, $\mathbf{v}_L$ in the RHS of the device characteristics of the capacitors and of the inductors yields equations of the form:

$\mathcal{C}(\dot{\mathbf{v}}_C) = f_C(\mathbf{v}_C, \mathbf{i}_L, \mathbf{e}, \mathbf{j})$

$\mathcal{L}(\dot{\mathbf{i}}_L) = f_L(\mathbf{v}_C, \mathbf{i}_L, \mathbf{e}, \mathbf{j})$

Premultiplying these equations respectively by $(\mathcal{C})^{-1}$ and $(\mathcal{L})^{-1}$ yields the state equations.

The above technique would work even if the nondynamic part of the network had controlled sources provided we can assume that the network can be solved uniquely after the capacitors have been replaced by voltage sources $\mathbf{v}_C$ and the inductors have been replaced by current sources $\mathbf{i}_L$. To relax the assumption about $C \cup \mathcal{E}$ containing no circuits and $\mathcal{L} \cup J$ containing no cutset we need some preparation. So we postpone the discussion of this case (see Section 8.5).

Exercise 5.18 *Write state equations for the network in Figure 5.14. Use capacitor voltages and inductor currents as state variables.*

Take $C = 2F$, $R_1 = 2\Omega$, $R_2 = 4\Omega$, $R_3 = 2\Omega$ $L = 4H$.

Figure 5.14: A Dynamic Network

5.8 Multiports in Electrical Networks

The notion of 'multiport' is fundamental to electrical network theory. Since the idea is intuitive and natural to this subject it is often difficult to appreciate its essential character, namely, that it is a topological concept. In this section we discuss electrical multiports in an informal way and outline their usual applications. Conventionally, one introduces a 'port' into an existing network $\mathcal{N}$ by specifying a pair of terminals at which connection can be made to another network with the provision that current entering at one of the terminals into $\mathcal{N}$ equals the current coming out of $\mathcal{N}$ at the other terminal.There may be many ports between one network and another.If we wish to study the effect of $\mathcal{N}$ on the external network when such port connections are made, we may do so by introducing norators, each between the two terminals of a port, resulting in a multiport. Observe that the introduction of a norator automatically ensures that the 'current entering equals current leaving' condition is satisfied. For the present an **electrical multiport** ('multiport' for short) is a network with some of its devices, which are norators, specified as **ports**. In Chapter 8 the idea is studied more formally in order to fully exploit it as a technique for solving electrical networks by decomposition.

5.8.1 An informal Description of Multiport Decomposition

Consider the network $\mathcal{N}$ of Figure 5.15

This network is made up of two subnetworks $\mathcal{N}_A$ and $\mathcal{N}_B$ which have two common nodes. We will assume that the devices in $\mathcal{N}_A$ and $\mathcal{N}_B$ are decoupled. Now it is clear that, topologically, the only way the currents in $\mathcal{N}_B$ can affect the currents in $\mathcal{N}_A$ (and vice versa) is by the KCE constraints at n_1, n_2. Since the net current entering $\mathcal{N}_A$ ($\mathcal{N}_B$) at n_1 must be the negative of the current entering at n_2, we may confine ourselves only to the KCE constraints at n_1. Similarly, topologically, the only way the voltages in $\mathcal{N}_B$ ($\mathcal{N}_A$) can affect the voltages in $\mathcal{N}_A$

Figure 5.15: A Network and its Multiport Decomposition

$(\mathcal{N}_B)$ is through the voltage $v = v_{n_1} - v_{n2}$. Thus, the constraints on all currents and voltages of $\mathcal{N}$ remain the same if we replace the network by the decomposition in the same figure. In this figure the dotted lines represent norators. The conditions $v_P = v_{P'}$, $i_P = -i_{P'}$, are imposed by the 'port connection diagram' which depicts the two norators in parallel.

More generally let the KCE of network $\mathcal{N}$ be

$$\left[\begin{array}{cc} \mathbf{A'}_{rA} & \mathbf{A'}_{rB} \end{array}\right]^{\mathbf{i}_A}_{\mathbf{i}_B} = \mathbf{0},$$

where $A \uplus B$ is the set of edges of the network.

Suppose these are **equivalent** in the variables $\mathbf{i}_A, \mathbf{i}_B$ to the following set of constraints with additional variables $\mathbf{i}_P, \mathbf{i}_{P'}$

$$[\mathbf{A}_{rA} \;\; \mathbf{A}_{rP}]^{\mathbf{i}_A}_{\mathbf{i}_P} = \mathbf{0},$$

$$[\mathbf{A}_{rB} \;\; \mathbf{A}_{rP'}]^{\mathbf{i}_B}_{\mathbf{i}_{P'}} = \mathbf{0},$$

$$[\mathbf{I} \; \mathbf{I}]^{\mathbf{i}_P}_{\mathbf{i}_{P'}} = \mathbf{0}.$$

Let $[\mathbf{A'}_{rA} \;\; \mathbf{A'}_{rB}]$, $[\mathbf{A}_{rA} \;\; \mathbf{A}_{rP}]$, $[\mathbf{A}_{rB} \;\; \mathbf{A}_{rP'}]$ be the reduced incidence matrices of graphs $\mathcal{G}$, $\mathcal{G}_{AP}$, $\mathcal{G}_{BP'}$ respectively. Then one can show (by using the 'Implicit Duality Theorem' of Chapter 7) that the KVE constraints

$$[\mathbf{B'}_A \;\; \mathbf{B'}_B]^{\mathbf{v}_A}_{\mathbf{v}_B} = \mathbf{0}$$

are equivalent in the variables $\mathbf{v}_A, \mathbf{v}_B$ to

$$[\mathbf{B}_A \quad \mathbf{B}_P]\begin{pmatrix}\mathbf{v}_A\\ \mathbf{v}_P\end{pmatrix} = \mathbf{0},$$
$$[\mathbf{B}_B \quad \mathbf{B}_{P'}]\begin{pmatrix}\mathbf{v}_B\\ \mathbf{v}_{P'}\end{pmatrix} = \mathbf{0},$$
$$[\mathbf{I} \quad -\mathbf{I}]\begin{pmatrix}\mathbf{v}_P\\ \mathbf{v}_{P'}\end{pmatrix} = \mathbf{0},$$

where $(\mathbf{B'}_A \quad \mathbf{B'}_B), (\mathbf{B}_A \quad \mathbf{B}_P), (\mathbf{B}_B \quad \mathbf{B}_{P'})$ are representative matrices of the current spaces of $\mathcal{G}, \mathcal{G}_{AP}, \mathcal{G}_{BP'}$ respectively.
Let devices in A and B be **decoupled** from each other. Let the 'electrical multiports' $\mathcal{N}_{AP}$, $\mathcal{N}_{BP'}$ (i.e., networks with norators in the 'port' edges of P, P') be defined on graphs, $\mathcal{G}_{AP}$, $\mathcal{G}_{BP'}$ with devices in A in $\mathcal{N}_{AP}$ same as in $\mathcal{N}$ and devices in B in $\mathcal{N}_{BP'}$ same as in $\mathcal{N}$. It is then clear that the KCE, KVE and device characteristics of the network $\mathcal{N}$ with graph $\mathcal{G}$ are equivalent as far as the current and voltage variables of the network $\mathcal{N}$ are concerned to those of the multiports $\mathcal{N}_{AP}$, $\mathcal{N}_{BP'}$, under the constraints imposed by the 'port connection diagram'

$$[\mathbf{I} \quad \mathbf{I}]\begin{pmatrix}\mathbf{i}_P\\ \mathbf{i}_{P'}\end{pmatrix} = \mathbf{0},$$
$$[\mathbf{I} \quad -\mathbf{I}]\begin{pmatrix}\mathbf{v}_P\\ \mathbf{v}_{P'}\end{pmatrix} = \mathbf{0}.$$

Observe that these constraints are imposed by a graph in which each port edge of P is parallel to a corresponding port edge of P'. In general, the port connection diagram would be more complicated.
There are two main applications for multiport decomposition - network simplification for theoretical purposes and network decomposition as a method of analysis. Here we restrict ourselves to the former.
In the electrical multiport $\mathcal{N}_{BP'}$ the device characteristic of B imposes, through KCE and KVE, constraints on the voltage and current vectors that can coexist in the norators P'. These constraints are transferred to the norators in P through the KCE and KVE of the port connection diagram. The result is that we have a compact representation of the effect in the network $\mathcal{N}$ of B upon A through the network $\mathcal{N}_{AP}$, where P now has the 'transferred' device characteristic of B.
Suppose the devices in B have a uniform character, example - all linear or affine such as linear positive resistors and voltage and current sources, all capacitors, all inductors etc. In such a case we will show that the characteristic that B projects on the norators P' in the multiport $\mathcal{N}_{BP'}$ is particularly simple.

5.8.2 Thevenin-Norton Theorem

Let $\mathcal{N}$,$\mathcal{N}_{AP}$,$\mathcal{N}_{BP'}$ be as in the previous subsection. Let B be composed of positive linear resistors, controlled sources, voltage sources and current sources. The constraints of network $\mathcal{N}_{BP'}$ would therefore have the form

$$(\mathbf{A}_{rB} \quad \mathbf{A}_{rP'})\begin{pmatrix}\mathbf{i}_B\\ \mathbf{i}_{P'}\end{pmatrix} = \mathbf{0}$$

$$(\mathbf{B}_B \quad \mathbf{B}_{P'})^{\mathbf{v}_B}_{\mathbf{v}_{P'}} = \mathbf{0}$$
$$(\mathbf{M}_{BR} \; \mathbf{N}_{BR})^{\mathbf{i}_{BR}}_{\mathbf{v}_{BR}} = \mathbf{0}$$
$$\mathbf{v}_{B\mathcal{E}} \; - \; \mathbf{e}_B = \mathbf{0}$$
$$\mathbf{i}_{B\mathcal{J}} \; - \; \mathbf{j}_B = \mathbf{0}.$$

Thus, these constraints have the form

$$[\mathbf{K}_B \vdots \mathbf{K}_{P'}]^{\mathbf{x}_B}_{\mathbf{x}_{P'}} = \mathbf{s}, \tag{5.7}$$

where $\mathbf{x}$ denotes both voltages and currents. By reversible row transformations we can reduce these equations to

$$\begin{bmatrix} \mathbf{K}_{BB} & \vdots & \mathbf{K}_{BP'} \\ \mathbf{0} & \vdots & \mathbf{K}_{P'P'} \end{bmatrix} \begin{matrix} \mathbf{x}_B \\ \mathbf{x}_{P'} \end{matrix} = \begin{bmatrix} \hat{\mathbf{s}}_B \\ \hat{\mathbf{s}}_{P'} \end{bmatrix}, \tag{5.8}$$

where the rows of $\mathbf{K}_{BB}$ are linearly independent.
It can be shown that as far as the variables $\mathbf{x}_{P'}$ are concerned the constraints of Equation 5.8 are equivalent to $\mathbf{K}_{P'P'} \; \mathbf{x}_{\mathbf{P'}} = \hat{\mathbf{s}}_{P'}$ (see Exercise 5.21). Let us, for clarity, rewrite these constraints as

$$\left(\; \mathbf{M} \quad \mathbf{N} \; \right)^{\mathbf{i}_{P'}}_{\mathbf{v}_{P'}} = \hat{\mathbf{s}}_{P'}$$

Now

$$\mathbf{i}_{P'} = -\mathbf{i}_P$$
$$\mathbf{v}_{P'} = \mathbf{v}_P$$

It is clear then that $\mathbf{v}_P, \mathbf{i}_P$ must satisfy the constraints

$$\left(\; -\mathbf{M} \quad \mathbf{N} \; \right) \begin{pmatrix} \mathbf{i}_P \\ \mathbf{v}_P \end{pmatrix} = \hat{\mathbf{s}}_{P'}.$$

Note that in the network $\mathcal{N}_{AP}$ if the norators in P were replaced by devices satisfying the characteristics above the constraints on voltages and currents in A must remain as in the original network $\mathcal{N}$.
Thus, *the constraints on voltages and currents of A in the network $\mathcal{N}$ are the same as those in $\mathcal{N}_{AP}$ provided the norators of P were replaced by devices with characteristic*

$$\left(\; -\mathbf{M} \quad \mathbf{N} \; \right)^{\mathbf{i}_P}_{\mathbf{v}_P} = \hat{\mathbf{s}}_{P'}. \tag{5.9}$$

This could be called the **Generalized Thevenin - Norton Theorem**. Observe that its validity does not depend on the type of devices present in A nor on whether the network $\mathcal{N}$ has a unique solution or even any solution. We have glossed over how to decompose into multiports, how many ports to choose, what should be the graphs $\mathcal{G}_{AP}$, $\mathcal{G}_{BP'}$ etc. These are topological issues. For the present we

Figure 5.16: 2-Port convention

assume these to be intuitively clear. Such questions are dealt with more carefully in Chapter 8.

Remark: A word on the usual convention. Suppose $\mathcal{N}_{BP'}$ is the 2-port shown in the Figure 5.16. Usually it is drawn without the norators P_1', P_2' but with the 'entering' current i_{P_1}, i_{P_2} shown. It is understood that these same currents leave at the primed terminals, but this is not always shown.
Often one talks of the multiport immittance matrix of $\mathcal{N}_{BP'}$ (corresponding to B being composed of linear static devices and sources). This would relate (v_{P_1}, v_{P_2}), to (i_{P_1}, i_{P_2}) (**not** to $(i_{P_1'}, i_{P_2'})$), as in Equation 5.9. When devices in P_1, P_2 in $\mathcal{N}_{AP}$ behave as in the above equation they represent B 'as seen' at its ports by A.

Exercise 5.19 *(k) Assume that the devices in B are linear positive resistors, voltage sources $\mathbf{e}_B$ and current sources $\mathbf{j}_B$.*

i. Suppose that $\mathcal{N}_{BP'}$ can be solved uniquely for arbitrary values of $\mathbf{i}_{P'}$, $\mathbf{e}_B$ and $\mathbf{j}_B$. In this case show that the Equation 5.9 reduces to

$$\begin{pmatrix} -\mathbf{R} & \mathbf{I} \end{pmatrix}^{\mathbf{i}_P}_{\mathbf{v}_P} = \hat{\mathbf{s}}_{P'},$$

where $\mathbf{R}$ is a positive semidefinite matrix.
Further show that $\hat{\mathbf{s}}_{P'}$, is the voltage that appears across P' when $\mathbf{i}_{P'}$ is equal to zero.

ii. Examine the case when $\mathcal{N}_{BP'}$ can be solved uniquely for arbitrary values of $\mathbf{v}_{P'}$, $\mathbf{e}_B$ and $\mathbf{j}_B$.

Exercise 5.20 *(k) Examine the case where B is composed only of*

i. capacitors and

ii. inductors.

Exercise 5.21 *Show that,* **as far as** *the variables* $\mathbf{x}_{P'}$ *are concerned Equation 5.8 is equivalent to* $\mathbf{K}_{P'P'}\ \mathbf{x}_{P'} = \hat{\mathbf{s}}_{P'}$.

5.9 Some Elementary Results of Network Theory

In this section we consider two elementary results of electrical network theory: Substitution Theorem and Superposition Theorem. The former is very general and has very little power but is useful in simplifying arguments. Superposition Theorem is fundamental to linear network theory. Two other basic results perhaps deserve a place here: v-shift and i-shift theorems. But we choose to study them in a more formal setting later.

Substitution Theorem

In its present form this is a relatively weak result. For the result to be applicable the network has to satisfy strong conditions.

Theorem 5.9.1 (Substitution Theorem) *Let* $\mathcal{N}$ *be a network with one of its branches, say* e_j *decoupled from the remaining devices in the device characteristic. Let the voltage (current) associated with branch* e_j *be known to be unique, equal to* v_j (i_j). *Let* $\mathcal{N}_s$ *be the network derived from* $\mathcal{N}$ *by replacing the device in* e_j *by a voltage source of value* v_j *(current source of value* i_j*) and leaving all other device characteristic constraints unaltered. If the solution of* $\mathcal{N}_s$ *is unique then so is that of* $\mathcal{N}$ *and both the networks have identical solutions.*

Proof : We consider only the voltage substitution case since current substitution can be handled similarly. The constraints of the network $\mathcal{N}$ imply that the voltage of branch e_j is v_j. Therefore, adding the constraint
'voltage of $e_j = v_j$'
does not alter the constraints of network $\mathcal{N}$.

Now if we delete the device characteristic constraint of e_j from this set of constraints we get the constraints of $\mathcal{N}_s$. These constraints are therefore, implied by the constraints of $\mathcal{N}$. If $\mathcal{N}_s$ has a unique solution $(\mathbf{v}, \mathbf{i})$ it means that the constraints of $\mathcal{N}_s$ imply that the voltage vector associated with the branches is necessarily $\mathbf{v}$ and that the current vector associated with the branches is necessarily $\mathbf{i}$. But then it follows that this fact is also implied by the constraints of $\mathcal{N}$, i.e., that $\mathcal{N}$ has the unique solution $(\mathbf{v}, \mathbf{i})$.

□

Superposition Theorem

Consider the set of equations

$$\mathbf{A}\mathbf{x} = \mathbf{b}.$$

If $\mathbf{x}_1$ is a vector solution to this equation when the right side is $\mathbf{b}_1$ and $\mathbf{x}_2$ is a solution when the right side is $\mathbf{b}_2$, it is clear that $\alpha\mathbf{x}_1 + \beta\mathbf{x}_2$ is a solution when $\alpha\mathbf{b}_1 + \beta\mathbf{b}_2$ is the right side. The same idea holds if the equations were

$$\left(\begin{array}{cc} \mathbf{A}_1 & \mathbf{A}_2 \end{array} \right)_{\mathbf{x}}^{\dot{\mathbf{x}}} = \mathbf{b}$$

In this case (under the same assumptions as above) we would have

$$\begin{aligned} \left(\begin{array}{cc} \mathbf{A}_1 & \mathbf{A}_2 \end{array} \right)_{\alpha\mathbf{x}_1+\beta\mathbf{x}_2}^{\alpha\dot{\mathbf{x}}_1+\beta\dot{\mathbf{x}}_2} &= \alpha\left(\begin{array}{cc} \mathbf{A}_1 & \mathbf{A}_2 \end{array} \right)_{\mathbf{x}_1}^{\dot{\mathbf{x}}_1} + \beta\left(\begin{array}{cc} \mathbf{A}_1 & \mathbf{A}_2 \end{array} \right)_{\mathbf{x}_2}^{\dot{\mathbf{x}}_2} \\ &= \alpha\mathbf{b}_1 + \beta\mathbf{b}_2. \end{aligned}$$

Thus, $\alpha\mathbf{x}_1 + \beta\mathbf{x}_2$ is a solution when $\alpha\mathbf{b}_1 + \beta\mathbf{b}_2$ is the right side provided $\mathbf{x}_1, \mathbf{x}_2$ are solutions respectively when $\mathbf{b}_1, \mathbf{b}_2$ are right sides.

The above ideas are clearly valid for linear static or dynamic networks since the constraints in these cases are composed of KCE, KVE which are always linear and device characteristics which are given to be linear. We therefore have

Theorem 5.9.2 (Superposition Theorem) *In a network $\mathcal{N}$ with voltage sources, current sources and linear static and dynamic devices, if solution $\mathbf{x}^1$ coexists with source vector $\mathbf{s}^1$ and solution $\mathbf{x}^2$ coexists with source vector $\mathbf{s}^2$ then solution $\alpha\mathbf{x}^1 + \beta\mathbf{x}^2$ coexists with source vector $\alpha\mathbf{s}^1 + \beta\mathbf{s}^2$.*

Usually the network might also be known to have a unique solution for arbitrary source values (i.e., for arbitrary right sides). The most common way of obtaining right sides is to start from a source distribution $(e_1, e_2 \cdots e_k, j_1, \cdots j_k)$ and decompose it into $(e_1, 0 \cdots 0, 0 \cdots 0) \cdots (0, \cdots 0, e_k, 0 \cdots 0)$, $(0 \cdots 0, j_1 \cdots 0) \cdots (0 \cdots 0, j_k)$. The solution for the original source distribution would then be the sum of the solutions for the 'decomposed' distributions. The advantages of this kind of decomposition is that when a voltage source is set to zero the effect is to have a short circuit in its place. Usually the current in this short circuit is not of immediate interest. So the end point of the device may be fused and the device removed as far as the rest of the variables are concerned. Similarly, when a current source is set to zero we can put an 'open circuit branch' in its place. If the voltage across this branch is not of interest, it may be deleted. In linear dynamical circuits uniqueness of solution would hold only if initial conditions are known. While applying superposition theorem we could either fix the intial conditions to be zero or linearly combine the initial conditions the same way as we combine the sources.

5.10 Notes

The reader interested in a comprehensive introduction to electrical network theory could refer to [Desoer+Kuh69] and [Chua+Desoer+Kuh87]. For numerical solution of networks convenient references are [Chua+Lin75], [McCalla88]. The first systematic treatment of the link between graphs and electrical networks was given in [Seshu+Reed61]. This is still an excellent reference.

5.11 Solutions of Exercises

In the interest of brevity we only give outlines for most of the solutions.

E 5.1:
i. The electrical network divides the universe into an 'external' and an 'internal' region. The network sets up a conservative field in the external region. Suppose $v(t)$ associated with a device is positive. If a positive current flows in the direction of the arrow, then, as far as the external region is concerned, positive charges are falling though a potential of $v(t)$ volts and hence losing energy to the device. So the power absorbed by the device must be positive, which agrees with the fact that $v(t)i(t)$ is positive.

ii. For a resistor we have, $v \cdot i = Ri^2$. This would be positive or negative depending on the sign of R. Connect the given source parallel to a resistor R. By Tellegen's Theorem the sum of the powers absorbed by all the devices in a network must be zero. So the source would absorb negative or positive power depending on whether R is positive or negative.

E 5.2:
i. If $\mathbf{i}$ is a possible current vector in the circuit, so would $\mathbf{i}+\mathbf{i}_o$ be, where $\mathbf{i}_o$ is the circuit vector corresponding to the voltage source circuit.

ii. Similar to the above. Use $\mathbf{v}_o$, the cutset vector corresponding to the cutset of current sources.

E 5.3: See Theorem 5.9.2.

E 5.4:
i. If a voltage is controlling, put an 'open circuit' branch in parallel and if a current is controlling, then put a 'short circuit' branch in series.

E 5.5:
i. Energy gained by the capacitor

$$\begin{aligned} = \int_{t_o}^{t_f} v(t)\ i(t)\ dt \quad &= \quad \int_{t_o}^{t_f} c\ v(t)\ \dot{v}(t)\ dt \\ &= \quad \int_{v_o}^{v_f} c\ v\ dv = \frac{1}{2}\ c\ v^2\ \left.\right|_{v_o}^{v_f}\ . \end{aligned}$$

E 5.6: Energy gained by the capacitor

$$= \int_{t_o}^{t} v(t)\ i(t)\ dt \quad = \quad \int_{t_o}^{t_f} \frac{d\hat{q}(v)}{dt} v(t)\ dt$$

$$= \int_{t_o}^{t_f} \frac{d\hat{q}}{dv}\frac{dv}{dt}v(t)\ dt = \int_{v_o}^{v_f} \frac{d\hat{q}}{dv}\ v\ dv.$$

E 5.7:
i. Energy gained by the coupled inductors

$$\begin{aligned} = \int_{t_o}^{t_f} \mathbf{v}^T(t)\ \mathbf{i}(t)\ dt &= \int_{t_o}^{t_f} \dot{\mathbf{i}}^T(t)\mathcal{L}\mathbf{i}(t)\ dt\ (\mathcal{L}\ is\ symmetric) \\ &= \int_{t_o}^{t_f} \frac{1}{2}\frac{d}{dt}(\mathbf{i}^T(t)\mathcal{L}\mathbf{i}(t))\ dt \\ &= \int_{i_o^T\mathcal{L}i_o}^{i_f^T\mathcal{L}i_f} \frac{1}{2}\ d\ (\mathbf{i}^T(t)\mathcal{L}\mathbf{i}(t)) \\ &= \frac{1}{2}[\mathbf{i}^T(t)\mathcal{L}\mathbf{i}(t)]_{\mathbf{i}_o}^{\mathbf{i}_f}. \end{aligned}$$

ii. Since $\mathcal{L}$ is positive definite $\frac{1}{2}\mathbf{i}_f^T\mathcal{L}\mathbf{i}_f$ is nonnegative.

E 5.8: Solution similar to that of Exercise 5.6.

E 5.9:
iv. We have

$$\begin{bmatrix} \mathbf{I} & \mathbf{K} \end{bmatrix} \begin{matrix} \mathbf{i}_B \\ \mathbf{i}_A \end{matrix} = \mathbf{0}$$

$$\begin{bmatrix} -\mathbf{K}^T & \mathbf{I} \end{bmatrix} \begin{matrix} \mathbf{v}_B \\ \mathbf{v}_A \end{matrix} = \mathbf{0}$$

$\mathbf{v}_R = \mathbf{v}_B$ and $\mathbf{i}_R = -\mathbf{i}_B$.
So $\mathbf{i}_R = \mathbf{K}\ \mathbf{i}_A$ and $\mathbf{v}_A = \mathbf{K}^T\mathbf{v}_B = \mathbf{K}^T\ \mathbf{R}\mathbf{K}\mathbf{i}_A$.

E 5.10:
ii. (Capacitor case) - We have, $v_2 = v_C,\ i_2 = -i_C,\ i_C = C\dot{v}_C$.
So $v_1 = -rC\dot{v}_C = (r^2C)\dot{i}_1$. So the behaviour is that of an inductor of value r^2C units.

(Voltage source case): $v_1 = ri_2, v_2 = -r\ i_1,\ v_2 = v_e$.
So $i_1 = -v_e/r$ and v_1 is unconstrained since i_2 is unconstrained. Thus, we have the behaviour of a current source.

E 5.13: Let $\mathbf{R}$ be the representative matrix of the current space (on set S) of the ideal transformer. Let this matrix have r rows. Corresponding to the j^{th} column we build r 2-port transformers $1j, \cdots, rj$ with the ij^{th} transformer corresponding to the ij^{th} entry of the matrix $\mathbf{R}$ and having turns ratio $1 : R(i,j)$.
We thus have $r \mid S \mid$ 2-port ideal transformers each with its pair of reference arrows. Next the primaries of all the transformers corresponding to the same column are put in parallel (tails of arrows together and heads of arrows together) and the secondaries correponding to the same row are put in series forming a circuit (with the polarity of the secondary winding of the ij^{th} transformer consistent with the sign of

$R(i, j)$). After these connections are made we are left with $| S |$ exposed ports one for each column (each being the parallel combination of primaries corresponding to the column). The voltage vectors that can exist on these ports form precisely the space orthogonal to the rows of $\mathbf{R}$.
To prove that the current vectors that can exist on these ports form the complementary orthogonal space requires the use of Theorem 7.1.1.

E 5.14: (At time t_n) resistor R would be unchanged, in place of voltage source $e(t)$ and current source $j(t)$ we put voltage source $e(t_n)$ and current source $j(t_n)$ respectively. The capacitor C is replaced by the composite device

$$\frac{C(v(t_n) - v(t_{n-1}))}{h} - i(t_n) = 0.$$

This device is made up of a conductance of $\frac{C}{h}$ mhos in series with a voltage source of value $v(t_{n-1})$ volts (arrow of $v(t_{n-1})$ along the arrow of $v(t_n)$). The inductor L is replaced by the composite device

$$\frac{L(i(t_n) - i(t_{n-1}))}{h} - v(t_n) = 0.$$

This device is made up of a resistance of $\frac{L}{h}\Omega$ in parallel with a current source of value $i(t_{n-1})$ amps (arrow of $i(t_{n-1})$ along the arrow of $i(t_n)$).

E 5.15: Second Order Gear (At time (t_n)) $R, e(t), j(t)$ are replaced by devices as in the 'Backward Euler case'). Capacitor C is replaced by the composite device

$$C(\frac{3}{2h}v(t_n)) - i(t_n) - C(\frac{3}{2h})(\frac{4}{3}v(t_{n-1}) - \frac{1}{3}v(t_{n-2})) = 0.$$

This is equivalent to a composite device which has $\frac{3C}{2h}$ mhos in series with a voltage source of value $(\frac{4}{3}v(t_{n-1}) - \frac{1}{3}v(t_{n-2}))$ volts with direction of arrow same for both devices as well as the composite device.
Inductor L is replaced by the composite device

$$L(\frac{3}{2h}i(t_n)) - v(t_n) - L(\frac{3}{2h})(\frac{4}{3}i(t_{n-1}) - \frac{1}{3}i(t_{n-2})) = 0.$$

This is equivalent to a composite device which has $\frac{3L}{2h}\Omega$ in parallel with a current source of value $(\frac{4}{3}i(t_{n-1}) - \frac{1}{3}i(t_{n-2}))$ amps with direction of arrow same for both devices as well as the composite device.

E 5.16: (At the j^{th} iteration) To avoid confusion the value of the variable x at the end of the $(j-1)^{th}$ iteration is denoted by $(x)_j$.
R_1 is replaced by the composite device

$$((v_{R_1})_{j+1} - (v_{R_1})_j) - 3(i_{R_1})_j^2((i_{R_1})_{j+1} - (i_{R_1})_j) + (v_{R_1})_j - (i_{R_1})_j^3 = 0.$$

The controlled source v_5 is replaced by

$$((v_5)_{j+1} - (v_5)_j) - (e^{i_1})_j((i_1)_{j+1} - (i_1)_j) + (v_5)_j - (e^{i_1})_j = 0.$$

(Note that the unknowns have subscript $(j+1)$).

E 5.19:

i. Suppose all internal sources in $\mathcal{N}_{BP'}$ are set to zero. It is clear then that the relation between $\mathbf{i}_{P'}, \mathbf{v}_{P'}$ would reduce to

$$\begin{pmatrix} \mathbf{M} & \mathbf{N} \end{pmatrix}\binom{\mathbf{i}_{P'}}{\mathbf{v}_{P'}} = \mathbf{0}$$

since the right side of Equation 5.8 is zero. Since $\mathcal{N}_{BP'}$ can be solved uniquely for arbitrary values of $\mathbf{i}_{P'}$, it follows that $\mathbf{v}_{P'}$ can be expressed uniquely in terms of $\mathbf{i}_{P'}$ in the above equation. Hence $\mathbf{N}$ must be invertible and we can write the equation equivalently as

$$\begin{pmatrix} \mathbf{R} & \mathbf{I} \end{pmatrix}\binom{\mathbf{i}_{P'}}{\mathbf{v}_{P'}} = \mathbf{0} \text{ or } \begin{pmatrix} -\mathbf{R} & \mathbf{I} \end{pmatrix}\binom{\mathbf{i}_{P}}{\mathbf{v}_{P}} = \mathbf{0}.$$

By Tellegen's Theorem

$$< \mathbf{v}_{BP'}, \mathbf{i}_{BP'} >= 0, \; i.e., \; < \mathbf{v}_B, \mathbf{i}_B >= - < \mathbf{v}_{P'}, \mathbf{i}_{P'} > .$$

But when the sources in B are set to zero it contains only positive resistors. Therefore $< \mathbf{v}_B, \mathbf{i}_B >= \sum v_j i_j$ over all resistors which is equal to $\sum R_j (i_j)^2$ over all resistors. Hence $< \mathbf{v}_B, \mathbf{i}_B > \geq 0$. Thus, $< \mathbf{v}_{P'}, \mathbf{i}_{P'} > \leq 0$ for all possible vectors $\mathbf{i}_{P'}$. Equivalently $(\mathbf{i}_{P'}^T)\mathbf{R}(\mathbf{i}_{P'}) \geq 0$ for all $\mathbf{i}_{P'}$. We conclude that $\mathbf{R}$ is positive semidefinite.

Next consider the situation where $\mathbf{i}_{P'} = \mathbf{0}$, i.e., the ports of $\mathcal{N}_{BP'}$ are open circuited. We then get

$$\begin{pmatrix} -\mathbf{R} & \mathbf{I} \end{pmatrix}\binom{\mathbf{0}}{\mathbf{v}_{P'}} = \hat{\mathbf{s}}_{P'},$$

i.e., $\mathbf{v}_{P'} = \hat{\mathbf{s}}_{P'}$.

ii. The arguments are identical for this case. We finally have

$$\begin{pmatrix} \mathbf{I} & -\mathbf{G} \end{pmatrix}\binom{\mathbf{i}_{P}}{\mathbf{v}_{P}} = \tilde{\mathbf{s}}_{P'},$$

where $\mathbf{G}$ is positive semidefinite. Clearly $\tilde{\mathbf{s}}_{P'}$ is equal to $\mathbf{i}_P$ when $\mathbf{v}_P = \mathbf{0}$.

E 5.20:

i. In this case we take the variables to be $\mathbf{i}, \dot{\mathbf{v}}$ and consider the constraints on them. The KV constraints would be satisfied by $\dot{\mathbf{v}}$. The constraints $\mathbf{v}_{B\mathcal{E}} - \mathbf{e}_B = \mathbf{0}$ would change to $\dot{\mathbf{v}}_{B\mathcal{E}} - \dot{\mathbf{e}}_B = \mathbf{0}$. So if we assume that $\mathbf{x}$ is made of $\mathbf{i}, \dot{\mathbf{v}}$ the constraints on $\mathbf{x}$ would be as in Equation 5.8. The constraints on $\mathbf{i}_{P'}, \dot{\mathbf{v}}_{P'}$ would be

$$\begin{pmatrix} \mathbf{M} & \mathbf{N} \end{pmatrix}\binom{\mathbf{i}_{P'}}{\dot{\mathbf{v}}_{P'}} = \hat{\mathbf{s}}_{P'}.$$

The arguments of the previous problem would go through if we use $\dot{\mathbf{v}}$ in place of $\mathbf{v}$ in the present case. Thus, if we assume that $\dot{\mathbf{v}}_{BP'}, \mathbf{i}_{BP'}$ can be determined uniquely for arbitrary values of $\dot{\mathbf{v}}_P$ and of internal sources $\dot{\mathbf{e}}_B, \mathbf{j}_B$ then it would turn out that

$$\begin{pmatrix} -\mathbf{I} & \mathbf{C} \end{pmatrix}\binom{\mathbf{i}_{P}}{\dot{\mathbf{v}}_{P}} = \hat{\mathbf{s}}_{P}$$

where $\mathbf{C}$ is positive semidefinite. When $\dot{\mathbf{v}}_P = \mathbf{0}$, i.e., when $\mathbf{v}_P$ is constant, $\mathbf{i}_P = -\hat{\mathbf{s}}_{P'}$.

ii. The inductor case can be handled similarly except that we work with $\frac{di}{dt}$ and $\mathbf{v}$.

E 5.21: It is clear that if $\mathbf{x}_B, \mathbf{x}_{P'}$ satisfy Equation 5.8 then $\mathbf{x}_{P'}$ would satisfy

$$\mathbf{K}_{P'P'}\mathbf{x}_{P'} = \hat{\mathbf{s}}_{P'}$$

On the other hand suppose $\tilde{\mathbf{x}}_{P'}$ satisfies this latter equation. Then one can find an $\tilde{\mathbf{x}}_B$ that satisfies

$$\begin{pmatrix} \mathbf{K}_{BB} & \mathbf{K}_{BP'} \end{pmatrix}^{\mathbf{x}_B}_{\tilde{\mathbf{x}}_{P'}} = \hat{\mathbf{s}}_B$$

since the rows of $\mathbf{K}_{BB}$ are given to be linearly independent. Thus, whenever $\mathbf{x}_B, \mathbf{x}_{P'}$ satisfies Equation 5.8, $\mathbf{x}_{P'}$ satisfies the reduced equation and any solution to the reduced equation can be extended to a solution of the 'enlarged' equation. This proves the required result.

Chapter 6

Topological Hybrid Analysis

6.1 Introduction

In this chapter we discuss a method of network decomposition that is a topological generalization of hybrid analysis (i.e., analysis where unknowns involve both voltages and currents). Our main result, Theorem 6.4.1 of Section 6.4, states that solving a network is equivalent to solving two derived subnetworks matching certain current and voltage boundary conditions. In order to state the result precisely we need to define electrical networks formally. This we do in Section 6.2. The implications of the result would be clearer if we understand the effect of sources on the structure of the constraints of a network. We discuss three results on this topic in Section 6.3. Two of these viz. 'v-shift' and 'i-shift' are well known basic results. Issues concerned with optimal application of Theorem 6.4.1 to writing network equations are discussed in Section 6.5. In that section we also discuss (in Subsection 6.5.3) the application of the method to the important special case of linear electrical networks.

6.2 Electrical Network: A Formal Description

In this section we give a formal description of a general electrical network. The formality concerns the definition of a general device characteristic and the notion of edges of the network being 'decoupled' in the device characteristic. This exercise has to be carried out in order to emphasize the topological nature of the results that follow in this chapter.

6.2.1 Static and Dynamic Electrical Networks

Definition 6.2.1 *An* **electrical network** $\mathcal{N}$ *on the set of edges* E *is an ordered pair* $(\mathcal{G}, \mathcal{D})$ *where* $\mathcal{G}$ *is a directed graph on the edge set* E *and* $\mathcal{D}$ *is a* **device characteristic** *(to be defined below). A* **generalized electrical network** $\mathcal{N}_g$ *on the set* E *is an ordered pair* $(\mathcal{V}, \mathcal{D})$ *where* $\mathcal{V}$ *is a vector space on* E *and* $\mathcal{D}$ *is a device characteristic on* E.

Definition 6.2.2 *The* **device characteristic** $\mathcal{D}$ *may be* **static** *or* **dynamic**. *A* **static device characteristic** $\mathcal{D}$ *on* E *is a collection of ordered pairs* $(\mathbf{v}, \mathbf{i})$ *where* $\mathbf{v}, \mathbf{i}$ *are vectors on* E. *A* **dynamic device characteristic** $\mathcal{D}$ *on* E *is a collection of ordered pairs* $(\mathbf{v}(\cdot), \mathbf{i}(\cdot))$, *where* $\mathbf{v}(\cdot)$, $\mathbf{i}(\cdot)$ *are functions on* $\Re$ *s.t.* $\mathbf{v}(t), \mathbf{i}(t)$, $t \in \Re$ *are vectors on* E. *We speak of* $\mathbf{v}(\cdot)$ *as the voltage part and* $\mathbf{i}(\cdot)$ *as the current part of an element* $(\mathbf{v}(\cdot), \mathbf{i}(\cdot))$ *of* $\mathcal{D}$.

A network is said to be **static** (**dynamic**) iff its device characteristic is **static** (**dynamic**).

Common Devices Revisited

We now list the usual electrical devices and redefine them in the light of our definition of device characteristic. For figures see Section 5.4.

i. Voltage source $\equiv \{(v(\cdot), i(\cdot)),\ v(t) = e(t) \quad \forall t \in \Re\}$

ii. Current source $\equiv \{(v(\cdot), i(\cdot)),\ i(t) = j(t) \quad \forall t \in \Re\}$.

iii. Linear resistor $\equiv \{(v(\cdot), i(\cdot)),\ v(t) = Ri(t) \quad \forall t \in \Re\}$.

iv. Nonlinear resistor $\equiv \{(v(\cdot), i(\cdot)),\ f(v(t), i(t)) = 0 \quad \forall t \in \Re\}$.

v. Linear capacitor $\equiv \{(v(\cdot), i(\cdot)),\ i(t) = C\dot{v}(t) \quad \forall t \in \Re\}$.

vi. Linear inductor $\equiv \{(v(\cdot), i(\cdot)),\ v(t) = L\dot{i}(t) \quad \forall t \in \Re\}$.

vii. Linear coupled inductors $\equiv \{(\mathbf{v}(\cdot), \mathbf{i}(\cdot)),\ \mathbf{v}(t) = \mathcal{L}\dot{\mathbf{i}}(t) \quad \forall t \in \Re\}$.

viii. Nonlinear capacitor $\equiv \{(v(\cdot), i(\cdot)),\ i(t) = \dot{q}(t), f(q(t), v(t)) = 0 \quad \forall t \in \Re\}$

ix. Nonlinear coupled inductors $\equiv \{(\mathbf{v}(\cdot), \mathbf{i}(\cdot)),\ \mathbf{v}(t) = \dot{\psi}(t), f(\psi(t), \mathbf{i}(t)) = \mathbf{0} \ \forall t \in \Re\}$

x. Current controlled voltage source $\equiv \{((v_1(\cdot),\ v_2(\cdot)),\ (i_1(\cdot),\ i_2(\cdot))),\ v_2(t) - r\, i_1(t) = 0,\ v_1(t) = 0 \quad \forall t \in \Re\}$.

xi. Norator $\equiv \{(v_1(\cdot), i_1(\cdot)) : v_1(\cdot), i_1(\cdot)$ real functions$\}$.

xii. Nullator $\equiv \{(v_1(\cdot), i_1(\cdot)) : v_1(\cdot) = i_1(\cdot) = 0\}$.

Figure 6.1: A Static and a Dynamic Network

The definitions of the other controlled sources are similar.

Example: Consider the static network $\mathcal{N}_s$ in Figure 6.1. The device characteristic of this network is the collection of ordered pairs $((v_1, v_2), (i_1, i_2))$ s.t. $v_1 - R_1 i_1 = 0$, $v_2 - e_2 = 0$.
For the dynamic network $\mathcal{N}_d$ in Figure 6.1, the device characteristic is the collection of pairs $((v_1(\cdot), v_2(\cdot)), (i_1(\cdot), i_2(\cdot)))$ s.t. $v_1 - L_1 \dot{i}_1 = 0$, $i_2 - C_2 \dot{v}_2 = 0$.

Remark: A static network can be modelled as a dynamic network using constant functions of time. So henceforth, we will speak only of dynamic networks. But usually we would write $(\mathbf{v}, \mathbf{i})$ instead of $(\mathbf{v}(\cdot), \mathbf{i}(\cdot))$ when we speak of a typical element of a device characteristic.

Definition 6.2.3 Solution of a dynamic network: *Let $\mathcal{N} = (\mathcal{G}, \mathcal{D})$. An ordered pair $(\mathbf{v}(\cdot), \mathbf{i}(\cdot))$ of vector functions of time on $E(\mathcal{G})$ is a solution of $\mathcal{N}$ iff*

$$\begin{aligned} \mathbf{v}(t) &\in \mathcal{V}_v(\mathcal{G}) \quad \forall t \in \Re \\ \mathbf{i}(t) &\in \mathcal{V}_i(\mathcal{G}) \quad \forall t \in \Re \\ (\mathbf{v}(\cdot), \mathbf{i}(\cdot)) &\in \mathcal{D}. \end{aligned}$$

6.2.2 Device Decoupling

Usually an electrical network has a device characteristic in which the constraints on sets of branches can be specified independently of each other. (The interdependence of currents and voltages of different branches arises in the **solution** through KCE and KVE). We introduce notation which would allow us to speak of this 'device decoupling'.

Definition 6.2.4 *Let $\mathcal{N} \equiv (\mathcal{G}, \mathcal{D})$ where $\mathcal{G}$ is a graph on E. Let $A, B \subseteq E$. Then, the* **section (A,B)** *of $\mathcal{D}$ denoted by $\mathcal{D}_{AB}$ is defined as follows: $\mathcal{D}_{AB} \equiv \{(\mathbf{v}/A, \mathbf{i}/B),$*

$(\mathbf{v}, \mathbf{i}) \in \mathcal{D}\}$. *We denote* $\mathcal{D}_{\emptyset B}$ *by* $\mathcal{D}_{Bi}$, $\mathcal{D}_{A\emptyset}$ *by* $\mathcal{D}_{Av}$ *and* $\mathcal{D}_{AA}$ *by* $\mathcal{D}_{A}$. *If* $(\mathbf{v}, \mathbf{i}) \in \mathcal{D}$ *we speak of* $\mathbf{v}/A$, $\mathbf{i}/B$, *respectively as the voltage part and current part of the section* (A, B) *of* $(\mathbf{v}, \mathbf{i})$. *When* $A = B$ *we write section* (A) *in place of section* (A, A).

Thus, the notion of section (A, B) allows us to focus attention on device constraints of subset A as far as voltages are concerned and on subset B as far as currents are concerned.

Definition 6.2.5 *Let* $A, B, P, Q \subseteq E$ *s.t.* $A \cap P = \emptyset$, $B \cap Q = \emptyset$. *Then, the product of* $\mathcal{D}_{AB}$ *and* $\mathcal{D}_{PQ}$, *denoted by* $\mathcal{D}_{AB} \times \mathcal{D}_{PQ}$, *is defined by* $\mathcal{D}_{AB} \times \mathcal{D}_{PQ} \equiv \{(\mathbf{v}, \mathbf{i}),$ $\mathbf{v} = \mathbf{v}_A \oplus \mathbf{v}_P$, $\mathbf{i} = \mathbf{i}_B \oplus \mathbf{i}_Q$, $(\mathbf{v}_A, \mathbf{i}_B) \in \mathcal{D}_{AB}$, $(\mathbf{v}_P, \mathbf{i}_Q) \in \mathcal{D}_{PQ}\}$.

Thus the elements of $\mathcal{D}_{AB} \times \mathcal{D}_{PQ}$ are obtained by taking elements $(\mathbf{v}_A, \mathbf{i}_B)$ of $\mathcal{D}_{AB}$ and $(\mathbf{v}_P, \mathbf{i}_Q)$ of $\mathcal{D}_{PQ}$, adjoining the voltage parts to get the voltage part and adjoining the current parts to get the current part of the corresponding element of the product. Suppose $\mathcal{D} = \mathcal{D}_{AB} \times \mathcal{D}_{PQ}$. Then this means that the voltages in A and currents in B are independent of the voltages and currents in P and Q respectively as far as the device characteristic is concerned.

Definition 6.2.6 *We say that* (A, B) *and* $(E - A, E - B)$ *are decoupled in the device characteristic iff* $\mathcal{D} = \mathcal{D}_{AB} \times \mathcal{D}_{(E-A)(E-B)}$. *More generally let* $A_1, \cdots, A_k$ $(B_1, \cdots, B_k)$ *be disjoint subsets of* E *whose union is* E *(void sets allowed) and let* $\mathcal{D} = \mathcal{D}_{A_1B_1} \times \cdots \times \mathcal{D}_{A_kB_k}$. *Then we say that* $(A_1, B_1) \cdots (A_k, B_k)$ *are decoupled in the device characteristic. If* $A_i = B_i$, $i = 1, \cdots, k$, *then we say that* $A_1, \cdots, A_k$ *are decoupled in the device characteristic.*

The following example describes a situation where the devices are **not** decoupled.

Example: Consider a network composed entirely of a current controlled current source. Let A be the singleton composed of the controlling short circuit branch and $E - A$, the singleton composed of the controlled current source branch.
$\mathcal{D}_A \equiv \{(0, i_1), i_1 \in \Re\}$
$\mathcal{D}_{E-A} \equiv \{(v_2, i_2), i_2 \in \Re\}$
However, $\mathcal{D} = \{((0, v_2), (i_1, \beta i_1)),\ i_1, v_2 \in \Re\}$. So $\mathcal{D} \neq \mathcal{D}_A \times \mathcal{D}_{E-A}$ as is to be expected since in $\mathcal{D}$, i_1, i_2 are linked by β.

A common situation is where a branch has no voltage constraints (e.g. a current source, the controlled element in a cccs or vccs) or has no current constraints (e.g. a voltage source, the controlled element in a ccvs or vcvs). We introduce special notation to speak of such situations.

Definition 6.2.7 $\delta_{AB} \equiv \{(\mathbf{v}_A, \mathbf{i}_B)$, $\mathbf{v}_A$ *is any vector on* A, $\mathbf{i}_B$ *is any vector on* $B\}$. $\delta_A \equiv \delta_{AA}$, $\delta_{A_v} \equiv \delta_{A\emptyset}$, $\delta_{A_i} \equiv \delta_{\emptyset A}$.

We say $A \subseteq E$ *is* **voltage unconstrained** *in* $\mathcal{D}$ *iff* $\mathcal{D} = \delta_{Av} \times \mathcal{D}_{(E-A)E}$. *A set* $B \subseteq E$ *is* **current unconstrained** *in* $\mathcal{D}$ *iff* $\mathcal{D} = \delta_{Bi} \times \mathcal{D}_{E(E-B)}$.

A set $A \subseteq E$ *is a* **dummy** *set, a set of* **norators**, *a set of* **unknown current** *sources or a set of* **unknown voltage** *sources iff* $\mathcal{D} = \mathcal{D}_{E-A} \times \delta_A$, *i.e., iff it is both voltage and current unconstrained.*

Figure 6.2: Decoupling in Device Characteristic

As remarked before, voltage sources are current unconstrained and current sources, voltage unconstrained. Dummy elements have to be introduced in order to speak of port elements and also to describe network decomposition methods. The following example illustrates device characteristic decoupling.

Example: Consider the network in Figure 6.2. This network has a graph whose edges are $\{e_0, e_1, \cdots, e_6\}$ with e_0 corresponding to J_0, e_1 to $\mathcal{E}_1$, e_2 to R_2 etc. We note that in the device characteristic $\{e_0\}, \{e_1\}$, $\{e_2\}$, $\{e_3\}, \{e_4\}$, $\{e_5, e_6\}$, are decoupled. Calling these sets $\mathcal{J}, \mathcal{E}, R_2, R_3, C, L$ we have $\mathcal{D} = \mathcal{D}_J \times \mathcal{D}_{\mathcal{E}} \times \mathcal{D}_{R_2} \times \mathcal{D}_{R_3} \times \mathcal{D}_C \times \mathcal{D}_L$. Note however, that $\mathcal{D}_L$ cannot be further split since $\dot{i}_{L_5}$, $\dot{i}_{L_6}$ are related to v_{L_5} and v_{L_6}. However the current source is voltage unconstrained and the voltage source is current unconstrained. So we have, $\mathcal{D}_{\mathcal{J}} = \mathcal{D}_{\mathcal{J}_i} \times \delta_{\mathcal{J}_v}$ and $\mathcal{D}_{\mathcal{E}} = \mathcal{D}_{\mathcal{E}_v} \times \delta_{\mathcal{E}_i}$.

6.3 Some Basic Topological Results

In the following pages we present three basic topological results on electrical network analysis. Two of these results viz. 'v-shift' and 'i-shift' are very well known. The third result, which we present first, should be regarded as part of network theory folklore.

6.3.1 Effect of Voltage Unconstrained and Current Unconstrained Devices on the Topological Constraints

The following result speaks of how presence of voltage sources and current sources affects the network (i.e., device characteristic + topological) constraints. Since voltage sources do not have current constraints on them their current variables can be suppressed from constraints involving other currents. A dual result is true for current sources. It turns out that the suppression can be achieved by using $\mathcal{G} \times (E - \mathcal{E})$ for KC constraints and $\mathcal{G} \cdot (E - \mathcal{J})$ for KV constraints, where $\mathcal{E}, \mathcal{J}$ stand for the voltage and current source edge sets. More formally we have

Theorem 6.3.1 *Let $\mathcal{N} \equiv (\mathcal{G}, \mathcal{D})$, where $\mathcal{G}$ is a graph on E. Let $A, B \subseteq E$ and let $\mathcal{D} = \delta_{AB} \times \mathcal{D}_{(E-A)(E-B)}$, i.e., devices in A are voltage unconstrained and devices in B are current unconstrained. Then,*
an ordered pair $(\mathbf{v}(\cdot), \mathbf{i}(\cdot))$ is a solution of $\mathcal{N}$ only if

i. $\mathbf{i}(t)/(E - B)$ *is a current vector of* $\mathcal{G} \times (E - B) \quad \forall t \in \Re$,

ii. $\mathbf{v}(t)/(E - A)$ *is a voltage vector of* $\mathcal{G} \cdot (E - A) \quad \forall t \in \Re$,

iii. $(\mathbf{v}(\cdot)/(E - A), \mathbf{i}(\cdot))/(E - B)) \in \mathcal{D}_{(E-A)(E-B)}$.

Conversely if $(\mathbf{v}_{E-A}, \mathbf{i}_{E-B}) \in \mathcal{D}_{(E-A)(E-B)}$ and if $\mathbf{v}_{E-A}, \mathbf{i}_{E-B}$ satisfy conditions (i) and (ii) above, there exists a solution $(\mathbf{v}(\cdot), \mathbf{i}(\cdot))$ of $\mathcal{N}$ s.t. $\mathbf{v}/(E - A) = \mathbf{v}_{E-A}$ and $\mathbf{i}/(E - B) = \mathbf{i}_{E-B}$.

Proof : Let $(\mathbf{v}(\cdot), \mathbf{i}(\cdot))$ be a solution of $\mathcal{N} \equiv (\mathcal{G}, \mathcal{D})$. Then $\mathbf{v}(t) \in \mathcal{V}_v(G)$, $\mathbf{i}(t) \in \mathcal{V}_i(G) \quad \forall t \in \Re$. Hence, by Theorems 3.4.4 and 3.4.5 we must have $\mathbf{v}(t)/(E - A)$, a voltage vector of $\mathcal{G} \cdot (E - A)$ and $\mathbf{i}(t)/(E - B)$, a current vector of $\mathcal{G} \times (E - B)$. Condition (iii) is satisfied by the definition of $\mathcal{D}_{(E-A)(E-B)}$.
Now let the ordered pair $(\mathbf{v}_{E-A}, \mathbf{i}_{E-B}) \in \mathcal{D}_{(E-A)(E-B)}$ and further satisfy conditions (i) and (ii) of the statement of the theorem. By the above mentioned theorems it is clear that there must exist $(\mathbf{v}(\cdot), \mathbf{i}(\cdot))$ s.t. $\mathbf{v}(t)$ is a voltage vector ($\mathbf{i}(t)$ is a current vector) of $\mathcal{G} \;\; \forall t \in \Re$ and s.t. $\mathbf{v}(\cdot)/E - A = \mathbf{v}_{E-A}(\cdot)$ and $\mathbf{i}(\cdot)/E - B = \mathbf{i}_{E-B}(\cdot)$. Since $\mathcal{D} = \delta_{AB} \times \mathcal{D}_{(E-A)(E-B)}$ it is clear that $(\mathbf{v}(\cdot), \mathbf{i}(\cdot)) \in \mathcal{D}$. Thus, $(\mathbf{v}(\cdot), \mathbf{i}(\cdot))$ is a solution of $\mathcal{N}$ as required.

□

An Application of Theorem 6.3.1

Suppose a static linear network is composed of current sources $\mathcal{J}$, voltage sources $\mathcal{E}$, and remaining devices R. Let the device characteristic be

$$\mathbf{v}_R = \mathbf{R}\,\mathbf{i}_R \tag{6.1}$$
$$\mathbf{i}_{\mathcal{J}} = \mathbf{j} \tag{6.2}$$
$$\mathbf{v}_{\mathcal{E}} = \mathbf{e}. \tag{6.3}$$

Then, by Theorem 6.3.1, the topological constraints on the above variables are as follows: $\mathbf{v}_R \oplus \mathbf{v}_{\mathcal{E}} \in \mathcal{V}_v(\mathcal{G} \cdot (E - \mathcal{J}))$, $\mathbf{i}_R \oplus \mathbf{i}_{\mathcal{J}} \in \mathcal{V}_i(\mathcal{G} \times (E - \mathcal{E}))$.

Let us write the constraints in the form of equations. We then have:

$$\begin{bmatrix} -\mathbf{I} & \mathbf{R} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} \\ \mathbf{B}_R & \mathbf{0} & \mathbf{0} & \mathbf{B}_{\mathcal{E}} \\ \mathbf{0} & \mathbf{A}_R & \mathbf{A}_{\mathcal{J}} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{v}_R \\ \mathbf{i}_R \\ \mathbf{i}_{\mathcal{J}} \\ \mathbf{v}_{\mathcal{E}} \\ \ \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{j} \\ \mathbf{e} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}$$

where $(\mathbf{B}_R \mathbf{B}_{\mathcal{E}})$ is a representative matrix of $\mathcal{V}_i(\mathcal{G} \cdot (E - \mathcal{J}))$ and $(\mathbf{A}_R\ \mathbf{A}_{\mathcal{J}})$ is the reduced incidence matrix of $\mathcal{G} \times (E - \mathcal{E})$. Use of Theorems 3.4.4 and 3.4.5 tells us that rows of $\mathbf{B}_R$ must belong to and span $\mathcal{V}_i(\mathcal{G} \times (E - \mathcal{E}) \cdot R)$ while rows of $\mathbf{A}_R$ must belong to and span $\mathcal{V}_v(\mathcal{G} \times (E - \mathcal{E}) \cdot R)$. If we may assume that in the graph $\mathcal{G}$ voltage sources do not form circuits and current sources do not form cutsets it can be seen that
$r(\mathbf{B}_R) = r(\mathbf{B}_R \mathbf{B}_{\mathcal{E}})$ and $r(\mathbf{A}_R) = r(\mathbf{A}_R \mathbf{A}_{\mathcal{J}})$.
In this case $\mathbf{B}_R$, $\mathbf{A}_R$ would be the representative matrices of the corresponding spaces and we can rewrite the equations shifting source terms to the right as follows:

$$\begin{bmatrix} \mathbf{I} & -\mathbf{R} \\ \mathbf{B}_R & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_R \end{bmatrix} \begin{bmatrix} \mathbf{v}_R \\ \mathbf{i}_R \\ \ \end{bmatrix} = \begin{pmatrix} \mathbf{0} \\ -\mathbf{B}_{\mathcal{E}} \mathbf{v}_{\mathcal{E}} \\ -\mathbf{A}_{\mathcal{J}} \mathbf{i}_{\mathcal{J}} \end{pmatrix}.$$

We see in this equation that the coefficient matrix of the unknowns $\mathbf{v}_R, \mathbf{i}_R$ is entirely dependent upon the section of the device characteristic to the set R and to the graph obtained by *shorting the voltage sources* and *opening the current sources*. This would be true whatever be the method of analysis we use: nodal, loop, hybrid etc.

Remark: We may state the following general rule
The structure of the constraints in an arbitrary network, as far as current variables other than the unconstrained variables are concerned, corresponds to the graph obtained by setting the voltage sources to zero (shorting them) and as far as the voltage variables other than the unconstrained variables are concerned, corresponds to the graph obtained by setting the current sources to zero (opening them). Norators may be either shorted or opened.

6.3.2 Voltage and Current shift

We next discuss the techniques of current shift (i-shift) and voltage shift (v-shift). These are methods of altering the source structure in the network leaving the other variables 'unaffected'. We need to define the notion of **equivalence** in order to make the term 'unaffected' precise.

Definition 6.3.1 *Let $\mathcal{N}_1 \equiv (\mathcal{G}_1, \mathcal{D}_1)$, $\mathcal{N}_2 \equiv (\mathcal{G}_2, \mathcal{D}_2)$, where $\mathcal{G}_1, \mathcal{G}_2$ are graphs on E_1, E_2 respectively. Let $A, B \subseteq E_1 \cap E_2$. We say that $\mathcal{N}_1, \mathcal{N}_2$ are* **equivalent**

in A_v, B_i *iff for any given solution* $(\mathbf{v}_1, \mathbf{i}_1)$ *of* $\mathcal{N}_1$ *(solution* $(\mathbf{v}_2, \mathbf{i}_2)$ *of* $\mathcal{N}_2$*) we can find a solution* $(\mathbf{v}_2, \mathbf{i}_2)$ *of* $\mathcal{N}_2$ *(solution* $(\mathbf{v}_1, \mathbf{i}_1)$ *of* $\mathcal{N}_1$*) s.t.* $\mathbf{v}_1/A = \mathbf{v}_2/A$ *and* $\mathbf{i}_1/B = \mathbf{i}_2/B$. *If* $A = B$ *we simply write '***equivalent** *in A'.*

The technique of i-shift is based on adding, without disturbing the incidence relationships of existing edges (i.e., the addition is 'soldering iron type') a circuit of current sources, all of the same value J and all of the same direction with respect to the orientation of the circuit. (The value J could be a function of the remaining variables). The following theorem essentially states that this procedure does not affect the remaining variables.

Theorem 6.3.2 *Let* $\mathcal{N}_1 \equiv (\mathcal{G}, \mathcal{D})$ *be a network.* $\mathcal{N}_2 \equiv (\mathcal{G}_1, \mathcal{D}_1)$ *be a network obtained from* $\mathcal{N}_1$ *by adding (*$\mathcal{G}$ *being a restriction of* $\mathcal{G}_1$*) a circuit of (controlled) current sources of value* J*, all of the same direction with respect to the orientation of the circuit. Let these additional devices be voltage unconstrained (but the value* J *may be dependent on the remaining variables). Then* $\mathcal{N}_1$ *and* $\mathcal{N}_2$ *are equivalent in* $E(\mathcal{G})$.

Proof : Let $(\mathbf{v}_1, \mathbf{i}_1)$ be a solution of $\mathcal{N}_2$. We denote $E(\mathcal{G})$ by E and $E(\mathcal{G}_1)$ by E_1. Then $\mathbf{v}_1/E \in (\mathcal{V}_v(\mathcal{G}_1)) \cdot E = \mathcal{V}_v(\mathcal{G}_1 \cdot E) = \mathcal{V}_v(\mathcal{G})$. Now $\mathbf{i}_1/(E_1 - E)$ is a vector with all entries equal to J. Let $\mathbf{i}_c$ be a circuit vector of $\mathcal{G}_1$ with $E_1 - E$ as its support. From the conditions of the theorem it is clear that we can assume $\mathbf{i}_c$ to have all entries equal to $+1$. It follows that $\mathbf{i}_1 - J\mathbf{i}_c$ has zero entries in $E_1 - E$ and further agrees with $\mathbf{i}_1/E$. Since $\mathbf{i}_1, \mathbf{i}_c \in \mathcal{V}_i(\mathcal{G}_1)$ so does $\mathbf{i}_1 - J\mathbf{i}_c$. Hence $\mathbf{i}_1/E \in \mathcal{V}_i(\mathcal{G}_1) \times E = \mathcal{V}_i(\mathcal{G}_1 \cdot E) = \mathcal{V}_i(\mathcal{G})$.
Now $(\mathbf{v}_1, \mathbf{i}_1) \in \mathcal{D}_1$. By the definition of $\mathcal{D}_1$ it is clear that $(\mathbf{v}_1/E, \mathbf{i}_1/E) \in \mathcal{D}$. Thus, $(\mathbf{v}_1/E, \mathbf{i}_1/E)$ is a solution of $\mathcal{N}_1$.
Next let $(\mathbf{v}, \mathbf{i})$ be a solution of $\mathcal{N}_1$. Since $\mathbf{v} \in \mathcal{V}_v(\mathcal{G}) = \mathcal{V}_v(\mathcal{G}_1 \cdot E)$ there exists $\mathbf{v}_1 \in (\mathcal{V}_v(\mathcal{G}_1))$ s.t. $\mathbf{v}_1/E = \mathbf{v}$. By the definition of $\mathcal{D}_1$, if $(\mathbf{v}, \mathbf{i}) \in \mathcal{D}$ and if $\mathbf{v}_1$ is on E_1 s.t. $\mathbf{v}_1/E = \mathbf{v}$, then there exists some $(\mathbf{v}_1, \mathbf{i}_1) \in \mathcal{D}$ s.t. $\mathbf{i}_1/E = \mathbf{i}$ and $\mathbf{i}_1(e) = J \ \ \forall e \in E_1 - E$.

Now $\mathbf{i}_1 - J\mathbf{i}_c$ where $\mathbf{i}_c$ is the abovementioned circuit vector of $\mathcal{G}_1$, takes zero value on $E_1 - E$ and agrees with $\mathbf{i}$ on E. Since $\mathbf{i} \in \mathcal{V}_i(\mathcal{G}) = (\mathcal{V}_i(\mathcal{G}_1)) \times E$, we must have $\mathbf{i}_1 - J\mathbf{i}_c \in \mathcal{V}_i(\mathcal{G}_1)$. We know that $\mathbf{i}_c \in \mathcal{V}_i(\mathcal{G}_1)$. Hence $\mathbf{i}_1 \in \mathcal{V}_i(\mathcal{G}_1)$.
We thus see that there is a solution $(\mathbf{v}_1, \mathbf{i}_1)$ of $\mathcal{N}_2$ s.t. $\mathbf{v} = \mathbf{v}_1/E$ and $\mathbf{i} = \mathbf{i}_1/E$. This proves that $\mathcal{N}_1$ and $\mathcal{N}_2$ are equivalent in E.

□

We illustrate the use of this theorem by an example.

Example: Consider the network $\mathcal{N}_1$ shown in Figure 6.3 (a). It has a current source J which we wish to shift. In (b) we add a directed circuit of current sources J_2 to obtain a network $\mathcal{N}_2$. One of these current sources is opposite and parallel to the source J. The networks in (a) and (b) are equivalent in the set of edges of $\mathcal{N}_1$.
Now we set $J_2 = J$. We find that the two parallel but oppositely directed sources constitute a directed current source circuit. So $\mathcal{N}_2$ could be thought of as obtained from $\mathcal{N}_3$ in (c) by addition of this circuit. Hence $\mathcal{N}_3, \mathcal{N}_2$ are equivalent in the

Figure 6.3: Current Shift

set of edges of $\mathcal{N}_3$. It follows that $\mathcal{N}_1$ and $\mathcal{N}_3$ are equivalent in the set of edges $\{e_1, e_2, e_3, e_4, e_5\}$ which is equal to (set of edges of $\mathcal{N}_1 - \{J\}$).
The reader should verify for himself that the source current leaving any node is the same before and after i-shift. Thus, in Figure 6.3, a current J was originally leaving a and entering b. This happens also after the shifting operation is carried out. After shifting the current source we see that at node c, J enters and leaves so that there is no net source current leaving.

Remark: Normally i-shift is performed as follows. We select a circuit containing the given current source J. We then introduce an oppositely directed current source in parallel with every branch of this circuit. We next delete the two equal but oppositely directed sources. For instance in the network $\mathcal{N}_1$ of Figure 6.3 we see that e_4, J, e_1 form a circuit. Equal but oppositely directed current sources to J are introduced in parallel with each of these branches. The branch that originally contained J now has in parallel an oppositely directed source of equal value. This combination is now deleted.
After i-shift we would be left with current sources in parallel with devices of $\mathcal{N}_1$.

When current sources appear in parallel with other devices, we can combine them with these devices so that the graph of the modified network has single edges in place of these parallel edges. For example, a resistor R in parallel with current source J can be regarded as a single device with characteristic $v = R(i - J)$. When this is done we note that the effect of i-shift on the graph of the network is simply to open the current source branches leaving the end-points in place.

The technique of voltage shift is based on adding by splitting nodes (i.e., pliers type entry) a cutset of voltage sources all of the same value e_S and all of the same direction relative to the orientation of the cutset. (The value e_S could be a function of the remaining variables). The following theorem essentially states that the procedure does not affect the remaining variables. The theorem is dual to Theorem 6.3.2. It is obtained by making the following dual substitutions:

circuit	$\leftrightarrow$	cutset
graph restriction	$\leftrightarrow$	graph contraction
voltage vector	$\leftrightarrow$	current vector

The proof again can be obtained by making the same substitutions on the proof of Theorem 6.3.2. We therefore omit it. (The reader should note that the plan for deriving the dual of a result about electrical networks is essentially the same as the one discussed in Section 3.7. This is because any statement about electrical networks is a statement about graphs and certain members of their voltage and current spaces. Of course, as we stated in the above mentioned section, not every such statement can be dualized.)

Theorem 6.3.3 *Let $\mathcal{N}_1 \equiv (\mathcal{G}, \mathcal{D})$ be a network. Let $\mathcal{N}_2 = (\mathcal{G}_1, \mathcal{D}_1)$ be a network obtained from $\mathcal{N}_1$ by adding ($\mathcal{G}$ being a contraction of $\mathcal{G}_1$) a cutset of (controlled) voltage sources of value e_S, all of the same direction with respect to the orientation of the cutset. Let these additional devices be current unconstrained (but the value e_S may be dependent on the remaining variables). Then $\mathcal{N}_1$ and $\mathcal{N}_2$ are equivalent in $E(\mathcal{G})$.*

We illustrate the use of this theorem by an example.

Example: Consider the network $\mathcal{N}_1$ shown in Figure 6.4 (a). It has a voltage source e_s which we wish to shift. In (b) we add a cutset of voltage sources e_{2s} to obtain a network $\mathcal{N}_2$. The entry of the cutset is 'pliers type', i.e., by splitting nodes. Hence if we contract the branches in the cutset we get back network $\mathcal{N}_1$. One of the added voltage sources is opposite and in series to the source e_s. The networks in (a) and (b) are equivalent in the set of edges of $\mathcal{N}_1$. Now we set $e_{2s} = e_s$. We find that the two series but oppositely directed sources constitute a directed voltage source cutset. So $\mathcal{N}_2$ could be thought of as obtained from $\mathcal{N}_3$ in (c) by 'pliers type' addition of this cutset. Hence $\mathcal{N}_3, \mathcal{N}_2$ are equivalent in the set of edges of $\mathcal{N}_3$. It follows that $\mathcal{N}_1$ and $\mathcal{N}_3$ are equivalent in the set of edges $\{e_1, e_2, e_3, e_4, e_5\}$ which is equal to (set of edges of $\mathcal{N}_1 - \{e_{2s}\}$).

Figure 6.4: Voltage Shift

Remark: Normally v-shift is performed as follows. We select a cutset containing the given voltage source e_S. We then introduce an oppositely directed voltage source in series with every branch of this cutset. We next contract the two equal but oppositely directed sources. For instance in the network $\mathcal{N}_1$ of Figure 6.4 we see that e_5, e_S, e_1 form a cutset. Equal but oppositely directed voltage sources to e_S are introduced in series with each of these branches. The branch that originally contained e_S now has in addition an oppositely directed source of equal value. This combination is now contracted.
Usually, after v-shift, we would be left with voltage sources in series with devices of $\mathcal{N}_1$. When voltage sources appear in series with other devices we can combine them with these devices so that the graph of the modified network has single edges in place of these series edges. For example, a resistor R, in series with a voltage source e_S, can be regarded as a single device with characteristic $v - e_S = Ri$. When this is done we note that the effect of v-shift on the graph of the network is simply to short the voltage sources (removing them after fusing the end points).

Exercise 6.1 *Let e_S be a voltage source and let C be a cutset containing it.*

i. *Visualize the results of v-shifting* e_S *into the remaining branches of the cutset. What happens to the cutset in the process?*

ii. *Suppose a collection of voltage sources formed a cutset. How would you detect such a cutset?*

iii. *How would you reduce the number of voltage sources without affecting the solution as far as the rest of the network is concerned?*

Exercise 6.2 *Dualize the previous problem and solve the dual problem.*

Exercise 6.3 *Let* $\mathcal{G}$ *be a graph on* $E = R \uplus \mathcal{E} \uplus \mathcal{J}$. *Let* $\mathcal{E}$ *contain no circuits and* $\mathcal{J}$*, no cutsets. Let* $(\mathbf{B}_R\ \mathbf{B}_{\mathcal{E}})$ *be a representative matrix of* $\mathcal{V}_i(\mathcal{G}\ .\ (E-\mathcal{J}))$ *and* $(\mathbf{A}_R\ \mathbf{A}_J)$ *be a representative matrix of* $\mathcal{V}_v(\mathcal{G} \times (E-\mathcal{E}))$. *Show that* $r(\mathbf{B}_R\ \mathbf{B}_{\mathcal{E}}) = r(\mathbf{B}_R)$ *and* $r(\mathbf{A}_R\ \mathbf{A}_J) = r(\mathbf{A}_R)$.

Exercise 6.4 *Let* $\mathcal{N} \equiv (\mathcal{G}, \mathcal{D})$ *where* $E(\mathcal{G}) = E$ *and* $E = R \uplus \mathcal{E} \uplus J$. *Let* $\mathcal{D} = \mathcal{D}_R \times \mathcal{D}_{\mathcal{E}} \times \mathcal{D}_J$ *with* $\mathcal{D}_R$ *being specified by* $\mathbf{v}_R = \mathbf{R}\mathbf{i}_R$ *or* $\mathbf{i}_R = \mathbf{G}\mathbf{v}_R$, $\mathcal{D}_{\mathcal{E}}$ *being specified by* $\mathbf{v}_{\mathcal{E}} = \mathbf{e}$ *and* $\mathcal{D}_J$ *by* $\mathbf{i}_{\mathcal{J}} = \mathbf{j}$.
Write nodal and loop type equations for this network assuming that $\mathcal{E}$ *does not contain circuits and that* $\mathcal{J}$ *does not contain cutsets. Show in both cases that the coefficient matrix structure is controlled by the graph* $\mathcal{G}\ .\ (E-\mathcal{J}) \times R$.

Exercise 6.5 *State Theorem 6.3.2 formally using the device characteristic notation.*

Exercise 6.6 *Let* $\mathcal{N} \equiv (\mathcal{G}, \mathcal{D})$ *where* $\mathcal{G}$ *is a graph on* E. *Let* $\mathcal{D} = \delta_{A_v} \times \mathcal{D}_{A_i} \times \mathcal{D}_{E-A}$, *where* $\mathcal{D}_{A_i} \equiv$ *a collection of ordered pairs* $(\phi, \mathbf{i}_A(.))$,
where $\mathbf{i}_A(t) \in \mathcal{V}_i(\mathcal{G}\ .\ A)\quad \forall t\ \in \Re$.
Let $\mathcal{N}_1 \equiv (\mathcal{G}\ .\ (E-A), \mathcal{D}_{E-A})$. *Show that* $\mathcal{N}$ *and* $\mathcal{N}_1$ *are equivalent in* $E-A$.

Exercise 6.7 *(Dual of the previous exercise)*
Let $\mathcal{N} \equiv (\mathcal{G}, \mathcal{D})$ *where* $\mathcal{G}$ *is a graph on* E. *Let* $\mathcal{D} = \delta_{A_i} \times \mathcal{D}_{A_v} \times \mathcal{D}_{E-A}$
where $\mathcal{D}_{A_v} \equiv$ *a collection of ordered pairs* $(\mathbf{v}_A(.), \phi)$
where $\mathbf{v}_A(t) \in \mathcal{V}_v(\mathcal{G} \times A)\quad \forall t \in \Re$.
Let $\mathcal{N}_1 \equiv (\mathcal{G} \times (E-A), \mathcal{D}_{E-A})$. *Show that* $\mathcal{N}$ *and* $\mathcal{N}_1$ *are equivalent in* $E-A$.

Exercise 6.8 *Let* $\mathcal{N} \equiv (\mathcal{G}, \mathcal{D})$, *where* $\mathcal{G}$ *is a graph on* E. *Let* $\mathcal{D} = \delta_A \times \mathcal{D}_{E-A}$. *Show that the constraints as far as* $\mathbf{i}_{E-A}$, $\mathbf{v}_{E-A}$ *are concerned, are:*

i. $\mathbf{i}_{E-A}(t) \in \mathcal{V}_i(\mathcal{G} \times A),\ \forall t \in \Re$

ii. $\mathbf{v}_{E-A}(t) \in \mathcal{V}_v(\mathcal{G} \cdot A),\ \forall t \in \Re$

iii. $(\mathbf{v}_{E-A}, \mathbf{i}_{E-A}) \in \mathcal{D}_{E-A}$.

Exercise 6.9 *Let* $\mathcal{D} = \mathcal{D}_A \times \mathcal{D}_{E-A}$, *where* $\mathcal{D}_A \equiv \{(\mathbf{0}_A, \mathbf{0}_A)\}$, *i.e., devices in* A *have zero voltage across and zero current through ('nullators'). Show that, the network constraints on* $\mathbf{v}_{E-A}, \mathbf{i}_{E-A}$ *are*
$\mathbf{i}_{E-A}(t) \in \mathcal{V}_i(\mathcal{G}\ .\ A)$
$\mathbf{v}_{E-A}(t) \in \mathcal{V}_v(\mathcal{G} \times A)$
$(\mathbf{v}_{E-A}, \mathbf{i}_{E-A}) \in \mathcal{D}_{E-A}$.

Figure 6.5: Accompanied Device

Exercise 6.10 *Suppose $\mathcal{E}$ is a set of voltage source branches so that $\mathcal{G} \times (E - \mathcal{E})$ has A and B as separators. Show that $\mathcal{E}$ can be v-shifted and absorbed in the device characteristic of the remaining elements so that the resulting network has graph $\mathcal{G} \times (E - \mathcal{E}) = \mathcal{G} \times A \oplus \mathcal{G} \times B$, hinging between the graphs being permitted.*

Exercise 6.11 *Suppose $\mathcal{J}$ is a set of current branches such that $\mathcal{G} \,.\, (E - \mathcal{J})$ has A and B as separators. Show that $\mathcal{J}$ can be i-shifted and absorbed in the device characteristic of the remaining elements so that the resulting network has graph $\mathcal{G} \,.\, (E - \mathcal{J}) = \mathcal{G} \,.\, A \oplus \mathcal{G} \,.\, B$, hinging between the graphs being permitted.*

Exercise 6.12 *Suppose all sources were shifted so as to 'accompany' nonsource branches. (A nonsource branch is said to be 'accompanied' if it appears composite with a series voltage source and a current source in parallel with the series combination as in Figure 6.5). Show that the graph of the resulting network is $\mathcal{G} \,.\, (E - \mathcal{J}) \times (E - (\mathcal{J} \cup \mathcal{E}))$.*

Exercise 6.13 *Show that norators can be both current and voltage shifted while nullators cannot be current or voltage shifted.*

Exercise 6.14 *Let A be a set of norators, i.e., $\mathcal{D} = \delta_A \times \mathcal{D}_{E-A}$. Let $\mathcal{G} \,.\, (E - A)$ $(\mathcal{G} \times (E - A))$ have $B_1, \cdots, B_k$ as separators. Let $\mathcal{D}_B = \mathcal{D}_{B_1} \times \cdots \times \mathcal{D}_{B_k}$. Show that we can perform i-shift on A (v-shift on A) so that the resulting network (in which the norator currents (norator voltages) accompany the branches in E-A) is on the graph $\mathcal{G} \,.\, B_1 \oplus \cdots \oplus \mathcal{G} \,.\, B_n$ $(\mathcal{G} \times B_1 \oplus \cdots \oplus \mathcal{G} \times B_n)$ and further show that in this network the only way in which the variables in the different B_i affect each other is through the norator current variables (norator voltage variables).*

6.4 A Theorem on Topological Hybrid Analysis

In this section we discuss a method of network decomposition that is a topological generalization of hybrid analysis (i.e., analysis where unknowns involve both volt-

Figure 6.6: Networks $\mathcal{N}$, $\mathcal{N}_{AL}$ and $\mathcal{N}_{BK}$

ages and currents). Our main result is Theorem 6.4.1. This result states that the solution of a network $\mathcal{N}$ is equivalent to the simultaneous solution of two derived networks $\mathcal{N}_{AL}$ and $\mathcal{N}_{BK}$ under appropriate boundary conditions. This we prove through a sequence of lemmas. We then discuss applications of this result. Theorem 6.4.1 is a formalization of the topological intuition behind G.Kron's Diakoptics [Kron63] and its popular version due to F.H. Branin [Branin62].

6.4.1 The Networks $\mathcal{N}_{AL}$ and $\mathcal{N}_{BK}$

Definition 6.4.1 *Let $\mathcal{N} \equiv (\mathcal{G}, \mathcal{D})$ be a network. Let $E(\equiv E(\mathcal{G}))$ be partitioned into $\{A, B\}$ such that $\mathcal{D} = \mathcal{D}_A \times \mathcal{D}_B$ (i.e., the devices in A and B are decoupled from each other). Two networks $\mathcal{N}_{AL}, \mathcal{N}_{BK}$ are derived from $\mathcal{N}$ as follows:*
Let $L \subseteq B$ be such that $\mathcal{G} \cdot (E - L)$ has $A, B - L$ as separators. Let $K \subseteq A$ be such that $\mathcal{G} \times (E - K)$ has $A - K, B$ as separators. The network $\mathcal{N}_{AL}$ has graph $\mathcal{G} \times (A \cup L)$ and device characteristic $\mathcal{D}_A \times \delta_L$ while the network $\mathcal{N}_{BK}$ has graph $\mathcal{G} \cdot (B \cup K)$ and hence, device characteristic $\mathcal{D}_B \times \delta_k$.

We illustrate the above construction through an example.
Example: Consider the network $\mathcal{N}$ in Figure 6.6. The partition A, B is indicated in the figure. We will assume that A, B are decoupled in the device characteristic. For instance, there are no controlled sources with controlling branch in A and controlled branch in B.

From the figure $A \equiv \{1,2,3,4,5\}$, $B \equiv \{6,7,8,9,10,11\}$. It can be seen that when $L \equiv \{6\}$ is opened $A, B-L$ become separators and when $K \equiv \{5\}$ is contracted $A-K, B$ become separators. Other possible candidates for L are $\{7,8\}$, $\{6,7,8\}, \{6,7,8,9,10,11\}$. Note that all candidates for L need not have the same size. Indeed all of B can be L. As can be seen, $\{6\}, \{7,8\}$ are both minimal choices for L.

Other possible candidates for K are $\{3,4\}, \{3,4,5\}, \{1,2,3,4,5\}$. All candidates for K need not have the same size. Indeed all of A can be K. As can be seen, $\{5\}, \{3,4\}$ are both minimal choices for K. The network $\mathcal{N}_{AL}$ is shown in Figure 6.6 with the choice $L \equiv \{6\}$. The device(s) in L are norators but it is convenient to think of them as unknown current sources. In $\mathcal{N}_{AL}$, the devices in A are the same as in $\mathcal{N}$. The network $\mathcal{N}_{BK}$ is shown in the same figure with the choice $K = \{5\}$. The device(s) in K are norators but it is convenient to think of them as unknown voltage sources. In $\mathcal{N}_{BK}$, the devices in B are the same as in $\mathcal{N}$. Observe that L, K are common to $\mathcal{N}_{AL}, \mathcal{N}_{BK}$. As will be seen, if we solve $\mathcal{N}_{AL}, \mathcal{N}_{BK}$ simultaneously, matching i_L and matching v_K in both networks we essentially solve $\mathcal{N}$.

The Main Theorem

We now state our main result [Narayanan75], [Narayanan79].

Theorem 6.4.1 *(The $\mathcal{N}_{AL} - \mathcal{N}_{BK}$ Theorem) : The ordered pair $(\mathbf{v}(\cdot), \mathbf{i}(\cdot))$ is a solution of $\mathcal{N}$ iff there exist solutions $(\mathbf{v}_{AL}(\cdot), \mathbf{i}_{AL}(\cdot))$ of $\mathcal{N}_{AL}$ and $(\mathbf{v}_{BK}(\cdot), \mathbf{i}_{BK}(\cdot))$ of $\mathcal{N}_{BK}$ s.t.*

$$\begin{aligned}
\mathbf{v}(\cdot)/A &= \mathbf{v}_{AL}(\cdot)/A, \\
\mathbf{v}(\cdot)/(B \cup K) &= \mathbf{v}_{BK}(\cdot) \\
\mathbf{i}(\cdot)/(A \cup L) &= \mathbf{i}_{AL}(\cdot), \\
\mathbf{i}(\cdot)/B &= \mathbf{i}_{BK}(\cdot)/B.
\end{aligned}$$

We prove the theorem through a series of lemmas and their corollaries.

Lemma 6.4.1 *: Let $\mathcal{V}$ be a vector space on S. Let S be partitioned into P, Q, T s.t. $\mathcal{V} \cdot (S-P)$ has Q, T as separators. Then vector $\mathbf{x} \in \mathcal{V}^{\perp}$ iff there exist vectors $\mathbf{x}_{PQ} \in (\mathcal{V} \times (P \cup Q))^{\perp}$ and $\mathbf{x}_{PT} \in (\mathcal{V} \times (P \cup T))^{\perp}$ s.t.*
$\mathbf{x}_{PQ}/P = \mathbf{x}_{PT}/P$ and $\mathbf{x}_{PQ}/Q \oplus \mathbf{x}_{PT} = \mathbf{x}$.

Proof : We select a representative matrix $\mathbf{R}$ for $\mathcal{V}$ in which $\mathcal{V} \times (P \cup Q)$ and $\mathcal{V} \times (P \cup Q) \cdot Q$ become visible:

$$\mathbf{R} = \begin{bmatrix} \mathbf{R}_{QQ} & \mathbf{R}_{QP} & \mathbf{0} \\ \mathbf{0} & \mathbf{R}_{PP} & \mathbf{0} \\ \mathbf{R}_{TQ} & \mathbf{R}_{TP} & \mathbf{R}_{TT} \end{bmatrix}, \tag{6.4}$$

rows of $\mathbf{R}_{QQ}, \mathbf{R}_{PP}, \mathbf{R}_{TT}$ being linearly independent.

Now $\mathcal{V} \cdot (S-P)$ has Q, T as separators. Therefore, $\mathbf{R}_{TQ}$ may be taken, without loss of generality, to be $\mathbf{0}$. Hence, if $\mathbf{x} \in \mathcal{V}^{\perp}$ we have

$$\begin{bmatrix} \mathbf{R}_{QQ} & \mathbf{R}_{QP} & \mathbf{0} \\ \mathbf{0} & \mathbf{R}_{PP} & \mathbf{0} \\ \mathbf{0} & \mathbf{R}_{TP} & \mathbf{R}_{TT} \end{bmatrix} \begin{bmatrix} \mathbf{x}/Q \\ \mathbf{x}/P \\ \mathbf{x}/T \end{bmatrix} = \mathbf{0}. \tag{6.5}$$

Now $\mathbf{x}$ satisfies the Equation 6.5 iff

$$\begin{bmatrix} \mathbf{R}_{QQ} & \mathbf{R}_{QP} \\ \mathbf{0} & \mathbf{R}_{PP} \end{bmatrix} \begin{bmatrix} \mathbf{x}/Q \\ \mathbf{x}/P \end{bmatrix} = \mathbf{0} \tag{6.6}$$

and

$$\begin{bmatrix} \mathbf{R}_{PP} & \mathbf{0} \\ \mathbf{R}_{TP} & \mathbf{R}_{TT} \end{bmatrix} \begin{bmatrix} \mathbf{x}/P \\ \mathbf{x}/T \end{bmatrix} = \mathbf{0}. \tag{6.7}$$

But from the structure of $\mathbf{R}$ it is seen that the coefficient matrix for Equation 6.6 is the representative matrix for $\mathcal{V} \times (P \cup Q)$ and the coefficient matrix for Equation 6.7 is the representative matrix for $\mathcal{V} \times (P \cup T)$. Hence, $\mathbf{x} \in \mathcal{V}^{\perp}$ iff $\mathbf{x}/(P \cup Q) \in (\mathcal{V} \times (P \cup Q))^{\perp}$ and $\mathbf{x}/(P \cup T) \in (\mathcal{V} \times (P \cup T))^{\perp}$. The lemma follows.

□

We now apply the above lemma to the voltage and current spaces of graphs.

Corollary 6.4.1 *Let $\mathcal{G}$ be a graph on E. Let E be partitioned into A, B and let $L \subseteq B$ s.t. $\mathcal{G}\ .\ (E - L)$ has $A, B - L$ as separators. Then $\mathbf{i}$ is a current vector of $\mathcal{G}$ iff there exist current vectors $\mathbf{i}_{AL}, \mathbf{i}_B$ of $\mathcal{G} \times (A \cup L), \mathcal{G} \times B$, respectively such that*

$$\begin{aligned} \mathbf{i}_{AL}/L &= \mathbf{i}_B/L \text{ and} \\ \mathbf{i}_{AL}/A \oplus \mathbf{i}_B &= \mathbf{i}. \end{aligned}$$

Proof : We take $\mathcal{V}$ to be $\mathcal{V}_v(\mathcal{G})$, P to be L, Q to be A and T to be $B - L$. We use the facts that $(\mathcal{V}_v(\mathcal{G})) \times (A \cup L) = \mathcal{V}_v(\mathcal{G} \times (A \cup L))$ and $(\mathcal{V}_v(\mathcal{G})) \times B = \mathcal{V}_v(\mathcal{G} \times B)$ and the strong form of Tellegen's Theorem. The corollary then follows from Lemma 6.4.1.

□

Corollary 6.4.2 *Let $\mathcal{G}$ be a graph on E. Let E be partitioned into A, B and let $K \subseteq A$ s.t. $\mathcal{G} \times (E - K)$ has $A - K, B$ as separators. Then, $\mathbf{v}$ is a voltage vector of $\mathcal{G}$ iff there exist voltage vectors $\mathbf{v}_A, \mathbf{v}_{BK}$ of $\mathcal{G}\ .\ A$, $\mathcal{G}\ .\ (B \cup K)$ respectively such that $\mathbf{v}_A/K = \mathbf{v}_{BK}/K$ and $\mathbf{v}_A \oplus (\mathbf{v}_{BK}/B) = \mathbf{v}$.*

Proof : Take $\mathcal{V}$ to be $\mathcal{V}_i(\mathcal{G})$, P to be K, Q to be $A - K$, T to be B. We use the facts that $\mathcal{V}_i(\mathcal{G} \times (E - K)) = (\mathcal{V}_i(\mathcal{G})) \cdot (E - K)$, $(\mathcal{V}_i(\mathcal{G})) \times (A) = \mathcal{V}_i(\mathcal{G}\ .\ A)$, $(\mathcal{V}_i(\mathcal{G})) \times (B \cup K) = \mathcal{V}_i(\mathcal{G}\ .\ (B \cup K))$ and the strong form of Tellegen's Theorem. The corollary then follows from Lemma 6.4.1.

□

We need another elementary lemma which follows directly from Theorem 3.4.1.

Lemma 6.4.2 *Let $\mathcal{G}$ be on E and let E be partitioned into A, B. Let $L \subseteq B$, $K \subseteq A$. Then*

i. $\mathcal{G} \,.\, (E - L)$ *has* $A, (B - L)$ *as separators iff* $\mathcal{G} \times (A \cup L) \cdot A \cong \mathcal{G} \,.\, A$ *(i.e., both the graphs have the same current and voltage space) and*

ii. $\mathcal{G} \times (E - K)$ *has* $A - K, B$ *as separators iff* $\mathcal{G} \,.\, (B \cup K) \times B \cong \mathcal{G} \times B$.

Proof of Theorem 6.4.1: Throughout this proof we write $\mathbf{x}$ in place of $\mathbf{x}(\cdot)$ and $\mathbf{x} \in \mathcal{V}$ in place of $\mathbf{x}(t) \in \mathcal{V} \ \ \forall t \in \Re$.
An ordered pair $(\mathbf{v}, \mathbf{i})$ is a solution of $\mathcal{N}$ only if

$$\begin{aligned}(\mathbf{v}/A, \mathbf{i}/A) \in \mathcal{D}_A, (\mathbf{v}/B, \mathbf{i}/B) \in \mathcal{D}_B, \mathbf{v}/A \in \mathcal{V}_v(\mathcal{G} \,.\, A),\\ \mathbf{v}/(B \cup K) \in \mathcal{V}_v(\mathcal{G} \,.\, (B \cup K)), \mathbf{i}/(A \cup L) \in \mathcal{V}_i(\mathcal{G} \times (A \cup L)),\\ \text{and } \mathbf{i}/B \in \mathcal{V}_i(\mathcal{G} \times B).\end{aligned}$$

(Using Theorems 3.4.4 and 3.4.5).
Now by Lemma 6.4.2 we have,

$$\mathcal{V}_v(\mathcal{G} \,.\, A) = \mathcal{V}_v(\mathcal{G} \times (A \cup L) \cdot A) \tag{6.8}$$
$$\mathcal{V}_i(\mathcal{G} \times B) = \mathcal{V}_i(\mathcal{G} \,.\, (B \cup K) \times B). \tag{6.9}$$

By Theorems 3.4.4 and 3.4.5 we have

$$\mathcal{V}_v(\mathcal{G} \times (A \cup L) \cdot A) = (\mathcal{V}_v(\mathcal{G} \times (A \cup L))) \cdot A \tag{6.10}$$
$$\mathcal{V}_i(\mathcal{G} \,.\, (B \cup K) \times B) = (\mathcal{V}_i(\mathcal{G} \,.\, (B \cup K))) \cdot B. \tag{6.11}$$

Thus $\mathbf{v}/A \in \mathcal{V}_v(\mathcal{G} \,.\, A)$ iff there exists $\mathbf{v}_{AL} \in \mathcal{V}_v(\mathcal{G} \times (A \cup L))$ s.t. $\mathbf{v}_{AL}/A = \mathbf{v}/A$.
Further $\mathbf{i}/B \in \mathcal{V}_i(\mathcal{G} \times B)$ iff there exists $\mathbf{i}_{BK} \in \mathcal{V}_i(\mathcal{G} \,.\, (B \cup K))$ s.t. $\mathbf{i}_{BK}/B = \mathbf{i}/B$.
We thus see that
$(\mathbf{v}/A, \mathbf{i}/A) \in \mathcal{D}_A$, $\mathbf{v}/A \in \mathcal{V}_v(\mathcal{G} \,.\, A)$ and $\mathbf{i}/A \cup L \in \mathcal{V}_i(\mathcal{G} \times (A \cup L))$
iff there exists $(\mathbf{v}_{AL}, \mathbf{i}_{AL}) \in \mathcal{D}_A \times \delta_L$
s.t. $\mathbf{v}_{AL} \in \mathcal{V}_v(\mathcal{G} \times (A \cup L)), \mathbf{i}_{AL} \in \mathcal{V}_i(\mathcal{G} \times (A \cup L))$ and
$\mathbf{v}/A = \mathbf{v}_{AL}/A, \mathbf{i}/A \cup L = \mathbf{i}_{AL}$, i.e.,
iff there exists a solution $(\mathbf{v}_{AL}, \mathbf{i}_{AL})$ of $\mathcal{N}_{AL}$ with

$$\mathbf{v}/A = \mathbf{v}_{AL}/A, \mathbf{i}/A \cup L = \mathbf{i}_{AL}. \tag{6.12}$$

Similarly we see that
$(\mathbf{v}/B, \mathbf{i}/B) \in \mathcal{D}_B$, $\mathbf{v}/B \cup K \in \mathcal{V}_v(\mathcal{G} \,.\, (B \cup K))$, $\mathbf{i}/B \in \mathcal{V}_i(\mathcal{G} \times B)$
iff there exists a solution $(\mathbf{v}_{BK}, \mathbf{i}_{BK})$ of $\mathcal{N}_{BK}$ with

$$\mathbf{v}/B \cup K = \mathbf{v}_{BK}, \mathbf{i}/B = \mathbf{i}_{BK}/B. \tag{6.13}$$

Thus $(\mathbf{v},\mathbf{i})$ is a solution of $\mathcal{N}$ only if there exist solutions $(\mathbf{v}_{AL},\mathbf{i}_{AL}),(\mathbf{v}_{BK},\mathbf{i}_{BK})$ of $\mathcal{N}_{AL}$,$\mathcal{N}_{BK}$ respectively which satisfy Equations 6.12 and 6.13.

Conversely let $(\mathbf{v}_{AL},\mathbf{i}_{AL}),(\mathbf{v}_{BK},\mathbf{i}_{BK})$ be solutions of $\mathcal{N}_{AL}$,$\mathcal{N}_{BK}$ respectively s.t.

$$\mathbf{v}_{AL}/K = \mathbf{v}_{BK}/K$$
$$\mathbf{i}_{AL}/L = \mathbf{i}_{BK}/L.$$

We then have

$$\mathbf{v}_{AL}/A \in \mathcal{V}_v(\mathcal{G} \times (A \cup L) \cdot A)\ (= \mathcal{V}_v(\mathcal{G}\ .\ A))$$
$$\mathbf{i}_{BK}/B \in \mathcal{V}_i(\mathcal{G}.(B \cup K) \times B)\ (= \mathcal{V}_i(\mathcal{G} \times B))$$

(using Lemma 6.4.2).
By Corollary 6.4.2 it follows that
$\mathbf{v} \equiv \mathbf{v}_{AL}/A \oplus \mathbf{v}_{BK}$ belongs to $\mathcal{V}_v(\mathcal{G})$ and by Corollary 6.4.1 it follows that
$\mathbf{i} \equiv \mathbf{i}_{BK}/B \oplus \mathbf{i}_{AL}$ belongs to $\mathcal{V}_i(\mathcal{G})$.
Since $(\mathbf{v}_{AL},\mathbf{i}_{AL}) \in \mathcal{D}_A \times \delta_L$ and $(\mathbf{v}_{BK},\mathbf{i}_{BK}) \in \mathcal{D}_B \times \delta_k$ it follows that
$(\mathbf{v}_{AL}/A \oplus \mathbf{v}_{BK}/B,\ \mathbf{i}_{AL}/A \oplus \mathbf{i}_{BK}/B) \in \mathcal{D}_A \times \mathcal{D}_B$.
Hence, $(\mathbf{v},\mathbf{i})$ is a solution of $\mathcal{N}$ such that Equations 6.12 and 6.13 hold.

□

Exercise 6.15 *State and prove the dual of Lemma 6.4.1.*

Exercise 6.16 *Theorem 6.4.1 involves matching both boundary voltages and boundary currents. Is it possible to get a similar topological result where the solution of $\mathcal{N}$ is stated to be equivalent to the solution of derived networks with (i) only boundary voltages matched (ii) only boundary currents matched?*

6.5 Structure of Constraints and Optimization

In this section we consider issues relevant to the application of the $\mathcal{N}_{AL} - \mathcal{N}_{BK}$ method. We begin by showing that the essential structure of the constraints in this method is controlled by the graphs $\mathcal{G}\ .\ A$ and $\mathcal{G} \times B$. Next we consider the problem of selecting minimal L and K. For linear networks we show that the solution of the network $\mathcal{N}$ can be achieved essentially by solving $(1+\xi(A,B))$ times a network with the same device characteristic but with graph $\mathcal{G}\ .\ A \oplus \mathcal{G} \times B$. We then discuss how to use this method to write hybrid equations for the network in a way which, without compromising on sparsity, exploits the advantages of good structure.

6.5.1 Essential Structure of the Constraints

We remind the reader that if a device is voltage (current) unconstrained then as far as KVE (KCE) is concerned we may open (short) the corresponding branch (Theorem 6.3.1).

The constraints for variables $\mathbf{v}_A, \mathbf{i}_A, \mathbf{i}_L$ in $\mathcal{N}_{AL}$ are

$$\begin{aligned}
(\mathbf{v}_A, \mathbf{i}_A) &\in \mathcal{D}_A, && (6.14)\\
(\mathbf{A}_{rA} \vdots \mathbf{A}_{rL})\begin{matrix}\mathbf{i}_A\\ \mathbf{i}_L\end{matrix} &= \mathbf{0} \qquad (KCE \ for \ \mathcal{G} \times (A \cup L)) && (6.15)\\
(\mathbf{A}_{rA}{}^T)\mathbf{v}_{nA} - \mathbf{v}_A &= \mathbf{0}. \qquad (KVL \ for \ \mathcal{G} \times (A \cup L) \cdot A), && (6.16)
\end{aligned}$$

where $(\mathbf{A}_{rA} \vdots \mathbf{A}_{rL})$ is a reduced incidence matrix of $\mathcal{G} \times (A \cup L)$. Clearly the rows of matrix $\mathbf{A}_{rA}$ span $\mathcal{V}_v((\mathcal{G} \times (A \cup L)) \cdot A)$ $(= \mathcal{V}_v(\mathcal{G} \times (A \cup L) \cdot A))$. We show later that if L is chosen minimally then $r(\mathcal{G} \times (A \cup L)) = r(\mathcal{G} \times (A \cup L) \cdot A)$. So in this case $\mathbf{A}_{rA}$ is a reduced incidence matrix for $\mathcal{G} \times (A \cup L) \cdot \ A$. Thus, if we were to shift the terms corresponding to $\mathbf{i}_L$ to the right, the structure of the constraints, as far as LHS is concerned is controlled by $\mathcal{G} \times (A \cup L) \cdot A$, i.e., by $\mathcal{G} \ . \ A$ (using Lemma 6.4.2).
The constraints for variables $\mathbf{v}_B, \mathbf{i}_B, \mathbf{v}_K$ in $\mathcal{N}_{BK}$ are

$$\begin{aligned}
(\mathbf{v}_B, \mathbf{i}_B) &\in \mathcal{D}_B, && (6.17)\\
(\mathbf{B}_B \vdots \mathbf{B}_K)\begin{matrix}\mathbf{v}_B\\ \mathbf{v}_K\end{matrix} &= \mathbf{0} \qquad (KVE \ for \ \mathcal{G} \ . \ (B \cup K)) && (6.18)\\
\mathbf{B}_B{}^T\mathbf{i}_{lB} - \mathbf{i}_B &= \mathbf{0}. \qquad (KCL \ for \ \mathcal{G} \ . \ (B \cup K) \times B) && (6.19)
\end{aligned}$$

The rows of matrix $\mathbf{B}_B$ span $\mathcal{V}_i(\mathcal{G} \ . \ (B \cup K)) \cdot B$ $(= \mathcal{V}_i(\mathcal{G} \ . \ (B \cup K) \times B))$
We show later that if K is chosen minimally $\nu(\mathcal{G} \ . \ (B \cup K)) = \nu(\mathcal{G} \ . \ (B \cup K) \times \ B)$. So in this case, $\mathbf{B}_B$ is a representative matrix for $\mathcal{V}_i(\mathcal{G} \ . \ (B \cup K) \times B)$. Thus, if we were to shift the terms corresponding to $\mathbf{v}_K$ to the right the structure of the constraints, as far as LHS is concerned, is controlled by $\mathcal{G} \ . \ (B \cup K) \times B$, i.e., by $\mathcal{G} \times B$ (using Lemma 6.4.2).

6.5.2 Selection of Minimal L and K

In the following pages we give simple necessary and sufficient conditions for the choice of minimal L and K in order that $\mathcal{G} \ . \ (E - L)$ has $A, B - L$ as separators and $\mathcal{G} \times (E - K)$ has $B, A - K$ as separators. We do not attempt to (globally) minimize the size of L and K. As we shall show later this problem is not relevant to our development.

Theorem 6.5.1 *Let $\mathcal{G} \ . \ (E - L)$ have $A, B - L$ as separators.
The subset L of B is minimal with respect to the property that $\mathcal{G} \ . \ (E - L)$ has $A, B - L$ as separators iff both the following conditions are satisfied.*

i. $\mathcal{G} \times (A \cup L)$ *has no self loops in* L, $\quad (*)$

ii. $r(\mathcal{G} \times (A \cup L)) = r(\mathcal{G} \ . \ A)$. $\quad (**)$

Proof : By Lemma 6.4.2, the minimality of L can equivalently be characterized with respect to the property that $\mathcal{G} \times (A \cup L) \cdot A \cong \mathcal{G} \ . \ A$.

Necessity
i. If $e \in L$ and is a self loop of $\mathcal{G} \times (A \cup L)$ then $\mathcal{G} \times (A \cup L) \times (A \cup (L-e)) = \mathcal{G} \times (A \cup L) \cdot (A \cup (L-e))$. Now the LHS is 2-isomorphic to $\mathcal{G} \times (A \cup (L-e))$. Hence, $\mathcal{G} \times (A \cup (L-e)) \cdot A \cong \mathcal{G} \times (A \cup L) \cdot A \cong \mathcal{G} \, . \, A$. Thus, L is not minimal.

ii. Suppose $r(\mathcal{G} \times (A \cup L)) > r(\mathcal{G} \, . \, A)$. Then, $r(\mathcal{G} \times (A \cup L)) > r(\mathcal{G} \times (A \cup L) \cdot A)$ and L contains a cutset C of $\mathcal{G} \times (A \cup L)$ and therefore of $\mathcal{G}$. Let $e \in C$. If we open the branches $L - e$ from the graph $\mathcal{G}$, it is clear that $C \cap (A \cup e)$ must contain a cutset of $\mathcal{G} \, . \, (E - (L - e))$, i.e., e is a coloop of $\mathcal{G} \, . \, (E - (L - e))$. We claim that $A, B - (L - e)$ are separators of this graph. Suppose otherwise. Let T be a circuit of $\mathcal{G} \, . \, (E - (L - e))$ which has nonvoid intersection with both A and $B - (L - e)$. Since e is a coloop of this graph, $e \notin T$. Then this circuit would remain in the graph $\mathcal{G} \, . \, (E - L)$ $(\cong \mathcal{G} \, . \, (E - (L - e)) \cdot (E - L))$, which contradicts the fact that A is a separator of the latter graph. We conclude that there is no such circuit in $\mathcal{G} \, . \, (E - (L - e))$ and therefore $(A, B - (L - e))$ are separators of this graph. Thus, L is not minimal.

Sufficiency
Suppose the conditions $(*)$ and $(**)$ are satisfied and yet L is not minimal. Then there exists $T \subseteq L$ s.t. $\mathcal{G} \times (A \cup (L-T)) \cdot A \cong \mathcal{G} \, . \, A$. Now, $r(\mathcal{G} \times (A \cup L)) = r(\mathcal{G} \times (A \cup (L-T))) + r(\mathcal{G} \times (A \cup L) \cdot T)$. Since L has no self loops in $\mathcal{G} \times (A \cup L)$, we must have $r(\mathcal{G} \times (A \cup L) \cdot T) \geq 1$. But this means that $r(\mathcal{G} \times (A \cup (L-T))) < r(\mathcal{G} \times (A \cup L))$ $= r(\mathcal{G} \, . \, A)$. But then $r(\mathcal{G} \times (A \cup (L-T)) \cdot A) \leq r(\mathcal{G} \times (A \cup (L-T))) < r(\mathcal{G} \, . \, A)$. We conclude that $\mathcal{G} \times (A \cup (L-T)) \cdot A$ cannot be 2-isomorphic to $\mathcal{G} \, . \, A$.

□

The conditions for minimality of K and the proof of these conditions are dual to those for the minimality of L, i.e., we make the following interchanges line by line: contraction with restriction, circuit with cutset, self loop with coloop, rank with nullity. We therefore, omit the proof of the following result.

Theorem 6.5.2 *The subset K of A is minimal with respect to the property that $\mathcal{G} \times (E - K)$ has $B - K, A$ as separators iff*

i. $\mathcal{G} \, . \, (B \cup K)$ has no coloops in K,

ii. $\nu(\mathcal{G} \, . \, (B \cup K)) = \nu(\mathcal{G} \times B)$.

Algorithms for Minimal L and K

We now give efficient algorithms for the construction of minimal L and K. We first give an algorithm which is easy to justify. Then refine it to a linear time algorithm.

ALGORITHM 6.1 First Algorithm for Min L,K
INPUT *Graph $\mathcal{G}$ with partition $A, E - A$ of edge set E.*

OUTPUT *Minimal sets L and K.*

STEP 1 *Select forest f_A of $\mathcal{G}\ .\ A$. Extend it to a forest f of $\mathcal{G}$.*

STEP 2 *Take L to be the subset of all coforest edges in B in whose f-circuits some edges of f_A lie. Take K to be the subset of all forest edges in A in whose f-cutsets some edges of $B - f$ lie (equivalently which lie in f-circuits of L).*

STOP

Justification for the Algorithm

The f-cutset matrix with respect to the forest f in the above algorithm has the following structure:

$$\begin{array}{cc} & \begin{array}{cccccc} f_A - K & K & \bar{f}_A & f_B & L & \bar{f}_B - L \end{array} \\ \mathbf{Q}_f = & \left[\begin{array}{cccccc} \mathbf{I} & \mathbf{0} & \mathbf{Q}_{AA} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{Q}_{KA} & \mathbf{0} & \mathbf{Q}_{KL} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{Q}_{BL} & \mathbf{Q}_{BB} \end{array}\right] \end{array} \tag{6.20}$$

We denote the intersection of $E - f$ with A, B by $\bar{f}_A, \bar{f}_B$ respectively. For convenience we have shortened the notation for the submatrix denoted by $\mathbf{Q}_{AA}, \mathbf{Q}_{BB}$. The indices for these submatrices should really be $(f_A - K)\bar{f}_A$ and $(f_B)(\bar{f}_B - L)$. Note that $\mathbf{Q_{BA}} = \mathbf{0}$, since f_A is a forest of $\mathcal{G}\ .\ A$. Note also that the matrix $\mathbf{Q}_{KL}$ has no zero columns. Otherwise the corresponding element e will not be in the f-cutset of any of the branches in K (equivalently the f-circuit of e would not intersect f_A). Similarly $\mathbf{Q}_{KL}$ has no zero rows. Otherwise the corresponding branch in K will not lie in the f-circuit of any branch in L. An application of Theorem 3.4.6 tells us that $(\mathcal{V}_v(\mathcal{G}))\cdot(E-L)\ (= \mathcal{V}_v(\mathcal{G}\ .\ (E-L)))$ has a representative matrix which can be obtained from $\mathbf{Q}_f$ by deleting the columns L. An examination of the representative matrix reveals that $A, B - L$ are separators for this space (equivalently for $(\mathcal{G}\ .\ (E-L))$. Further, $(\mathcal{V}_v(\mathcal{G})) \times (E-K)\ (= \mathcal{V}_v(\mathcal{G} \times (E-K)))$ has a representative matrix which can be obtained from $\mathbf{Q}_f$ by deleting the rows and columns corresponding to K. An examination of this representative matrix reveals that $A - K, B$ are separators for this space (equivalently for $(\mathcal{G} \times (E-K))$.
We next prove minimality of L and K. Another application of Theorem 3.4.6 gives the following matrix $\mathbf{Q}_1$ as the representative matrix for $\mathcal{V}_v(\mathcal{G} \times (A \cup L))$

$$\begin{array}{cc} & \begin{array}{cccc} f_A - K & K & \bar{f}_A & L \end{array} \\ \mathbf{Q}_1 = & \left[\begin{array}{cccc} \mathbf{I} & \mathbf{0} & \mathbf{Q}_{AA} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{Q}_{KA} & \mathbf{Q}_{KL} \end{array}\right] \end{array} \tag{6.21}$$

Since $\mathbf{Q}_{KL}$ has no zero columns, no edge in L is a self loop in $\mathcal{G} \times (A \cup L)$. Deletion of columns L gives the representative matrix for $\mathcal{V}_v(\mathcal{G} \times (A \cup L)\cdot A)(= \mathcal{V}_v(\mathcal{G}\ .\ A))$. It is clear that this latter matrix and $\mathbf{Q}_1$ have the same rank (because of the presence of the identity matrix). Similarly we can see that the following matrix $\mathbf{Q}_2$ is the representative matrix for $\mathcal{V}_v(\mathcal{G} \,.\, (B \cup K))$.

$$\mathbf{Q}_2 = \begin{array}{cccc} K & f_B & L & \bar{f}_B - L \\ \end{array}$$

$$\mathbf{Q}_2 = \left[\begin{array}{cccc} \mathbf{I} & \mathbf{0} & \mathbf{Q}_{KL} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{Q}_{BL} & \mathbf{Q}_{BB} \end{array} \right] \qquad (6.22)$$

Since $\mathbf{Q}_{KL}$ has no zero rows, no edge in K is a coloop in $\mathcal{G} \,.\, (B \cup K)$. Deletion of the rows and columns corresponding to K gives the representative matrix for $\mathcal{V}_v(\mathcal{G} \,.\, ((B \cup K)) \times B)(= \mathcal{V}_v(\mathcal{G} \times B))$. This matrix and $\mathbf{Q}_2$ have the same nullity because of the complement of the identity matrix columns being the same. We thus see that L and K satisfy the conditions of minimality in Theorems 6.5.1 and 6.5.2.

We next refine the above algorithm.

ALGORITHM 6.2 Fast Algorithm for Min L,K.

INPUT *Graph $\mathcal{G}$ with partition $A, E - A$ of edge set E.*

OUTPUT *Minimal sets L and K.*

STEP 1 *Select a forest f_A of $\mathcal{G} \,.\, A$. Extend it to a forest f of $\mathcal{G}$.*

STEP 2 *Let $f_B \equiv f \cap B$. Construct $\mathcal{G} \times (E - f_B)$. Take L to be the subset of non selfloops contained in B in this graph.*

STEP 3 *Construct $\mathcal{G} \,.\, (f_A \cup B)$. Take K to be the subset of noncoloops contained in A in this graph.*

STOP

Justification for Algorithm 6.2

In both the Algorithms 6.1 and 6.2 the sets f_A, f_B are constructed identically. We will show that the sets L, K selected according to Algorithm 6.1 are identical to the corresponding sets selected according to Algorithm 6.2. Select L, K according to Algorithm 6.1. To avoid confusion let us call these sets L', K'. In the graph $\mathcal{G} \times (E - f_B), f_A$ is a forest. Now the f-cutsets of branches in K remain as f-cutsets in $\mathcal{G} \times (E - f_B)$. The union of these f-cutsets intersects $B - f_B$ in L'. No branch in L' can therefore be a selfloop. The remaining branches in $B - f_B$ lie in no f-cutset. Hence, they must be selfloops. Thus, L' would also be output as the set L by Algorithm 6.2. In the graph $\mathcal{G} \,.\, (f_A \cup B)$, f is still a forest. The f-circuits of branches in L' remain as such in $\mathcal{G} \,.\, (f_A \cup B)$. The union of these f-circuits intersects f_A in K'. No branch in K' can, therefore, be a coloop. The remaining branches in f_A lie in no f-circuit. Hence, they must be coloops. Thus, the set K' would also be output as the set K by Algorithm 6.2.

Complexity of Algorithm 6.2

Building the forest f is $O(\mid V \mid + \mid E \mid)$. Building graphs $\mathcal{G} \times (E - f_B)$, $\mathcal{G} \ . \ (f_A \cup B)$ is $O(\mid V \mid + \mid E \mid)$. Selection of self loops and coloops in these graphs in $O(\mid E \mid)$. Thus the overall complexity is $O(\mid V \mid + \mid E \mid)$.
Our algorithms help us to find minimal L and K but the sets may not be minimum in size. It is, however, not necessary to find such minimum sets for, as we show below, as long as L and K are minimal, we can work, in place of $\mathbf{i}(\cdot)/L$, $\mathbf{v}(\cdot)/K$, with $\mathbf{Q}_L(\mathbf{i}(\cdot)/L)$, $\mathbf{B}_K(\mathbf{v}(\cdot)/K)$, where $\mathbf{Q}_L$, $\mathbf{B}_K$ are representative matrices of $\mathcal{V}_v(\mathcal{G} \times (A \cup L) \cdot L), \mathcal{V}_i(\mathcal{G} \ . \ (B \cup K) \times K$ respectively. This will allow us to reduce the number of current and voltage 'boundary' variables to $\xi(A, B) \equiv$ $r(\mathcal{G} \ . \ A) - r(\mathcal{G} \times A) = r(\mathcal{V}_v(\mathcal{G} \times (A \cup L) \cdot L)) = r(\mathcal{V}_i(\mathcal{G} \ . \ (B \cup K) \times K))$.

Theorem 6.5.3 *In Theorem 6.4.1 the conditions*

$$\begin{aligned} \mathbf{v}(\cdot)/A &= \mathbf{v}_{AL}(\cdot)/A, \\ \mathbf{v}(\cdot)/(B \cup K) &= \mathbf{v}_{BK}(\cdot) \\ \mathbf{i}(\cdot)/(A \cup L) &= \mathbf{i}_{AL}(\cdot), \\ \mathbf{i}(\cdot)/B &= \mathbf{i}_{BK}(\cdot)/B. \end{aligned}$$

can be replaced by

$$\begin{aligned} \mathbf{v}(\cdot)/A &= \mathbf{v}_{AL}(\cdot)/A, \\ \mathbf{v}(\cdot)/B &= \mathbf{v}_{BK}(\cdot)/B \\ \mathbf{i}(\cdot)/A &= \mathbf{i}_{AL}(\cdot)/A, \\ \mathbf{i}(\cdot)/B &= \mathbf{i}_{BK}(\cdot)/B \\ \mathbf{Q}_L(\mathbf{i}(\cdot)/L) &= \mathbf{Q}_L(\mathbf{i}_{AL}(\cdot)/L) \\ \mathbf{B}_K(\mathbf{v}(\cdot)/K) &= \mathbf{B}_K(\mathbf{v}_{BK}(\cdot)/K), \end{aligned}$$

where $\mathbf{Q}_L$ *is a representative matrix of* $\mathcal{V}_v(\mathcal{G} \times (A \cup L) \cdot L)$, $\mathbf{B}_K$ *is a representative matrix of* $\mathcal{V}_i(\mathcal{G} \ . \ (B \cup K) \times K)$.

Proof : Consider the constraints of $\mathcal{N}_{AL}$ as far as $\mathbf{v}_A, \mathbf{i}_A, \mathbf{i}_L$ are concerned in Subsection 6.5.1. We notice that the only constraints involving $\mathbf{i}_L$ are the KCE of $\mathcal{G} \times (A \cup L)$:

$$\left(\ \mathbf{A}_{rA} \mathbf{A}_{rL} \ \right) \begin{bmatrix} \mathbf{i}_A \\ \mathbf{i}_L \end{bmatrix} = \mathbf{0}.$$

It is clear that $\mathbf{i}_L$ can be replaced by $\mathbf{i}_L{}'$ so long as $\mathbf{A}_{rL}\mathbf{i}_L = \mathbf{A}_{rL}\mathbf{i}_L{}'$. The rows of $\mathbf{A}_{rL}$ span $\mathcal{V}_v(\mathcal{G} \times (A \cup L) \cdot L)$. This proves the part of the statement concerning $\mathbf{i}_L$. Similarly the constraints of $\mathcal{N}_{BK}$ as far as $\mathbf{v}_B, \mathbf{i}_B, \mathbf{v}_K$ are concerned (in the same subsection) involve $\mathbf{v}_K$ only in the KVE of $\mathcal{G} \ . \ (B \cup K)$:

$$\left(\ \mathbf{B}_K \mathbf{B}_B \ \right) \begin{bmatrix} \mathbf{v}_K \\ \mathbf{v}_B \end{bmatrix} = \mathbf{0}.$$

It is clear that $\mathbf{v}_K$ can be replaced by $\mathbf{v}_K{}'$ so long as $\mathbf{B}_K\mathbf{v}_K = \mathbf{B}_K\mathbf{v}_K{}'$. The rows of $\mathbf{B}_K$ span $\mathcal{V}_i(\mathcal{G} \ . \ (B \cup K) \times K)$. This proves the part of the statement concerning $\mathbf{v}_K$.

□

6.5.3 Solution of Linear Networks by Topological Hybrid Analysis

The $\mathcal{N}_{AL}, \mathcal{N}_{BK}$ Theorem can be used to alter the topology of the network during analysis at a certain cost. Below we show that by using this method a linear network $\mathcal{N}$ can be solved as though it has the topology $\mathcal{G} \cdot A \oplus \mathcal{G} \times B$ but at the cost of solving it $(1 + \xi(A, B))$ times, where $\xi(A, B) \equiv r(\mathcal{G} \cdot A) - r(\mathcal{G} \times A)$.

Let L and K be chosen minimally. In the network $\mathcal{N}_{AL}$ the devices in L are norators. However, we are interested in the variables $\mathbf{i}_L$. Hence, we write down the constraints for $\mathbf{v}_A, \mathbf{i}_A, \mathbf{i}_L$ in $\mathcal{N}_{AL}$ using the notation of Subsection 6.5.1. By Theorem 6.3.1 these are

$$\begin{pmatrix} \mathbf{A}_{rA} & \mathbf{A}_{rL} \end{pmatrix} \begin{bmatrix} \mathbf{i}_A \\ \mathbf{i}_L \end{bmatrix} = \mathbf{0}. \tag{6.23}$$

$$\begin{pmatrix} \mathbf{M}_A & \mathbf{N}_A \end{pmatrix} \begin{bmatrix} \mathbf{i}_A \\ \mathbf{v}_A \end{bmatrix} = \mathbf{s}_A \tag{6.24}$$

$$\mathbf{A}_{rA}{}^T \mathbf{v}_{nA} - \mathbf{v}_A = \mathbf{0}. \tag{6.25}$$

Let $\mathbf{A}_{rLt}$ be the submatrix of $\mathbf{A}_{rL}$ composed of a maximal linearly independent subset of columns of $\mathbf{A}_{rL}$ (i.e., L_t is a forest of $\mathcal{G} \times (A \cup L) \cdot L$). Then $\mathbf{A}_{rL} = \mathbf{A}_{rLt}\mathbf{P_L}$ for some matrix $\mathbf{P_L}$.
So $\mathbf{A}_{rL}\mathbf{i}_L = \mathbf{A}_{rLt}(\mathbf{P}_L\mathbf{i}_L)$. Let $\mathbf{i}'_{L_t} \equiv \mathbf{P}_L\mathbf{i}_L$. Hence the above KCE conditions can be rewritten as

$$\begin{pmatrix} \mathbf{A}_{rA} & \mathbf{A}_{rLt} \end{pmatrix} \begin{bmatrix} \mathbf{i}_A \\ \mathbf{i}'_{L_t} \end{bmatrix} = \mathbf{0}.$$

We assume that the network can be solved uniquely for arbitrary values of $\mathbf{i}_L$ and therefore of $\mathbf{i}'_{L_t}$. We can solve this network repeatedly for the following conditions:

i. Keep the sources $\mathbf{s}_A$ active, set all $\mathbf{i}'_{L_t}$ to zero

ii. Set the sources $\mathbf{s}_A$ to zero and let $\mathbf{i}'_{L_t}$ take values as the columns of the unit matrix of size $\mid L_t \mid$.

Thus by solving the network for $(1+ \mid L_t \mid)$ source conditions we can write

$$\begin{bmatrix} \mathbf{v}_A \\ \mathbf{i}_A \end{bmatrix} = \begin{bmatrix} \mathbf{H}_{vA} \\ \mathbf{H}_{iA} \end{bmatrix} \mathbf{i}'_{L_t} + \begin{bmatrix} \mathbf{v}_A{}^o \\ \mathbf{i}_A{}^o \end{bmatrix} \tag{6.26}$$

We write the constraints for $\mathcal{N}_{BK}$ as far as the variables $\mathbf{v}_B$, $\mathbf{i}_B$, $\mathbf{v}_K$ are concerned in the form (once again using the notation of Subsection 6.5.1)

$$\begin{pmatrix} \mathbf{M}_B & \mathbf{N}_B \end{pmatrix} \begin{bmatrix} \mathbf{i}_B \\ \mathbf{v}_B \end{bmatrix} = \mathbf{s}_B, \tag{6.27}$$

$$\mathbf{B}_B{}^T \mathbf{i}_{l_B} - \mathbf{i}_B = \mathbf{0}, \tag{6.28}$$

$$\begin{pmatrix} \mathbf{B}_B & \mathbf{B}_K \end{pmatrix} \begin{bmatrix} \mathbf{v}_B \\ \mathbf{v}_K \end{bmatrix} = \mathbf{0}, \tag{6.29}$$

Let $\mathbf{B}_{Kc}$ be composed of a maximal linearly independent subset of columns of $\mathbf{B}_K$ (i.e., K_c is a coforest of $\mathcal{G} \,.\, (B \cup K) \times K$).
Let $\mathbf{B}_K = \mathbf{B}_{Kc}\mathbf{P}_K$. So $\mathbf{B}_K\mathbf{v}_K = \mathbf{B}_{Kc}(\mathbf{P}_k\mathbf{v}_K)$. Let $\mathbf{v}'_{K_c} \equiv \mathbf{P}_k\mathbf{v}_K$. Then the above KVE conditions can be rewritten as

$$\begin{pmatrix} \mathbf{B}_B & \mathbf{B}_{Kc} \end{pmatrix} \begin{bmatrix} \mathbf{v}_B \\ \mathbf{v}'_{K_c} \end{bmatrix} = \mathbf{0}.$$

We assume that the network can be solved uniquely for arbitrary values of $\mathbf{v}_K$ and therefore of $\mathbf{v}'_{K_c}$. We can solve this network repeatedly for the following conditions:

i. Keep the sources $\mathbf{s}_B$ active, set all $\mathbf{v}'_{K_c}$ to zero

ii. Set the sources $\mathbf{s}_B$ to zero and let $\mathbf{v}'_{K_c}$ take values as the columns of the unit matrix of size $\mid K_c \mid$.

Thus by solving the network for $(1+ \mid K_c \mid)$ source conditions we can write

$$\begin{bmatrix} \mathbf{v}_B \\ \mathbf{i}_B \end{bmatrix} = \begin{bmatrix} \mathbf{H}_{vB} \\ \mathbf{H}_{iB} \end{bmatrix} \mathbf{v}'_{K_c} + \begin{bmatrix} \mathbf{v}_B{}^o \\ \mathbf{i}_B{}^o \end{bmatrix} \tag{6.30}$$

From Equations 6.26, 6.30 we can extract the following:

$$\mathbf{v}_K = \mathbf{H}_{vK}\mathbf{i}'_{L_t} + \mathbf{v}_K{}^o \tag{6.31}$$

$$\mathbf{i}_L = \mathbf{H}_{iL}\mathbf{v}'_{K_c} + \mathbf{i}_L{}^o \tag{6.32}$$

From which we get by premultiplication respectively by $\mathbf{B}_K$ and $\mathbf{A}_{rL}$

$$\mathbf{B}_K\mathbf{v}_K = \mathbf{B}_K\mathbf{H}_{vK}\mathbf{i}'_{L_t} + \mathbf{B}_K\mathbf{v}_K{}^o \tag{6.33}$$

$$\mathbf{A}_{rL}\mathbf{i}_L = \mathbf{A}_{rL}\mathbf{H}_{iL}\mathbf{v}'_{K_c} + \mathbf{A}_{rL}\mathbf{i}_L{}^o \tag{6.34}$$

But

$$\mathbf{B}_K\mathbf{v}_K = \mathbf{B}_{K_c}\mathbf{v}'_{K_c}$$

and

$$\mathbf{A}_{rL}\mathbf{i}_L = \mathbf{A}_{rL_t}\mathbf{i}'_{L_t}.$$

So we get

$$\begin{pmatrix} \mathbf{B}_{Kc} & -\mathbf{B}_K\mathbf{H}_{vK} \\ -\mathbf{A}_{rL}\mathbf{H}_{iL} & \mathbf{A}_{rLt} \end{pmatrix} \begin{bmatrix} \mathbf{v}'_{K_c} \\ \mathbf{i}'_{L_t} \end{bmatrix} = \begin{pmatrix} \mathbf{B}_K\mathbf{v}_K{}^o \\ \mathbf{A}_{rL}\mathbf{i}_L{}^o \end{pmatrix} \tag{6.35}$$

If the given network has a unique solution the above equations also have a unique solution from which we get values of $\mathbf{v}'_{K_c}$ and $\mathbf{i}'_{L_t}$. Substitution of these values in Equations 6.26 and 6.30 yields the solution $(\mathbf{v},\mathbf{i})$. Observe that, provided L, K

have been chosen minimally,

$$\begin{aligned} \mid L_t \mid &= r(\mathcal{G} \times (A \cup L) \cdot L) \\ &= r(\mathcal{G} \times (A \cup L)) - r(\mathcal{G} \times (A \cup L) \times A) \\ &= r(\mathcal{G} \ . \ A) - r(\mathcal{G} \times A) \\ &= \xi(A, B) \end{aligned} \tag{6.36}$$

and

$$\begin{aligned} \mid K_c \mid &= \nu(\mathcal{G} \ . \ (B \cup K) \times K) \\ &= \nu(\mathcal{G} \ . \ (B \cup K)) - \nu(\mathcal{G} \ . \ (B \cup K) \cdot B) \\ &= \nu(\mathcal{G} \times B) - \nu(\mathcal{G} \ . \ B) \\ &= r(\mathcal{G} \ . \ A) - r(\mathcal{G} \times A) \\ &= \xi(A, B). \end{aligned} \tag{6.37}$$

Thus, by the above method we need to solve the linear network on graph $\mathcal{G} \ . \ A \oplus \mathcal{G} \times B$ for $(\xi(A,B)+1)$ source conditions.(The effort required to solve Equation 6.35 can usually be neglected in comparison, since $\xi(A,B)$ would be much smaller than $r(\mathcal{G} \cdot A), \nu(\mathcal{G} \times B)$ if A, B have been properly chosen.) The value $\xi(A,B)$ represents the number of essential linkages between A and B.

6.5.4 Decomposition procedure for $\mathcal{N}_{AL}, \mathcal{N}_{BK}$

In practice the partition A, B would have been chosen so that $\mathcal{G} \ . \ A \oplus \mathcal{G} \times B$ has several 2-connected components which are decoupled in the device characteristic of the network. However, this structure is revealed only when the 'artificial' sources in $\mathbf{i}_L, \mathbf{v}_K$ are set to zero. It is better therefore, that these sources are (current or voltage) shifted in such a way that the 2-connected components of the resulting network correspond to the 2-connected components of $\mathcal{G} \ . \ A \oplus \mathcal{G} \times B$.

When a connected network has more than one 2-connected components which are decoupled in the device characteristic, it is equivalent as far as **branch** voltages and currents are concerned to the network obtained by cutting it at the hinges separating the 2-connected components. While writing equations it is better to separate them in this manner. This would ensure that the constraints appear in decoupled blocks.

The following procedure allows us to convert the problem of solving $\mathcal{N}_{AL}, \mathcal{N}_{BK}$ to one of solving subnetworks each on a 2-connected component of $\mathcal{G} \ . \ A$ or $\mathcal{G} \times B$. Observe that the procedure does not depend on the type of devices in the network.

STEP 1: Select a forest L_t of $\mathcal{G} \times (A \cup L) \cdot L$. Current shift the unknown current sources in $L - L_t$ in the network $\mathcal{N}_{AL}$ into L_t. The resulting current sources $\mathbf{i}'_{L_t}$ are appropriate linear combinations of the current sources $\mathbf{i}_L$.

STEP 2: Current shift each of the current sources $\mathbf{i}'_{L_t}$ across edges of paths within $\mathcal{G} \ . \ A$ such that every shifted source is across vertices within the same 2-connected

Figure 6.7: To illustrate decomposition of $\mathcal{N}_{AL}$

component. Cut the network at the hinges so that the resulting network has as its connected components, the 2-connected components of $\mathcal{G} \cdot A$.

We thus have a number of networks on the 2-connected components of $\mathcal{G} \cdot A$. Figure 6.7 illustrates the procedure. The unknown current sources i_1, i_2, i_3 make up $\mathbf{i}_L$.

STEP 3: Select a coforest K_c of $\mathcal{G} \cdot (B \cup K) \times K$. The branches in $K - K_c$ form a forest of $\mathcal{G} \cdot (B \cup K) \times K$. The fundamental cutsets of this forest remain as cutsets even in $\mathcal{G} \cdot (B \cup K)$. In each of these cutsets do the following: If $e \in K - K_c$, shift v_e into the remaining branches of the cutset while contracting e. Thus, v-shift all the unknown voltage sources of $K - K_c$ in the network $\mathcal{N}_{BK}$ into K_c. The resulting voltage sources $\mathbf{v'}_{K_c}$ are an appropriate linear combination of the voltage sources $\mathbf{v}_K$.

STEP 4: Voltage shift each of the voltage sources $\mathbf{v'}_{K_c}$ so that they move to positions in series with edges of $\mathcal{G} \times B$. Every shifted source now lies entirely within a 2-connected component of $\mathcal{G} \times B$. Cut the network at the hinges so that these 2-connected components of $\mathcal{G} \times B$ become connected components of the resulting network. The resulting network can therefore be thought of as a number of networks on the 2-connected components of $\mathcal{G} \times B$. Figure 6.8 illustrates the

Figure 6.8: To illustrate decomposition of $\mathcal{N}_{BK}$

procedure. The unknown voltage sources v_1, v_2, v_3 make up $\mathbf{v}_K$. Thus, at the end of the procedure we have networks on 2-connected components of $\mathcal{G} \ . \ A \oplus \mathcal{G} \times B$ linked by the variables in $\mathbf{i}'_{L_t}, \mathbf{v}'_{K_c}$.

6.5.5 Hybrid Analysis Equations for Linear Networks

An immediate application of the $\mathcal{N}_{AL} - \mathcal{N}_{BK}$ method is in the convenient formulation of hybrid analysis equations. We assume $\mathcal{D}_A, \mathcal{D}_B$ have the form

$$\mathbf{i}_A - \mathbf{G}\mathbf{v}_A = \mathbf{s}_A \tag{6.38}$$

$$\mathbf{v}_B - \mathbf{R}\mathbf{i}_B = \mathbf{s}_B \tag{6.39}$$

We now write nodal type equations for $\mathcal{N}_{AL}$ and loop type equations for $\mathcal{N}_{BK}$. First using the equations for the variables $\mathbf{v}_A, \mathbf{i}_A, \mathbf{i}_L$ in Subsection 6.5.1 we get

$$\mathbf{A}_{rA}\mathbf{i}_A + \mathbf{A}_{rL}\mathbf{i}_L = \mathbf{0} \qquad (\ KCE\ for\ \mathcal{G} \times (A \cup L)) \tag{6.40}$$

$$\mathbf{A}_{rA}\mathbf{G}\mathbf{v}_A + \mathbf{A}_{rA}\mathbf{s}_A + \mathbf{A}_{rL}\mathbf{i}_L = \mathbf{0} \qquad (by\ device\ characteristic) \tag{6.41}$$

$$\mathbf{A}_{rA}\mathbf{G}{\mathbf{A}_{rA}}^T\mathbf{v}_{n_A} + \mathbf{A}_{rL}\mathbf{i}_L = -\mathbf{A}_{rA}\mathbf{s}_A \qquad (by\ KVL\ for\ \mathcal{G} \ . \ A) \tag{6.42}$$

Next using the equations for the variables $\mathbf{v}_B, \mathbf{i}_B, \mathbf{v}_K$ in the above mentioned subsection, we get

$$\mathbf{B}_B\mathbf{v}_B + \mathbf{B}_K\mathbf{v}_K = \mathbf{0} \qquad (\ KVE\ for\ \mathcal{G}\ .\ (B \cup K)) \tag{6.43}$$
$$\mathbf{B}_B\mathbf{R}\mathbf{i}_B + \mathbf{B}_B\mathbf{s}_B + \mathbf{B}_K\mathbf{v}_K = \mathbf{0} \qquad (\textit{by device characteristic}) \tag{6.44}$$
$$\mathbf{B}_B\mathbf{R}\mathbf{B}_B{}^T\ \mathbf{i}_{l_B} + \mathbf{B}_K\mathbf{v}_K = -\mathbf{B}_B\mathbf{s}_B \qquad (\textit{by } KCL\ for\ \mathcal{G} \times B) \tag{6.45}$$

Next we need to match $\mathbf{i}_L$ from both networks and $\mathbf{v}_K$ from both networks. This is done by writing $\mathbf{i}_L$ in $\mathcal{N}_{BK}$ in terms of $\mathbf{i}_{l_B}$ and substituting this expression in place of $\mathbf{i}_L$ in $\mathcal{N}_{AL}$. Similarly $\mathbf{v}_K$ is written in terms of $\mathbf{v}_{n_A}$ in $\mathcal{N}_{AL}$ and substituted in $\mathcal{N}_{BK}$.
Let $\mathbf{A}_{rK}$ be the submatrix made up of columns of $\mathbf{A}_{rA}$ corresponding to K and let $\mathbf{B}_L$ be the submatrix made up of columns of $\mathbf{B}_B$ corresponding to L. We then have

$$\begin{aligned} \mathbf{v}_K &= (\mathbf{A}_{rK})^T\mathbf{v}_{n_A} \\ \mathbf{i}_L &= \mathbf{B}_L^T\mathbf{i}_{l_B}. \end{aligned}$$

Thus, the final hybrid analysis equations for $\mathcal{N}$ are:

$$\begin{bmatrix} \mathbf{A}_{rA}\mathbf{G}\mathbf{A}_{rA}{}^T & \mathbf{A}_{rL}\mathbf{B}_L^T \\ \mathbf{B}_K\mathbf{A}_{rK}{}^T & \mathbf{B}_B\mathbf{R}\mathbf{B}_B{}^T \end{bmatrix} \begin{bmatrix} \mathbf{v}_{nA} \\ \mathbf{i}_{lB} \end{bmatrix} = \begin{bmatrix} -\mathbf{A}_{rA}\mathbf{s}_A \\ -\mathbf{B}_B\mathbf{s}_B \end{bmatrix} \tag{6.46}$$

We now briefly discuss the structure of the hybrid analysis equations.

Positive Definiteness of the Coefficient Matrix

Theorem 6.5.4 : *If* $\mathbf{G}$, $\mathbf{R}$ *are positive definite and* L, K *are chosen minimally, then the coefficient matrix of the hybrid analysis equations is positive definite.*

To prove the theorem we need a couple of preliminary lemmas.

Lemma 6.5.1

$$(\mathbf{A}_{rL}\mathbf{B}_L{}^T)^T = -(\mathbf{B}_K\mathbf{A}_{rK}{}^T),$$

where $(\mathbf{A}_{rA}\mathbf{A}_{rL})$, $(\mathbf{B}_B\mathbf{B}_K)$ *are representative matrices respectively of* $\mathcal{V}_v(\mathcal{G}\times(A \cup L))$ *and* $\mathcal{V}_i(\mathcal{G}\ .\ (B \cup K))$ *and* $\mathbf{A}_{rK}, \mathbf{A}_{rL}$ *are submatrices of* $(\mathbf{A}_{rA}\mathbf{A}_{rL})$ *composed of columns corresponding to* K, L *respectively and* $\mathbf{B}_L, \mathbf{B}_K$ *are submatrices of* $(\mathbf{B}_B\ \mathbf{B}_K)$ *corresponding to* L, K *respectively.*

Proof : We observe that rows of $(\mathbf{A}_{rL}\ \mathbf{A}_{rK})$ span

$$(\mathcal{V}_v(\mathcal{G} \times (A \cup L))) \cdot (K \cup L) = \mathcal{V}_v(\mathcal{G} \times (A \cup L) \cdot (K \cup L))$$

and the rows of $(\mathbf{B}_L\mathbf{B}_K)$ span

$$(\mathcal{V}_i(\mathcal{G}\ .\ (B \cup K))) \cdot (K \cup L) = \mathcal{V}_i(\mathcal{G}\ .\ (B \cup K) \times (K \cup L)).$$

Now,

$$\mathcal{G} \times (A \cup L) \cdot (K \cup L) \cong \mathcal{G} \ . \ (B \cup K) \times (K \cup L).$$

Hence, by Tellegen's Theorem we have

$$\begin{pmatrix} \mathbf{A}_{rL} & \mathbf{A}_{rK} \end{pmatrix} \begin{pmatrix} {\mathbf{B}_L}^T \\ {\mathbf{B}_K}^T \end{pmatrix} = \mathbf{0}, \qquad (6.47)$$

i.e.,
$$\mathbf{A}_{rL}{\mathbf{B}_L}^T = -\mathbf{A}_{rK}{\mathbf{B}_K}^T = -(\mathbf{B}_K{\mathbf{A}_{rK}}^T)^T.$$

□

We also need the following elementary lemma whose simple proof is omitted.

Lemma 6.5.2 *Let* $\mathbf{Y}, \mathbf{Z}$ *be positive definite. Then* $\begin{bmatrix} \mathbf{Y} & \mathbf{H} \\ -\mathbf{H}^T & \mathbf{Z} \end{bmatrix}$ *is positive definite.*

Proof of the Theorem 6.5.4 : The rows of $\mathbf{A}_{rA}$ span

$$(\mathcal{V}_v(\mathcal{G} \times (A \cup L))) \cdot A \ (= \mathcal{V}_v(\mathcal{G} \times (A \cup L) \cdot A)).$$

The rows are linearly independent since the number of rows equals $r(\mathcal{G} \times (A \cup L))$ which by minimality of L is equal to $r(\mathcal{G} \times (A \cup L) \cdot A)$ (by Theorems 6.5.1 and 6.5.2). By a similar argument we see that $\mathbf{B}_B$ is a representative matrix of $\mathcal{V}_i(\mathcal{G} \ . \ (B \cup K) \times B)$ and, therefore, has linearly independent rows. Since $\mathbf{G}, \mathbf{R}$ are given to be positive definite, it follows that $\mathbf{A}_{rA}\mathbf{G}{\mathbf{A}_{rA}}^T$, $\mathbf{B}_B\mathbf{R}{\mathbf{B}_B}^T$ are positive definite (see Problem 2.10).
The theorem follows by Lemma 6.5.2.

□

For iterative methods of solution of linear equations (such as the conjugate gradient method) the coefficient matrix has to be positive definite in order that convergence be guaranteed. In practice these methods work well even if only a large principal diagonal submatrix is positive definite. The coefficient matrix that one obtains in hybrid analysis appears to satisfy these conditions to the extent that the network permits.(Compare remarks on the MNA coefficient matrix in Subsection 5.5.3).

The zero-nonzero structure of the coefficient matrix

The zero-nonzero structure of the hybrid analysis matrix is largely dependent on the zero-nonzero matrix structure of the matrices $(\mathbf{A}_{rA}\mathbf{G}{\mathbf{A}_{rA}}^T)$ and $\mathbf{B}_B\mathbf{R}{\mathbf{B}_B}^T$. If $\mathcal{G} \ . \ A$ is in several 2-connected components which are decoupled in the device characteristic we should first use the procedure described earlier of shifting the sources so that we have a number of subnetworks each on one of these 2-connected components. For each of these graphs we construct the reduced incidence matrix omitting one node as the datum node and write KCE using the shifted sources.

The coefficient matrix of the KCE would have the form

$$\begin{bmatrix} \mathbf{A}_{rA1} & \cdots & \mathbf{0} & \mathbf{A}_{1L} \\ & \ddots & & \vdots \\ \mathbf{0} & \cdots & \mathbf{A}_{rAk} & \mathbf{A}_{kL} \end{bmatrix}$$

and $(\mathbf{A}_{rA}\mathbf{G}\mathbf{A}_{rA}{}^T)$ would appear as a block diagonal matrix. Each block would be the coefficient matrix for nodal analysis of the corresponding 2-connected component and would have good sparsity. If $\mathcal{G} \times B$ is in several 2-connected components which are decoupled in the device characteristic we can similarly choose $(\mathbf{B}_B\ \mathbf{B}_K)$ so that it has the form

$$\begin{bmatrix} \mathbf{B}_{B1} & \cdots & \mathbf{0} & \mathbf{B}_{1K} \\ & \ddots & & \vdots \\ \mathbf{0} & \cdots & \mathbf{B}_{Bk} & \mathbf{B}_{kK} \end{bmatrix}.$$

The matrix $\mathbf{B}_B\mathbf{R}\mathbf{B}_B{}^T$ would appear in the block diagonal form. In this case however, the sparsity depends on the choice of the representative matrix $[\mathbf{B}_{Bi}\ \ \mathbf{B}_{iK}]$. If the individual 2-connected components were planar then mesh matrix is the best choice. Otherwise one could use a technique of planar slicing which seems to perform well [Ovalekar+Narayanan92] (see Problem 3.24).

6.5.6 The Hybrid Rank

A natural question that arises in connection with hybrid analysis is the following: 'When would the size of the coefficient matrix be the minimum? '
We have seen that the number of rows of $\mathbf{A}_{rA}$ is $r(\mathcal{G}\ .\ A)$ while that of $\mathbf{B}_B$ is $\nu(\mathcal{G} \times B)$. Thus, we have to look for a partition (A, B) that minimises $r(\mathcal{G}\ .\ A) + \nu(\mathcal{G} \times B)$. This is the principal partition problem that is dealt with in detail later in Chapter 11 and Section 15.2 of Chapter 15. (The minimum value is called the **hybrid rank of** $\mathcal{G}$). Here we only mention a couple of interesting facts.

i. Call two forests of $\mathcal{G}$ maximally distant if the size of their union is the largest possible. Let this maximum size be $r_2(\mathcal{G})$. Then

$$\min_{A \subseteq E, B = E - A} (r(\mathcal{G}\ .\ A) + \nu(\mathcal{G} \times B)) = r_2(\mathcal{G}) - r(\mathcal{G}).$$

ii. If (A', B'), $(A", B")$ both corresponds to the hybrid rank so do $(A' \cup A", B' \cap B")$ and $(A' \cap A", B' \cup B")$.

Thus, there is a unique (A, B) which maximizes size of A and minimizes size of B and a unique (A, B) which takes extreme value in the other direction.

There is a variation of the hybrid rank problem which is also relevant here. Suppose the devices in the network are partitioned into blocks within each of which

Figure 6.9: Diakoptics and Codiakoptics

there is device characteristic coupling. Suppose further that in each of the blocks the characteristic is available in both the conductance and the resistance form, i.e., $\mathbf{v}_B - \mathbf{R}_b\mathbf{i}_B = \mathbf{s}_b$ or $\mathbf{G}_b\mathbf{v}_B - \mathbf{i}_B = \hat{\mathbf{s}}_b$. When we partition the devices into A and B, we would not like to split the blocks of the above partition. So the problem to be solved becomes 'minimize $r(\mathcal{G} \,.\, A) + \nu(\mathcal{G} \times B)$ under the condition that A and B are unions of blocks of a given partition Π of $E(\mathcal{G})$'. It turns out this is the hybrid rank problem (equivalently the membership problem) for polymatroids. It is possible to solve this problem by a variation of the techniques used for solving the original hybrid rank problem (see Section 15.3).

Exercise 6.17 *When hybrid analysis is used as a technique of network decomposition we usually concentrate on getting a good structure for $\mathcal{G} \,.\, A$ or for $\mathcal{G} \times B$ but not necessarily for both. For both the networks with graphs in Figure 6.9 assume that branches have the characteristic $\mathbf{G}(\mathbf{v} - \mathbf{e}) - (\mathbf{i} - \mathbf{j}) = \mathbf{0}$, where $\mathbf{G}$ is an invertible diagonal matrix. For the first of these graphs take the dotted branches to be B and for the second of these graphs take them to be A. Now write by inspection hybrid equations for the two problems which reveal the fact that $\mathcal{G} \,.\, A$ $(\mathcal{G} \times B)$ has three 2-connected components.*

Exercise 6.18 *Let*

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{pmatrix}$$

Suppose $\mathbf{A}_{11}$ is invertible. Obtain the inverse of $\mathbf{A}$ in terms of $\mathbf{A}_{11}^{-1}$ and any other appropriate matrix. (This is one of the 'Householder inverses').

Exercise 6.19 *Let*

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} & \mathbf{A}_{13} & \mathbf{0} \\ \mathbf{A}_{21} & \mathbf{A}_{22} & \mathbf{A}_{23} & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_{32} & \mathbf{A}_{33} & \mathbf{A}_{34} \\ \mathbf{0} & \mathbf{A}_{42} & \mathbf{A}_{43} & \mathbf{A}_{44} \end{pmatrix}$$

Suppose

$$\begin{pmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{pmatrix} \textit{and} \begin{pmatrix} \mathbf{A}_{33} & \mathbf{A}_{34} \\ \mathbf{A}_{43} & \mathbf{A}_{44} \end{pmatrix}$$

are invertible. Obtain the inverse of $\mathbf{A}$ *in terms of the inverses of these matrices.*

Exercise 6.20 *Let* $\mathbf{A} = \mathbf{B} + \mathbf{CD}$ *and let* $\mathbf{A}$,$\mathbf{B}$ *be invertible. Obtain the inverse of* $\mathbf{A}$ *in terms of the inverse of* $\mathbf{B}$. *Relate this to the previous problem.*

Exercise 6.21

i. Show that the second condition of Theorem 6.5.1 is equivalent to 'L contains no cutsets in $\mathcal{G} \times (A \cup L)$*'.*

ii. Derive an equivalent condition to the second condition of Theorem 6.5.2 that is dual to the above.

Exercise 6.22 *We use the notation of Theorem 6.4.1.*

i. If L *contains no cutsets of* $\mathcal{G}$ *show that*

$$r(\mathcal{G} \times (A \cup L) \cdot L) = r(\mathcal{G} \ . \ A) - r(\mathcal{G} \times A) = \xi(A, B).$$

ii. If K *contains no circuits of* G *show that* $\nu(\mathcal{G} \ . \ (B \cup K) \times K) = \xi(A, B)$.

iii. In addition to the above if L *contains no circuits of* $\mathcal{G} \times (A \cup L)$ *and* K *contains no cutsets of* $\mathcal{G} \ . \ (B \cup K)$, *show that* L *and* K *are disjoint forests (coforests) of*

$$\mathcal{G} \times (A \cup L) \cdot (K \cup L) \ (= \mathcal{G} \ . \ (B \cup K) \times (K \cup L))$$

6.6 Notes

Most of the material in this chapter is taken from [Narayanan78]. The work arose in an attempt to understand Kron's Diakoptics [Kron63] from a topological point of view. In a sense these ideas are contained already in Branin's interpretation (hybrid analysis) of Kron's work [Branin62]. Our work, however, is in a different direction from that of Amari [Amari62], who has given an algebraic topological interpretation of Kron's work. In the literature there is sometimes an attempt made to derive hybrid analysis using Substitution Theorem. This is impossible, since that theorem needs a guarantee that the network be uniquely solvable after substitution. In fact, provided the device characteristics can be partitioned into conductance type and resistance type devices, hybrid analysis equations are valid equations for the network, irrespective of whether the network has a unique solution or even has a solution at all.

6.7 Solutions of Exercises

E 6.1: **i**. The cutset would disappear since one of its branches, viz. the voltage source, would get shorted during v-shift.

ii. Contract all branches other than voltage sources. A cutset in the resulting graph would be a cutset in the original graph.

iii. In the above contracted graph $\mathcal{G}_1$, the branches incident at each node would be a crossing edge set. Take any one such set. Observe that this is also a crossing edge set of the original graph. Do a v-shift in the original graph of one of its branches (each of which is a voltage source) into the others in the crossing edge set.Contract the shifted voltage source in the graph $\mathcal{G}_1$ to get a graph $\mathcal{G}_2$. Repeat this procedure until only a collection of self loops is left in the contracted graph.During this process the original graph would have some of its voltage sources shorted so that no purely voltage source cutsets are left.The new network and the old are equivalent as far as the devices other than the contracted voltage sources are concerned.

E 6.2: ((ii) Suppose a collection of current sources formed a circuit. How would you detect such a circuit?(iii) How would you reduce the number of current sources without affecting the solution as far as the rest of the network is concerned?)

ii. Open all branches other than current sources. A circuit in the resulting graph would be a circuit in the original graph.

iii. In the above reduced graph $\mathcal{G}_1$ consider any circuit. Observe that this is also a circuit of the original graph. Do an i-shift (in the original graph) of one of its branches (each of which is a current source) into the others in the circuit. Open the shifted current source in the graph $\mathcal{G}_1$ to get a graph $\mathcal{G}_2$. Repeat this procedure until only a collection of coloops is left in the reduced graph.During this process the original graph would have some of its current sources opened so that no purely current source circuits are left.The new network and the old are equivalent as far as the devices other than the deleted current sources are concerned.

E 6.3: The set $\mathcal{E}$ contains no circuits of $\mathcal{G}$. So it contains no circuits of $\mathcal{G} \, . \, (E-\mathcal{J})$ either. Now the column dependence structure of all representative matrices of a vector space is the same. Hence, for rank computations we may select the matrix $(\mathbf{B}_R\mathbf{B}_{\mathcal{E}})$ to be an f-circuit matrix of $\mathcal{G} \, . \, (E-\mathcal{J})$. The set $\mathcal{E}$ contains no circuit and hence can be included in some forest. We select one f-circuit matrix with respect to such a forest. Now $\mathbf{B}_R$ contains an identity submatrix with the same number of rows.
So, $r(\mathbf{B}_R \; \mathbf{B}_{\mathcal{E}}) = r(\mathbf{B}_R)$. The proof is dual to the above for '$r(\mathbf{A}_R \; \mathbf{A}_J) = r(\mathbf{A}_R)$'.

E 6.4: Let $(\mathbf{A}_R \; \mathbf{A}_J)$ be a reduced incidence matrix for $\mathcal{G} \times (E-\mathcal{E})$. We remind the reader that the reduced incidence matrix is obtained from the incidence matrix by omitting one row per component. Since $\mathcal{J}$ contains no cutset of $\mathcal{G}$ the rows of $\mathbf{A}_R$ must be linearly independent. It is then clear that $\mathbf{A}_R$ is the reduced incidence matrix for $\mathcal{G} \times (E-\mathcal{E}) \cdot R = \mathcal{G} \, . \, (E-\mathcal{J}) \times R$. We can select a representative matrix

$\mathbf{A}_{\mathcal{E}R}$ for $\mathcal{V}_v(\mathcal{G} \,.\, (E-\mathcal{J}))$ as follows:

$$\begin{array}{c} \quad\quad\quad\quad \mathcal{E} \quad\quad R \\ \mathbf{A}_{\mathcal{E}R} = \left[\begin{array}{cc} \mathbf{I} & \mathbf{Q}_{\mathcal{E}R} \\ \mathbf{0} & \mathbf{A}_R \end{array} \right]. \end{array} \tag{6.48}$$

Here $(\mathbf{I}\ \ \mathbf{Q}_{\mathcal{E}R})$ is obtained by selecting a forest of $\mathcal{G} \,.\, (E-\mathcal{J})$ containing $\mathcal{E}$ and taking f-cutset vectors corresponding to the edges in $\mathcal{E}$. The above is a representative matrix because

i. its rows are linearly independent and belong to $\mathcal{V}_v(\mathcal{G} \,.\, (E-\mathcal{J}))$ (note that $\mathbf{A}_R$ is a representative matrix for $\mathcal{V}_v(\mathcal{G} \,.\, (E-\mathcal{J}) \times R) = (\mathcal{V}_v(\mathcal{G} \,.\, (E-\mathcal{J}))) \times R$

ii. the number of rows $= r(\mathcal{G} \,.\, (E-\mathcal{J}) \times R) + r(\mathcal{G} \,.\, (E-\mathcal{J}) \cdot \mathcal{E})$ $= r(\mathcal{G} \,.\, (E-\mathcal{J}))$.

By Theorem 6.3.1 the constraints on $\mathbf{v}_R, \mathbf{i}_R, \mathbf{v}_{\mathcal{E}}, \mathbf{i}_{\mathcal{J}}$ are

$$\begin{array}{rcll} \left[\begin{array}{cc} \mathbf{A}_R & \mathbf{A}_J \end{array} \right] \left[\begin{array}{c} \mathbf{i}_R \\ \mathbf{i}_{\mathcal{J}} \end{array} \right] & = & \mathbf{0} & (KCE\ for\ \mathcal{G} \times (E-\mathcal{E})) \\ \left[\begin{array}{c} \mathbf{v}_{\mathcal{E}} \\ \mathbf{v}_R \end{array} \right] & = & \left[\begin{array}{cc} \mathbf{I} & \mathbf{0} \\ \mathbf{Q}_{\mathcal{E}R}^T & \mathbf{A}_R^T \end{array} \right] \left[\begin{array}{c} \mathbf{v}_{\mathcal{E}} \\ \mathbf{v}_n \end{array} \right] & (KVL\ for\ \mathcal{G} \,.\, (E-\mathcal{J})). \end{array}$$

Next $\mathbf{v}_{\mathcal{E}} = \mathbf{e}$, $\mathbf{i}_{\mathcal{J}} = \mathbf{j}$ and $\mathbf{G}\mathbf{v}_R - \mathbf{i}_R = \mathbf{0}$, where $\mathbf{G} = \mathbf{R}^{-1}$.
Hence we have

$$\begin{array}{rcl} \mathbf{A}_R\, \mathbf{i}_R & = & -\mathbf{A}_J\mathbf{j} \\ \mathbf{A}_R\mathbf{G}\mathbf{v}_R & = & -\mathbf{A}_J\mathbf{j} \\ \mathbf{A}_R\mathbf{G}[\mathbf{Q}_{\mathcal{E}R}^T\ \ \mathbf{A}_R^T] \left[\begin{array}{c} \mathbf{v}_{\mathcal{E}} \\ \mathbf{v}_n \end{array} \right] & = & -\mathbf{A}_J\mathbf{j} \\ (\mathbf{A}_R\mathbf{G}\mathbf{A}_R{}^T)\mathbf{v}_n & = & -\mathbf{A}_J\mathbf{j} - \mathbf{A}_R\mathbf{G}\mathbf{Q}_{\mathcal{E}R}^T\mathbf{e}. \end{array}$$

As required we observe that the coefficient matrix structure is controlled by $\mathbf{A}_R$ which is the reduced incidence matrix of $\mathcal{G} \,.\, (E-\mathcal{J}) \times R$.
The loop type equations are written dual to the above. The important step is to write a representative matrix $\hat{\mathbf{B}}_{JR}$ for $\mathcal{G} \times (E-\mathcal{E})$ in form shown below:

$$\begin{array}{c} \quad\quad\quad\quad \mathcal{J} \quad\quad R \\ \hat{\mathbf{B}}_{JR} = \left[\begin{array}{cc} \mathbf{I} & \mathbf{B}_{JR} \\ \mathbf{0} & \mathbf{B}_R \end{array} \right]. \end{array} \tag{6.49}$$

Here $\mathbf{B}_R$ is a representative matrix for
$(\mathcal{V}_i(\mathcal{G} \times (E-\mathcal{E}))) \times R \ \ = \mathcal{V}_i(\mathcal{G} \times (E-\mathcal{E}) \cdot R) = \mathcal{V}_i(\mathcal{G} \,.\, (E-\mathcal{J}) \times R))$.
The rows of $(\mathbf{I}\ \mathbf{B}_{JR})$ are obtained by selecting a forest of $\mathcal{G} \times (E-\mathcal{E})$ that does not intersect $\mathcal{J}$ and taking f-circuit vectors corresponding to the edges in $\mathcal{J}$. The

final equations we get are $(\mathbf{B}_R\ \mathbf{R}\ \mathbf{B}_R{}^T)\mathbf{i}_l = -\mathbf{B}_{\mathcal{E}}\mathcal{E} - \mathbf{B}_R\mathbf{R}\mathbf{B}_{JR}^T\ \mathbf{j}$.
Once again the coefficient matrix structure is controlled by $\mathbf{B}_R$ which is a representative matrix for $\mathcal{V}_i(\mathcal{G}\ .\ (E-\mathcal{J})\times R)$.

E 6.5: Let $\mathcal{N} \equiv (\mathcal{G},\mathcal{D})$ be a network.
Let $\mathcal{N}_1 \equiv (\mathcal{G}_1,\mathcal{D}_1)$ be defined as follows:

i. Let $E_1 \equiv E(\mathcal{G}_1), E \equiv E(\mathcal{G})$ such that $E \subseteq E_1$ and $\mathcal{G}_1 \cdot E = \mathcal{G}$,

ii. $E_1 - E$ is a circuit of $\mathcal{G}_1$ and its edges have the the same direction relative to the orientation of the circuit,

iii. $\mathcal{D}_1 = \delta_{(E_1-E)v} \times \mathcal{D}_{Ev(E1)i}$ (i.e., elements of $E_1 - E$ are voltage unconstrained).

iv. $(\mathbf{v}_1,\mathbf{i}_1) \in \mathcal{D}_1$ only if $(\mathbf{v}_1/E,\mathbf{i}_1/E) \in \mathcal{D}$ and $\mathbf{i}_1/(E_1-E)$ has all its entries equal to J.

Then $\mathcal{N}$ and $\mathcal{N}_1$ are equivalent in E.

E 6.6: The proof is identical to that of Theorem 6.3.2 except that in place of $J\mathbf{i}_c$ we use a vector $\hat{\mathbf{i}} = \mathbf{0}_{E-A} + \mathbf{i}_A$.

E 6.7: Dual to the above.

E 6.8: Let $\mathbf{v},\mathbf{i}$ be a solution of $\mathcal{N}$. We see that $(\mathbf{v}/(E-A),\mathbf{i}/(E-A))$ satisfy constraints (i), (ii) and (iii). On the other hand let $(\mathbf{v}_{E-A},\mathbf{i}_{E-A})$ satisfy constraints (i), (ii) and (iii). Then there exist $\mathbf{v}(\cdot)$ s.t. $\mathbf{v}(t) \in \mathcal{V}_v(\mathcal{G}), \forall t \in \Re$ and $\mathbf{v}/E-A = \mathbf{v}_{E-A}$ and there exist $\mathbf{i}(\cdot)$ s.t. $\mathbf{i}(t) \in v_i(\mathcal{G}), \forall t \in \Re$ and $\mathbf{i}/E-A = \mathbf{i}_{E-A}$. Since $(\mathbf{v}_{E-A},\mathbf{i}_{E-A}) \in \mathcal{D}_{E-A}$ and $\mathcal{D} = \delta_A \times \mathcal{D}_{E-A}$, we conclude that $\mathbf{v},\mathbf{i} \in \mathcal{D}$. Thus, if $(\mathbf{v}_{E-A},\mathbf{i}_{E-A})$ satisfy (i), (ii) and (iii). there exists a solution $\mathbf{v},\mathbf{i}$ of $\mathcal{N}$ s.t.

$$\mathbf{v}_{E-A} = \mathbf{v}/(E-A)$$
$$\mathbf{i}_{E-A} = \mathbf{i}/(E-A).$$

E 6.9: The result follows from the facts that

$$\mathcal{D} \equiv \mathcal{D}_A \times \mathcal{D}_{E-A}, \mathcal{D}_A \equiv \{(\mathbf{0}_A,\mathbf{0}_A)\}$$
$$\mathcal{V}_i(\mathcal{G}\ .\ A) = (\mathcal{V}_i(\mathcal{G})) \times A$$
$$\mathcal{V}_v(\mathcal{G} \times A) = (\mathcal{V}_v(\mathcal{G})) \times A.$$

E 6.10: When we v-shift a source and put it in series with existing devices, absorbing it into their characteristics, the effect on the graph is that the branch is contracted fusing its endpoints. So if $\mathcal{G} \times (E-\mathcal{E})$ has separators A, B, these sets would appear as separators after v-shift.

E 6.11: Dual to the previous problem.

E 6.12: Combination of the previous two problems.

E 6.13: In order to i-shift (v-shift) a device all that is needed is that the device be voltage unconstrained (current unconstrained). So norators can be both v-shifted as well as i-shifted. Nullators, however, have both voltage and current constraints. So they cannot be v- or i-shifted.

E 6.14: Norators can be both v- as well as i-shifted. When i-shifted a norator $\mathcal{D}_1$ can be thought of as an 'unknown' current source of value $i_{\mathcal{D}_1}$. After i-shifting this parameter can be absorbed into the characteristic of the device in parallel e.g. if the characteristic of the device is $f(i,v) = 0$, the characteristic of the parallel combination would be $f(i - i_{\mathcal{D}_1}, v) = 0$ if we assume that the original device and $i_{\mathcal{D}_1}$ have the same direction. (In the second characteristic i, v represent the current and voltage associated with the combined device). We have seen that i-shifting a device results in graph restriction. So if $\mathcal{G} \cdot (E-A)$ has $B_1, \cdots B_k$ as separators and if $\mathcal{D}_B = \mathcal{D}_{B_1} \times \cdots \times \mathcal{D}_{B_k}$, we have a set of networks on $\mathcal{G} \cdot B_1 \oplus \cdots \oplus \mathcal{G} \cdot B_k$ with only the parameters $i_{D_1}, \cdots, i_{D_n}$ (the currents of norators in A) linking the networks in the device characteristic. The discussion for v-shift is dual to the above.

E 6.15: The statement of the dual of Lemma 6.4.1

Lemma 6.7.1 *: Let $\mathcal{V}$ be a vector space on S. Let S be partitioned into P, Q, T s.t. $\mathcal{V} \times (S-P)$ has Q, T as separators. Then a vector $\mathbf{x} \in \mathcal{V}$ iff there exist vectors $\mathbf{x}_{PQ} \in \mathcal{V} \cdot (P \cup Q)$ and $\mathbf{x}_{PT} \in \mathcal{V} \cdot (P \cup T)$ s.t.*
$\mathbf{x}_{PQ}/P = \mathbf{x}_{PT}/P$ and $\mathbf{x}_{PQ}/Q \oplus \mathbf{x}_{PT} = \mathbf{x}$.

The proof is by use of vector space duality: Replace $\mathcal{V}$ in the statement of Lemma 6.4.1 throughout by $\mathcal{V}^{\perp}$. Next write $\mathcal{V}^{\perp} \cdot (S-P)$ as $(\mathcal{V} \times (S-P))^{\perp}$, $(\mathcal{V}^{\perp} \times (P \cup Q))^{\perp}$ as $(\mathcal{V} \cdot (P \cup Q))$ and $(\mathcal{V}^{\perp} \times (P \cup T))^{\perp}$ as $(\mathcal{V} \cdot (P \cup T))$. Observe that the separators of a vector space and of its orthogonal complement are identical, that the orthogonal complement of the orthogonal complement is the original vector space and rewrite the statement in terms of $\mathcal{V}$.

E 6.16: Matching only boundary voltages or only boundary currents would affect only KVL or only KCL constraints. So such a result is impossible unless device characteristics were used.

E 6.17: Outline (Figure 6.9(a)) Treat branches $1, 2, 3$ as current sources and i-shift them so that 1 goes across $(a, c), (c, g)$, 2 goes across $(b, c), (c, d)$ and 3 goes across $(f, c), (c, h)$. Cut the network at the hinge so that we have three networks for each of which write nodal equations in terms of actual sources in branches other than $1, 2, 3$ and the 'new' current sources i_1, i_2, i_3. Now write the additional equations

$$i_1 = g_1(v_1 - e_1) + j_1 = g_1(v_a^1 - v_c^1 + v_c^3 - v_g^3 - e_1) + j_1$$
$$i_2 = g_2(v_2 - e_2) + j_2 = g_2(v_b^1 - v_c^1 + v_c^2 - v_d^2 - e_2) + j_2$$
$$i_3 = g_3(v_3 - e_3) + j_3 = g_3(v_f^2 - v_c^2 + v_c^3 - v_h^3 - e_3) + j_3$$

If c had been taken as the datum node in each of the three 'split' networks then v_c^1, v_c^2, v_c^3 can be taken to be zero. Here superscripts $1, 2, 3$ refer to the three 'split' networks.

E 6.18: Let

$$\begin{pmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{pmatrix} \begin{matrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{matrix} = \begin{pmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \end{pmatrix}, \qquad (*)$$

where the RHS is arbitrary . If

$$\begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{K} \end{pmatrix} \begin{pmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \end{pmatrix}, \qquad (**)$$

then $\mathbf{K}$ must be the inverse of $\mathbf{A}$.
From the first set of rows of $(*)$ we get

$$\mathbf{x}_1 = \mathbf{A}_{11}^{-1}(\mathbf{b}_1 - \mathbf{A}_{12}\mathbf{x}_2)$$

Substituting in the second set of rows of $(*)$ we get

$$\mathbf{A}_{22}\mathbf{x}_2 = \mathbf{b}_2 - \mathbf{A}_{21}\mathbf{A}_{11}^{-1}(\mathbf{b}_1 - \mathbf{A}_{12}\mathbf{x}_2)$$

Let $\mathbf{A}'_{22} \equiv (\mathbf{A}_{22} - \mathbf{A}_{21}\mathbf{A}_{11}^{-1}\mathbf{A}_{12})$. Then

$$\mathbf{x}_2 = (\mathbf{A}'_{22})^{-1}(\mathbf{b}_2 - \mathbf{A}_{21}\mathbf{A}_{11}^{-1}\mathbf{b}_1)$$

and

$$\mathbf{x}_1 = \mathbf{A}_{11}^{-1}(\mathbf{b}_1 - \mathbf{A}_{12}(\mathbf{A}'_{22})^{-1}(\mathbf{b}_2 - \mathbf{A}_{21}\mathbf{A}_{11}^{-1}\mathbf{b}_1)).$$

(Note that the invertibility of $\mathbf{A}'_{22}$ follows from the invertibility of $\mathbf{A}$ and $\mathbf{A}_{11}$). Now we rewrite these equations in the form $(**)$, which yields $\mathbf{K}$.

E 6.19: Similar to the solution above.

E 6.20: Consider the solution of the equation

$$\mathbf{Bx} + \mathbf{Cy} = \mathbf{b}$$

$$-\mathbf{Dx} + \mathbf{y} = \mathbf{0}$$

Now if we obtain $\mathbf{x} = \mathbf{Kb}$ then $\mathbf{K}$ must be the inverse of $\mathbf{A}$. To obtain $\mathbf{K}$ we use the method of solution of Exercise 6.18.
To see that the previous problem is a special case of this take

$$\mathbf{B} \equiv \begin{pmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} & \mathbf{0} & \mathbf{0} \\ \mathbf{A}_{21} & \mathbf{A}_{22} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{A}_{33} & \mathbf{A}_{34} \\ \mathbf{0} & \mathbf{0} & \mathbf{A}_{43} & \mathbf{A}_{44} \end{pmatrix}$$

$$\mathbf{D} \equiv \begin{bmatrix} \mathbf{0} & \mathbf{A}'_3 \\ \mathbf{A}'_2 & \mathbf{0} \end{bmatrix},$$

where rows of $\mathbf{A}'_3, \mathbf{A}'_2$ span, respectively, rows of $\begin{pmatrix} \mathbf{A}_{13} \\ \mathbf{A}_{23} \end{pmatrix}, \begin{pmatrix} \mathbf{A}_{32} \\ \mathbf{A}_{42} \end{pmatrix}$. Since by linearly combining rows of $\mathbf{D}$ we can get $\mathbf{A}' \equiv \begin{pmatrix} \mathbf{0} & \mathbf{A}_{13} \\ \mathbf{0} & \mathbf{A}_{23} \\ \mathbf{A}_{32} & \mathbf{0} \\ \mathbf{A}_{42} & \mathbf{0} \end{pmatrix}$ So we can find $\mathbf{C}$ s.t. $\mathbf{CD} \equiv \mathbf{A}'$. (It is preferable that $\mathbf{C}$ be found by inspection where possible. It is not essential that rows of $\mathbf{D}$ be linearly independent).

E 6.21:
i. L contains no cutsets in $\mathcal{G} \times (A \cup L)$ iff L can be included in a coforest of $\mathcal{G} \times (A \cup L)$, i.e., iff $r(\mathcal{G} \times (A \cup L) \cdot A) = r(\mathcal{G} \times (A \cup L))$. (In the present case

$$\mathcal{G} \times (A \cup L) \cdot A \cong \mathcal{G} \,.\, (E - L) \times A \cong \mathcal{G} \,.\, (E - L) \cdot A \cong \mathcal{G} \,.\, A)$$

ii. K contains no circuits in $\mathcal{G} \,.\, (B \cup K)$ iff K can be included in a forest of $\mathcal{G} \,.\, (B \cup K)$, i.e., iff $\nu(\mathcal{G} \,.\, (B \cup K) \times B) = \nu(\mathcal{G} \,.\, (B \cup K))$.
(In the present case

$$\mathcal{G} \,.\, (B \cup K) \times B \cong \mathcal{G} \times (E - K) \cdot B \cong \mathcal{G} \times (E - K) \times B \cong \mathcal{G} \times B)$$

E 6.22: i. We have

$$\mathcal{G} \times (A \cup L) \cdot A \cong \mathcal{G} \,.\, A \text{ by definition of } L$$

Now

$$r(\mathcal{G} \times (A \cup L)) = r(\mathcal{G} \times (A \cup L) \times A) + r(\mathcal{G} \times (A \cup L) \cdot L)$$

Now if L contains no cutsets in $\mathcal{G}$, it has no cutsets in $\mathcal{G} \times (A \cup L)$. Hence,

$$r(\mathcal{G} \times (A \cup L)) = r(\mathcal{G} \times (A \cup L) \cdot A) = r(\mathcal{G} \,.\, A)$$

The result follows.

ii. This is dual to the above.

iii. If L contains no circuit of $\mathcal{G} \times (A \cup L)$ then L is a subforest of $\mathcal{G} \times (A \cup L) \cdot (K \cup L)$. Dually K is a subcoforest of $\mathcal{G} \,.\, (B \cup K) \times (K \cup L) (= \mathcal{G} \times (A \cup L) \cdot (K \cup L))$. We conclude that L is a forest and K is a coforest of $\mathcal{G} \times (A \cup L) \cdot (K \cup L)$.
Next $\mathcal{G} \times (A \cup L) \cdot A \cong \mathcal{G} \,.\, A$ (by definition of L). Since K contains no circuits of $\mathcal{G}$ it also contains no circuits in $\mathcal{G} \,.\, A$ and, therefore, in $\mathcal{G} \times (A \cup L)$ and $\mathcal{G} \times (A \cup L) \cdot (K \cup L)$. Thus K is a subforest of $\mathcal{G} \times (A \cup L) \cdot (K \cup L)$. Dually L is a subcoforest of $\mathcal{G} \,.\, (B \cup K) \times (K \cup L) (\cong \mathcal{G} \times (A \cup L) \cdot K \cup L))$. So K is a forest and L, a coforest of $\mathcal{G} \times (A \cup L) \cdot (K \cup L)$. The required result now follows.

Chapter 7

The Implicit Duality Theorem and Its Applications

In this chapter we discuss a useful result for constructing complementary orthogonal spaces to implicitly defined vector spaces. The result has wide applications in Network and Systems theory. Its form permits generalization to other situations such as inequality and integrality systems. In the first half of this chapter we discuss the vector space version of the theorem and its applications. Later, for completeness, we discuss the polarity and integrality versions of the result.

7.1 The Vector Space Version

The Implicit Duality Theorem is, in a sense, a generalization of the fact that contraction and restriction of vector spaces are dual operations. The result is stated in terms of the notion of a generalized minor, defined below. Also, keeping in mind the other versions of the theorem we introduce appropriate notation.

Definition 7.1.1 *[Narayanan87] Let S, P be disjoint finite sets. Let $\mathcal{K}_{SP}$,$\mathcal{K}_P$ be collections of vectors defined on $S \uplus P$, P, respectively. The* **generalized minor of $\mathcal{K}_{SP}$ through $\mathcal{K}_P$**, *denoted by $\mathcal{K}_{SP} \leftrightarrow \mathcal{K}_P$ is defined by*
$\mathcal{K}_{SP} \leftrightarrow \mathcal{K}_P \equiv \{\mathbf{f}_S : \mathbf{f}_S = \mathbf{f}_S P/S, \;\; where \;\; \mathbf{f}_S P \in \mathcal{K}_{SP} \;\; s.t. \;\; \mathbf{f}_S P/P \in \mathcal{K}_P\}$. *If $\mathcal{K}_{SP} \leftrightarrow \mathcal{K}_P = \mathcal{K}_S$ we say that $\mathcal{K}_{SP}$* **is an extension of $\mathcal{K}_S$ through $\mathcal{K}_P$.**

Remark: i. Throughout this chapter S, P would denote disjoint sets. All sets dealt with would be finite unless otherwise stated.

ii. For the dot product and inner product operations we use the same symbol '$< \cdot, \cdot >$' - but in most cases we deal with the dot product. If inner product is intended this is mentioned. In Section 7.2 the 'q-bilinear' operation (a generalization of dot product and inner product) is defined and denoted also by $< \cdot, \cdot >$. This should however cause no confusion since discussion on this operation in the general sense is confined to that section.

iii. We abuse the notation in the following sense: even when we refer to two different dot product operations say one on $\mathcal{V}_1 \times \mathcal{V}_1$ and the other on $\mathcal{V}_2 \times \mathcal{V}_2$, where $\mathcal{V}_1, \mathcal{V}_2$ are spaces on different sets, we would use the same symbol $< \cdot, \cdot >$. When $\mathbf{f}_1$ is on S_1 and $\mathbf{f}_2$ is on S_2 then we take the dot product of $\mathbf{f}_1$ with $\mathbf{f}_2$, also denoted by $< \mathbf{f}_1, \mathbf{f}_2 >$, to be

$$\sum_{e \in S_1 \cap S_2} \mathbf{f}_1(e)\mathbf{f}_2(e).$$

By definition, the dot product is zero when $S_1 \cap S_2 = \emptyset$.

For notational convenience we generalize the usual ideas of addition of vectors and of collections of vectors below. We also extend the definition of contraction and restriction to arbitrary collections of vectors.

Let us define **addition** of vectors $\mathbf{f}_{S_1}, \mathbf{f}_{S_2}$ on distinct sets by

$$\begin{aligned}(f_{S_1} + f_{S_2})(e) &= f_{S_1}(e) + f_{S_2}(e), e \in S_1 \cap S_2 \\ &= f_{S_1}(e), e \in S_1 - S_2 \\ &= f_{S_2}(e), e \in S_2 - S_1.\end{aligned}$$

When S_1, S_2 are disjoint sets we usually write $(\mathbf{f}_{S_1} \oplus \mathbf{f}_{S_2})$ in place of $(\mathbf{f}_{S_1} + \mathbf{f}_{S_2})$. **Addition** of collections of vectors $\mathcal{K}_{S_1}, \mathcal{K}_{S_2}$ is denoted as usual by $\mathcal{K}_{S_1} + \mathcal{K}_{S_2}$ and is defined by

$$\mathcal{K}_{S_1} + \mathcal{K}_{S_2} \equiv \{\mathbf{f}_{S_1} + \mathbf{f}_{S_2} : \mathbf{f}_{S_1} \in \mathcal{K}_{S_1}, \mathbf{f}_{S_2} \in \mathcal{K}_{S_2}\}.$$

Once again, if S_1, S_2 are disjoint sets we usually write $\mathcal{K}_{S_1} \oplus \mathcal{K}_{S_2}$ in place of $\mathcal{K}_{S_1} + \mathcal{K}_{S_2}$. For convenience we define $\mathcal{K}_{S_1} - \mathcal{K}_{S_2}$ by

$$\mathcal{K}_{S_1} - \mathcal{K}_{S_2} \equiv \{\mathbf{f}_{S_1} - \mathbf{f}_{S_2} : \mathbf{f}_{S_1} \in \mathcal{K}_{S_1}, \mathbf{f}_{S_2} \in \mathcal{K}_{S_2}\}.$$

It is clear that if $\mathcal{K}_{S_1}, \mathcal{K}_{S_2}$ are vector spaces on S_1, S_2 respectively then $\mathcal{K}_{S_1} - \mathcal{K}_{S_2}$ is equal to $\mathcal{K}_{S_1} + \mathcal{K}_{S_2}$ and is a vector space on $S_1 \cup S_2$.
On the other hand, if $\mathcal{K}_{S_1}, \mathcal{K}_{S_2}$ are **cones** on S_1, S_2 respectively (a cone is a collection of vectors such that nonnegative linear combinations of vectors in the collection remain in the collection) then $\mathcal{K}_{S_1} + \mathcal{K}_{S_2}$, $\mathcal{K}_{S_1} - \mathcal{K}_{S_2}$ are distinct but are still cones on $S_1 \cup S_2$.
For any collection $\mathcal{K}_{S_1}$ of vectors S_1 and $T \subseteq S_1$, let

$$\mathcal{K}_{S_1} \cdot T \equiv \{\mathbf{f}_T : \mathbf{f}_T = \mathbf{f}/T, \text{ where } \mathbf{f} \in \mathcal{K}_{S_1}\}$$

and let

$$\mathcal{K}_{S_1} \times T = \{\mathbf{f}_T : \mathbf{f}_T = \mathbf{f}/T, \text{ where } \mathbf{f} \in \mathcal{K}_{S_1} \text{ and } \mathbf{f}/(\mathbf{S}_1 - \mathbf{T})(\cdot) = \mathbf{0}\}.$$

We then have the following simple lemma whose routine proof we omit.

Lemma 7.1.1 *Let $\mathcal{K}_{SP}, \mathcal{K}_P$ be collection of vectors on $S \uplus P, P$ respectively. Then, $\mathcal{K}_{SP} \leftrightarrow \mathcal{K}_P = (\mathcal{K}_{SP} - \mathcal{K}_P) \times S$.*

7.1.1 The Implicit Duality Theorem: Orthogonality Case

Let $\mathcal{K}_P$ be a vector space whose representative matrix $\mathbf{R}_P$ is as follows:

$$\begin{array}{cc} & \begin{array}{ccc} P_1 & & P_2 \end{array} \\ \mathbf{R}_P \equiv & \left(\begin{array}{ccc} \mathbf{I} & \vdots & \mathbf{0} \end{array} \right), \end{array}$$

where $P_1 \uplus P_2 = P$. It is then clear that

$$\begin{aligned} \mathcal{K}_{SP} \leftrightarrow \mathcal{K}_P &= \mathcal{K}_{SP} \cdot (S \cup P_2) \times S \\ &= \mathcal{K}_{SP} \times (S \cup P_1) \cdot S. \end{aligned}$$

Further $\mathcal{K}_P^{\perp}$ has the representative matrix

$$\begin{array}{cc} & \begin{array}{ccc} P_1 & & P_2 \end{array} \\ \mathbf{Q}_P \equiv & \left(\begin{array}{ccc} \mathbf{0} & \vdots & \mathbf{I} \end{array} \right). \end{array}$$

Hence, in this case

$$\mathcal{K}_{SP}^{\perp} \leftrightarrow \mathcal{K}_P^{\perp} = \mathcal{K}_{SP}^{\perp} \cdot (S \cup P_1) \times S = (\mathcal{K}_{SP} \leftrightarrow \mathcal{K}_P)^{\perp}$$

(using Corollary 3.4.1).
We now show that this result is true for arbitrary vector spaces.

Theorem 7.1.1 *(The Implicit Duality Theorem) Let $\mathcal{V}_{SP}, \mathcal{V}_P$ be vector spaces on $S \uplus P, P$ respectively. Then,*

$$(\mathcal{V}_{SP} \leftrightarrow \mathcal{V}_P)^{\perp} = \mathcal{V}_{SP}^{\perp} \leftrightarrow \mathcal{V}_P^{\perp}.$$

We need the following lemma for the proof of this theorem.

Lemma 7.1.2 $\mathbf{A}\ \mathbf{z} = \mathbf{b}$ *has a solution iff* $(\mathbf{A}^T\sigma = \mathbf{0}) \Rightarrow (\mathbf{b}^T\sigma = \mathbf{0})$.

Proof of the Lemma:
Let $\mathcal{V}_A$ be the space spanned by the columns of $\mathbf{A}$. Since $(\mathcal{V}_A^{\perp})^{\perp} = \mathcal{V}_A$ (Theorem 2.2.5), a vector $\mathbf{b}$ belongs to $\mathcal{V}_A$ iff it is orthogonal to all vectors in $\mathcal{V}_A^{\perp}$, i.e., iff $(\mathbf{A}^T\sigma = \mathbf{0}) \Rightarrow (\mathbf{b}^T\sigma = \mathbf{0})$.

□

Proof of Theorem 7.1.1: Let

$$\begin{array}{cc} \begin{array}{cc} S & P \end{array} & P \\ (\mathbf{A}_S\ \mathbf{A}_P), & (\hat{\mathbf{A}}_P) \end{array}$$

be the representative matrices of $\mathcal{V}_{SP}$, $\mathcal{V}_P$ respectively. A vector $\mathbf{x}_S$ belongs to $\mathcal{V}_{SP} \leftrightarrow \mathcal{V}_P$ iff there exist vectors λ_1, λ_2 s.t.

$$\left[\begin{array}{cc} \mathbf{A}_S{}^T & \mathbf{0} \\ \mathbf{A}_P{}^T & \hat{\mathbf{A}}_P^T \end{array} \right] \left[\begin{array}{c} \lambda_1 \\ \lambda_2 \end{array} \right] = \left[\begin{array}{c} \mathbf{x}_S \\ \mathbf{0} \end{array} \right].$$

By lemma 7.1.2, this happens
iff

$$\begin{pmatrix} \mathbf{A}_S & \mathbf{A}_P \\ \mathbf{0} & \hat{\mathbf{A}}_P \end{pmatrix} \begin{bmatrix} \mathbf{y}_S \\ \mathbf{y}_P \end{bmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \end{pmatrix} \Rightarrow (\mathbf{x}_S^T \mathbf{y}_S = \mathbf{0})$$

i.e., iff

$$(\mathbf{y}_S \in (\mathcal{V}_{SP}^{\perp} \leftrightarrow \mathcal{V}_P^{\perp})) \Rightarrow (\mathbf{x}_S^T \mathbf{y}_S = 0).$$

Thus, $\mathcal{V}_{SP} \leftrightarrow \mathcal{V}_P = (\mathcal{V}_{SP}^{\perp} \leftrightarrow \mathcal{V}_P^{\perp})^{\perp}$. It is clear that $(\mathcal{V}_{SP}^{\perp} \leftrightarrow \mathcal{V}_P^{\perp})$ is a vector space. By Theorem 2.2.5

$$\mathcal{V}_{SP}^{\perp} \leftrightarrow \mathcal{V}_P^{\perp} = (\mathcal{V}_{SP}^{\perp} \leftrightarrow \mathcal{V}_P^{\perp})^{\perp\perp} = (\mathcal{V}_{SP} \leftrightarrow \mathcal{V}_P)^{\perp}.$$

□

Remark: The fact that vectors in $\mathcal{V}_{SP}^{\perp} \leftrightarrow \mathcal{V}_P^{\perp}$ and $\mathcal{V}_{SP} \leftrightarrow \mathcal{V}_P$ are orthogonal is easy to see but of limited value. For almost all applications it is necessary to show that the two spaces are complementary orthogonal.

Example: Suppose spaces $\mathcal{V}$ and $\mathcal{V}'$ are solution spaces respectively of

$$[\mathbf{A}_1|\mathbf{A}_2] \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} = \mathbf{0}, [\mathbf{B}_1|\mathbf{B}_2] \begin{bmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \end{bmatrix} = \mathbf{0}.$$

If the two spaces are complementary orthogonal and if we know that $\mathbf{x}_1 = \mathbf{K}\mathbf{x}_2$, we can conclude that $\mathbf{y}_2 = -\mathbf{K}^{\mathbf{T}}\mathbf{y}_1$. This conclusion cannot be reached if we only knew that the two spaces are merely orthogonal.

Complementary orthogonality of the concerned spaces is critical in the derivation of adjoints (see Subsection 7.3.4).

We observe that the proof of the Implicit Duality Theorem depends on two facts:

- The spaces that we deal with satisfy $(\mathcal{V}^{\perp})^{\perp} = \mathcal{V}$, and
- they have finite bases or, more generally, are finitely generated.

In order to generalize Theorem 7.1.1 we could look for situations where the above two conditions hold. Lemma 7.1.2 has a conical version (Farkas Lemma) and an integral version (a variation of Van der Waerden's Theorem). In both these cases the collections of vectors of interest are finitely generated. So our initial proofs assume this. For completeness we later give a proof which works even when 'finitely generated' is not assumed (Problem 7.14).

For the following exercises take $\mathcal{V}_K, \mathcal{V}_{KL}$, where K, L are disjoint sets, to be vector spaces on $K, K \uplus L$ respectively.

Exercise 7.1 *If $S_1 \cap S_2 = \emptyset$, and $\mathcal{V}_1, \mathcal{V}_2$ denote the vector spaces $\mathcal{V}_{S_1}, \mathcal{V}_{S_2}$, show that*
i. $(\mathcal{V}_1 \oplus \mathcal{V}_2)^\perp = \mathcal{V}_1^\perp \oplus \mathcal{V}_2^\perp$,
ii. $(\mathcal{V}_1 \oplus \mathcal{V}_2) \leftrightarrow \mathcal{V}_2 = \mathcal{V}_1$.

Exercise 7.2 *Let $\mathcal{V}_1, \mathcal{V}_2$ be vector spaces on $S \uplus P$ and let $\mathcal{V}_P, \mathcal{V}_P^2$ be vector spaces on P. Show that*
i. $(\mathcal{V}_1 + \mathcal{V}_2) \leftrightarrow \mathcal{V}_P \supseteq (\mathcal{V}_1 \leftrightarrow \mathcal{V}_P) + (\mathcal{V}_2 \leftrightarrow \mathcal{V}_P)$
ii. $(\mathcal{V}_1 \cap \mathcal{V}_2) \leftrightarrow \mathcal{V}_P \subseteq (\mathcal{V}_1 \leftrightarrow \mathcal{V}_P) \cap (\mathcal{V}_2 \leftrightarrow \mathcal{V}_P)$
iii. $(\mathcal{V}_1 \leftrightarrow (\mathcal{V}_P + \mathcal{V}_P^2)) \supseteq (\mathcal{V}_1 \leftrightarrow \mathcal{V}_P) + (\mathcal{V}_1 \leftrightarrow \mathcal{V}_P^2)$
iv. $(\mathcal{V}_1 \leftrightarrow (\mathcal{V}_P \cap \mathcal{V}_P^2)) \subseteq (\mathcal{V}_1 \leftrightarrow \mathcal{V}_P) \cap (\mathcal{V}_2 \leftrightarrow \mathcal{V}_P^2)$
v. $(\mathcal{V}_1 \leftrightarrow \mathcal{V}_P)^\perp = (\mathcal{V}_1 - \mathcal{V}_P)^\perp \cdot S$,
where $(\mathcal{V}_1 - \mathcal{V}_P)$ is a vector space on $S \uplus P$.

Exercise 7.3 Changes in the spaces which leave the generalized minor unaltered:
Show that
i. $\mathcal{V}_{SP} \leftrightarrow \mathcal{V}_P = \mathcal{V}_{SP} \leftrightarrow (\mathcal{V}_P \cap (\mathcal{V}_{SP} \cdot P))$
ii. $\mathcal{V}_{SP} \leftrightarrow \mathcal{V}_P = (\mathcal{V}_{SP} \cap (\mathcal{V}_{SP} \cdot S \oplus \mathcal{V}_P)) \leftrightarrow \mathcal{V}_P$
iii. *Let $\hat{\mathcal{V}}_P$ be a vector space on P s.t. $\hat{\mathcal{V}}_P - \mathcal{V}_{SP} \times P = \mathcal{V}_P - \mathcal{V}_{SP} \times P$.*
Then $\mathcal{V}_{SP} \leftrightarrow \hat{\mathcal{V}}_P = \mathcal{V}_{SP} \leftrightarrow \mathcal{V}_P$.

Exercise 7.4 Duality of contraction and restriction:
Let $Q \subseteq T \subseteq S_1$. Prove using the Implicit Duality Theorem
i. $(\mathcal{V} \cdot T)^\perp = \mathcal{V}^\perp \times T$
ii. $(\mathcal{V} \times T)^\perp = \mathcal{V}^\perp \cdot T$
iii. $(\mathcal{V} \times T \cdot Q)^\perp = \mathcal{V}^\perp \cdot T \times Q$.

Exercise 7.5 When can a space be a generalized minor of another? *Let $\mathcal{V}_{SP}, \mathcal{V}_S$ be vector spaces on $S \uplus P, S$ respectively. Then there exists a vector space $\mathcal{V}_P$ on P s.t. $\mathcal{V}_{SP} \leftrightarrow \mathcal{V}_P = \mathcal{V}_S$*
iff $\mathcal{V}_{SP} \times S \subseteq \mathcal{V}_S \subseteq \mathcal{V}_{SP} \cdot S$.

7.1.2 Matched and Skewed Sums

For applications such as the decomposition of multiports it would be useful to have an extension of the '$\leftrightarrow$' notation to the situation where one of the underlying sets is not contained in the other.

Definition 7.1.2 *Let $\mathcal{K}_{S_1}, \mathcal{K}_{S_2}$ be collections of vectors on sets S_1, S_2 respectively. Then the* **matched sum** *$\mathcal{K}_{S_1} \leftrightarrow \mathcal{K}_{S_2}$ is defined by*

$$\mathcal{K}_{S_1} \leftrightarrow \mathcal{K}_{S_2} \equiv \{\mathbf{f} : \mathbf{f} = \mathbf{f}_1/(S_1 - S_2) \oplus \mathbf{f}_2/(S_2 - S_1), \text{ where } \mathbf{f}_1 \in \mathcal{K}_{S_1}, \mathbf{f}_2 \in \mathcal{K}_{S_2} \text{ and } \mathbf{f}_1/S_1 \cap S_2 = \mathbf{f}_2/S_1 \cap S_2\}.$$

The **skewed sum** *$\mathcal{K}_{S_1} \rightleftharpoons \mathcal{K}_{S_2}$ is defined by*

$$\mathcal{K}_{S_1} \rightleftharpoons \mathcal{K}_{S_2} \equiv \{\mathbf{f} : \mathbf{f} = \mathbf{f}_1/(S_1 - S_2) \oplus \mathbf{f}_2/(S_2 - S_1), \text{ where } \mathbf{f}_1 \in \mathcal{K}_{S_1}, \mathbf{f}_2 \in \mathcal{K}_{S_2} \text{ and } \mathbf{f}_1/S_1 \cap S_2 = -\mathbf{f}_2/S_1 \cap S_2\}.$$

The reader may verify that if $S_1,\ S_2$ are disjoint, the matched sum and skewed sum both correspond to direct sum. If $S_2 \subseteq S_1$, they correspond to generalised minor.

We now have the following useful corollary to the Implicit Duality Theorem.

Corollary 7.1.1 *Let $\mathcal{V}_{S_1}, \mathcal{V}_{S_2}$ be vector spaces on sets S_1, S_2 respectively and let $< \cdot, \cdot >$ be the usual dot product operation. Then, $(\mathcal{V}_{S_1} \leftrightarrow \mathcal{V}_{S_2})^\perp = (\mathcal{V}_{S_1}^\perp \rightleftharpoons \mathcal{V}_{S_2}^\perp)$.*

Proof : Let P_1, P_2 be disjoint copies of $S_1 \cap S_2$, with $e \in S_1 \cap S_2$ corresponding to e_1 in P_1 and e_2 in P_2. Let $S_1{}' \equiv (S_1 - S_2) \cup P_1$ and let $S_2{}' \equiv (S_2 - S_1) \cup P_2$. Let $\mathcal{V}'_{S_1}, \mathcal{V}'_{S_2}$ be copies of $\mathcal{V}_{S_1}, \mathcal{V}_{S_2}$ built on $S_1{}', S_2{}'$ respectively as follows:

$$\mathcal{V}'_{S_1} \equiv \{\mathbf{f}' : \mathbf{f}'/S_1 - S_2 = \mathbf{f}/S_1 - S_2,\ \ \mathbf{f}'(e_1) = \mathbf{f}(e), \forall e \in S_1 \cap S_2,\ \mathit{for\ some}\ \mathbf{f} \in \mathcal{V}_{S_1}\}$$

and $\mathcal{V}'_{S_2}$ is defined similarly with respect to $\mathcal{V}_{S_2}$. Let $\mathcal{V}_{12}$ be the vector space on $P_1 \uplus P_2$ with the representative matrix

$$\begin{array}{cc} P_1 & P_2 \\ \multicolumn{2}{c}{\left[\begin{array}{cc} \mathbf{I} & \mathbf{I} \end{array}\right]} \end{array}$$

in which each row has a $+1$ in the columns of elements corresponding to one element of $S_1 \cap S_2$. It is now clear that

$$\mathcal{V}_{S_1} \leftrightarrow \mathcal{V}_{S_2} = (\mathcal{V}'_{S_1} \oplus \mathcal{V}'_{S_2}) \leftrightarrow \mathcal{V}_{12}$$

and therefore,

$$(\mathcal{V}_{S_1} \leftrightarrow \mathcal{V}_{S_2})^\perp = ((\mathcal{V}'_{S_1} \oplus \mathcal{V}'_{S_2}) \leftrightarrow \mathcal{V}_{12})^\perp$$

The RHS of the above equation, by Theorem 7.1.1, reduces to $(\mathcal{V}'_{S_1} \oplus \mathcal{V}'_{S_2})^\perp \leftrightarrow \mathcal{V}_{12}^\perp$. Since

$$(\mathcal{V}'_{S_1} \oplus \mathcal{V}'_{S_2})^\perp = (\mathcal{V}'_{S_1})^\perp \oplus (\mathcal{V}'_{S_2})^\perp,$$

this reduces to

$$((\mathcal{V}'_{S_1})^\perp \oplus (\mathcal{V}'_{S_2})^\perp) \leftrightarrow \mathcal{V}_{12}^\perp.$$

Thus, a vector $\mathbf{f}$ on $(S_1 - S_2) \cup (S_2 - S_1)$ belongs to the RHS iff there exist vectors $\mathbf{f}_1{}' \in (\mathcal{V}'_{S_1})^\perp$, $\mathbf{f}_2{}' \in (\mathcal{V}'_{S_2})^\perp$, s.t. $\mathbf{f}_1{}'/(S_1 - S_2) = \mathbf{f}/(S_1 - S_2)$, $\mathbf{f}_2{}'/(S_2 - S_1) = \mathbf{f}/(S_2 - S_1)$ and $\mathbf{f}_1{}'/P_1 \oplus \mathbf{f}_2{}'/P_2 \in \mathcal{V}_{12}^\perp$. Now the last condition is equivalent to $\mathbf{f}_1{}'(e_1) = -\mathbf{f}_2{}'(e_2)\ \forall e \in S_1 \cap S_2$, since $\mathcal{V}_{12}^\perp$ has the representative matrix

$$\begin{array}{cc} P_1 & P_2 \\ \multicolumn{2}{c}{\left[\begin{array}{cc} \mathbf{I} & -\mathbf{I} \end{array}\right].} \end{array}$$

Now $(\mathcal{V}'_{S_1})^\perp, (\mathcal{V}'_{S_2})^\perp$ are copies of $\mathcal{V}_{S_1}^\perp$ and $\mathcal{V}_{S_2}^\perp$. Thus,

$$(\mathcal{V}'^\perp_{S_1} \oplus \mathcal{V}'^\perp_{S_2}) \leftrightarrow (\mathcal{V}_{12})^\perp = (\mathcal{V}_{S_1}^\perp \rightleftharpoons \mathcal{V}_{S_2}^\perp)$$

and the corollary follows.

□

Exercise 7.6 *Let $\mathcal{V}_{AB}$ denote a vector space on $A \uplus B$, and $\mathcal{V}_{A1A2\cdots AK}$, a vector space on $A_1 \uplus A_2 \cdots \uplus A_k$. Show that*
i. Restricted associativity:

$$\begin{aligned}(\mathcal{V}_{ST} \leftrightarrow \mathcal{V}_{TP}) \leftrightarrow \mathcal{V}_{PQ} &= \mathcal{V}_{ST} \leftrightarrow (\mathcal{V}_{TP} \leftrightarrow \mathcal{V}_{PQ}) \\ &= (\mathcal{V}_{ST} \oplus \mathcal{V}_{PQ}) \leftrightarrow \mathcal{V}_{TP},\end{aligned}$$

if S, T, P, Q are pairwise disjoint and repeat for skewed sum;
ii.

$$\begin{aligned}(\mathcal{V}_{S_1T_1} \leftrightarrow \mathcal{V}_{T_1P_1}) \oplus (\mathcal{V}_{S_2T_2} \leftrightarrow \mathcal{V}_{T_2P_2}) &= (\mathcal{V}_{S_1T_1} \oplus \mathcal{V}_{S_2T_2}) \leftrightarrow (\mathcal{V}_{T_1P_1} \oplus \mathcal{V}_{T_2P_2}) \\ &= (\mathcal{V}_{S_1T_1} \oplus \mathcal{V}_{T_2P_2}) \leftrightarrow (\mathcal{V}_{S_2T_2} \oplus \mathcal{V}_{T_1P_1}),\end{aligned}$$

if $S_1, T_1, S_2, T_2, P_1, P_2$ are pairwise disjoint and repeat for skewed sum;
iii.

$$(\mathcal{V}_{S_1T_1} \oplus \cdots \oplus \mathcal{V}_{S_nT_n}) \leftrightarrow \mathcal{V}_{T_1T_2\cdots T_n} = (\mathcal{V}_{S_1T_1} \leftrightarrow \mathcal{V}_{T_1T_2\cdots T_n}) \leftrightarrow (\mathcal{V}_{S_2T_2} \oplus \cdots \oplus \mathcal{V}_{S_nT_n}),$$

where the S_i, T_i are all pairwise disjoint, and repeat for skewed sum.

Exercise 7.7 Compatibility:
An ordered pair $(\mathcal{V}_{S_1T}, \mathcal{V}_{S_2T})$, where S_1, S_2, T are pairwise disjoint and $\mathcal{V}_{S_1T}, \mathcal{V}_{S_2T}$ are vector spaces, is said to be **compatible** *iff*

$$\begin{aligned}&\mathcal{V}_{S_1T} \cdot T \supseteq \mathcal{V}_{S_2T} \cdot T \\ \text{and} \quad &\mathcal{V}_{S_1T} \times T \subseteq \mathcal{V}_{S_2T} \times T.\end{aligned}$$

Show that
i. $(\mathcal{V}_{S_1T}, \mathcal{V}_{S_2T})$ *is compatible iff* $(\mathcal{V}^{\perp}_{S_1T}, \mathcal{V}^{\perp}_{S_2T})$ *is;*
ii. *if* $(\mathcal{V}_{S_1T}, \mathcal{V}_{S_2T})$ *is compatible, then* $\mathcal{V}_{S_1T} \leftrightarrow (\mathcal{V}_{S_1T} \leftrightarrow \mathcal{V}_{S_2T}) = \mathcal{V}_{S_2T}$*;*
iii. *if* $(\mathcal{V}_{S_1T}, \mathcal{V}_{S_2T})$ *is compatible then* $\mathcal{V}_{S_1T} \rightleftharpoons (\mathcal{V}_{S_1T} \rightleftharpoons \mathcal{V}_{S_2T}) = \mathcal{V}_{S_2T}$.

7.2 *Quasi Orthogonality

It is convenient to focus attention on the essential properties of 'dot product' and 'orthogonality' which are needed for a result like Theorem 7.1.1 to hold. This would help us in generating other versions of the theorem. We do this through our definition of a 'q-bilinear operation' and 'q-orthogonality'.

For the following discussion $< \cdot, \cdot >$ would be a **quasi bilinear** (**q-bilinear** for short) operation.

Definition 7.2.1 *Let $\mathcal{X}$ be a vector space over the scalar field $\mathcal{F}$. A* **q-bilinear operation** *$< \cdot, \cdot >$ on the collection of all ordered pairs of vectors in $\mathcal{X}$ takes values in $\mathcal{F}$ and satisfies the following conditions:*

$$i. \quad < \alpha\, \mathbf{f} + \beta\, \mathbf{g}, \mathbf{h} >= \alpha < \mathbf{f}, \mathbf{h} > + \beta < \mathbf{g},\ \mathbf{h} >$$

$$ii. \quad <\mathbf{h}, \mathbf{f}+\mathbf{g}>=<\mathbf{h},\ \mathbf{f}>+<\mathbf{h},\ \mathbf{g}>$$

for all vectors $\mathbf{f},\ \mathbf{g},\ \mathbf{h}$ *in* $\mathcal{X}$ *and scalars* $\alpha,\ \beta$.

Remark:

i. The reader would notice that the second condition in the above definition differs from the usual inner product as well as dot product definitions, being weaker. We need a definition which includes these operations as special cases. Further, this weaker condition is adequate for our purposes.

ii. Note that the definition implies that $<\mathbf{f}, \mathbf{0}>=<\mathbf{0}, \mathbf{f}>=0$.

iii. In this section, unless otherwise stated, $<\cdot,\cdot>$ would always denote a q-bilinear operation.

Definition 7.2.2 *Let* $\mathcal{X}$ *be a vector space over the field* $\mathcal{F}$. *Let* $<\cdot,\cdot>$ *be a q-bilinear operation on ordered pairs of vectors in* $\mathcal{X}$. *Let* $\mathcal{A}$ *be a proper subset of* $\mathcal{F}$ *closed under addition and further let it satisfy (i)* $0 \in \mathcal{A}$, *(ii)* $<\mathbf{f}, \mathbf{g}>\in \mathcal{A} \Rightarrow <\mathbf{g}, \mathbf{f}>\in \mathcal{A}$ *for all vectors* $\mathbf{g}, \mathbf{f} \in \mathcal{X}$. *Vectors* $\mathbf{f}, \mathbf{g} \in \mathcal{X}$ *are* **orthogonal** *iff* $<\mathbf{f}, \mathbf{g}>=<\mathbf{g}, \mathbf{f}>=0$ *and* **q-orthogonal** *iff* $<\mathbf{f}, \mathbf{g}>\in \mathcal{A}$.

We mention some examples of q-orthogonality.

- Clearly orthogonality is a special case of q-orthogonality. Here $\mathcal{A} \equiv \{0\}$.
- We say two real vectors $\mathbf{x}, \mathbf{y}$ are polar iff their dot product is nonpositive. So polarity is a special case of q-orthogonality taking $\mathcal{A}$ to be the set of all nonpositive real numbers.
- We say two real vectors $\mathbf{x}, \mathbf{y}$ are integrally dual iff their dot product is an integer. In this case we take $\mathcal{A}$ to be the set of all integers so that integral duality becomes a special case of q-orthogonality.

The vector $\mathbf{0}$ is easily seen to be orthogonal to all vectors in $\mathcal{X}$ and is therefore also q-orthogonal to them.
For a collection $\mathcal{K} \subseteq \mathcal{X}, \mathcal{K}^{\perp}$ would denote the collection of all vectors orthogonal to every vector in $\mathcal{K}$, and $\mathcal{K}^{*}$ would denote the collection of all vectors q-orthogonal to every vector in $\mathcal{K}$. We now have the following simple lemma.

Lemma 7.2.1 *Let* $\mathcal{V}$ *be a subspace of* $\mathcal{X}$. *Then*

$$\mathcal{V}^{*}=\mathcal{V}^{\perp}.$$

Proof : Let $\mathbf{f} \in \mathcal{V}$ and $\mathbf{g} \in \mathcal{V}^{*}$ s.t. $<\mathbf{f}, \mathbf{g}>=\alpha \neq 0$. Let $\beta \in \mathcal{F}-\mathcal{A}$. Now $\beta \alpha^{-1} \mathbf{f} \in \mathcal{V}$. Further $<(\beta \alpha^{-1}) \mathbf{f}, \mathbf{g}>=\beta \alpha^{-1} \alpha=\beta \notin \mathcal{A}$. Thus, $\mathbf{g} \notin \mathcal{V}^{*}$, which is a contradiction. We conclude that $\alpha=0$ and therefore, $\mathbf{g} \in \mathcal{V}^{\perp}$.

□

Let us say that a collection of vectors $\mathcal{K} \subseteq \mathcal{X}$ is **closed** under q-orthogonality iff $(\mathcal{K}^{*})^{*}=\mathcal{K}$. We then have

Lemma 7.2.2 *Let* $\mathcal{K}_1, \mathcal{K}_2 \subseteq \mathcal{X}$ *with* $\mathbf{0} \in \mathcal{K}_1 \cap \mathcal{K}_2$. *Then* *i.* $(\mathcal{K}_1 + \mathcal{K}_2)^* = {\mathcal{K}_1}^* \cap {\mathcal{K}_2}^*$,

ii. *If* $\mathcal{K}_1, \mathcal{K}_2, {\mathcal{K}_1}^* + {\mathcal{K}_2}^*$ *are closed under q-orthogonality, then*

$$(\mathcal{K}_1 \cap \mathcal{K}_2)^* = {\mathcal{K}_1}^* + {\mathcal{K}_2}^*.$$

Proof :
i. Let $\mathbf{g} \in (\mathcal{K}_1 + \mathcal{K}_2)^*$. Since $\mathbf{0} \in \mathcal{K}_1 \cap \mathcal{K}_2$, we must have $\mathbf{g} \in {\mathcal{K}_1}^*$ as well as $\mathbf{g} \in {\mathcal{K}_2}^*$. Next let $\mathbf{g}' \in {\mathcal{K}_1}^* \cap {\mathcal{K}_2}^*$. Let $\mathbf{f}_1 \in \mathcal{K}_1, \mathbf{f}_2 \in \mathcal{K}_2$. We have $< \mathbf{f}_1 + \mathbf{f}_2,\ \mathbf{g}' > = < \mathbf{f}_1,\ \mathbf{g}' > + < \mathbf{f}_2,\ \mathbf{g}' >$. Now $< \mathbf{f}_1,\ \mathbf{g}' >, < \mathbf{f}_2,\ \mathbf{g}' > \in \mathcal{A}$. Since $\mathcal{A}$ is closed under addition it follows that $< \mathbf{f}_1 + \mathbf{f}_2,\ \mathbf{g}' > \in \mathcal{A}$. Thus, $\mathbf{g}'$ is q-orthogonal to every vector in $\mathcal{K}_1 + \mathcal{K}_2$. We conclude

$$(\mathcal{K}_1 + \mathcal{K}_2)^* = {\mathcal{K}_1}^* \cap {\mathcal{K}_2}^*.$$

ii. We have

$$\begin{aligned}({\mathcal{K}_1}^* + {\mathcal{K}_2}^*)^* &= ({\mathcal{K}_1}^*)^* \cap ({\mathcal{K}_2}^*)^* \\ &= \mathcal{K}_1 \cap \mathcal{K}_2.\end{aligned}$$

Hence, $(({\mathcal{K}_1}^* + {\mathcal{K}_2}^*)^*)^* = (\mathcal{K}_1 \cap \mathcal{K}_2)^*$
i.e., $(\mathcal{K}_1 \cap \mathcal{K}_2)^* = {\mathcal{K}_1}^* + {\mathcal{K}_2}^*$, since $({\mathcal{K}_1}^* + {\mathcal{K}_2}^*)$ is closed under q-orthogonality.

□

Lemma 7.2.3 *Let* $\mathcal{K}_1, \mathcal{K}_2 \subseteq \mathcal{X}$. *Then the following hold.*

i. ${\mathcal{K}_1}^{**} \supseteq \mathcal{K}_1$.

ii. *If* $\mathcal{K}_1 \supseteq \mathcal{K}_2$ *then* ${\mathcal{K}_1}^* \subseteq {\mathcal{K}_2}^*$.

iii. ${\mathcal{K}_1}^*$ *is closed under q-orthogonality.*

iv. *If* $\mathcal{K}_1, \mathcal{K}_2$ *are closed then* $\mathcal{K}_1 \cap \mathcal{K}_2$ *is closed under q-orthogonality.*

Proof :
i.,ii. are immediate from the definition of q-orthogonality.

iii. We have $({\mathcal{K}_1}^*)^{**} \supseteq {\mathcal{K}_1}^*$.
Next $(({\mathcal{K}_1}^*)^*)^* = ({\mathcal{K}_1}^{**})^* \subseteq {\mathcal{K}_1}^*$, since ${\mathcal{K}_1}^{**} \supseteq \mathcal{K}_1$. The result follows.

iv. $(\mathcal{K}_1 \cap \mathcal{K}_2)^* \supseteq {\mathcal{K}_1}^*$,
$(\mathcal{K}_1 \cap \mathcal{K}_2)^* \supseteq {\mathcal{K}_2}^*$, since $\mathcal{K}_1 \cap \mathcal{K}_2 \subseteq \mathcal{K}_1, \mathcal{K}_2$.
Hence, $((\mathcal{K}_1 \cap \mathcal{K}_2)^*)^* \subseteq ({\mathcal{K}_1}^*)^* \cap ({\mathcal{K}_2}^*)^* = \mathcal{K}_1 \cap \mathcal{K}_2$. But for any $\mathcal{K}$, we must have $\mathcal{K}^{**} \supseteq \mathcal{K}$. Hence, $(\mathcal{K}_1 \cap \mathcal{K}_2)^{**} \supseteq \mathcal{K}_1 \cap \mathcal{K}_2$. The result follows.

□

In the following lemma $\mathbf{x}_P$ denotes a vector on P.

Lemma 7.2.4 *Let* $\mathcal{K}_S$ *be a collection of vectors on* S *and let* $T \subseteq S$. *Then* $(\mathcal{K}_S \cdot T)^* = {\mathcal{K}_S}^* \times T$, *if, for all* $\mathbf{f}_T, \mathbf{f}_{S-T}, \mathbf{g}_T, \mathbf{g}_{S-T}$, *we have* $< \mathbf{f}_T \oplus \mathbf{f}_{S-T}, \mathbf{g}_T \oplus \mathbf{g}_{S-T} > = < \mathbf{f}_T,\ \mathbf{g}_T > + < \mathbf{f}_{S-T},\ \mathbf{g}_{S-T} >$.

Proof : Let $\mathbf{g}_T \in (\mathcal{K}_S \cdot T)^*$. Let $\mathbf{f}_S \in \mathcal{K}_S$. Then
$< \mathbf{f}_S, \mathbf{g}_T \oplus \mathbf{0}_{S-T} >=< \mathbf{f}_T \oplus \mathbf{f}_{S-T}, \mathbf{g}_T \oplus \mathbf{0}_{S-T} > = < \mathbf{f}_T, \mathbf{g}_T > \in \mathcal{A}$.
Hence, $\mathbf{g}_T \in \mathcal{K}_S{}^* \times T$.
On the other hand let $\mathbf{g}_T \in \mathcal{K}_S{}^* \times T$. Then $\mathbf{g}_T \oplus \mathbf{0}_{S-T} \in \mathcal{K}_S{}^*$. If $\mathbf{f}_T \in \mathcal{K}_S \cdot T$, there exists $\mathbf{f}_{S-T}$ s.t. $\mathbf{f}_T \oplus \mathbf{f}_{S-T} \in \mathcal{K}_S$. We then have $< \mathbf{f}_T, \mathbf{g}_T >$
$=< \mathbf{f}_T \oplus \mathbf{f}_{S-T}, \mathbf{g}_T \oplus \mathbf{0}_{S-T} > \in \mathcal{A}$. Hence, $\mathbf{g}_T \in (\mathcal{K}_S \cdot T)^*$.

□

7.3 Applications of the Implicit Duality Theorem

In this section we list applications of the Implicit Duality Theorem. Some of these are discussed in more detail in subsequent chapters.

7.3.1 Ideal Transformer Connections

An ideal transformer on a set of ports S is a pair $(\mathcal{V}_S, \mathcal{V}_S^\perp)$ of complementary orthogonal spaces on S. The constraints of the transformer are: $\mathbf{v}_S \in \mathcal{V}_S, \mathbf{i}_S \in \mathcal{V}_S^\perp$ where $\mathbf{v}_S, \mathbf{i}_S$ are the port voltage and current vectors. For example, the two port ideal transformer satisfies: $v_1 = nv_2, i_1 = -\frac{1}{n}i_2$. Thus, the voltage space has the representative matrix $(n : 1)$ and the current space has the representative matrix $(1 : -n)$. The two spaces are clearly complementary orthogonal. It is a well known fact in network theory that

> ($\dashv\vdash$) *if a set of 2-port transformers are connected together in an arbitrary manner and some ports exposed then on the exposed set of ports the permissible voltage and current vectors form complementary orthogonal spaces*

(see for instance [Belevitch68]). Usually, however, only one half of this fact is proved, namely, that the current and voltage vectors are orthogonal. We prove the result using the Implicit Duality Theorem.
First we observe that if $\mathcal{G}$ is a graph then, by Tellegen's Theorem, $(\mathcal{V}_v(\mathcal{G}), \mathcal{V}_i(\mathcal{G}))$ constitutes an ideal transformer. Next, if $(\mathcal{V}_{SQ}, \mathcal{V}_{SQ}^\perp), (\mathcal{V}_Q, \mathcal{V}_Q^\perp)$ are ideal transformers with $\mathcal{V}_{SQ}, \mathcal{V}_Q$ denoting vector spaces on $S \uplus Q,\ Q$ respectively, then equivalent to the Implicit Duality Theorem is the statement that $(\mathcal{V}_{SQ} \leftrightarrow \mathcal{V}_Q, \mathcal{V}_{SQ}^\perp \leftrightarrow \mathcal{V}_Q^\perp)$ is an ideal transformer. For the following discussion we take
$Q \equiv E(\mathcal{G}),\ S \uplus Q \equiv \biguplus P_j$, i.e., $S \equiv (\biguplus P_j - Q)$.
Now let the j^{th} ideal transformer be $(\mathcal{V}_{P_j}, \mathcal{V}_{P_j}^\perp)$ and let the P_j's be all disjoint. This disconnected set of transformers constitutes the ideal transformer $(\bigoplus_j \mathcal{V}_{P_j}, \bigoplus_j \mathcal{V}_{P_j}^\perp)$. Let the set of ports which are connected together according to the graph $\mathcal{G}$ be Q. The set of exposed ports is $\biguplus P_j - Q$. If the ports Q form the edges of a graph $\mathcal{G}$, then on the set P we are imposing the constraints of the ideal transformer $(\mathcal{V}_v(\mathcal{G}), \mathcal{V}_i(\mathcal{G}))$. The voltage vectors that can exist on ports $\biguplus P_j - Q$ are precisely

those in $(\bigoplus_j \mathcal{V}_{P_j}) \leftrightarrow \mathcal{V}_v(\mathcal{G})$ and the current vectors that exist on ports $\biguplus P_j - Q$ are precisely those in $(\bigoplus_j \mathcal{V}^{\perp}_{P_j}) \leftrightarrow \mathcal{V}_i(\mathcal{G})$. By the Implicit Duality Theorem the last two spaces are complementary orthogonal. Equivalently,

$$\left((\bigoplus_j \mathcal{V}_{P_j}) \leftrightarrow \mathcal{V}_v(\mathcal{G}) \ , \ (\bigoplus_j \mathcal{V}^{\perp}_{P_j}) \leftrightarrow \mathcal{V}_i(\mathcal{G}) \right)$$

is an ideal transformer.

Further, the fact ($\dashv\vdash$) implies the Implicit Duality Theorem and is therefore, equivalent to it. For, by connecting 2-port ideal transformers and exposing ports appropriately it can be seen that any ideal transformer can be built (see for instance Exercise 7.8). Thus, $(\mathcal{V}_{SP}, \mathcal{V}^{\perp}_{SP})$, $(\mathcal{V}_P, \mathcal{V}^{\perp}_P)$ can be built this way. By plugging the 'P ports' of the first ideal transformer with the second (using ($\dashv\vdash$) again) we get another ideal transformer namely $(\mathcal{V}_{SP} \leftrightarrow \mathcal{V}_P, \ \mathcal{V}^{\perp}_{SP} \leftrightarrow \mathcal{V}^{\perp}_P)$. We conclude that $\mathcal{V}^{\perp}_{SP} \leftrightarrow \mathcal{V}^{\perp}_P = (\mathcal{V}_{SP} \leftrightarrow \mathcal{V}_P)^{\perp}$.

Exercise 7.8 To create a 'graph' using ideal transformers: *Using only 1:1 2-port ideal transformers show how to build an ideal transformer* $(\mathcal{V}_v(\mathcal{G}), \mathcal{V}_i(\mathcal{G}))$ *where* $\mathcal{G}$ *is a specified graph.*

Exercise 7.9 Ideal transformers cannot be connected inconsistently: *Our discussion implies that ideal transformers cannot be connected inconsistently. What would happen if we connect two 2-port transformers of different turns ratio in parallel?*

Exercise 7.10 Effect of an ideal transformer on remaining edges: *Consider an electrical network with graph* $\mathcal{G}$. *Let* $P \subseteq E(\mathcal{G})$ *be the ports of an ideal transformer* $(\mathcal{V}_P, \mathcal{V}^{\perp}_P)$. *What would be the voltage and current constraints on* $(E(\mathcal{G}) - P)$? **Equivalently** *what is the ideal transformer to whose ports the remaining devices of the network are connected?*

7.3.2 Multiport Decomposition

An electrical network can often be conveniently visualized as being made up of a number of multiports whose ports are connected together according to a connection diagram. In the literature it is often not clear that this is essentially a topological notion. We dwell at greater length on this important concept in a separate chapter. Here we merely outline the basic idea.
Let $\mathcal{G}$ be the graph of the electrical network. Let $E \equiv E(\mathcal{G})$ be partitioned into $E_1, \cdots, E_k$. Let P, a set disjoint from E, be partitioned into $P_1, \cdots, P_k$. Let $\mathcal{V}_{E_iP_i}$ be a vector space on $E_i \cup P_i, i = 1, \cdots, k$ and let $\mathcal{V}_P$ be a vector space on P. Then, $(\mathcal{V}_{E_1P_1}, \cdots, \mathcal{V}_{E_kP_k}; \mathcal{V}_P)$ is a **multiport decomposition** of $\mathcal{V}_v(\mathcal{G})$ iff

$$(\bigoplus \mathcal{V}_{E_iP_i}) \leftrightarrow \mathcal{V}_P = \mathcal{V}_v(\mathcal{G}).$$

Usually the spaces $\mathcal{V}_{E_iP_i}$ would be voltage spaces of graphs $\mathcal{G}_{E_iP_i}$ respectively (this is not necessary as we point out in the next chapter). In such a case, we have

Figure 7.1: Multiport Decomposition of a Graph

voltage vectors $\mathbf{v}_{E_1P_1}, \cdots, \mathbf{v}_{E_kP_k}$ of $\mathcal{G}_{E_1P_1}, \cdots, \mathcal{G}_{E_kP_k}$ respectively. s.t. $\mathbf{v}_{E_1P_1}/P_1 \oplus \cdots \oplus \mathbf{v}_{E_kP_k}/P_k$ belongs to $\mathcal{V}_P$ (i.e., their port voltages match) iff $\mathbf{v}_{E_1P_1}/E_1 \oplus \cdots \oplus \mathbf{v}_{E_kP_k}/E_k$ belongs to $\mathcal{V}_v(\mathcal{G})$.

Let us call a graph $\mathcal{G}_{E_1P_1}$ on $E_1 \uplus P_1$ with set P_1 specified as 'ports' as a multiport (i.e., the multiport is the pair $(\mathcal{G}_{E_1P_1}, P_1)$).
In Figure 7.1, the graph $\mathcal{G}$ is decomposed into multiports $\mathcal{G}_{E_1P_1}, \mathcal{G}_{E_2P_2}$ connected according to the port connection diagram $\mathcal{G}_{P_1P_2}$. A voltage vector $(v_1, v_2, v_3, v_4, v_5, v_6)$ belongs to $\mathcal{V}_v(\mathcal{G})$ iff we can find vectors (v_{11}, v_{12}), (v_{21}, v_{22}) s.t. $(v_1, v_2, v_{11}, v_{12}) \in \mathcal{V}_v(\mathcal{G}_{E_1P_1})$, $(v_3, v_4, v_5, v_6, v_{21}, v_{22}) \in \mathcal{V}_v(\mathcal{G}_{E_2P_2})$ and $(v_{11}, v_{12}, v_{21}, v_{22}) \in \mathcal{V}_v(\mathcal{G}_{P_1P_2})$.
In our notation we say, if the above condition is satisfied, that

$$(\mathcal{V}_v(\mathcal{G}_{E_1P_1}), \mathcal{V}_v(\mathcal{G}_{E_2P_2}); \mathcal{V}_v(\mathcal{G}_{P_1P_2}))$$

is a multiport decomposition of $\mathcal{V}_v(\mathcal{G})$.
We expect intuitively that the multiport decomposition represented through graphs should work for both voltages and currents. This fact requires proof. Essentially we need to show that $(\mathcal{V}_i(\mathcal{G}_{E_1P_1}), \mathcal{V}_i(\mathcal{G}_{E_2P_2}); \mathcal{V}_i(\mathcal{G}_{P_1P_2}))$ is a multiport decomposition of $\mathcal{V}_i(\mathcal{G})$, whenever $(\mathcal{V}_v(\mathcal{G}_{E_1P_1}), \mathcal{V}_v(\mathcal{G}_{E_2P_2}); \mathcal{V}_v(\mathcal{G}_{P_1P_2}))$ is a multiport decomposition of $\mathcal{V}_v(\mathcal{G})$. Now, $\mathcal{V}_v(\mathcal{G}) = (\mathcal{V}_v(\mathcal{G}_{E_1P_1}) \oplus \mathcal{V}_v(\mathcal{G}_{E_2P_2})) \leftrightarrow \mathcal{V}_v(\mathcal{G}_{P_1P_2})$. Hence, by Implicit Duality Theorem

$$(\mathcal{V}_v(\mathcal{G}))^{\perp} = ((\mathcal{V}_v(\mathcal{G}_{E_1P_1}))^{\perp} \oplus (\mathcal{V}_v(\mathcal{G}_{E_2P_2}))^{\perp}) \leftrightarrow (\mathcal{V}_v(\mathcal{G}_{P_1P_2}))^{\perp}.$$

i.e., $\mathcal{V}_i(\mathcal{G}) = (\mathcal{V}_i(\mathcal{G}_{E_1P_1}) \oplus \mathcal{V}_i(\mathcal{G}_{E_2P_2})) \leftrightarrow \mathcal{V}_i(\mathcal{G}_{P_1P_2})$, as required.

When an idea is intuitive in terms of graphs why bring in vector spaces? Here are some reasons:

- The graph version is misleading: if we actually connect the multiports along their ports according to the connection diagram, we usually would not get a graph with the same voltage space as $\mathcal{G}$.
- It is inadequate for optimization purposes: for instance, we can usually reduce the number of port edges if we formulate decomposition as a vector space problem.
- While analysing the network, the vector spaces that we work with need not necessarily be associated with graphs - it is sufficient that their representative matrices be sparse and, preferably, $0, 1$.

7.3.3 Topological Transformation Of Electrical Networks

A general way of looking at Network Analysis through Decomposition is to view it as a way of modifying network structure: a desired structure is imposed on the network at the cost of additional variables and additional constraints [Narayanan87]. We give a sketch of the method. The technique is valid for arbitrary networks. For linear networks, we can do more: we can prove bounds on the effort involved, making certain additional assumptions.
Let the given network have graph $\mathcal{G}$ and device characteristic $\mathbf{N}(\mathbf{v}_E - \mathbf{e}) + \mathbf{M}(\mathbf{i}_E - \mathbf{j}) = \mathbf{0}$, where $\mathbf{v}$, $\mathbf{i}$ are the branch voltage and current vectors, $\mathbf{e}$,$\mathbf{j}$ are the voltage source and current source vectors. Each branch is composite made up of a device in series with a voltage source, the combination being in parallel with a current source.

Network analysis entails the solution of the following constraints:

$$\mathbf{A}_r \mathbf{i}_E = \mathbf{0}$$

$$\mathbf{v}_E - \mathbf{A}_r{}^T \mathbf{v}_n = \mathbf{0}$$

$$\mathbf{N}(\mathbf{v}_E - \mathbf{e}) + \mathbf{M}(\mathbf{i}_E - \mathbf{j}) = \mathbf{0}.$$

Suppose that the desired structure is a graph $\mathcal{G}'$ on the same set of edges as $\mathcal{G}$. Let $\mathcal{V}, \mathcal{V}'$ be the voltage spaces of graphs $\mathcal{G}, \mathcal{G}'$ respectively. We look for a space $\mathcal{V}_{EP}$ s.t. $\mathcal{V} = \mathcal{V}_{EP} \leftrightarrow \mathcal{V}_P$ and $\mathcal{V}' = \mathcal{V}_{EP} \leftrightarrow \mathcal{V}'_P$, i.e., $\mathcal{V}_{EP}$ is an **extension** of both $\mathcal{V}$ and $\mathcal{V}'$. It is desirable that $\mid P \mid$ is minimized since, as we shall show, each element of P is associated with an additional variable.
The space $\mathcal{V}_{EP}$ can be built using $\mathcal{V} + \mathcal{V}'$ as follows. Let $\begin{bmatrix} \mathbf{R}_1 \\ \mathbf{R}_\cap \end{bmatrix}$, $\begin{bmatrix} \mathbf{R}_\cap \\ \mathbf{R}_2 \end{bmatrix}$, $\mathbf{R}_\cap$ be the representative matrices of $\mathcal{V}, \mathcal{V}', \mathcal{V} \cap \mathcal{V}'$ respectively. We take the representative matrix of $\mathcal{V}_{EP}$ to be

$$\begin{array}{c} \begin{array}{ccc} E & P_1 & P_2 \end{array} \\ \left[\begin{array}{ccc} \mathbf{R}_1 & \mathbf{I} & \mathbf{0} \\ \mathbf{R}_\cap & \mathbf{0} & \mathbf{0} \\ \mathbf{R}_2 & \mathbf{0} & \mathbf{I} \end{array}\right] \end{array},$$

that of $\mathcal{V}'_P$ to be

$$\begin{array}{c} \begin{array}{cc} P_1 & P_2 \end{array} \\ \left[\begin{array}{cc} \mathbf{0} & \mathbf{I} \end{array}\right] \end{array}$$

and that of $\mathcal{V}_P$ to be

$$\begin{array}{c} \begin{array}{cc} P_1 & P_2 \end{array} \\ \left[\begin{array}{cc} \mathbf{I} & \mathbf{0} \end{array}\right] \end{array}.$$

It is then immediate that $\mathcal{V}_{EP} \leftrightarrow \mathcal{V}_P = \mathcal{V}$ and $\mathcal{V}_{EP} \leftrightarrow \mathcal{V}'_P = \mathcal{V}'$. By a suitable row transformation, the representative matrix of $\mathcal{V}_{EP}$ can be put in the form (for an appropriate representative matrix $\mathbf{A}'_r$ of $\mathcal{V}'$)

$$\begin{array}{c} \begin{array}{ccc} E & P_1 & P_2 \end{array} \\ \left[\begin{array}{ccc} \mathbf{R}_1 & \mathbf{I} & \mathbf{0} \\ \mathbf{A}_r{}' & \mathbf{0} & \mathbf{R}_{P_2} \end{array}\right] \end{array},$$

KVE for $\mathcal{G}$ can be written as:

$$\left[\begin{array}{c} \mathbf{v}_E \\ \mathbf{v}_{P_1} \\ \mathbf{v}_{P_2} \end{array}\right] - \left[\begin{array}{cc} \mathbf{R}_1^T & (\mathbf{A}_r{}')^T \\ \mathbf{I} & \mathbf{0}_T \\ \mathbf{0} & \mathbf{R}_{P_2}^T \end{array}\right] \begin{array}{c} \lambda_1 \\ \mathbf{v}_n{}' \end{array} = \left[\begin{array}{c} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{array}\right] \tag{7.1}$$

$$\left[\begin{array}{c} \mathbf{v}_{P_1} \\ \mathbf{v}_{P_2} \end{array}\right] - \left[\begin{array}{c} \mathbf{I} \\ \mathbf{0} \end{array}\right] \lambda_2 = \left[\begin{array}{c} \mathbf{0} \\ \mathbf{0} \end{array}\right] \tag{7.2}$$

As far as $\mathbf{v}_E$ is concerned the above is equivalent to

$$\left[\begin{array}{c} \mathbf{v}_E \\ \mathbf{0} \end{array}\right] - \left[\begin{array}{cc} \mathbf{R}_1^T & (\mathbf{A}_r{}')^T \\ \mathbf{0} & \mathbf{R}_{P_2}^T \end{array}\right] \begin{array}{c} \mathbf{v}_{P_1} \\ \mathbf{v}_n{}' \end{array} = \left[\begin{array}{c} \mathbf{0} \\ \mathbf{0} \end{array}\right] \tag{7.3}$$

By the Implicit Duality Theorem we have,

$$\mathcal{V}_{EP}^\perp \leftrightarrow \mathcal{V}_P^\perp = \mathcal{V}^\perp.$$

So the KCE of $\mathcal{G}$ can be written as

$$\left[\begin{array}{ccc} \mathbf{R}_1 & \mathbf{I} & \mathbf{0} \\ \mathbf{A}_r{}' & \mathbf{0} & \mathbf{R}_{P_2} \end{array}\right] \begin{array}{c} \mathbf{i}_E \\ \mathbf{i}_{P_1} \\ \mathbf{i}_{P_2} \end{array} = \left(\begin{array}{c} \mathbf{0} \\ \mathbf{0} \end{array}\right) \tag{7.4}$$

$$\begin{pmatrix} \mathbf{I} & \mathbf{0} \end{pmatrix} \begin{matrix} \mathbf{i}_{P_1} \\ \mathbf{i}_{P_2} \end{matrix} = \mathbf{0} \tag{7.5}$$

i.e.,

$$\begin{bmatrix} \mathbf{R}_1 & \mathbf{0} \\ \mathbf{A}_r{}' & \mathbf{R}_{P_2} \end{bmatrix} \begin{matrix} \mathbf{i}_E \\ \mathbf{i}_{P_2} \end{matrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \end{pmatrix} \tag{7.6}$$

Thus, the overall constraints can be written as

$$\begin{bmatrix} \mathbf{0} & -(\mathbf{A}_r{}')^T & \mathbf{I} & \vdots & \mathbf{0} & -\mathbf{R}_1^T \\ \mathbf{A}_r{}' & \mathbf{0} & \mathbf{0} & \vdots & \mathbf{R}_{P_2} & \mathbf{0} \\ \mathbf{M} & \mathbf{0} & \mathbf{N} & \vdots & \mathbf{0} & \mathbf{0} \\ \cdots & \cdots & \cdots & \vdots & \cdots & \cdots \\ \mathbf{0} & -\mathbf{R}_{P_2}^T & \mathbf{0} & \vdots & \mathbf{0} & \mathbf{0} \\ \mathbf{R}_1 & \mathbf{0} & \mathbf{0} & \vdots & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{matrix} \mathbf{i}_E \\ \mathbf{v'}_n \\ \mathbf{v}_E \\ \mathbf{i}_{P_2} \\ \mathbf{v}_{P_1} \end{matrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{Mj} + \mathbf{Ne} \\ \cdots \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} \tag{7.7}$$

Notice in Equation 7.7 that the border of the coefficient matrix has size equal to $\mid P_2 \mid + \mid P_1 \mid = \mid P \mid$. The **core of the matrix**, i.e., the left hand top corner of the matrix, is precisely the **constraint coefficient matrix of the 'new' network** with graph $\mathcal{G}'$ but device characteristic same as the original network. Let $\mathcal{G}$ and $\mathcal{G}'$ be near each other in the sense that $r(\mathcal{V}_v(\mathcal{G}) + \mathcal{V}_v(\mathcal{G}'))$ - $r(\mathcal{V}_v(\mathcal{G}) \cap \mathcal{V}_v(\mathcal{G}'))$ is very small in comparison with the ranks and nullities of the spaces involved. Then the form of Equation 7.7 permits us to solve network $\mathcal{N}$ by solving the network $\mathcal{N}'$ for appropriate source distributions, $\mid P \mid +1$ times. There is an additional set of equations of size $\mid P \mid \times \mid P \mid$ that has to be solved after this to complete the solution. (See Exercise 7.15).

Exercise 7.11 Minimum common extension of two vector spaces: *Let $\mathcal{V}, \mathcal{V}'$ be vector spaces on S. Let P be disjoint from S. Let $\mathcal{V}_{SP}$ be a vector space on $S \uplus P$, and $\mathcal{V}_P, \mathcal{V}'_P$, be vector spaces on P. Let $\mathcal{V}_{SP} \leftrightarrow \mathcal{V}_P = \mathcal{V}, \mathcal{V}_{SP} \leftrightarrow \mathcal{V}'_P = \mathcal{V}'$. Show that when the above conditions are satisfied, $\mid P \mid$ is minimum iff it equals*

$$r(\mathcal{V}' + \mathcal{V}) - r(\mathcal{V}' \cap \mathcal{V}).$$

Exercise 7.12 Minimum extension of graphic spaces not always graphic: *Construct a simple example for which the minimum extension $\mathcal{V}_{EP}$ of $\mathcal{V}_v(\mathcal{G}), \mathcal{V}_v(\mathcal{G}')$, where $\mathcal{G}, \mathcal{G}'$ are given graphs, is not the voltage space of a graph.*

Exercise 7.13 *Suppose $\mathcal{G}'$ is made up only of coloops (selfloops). What would the matrices $\mathbf{R}_1, \mathbf{R}_{P_2}$ be?*

Exercise 7.14 *[Narayanan80],[Kajitani+Sakurai+Okamoto]* **A metric on graphs on a given set of edges:** *Define the* **distance** *between two graphs $\mathcal{G}$ and $\mathcal{G}'$ s.t. $E(\mathcal{G}) = E(\mathcal{G}')$ by $d(\mathcal{G}, \mathcal{G}') \equiv r(\mathcal{V}_v(\mathcal{G}) + \mathcal{V}_v(\mathcal{G}')) - r(\mathcal{V}_v(\mathcal{G}) \cap \mathcal{V}_v(\mathcal{G}'))$.*
i. *Show that $d(\cdot, \cdot)$ is a* **metric** *on the space of all graphs with edge set $E(\mathcal{G})$, i.e., $d(\mathcal{G}, \mathcal{G}) = 0$, $d(\mathcal{G}, \mathcal{G}') = d(\mathcal{G}', \mathcal{G})$, $d(\mathcal{G}, \mathcal{G}') + d(\mathcal{G}', \mathcal{G}") \geq d(\mathcal{G}', \mathcal{G}")$.*
ii. *Define $d(\mathcal{V}, \mathcal{V}') \equiv r(\mathcal{V} + \mathcal{V}') - r(\mathcal{V} \cap \mathcal{V}')$, where $\mathcal{V}, \mathcal{V}'$ are vector spaces on E.*
(a) *Show that $d(.,.)$ is a metric on the collection of all vector spaces on E.*
(b) *Show that $d(\mathcal{V}, \mathcal{V}') = d(\mathcal{V}^{\perp}, (\mathcal{V}')^{\perp})$*

Exercise 7.15 To solve $\mathcal{N}$ as though it has the structure of $\mathcal{N}'$: *Let $\mathcal{N}, \mathcal{N}'$ be networks on graphs $\mathcal{G}, \mathcal{G}'$ respectively with the same device characteristic*

$$\mathbf{M}(\mathbf{i}_E - \mathbf{j}) + \mathbf{N}(\mathbf{v}_E - \mathbf{e}) = \mathbf{0}.$$

Assume that $d(\mathcal{G}, \mathcal{G}') \ll min(r(\mathcal{G}), r(\mathcal{G}'))$ and that $\mathcal{N}$ and $\mathcal{N}'$ can be solved uniquely for arbitrary sources. Show that the solution of $\mathcal{N}$ can be obtained essentially by solving $\mathcal{N}'$, $(d(\mathcal{G}, \mathcal{G}') + 1)$ times, for appropriate choices of sources.

7.3.4 The Adjoint of a Linear System

The notion of adjoint of a linear system has extensive theoretical and practical applications. For simplicity, here we confine ourselves to nondynamic systems defined through flow graphs. Also we do not speak of the utility of this notion. Our aim here is only to show how the Implicit Duality Theorem helps us to derive the adjoint in a simple way. A word of caution is in order here. Very often, while deriving the adjoint, authors only show that the original (input, output) vector is orthogonal to the (output, input) vector of the adjoint. This is insufficient. In order to show that the inputs to the adjoint can be chosen **independently**, complementary orthogonality of the concerned spaces has to be shown. The easiest route is through the Implicit Duality Theorem (for the dynamic case based on flow graphs as well as networks and for exposition of such ideas see [Narayanan86b]).

Nondynamic Systems Based On Flow Graphs

The system is composed of a number of **blocks** of the kind '$\mathbf{z}_y = \mathbf{K}\mathbf{z}_u$'. The vector inputs of these blocks are outputs of 'summers' while the outputs of the blocks are inputs to 'connection points'. The **summer** is a device which outputs the vector sum of its vector inputs. The **connection point** is a device each of whose vector outputs is equal to its vector input. (See Figure 7.2.) The overall inputs $\mathbf{u}$ of the system are inputs to summers while the outputs $\mathbf{y}$ are outputs of connection points. Let us suppose the overall input output relation is of the form $\mathbf{y} = \mathbf{K}\mathbf{u}$. We prove the following:

Theorem 7.3.1 *(k) If in the original system the following transformations are made, then the resulting system would satisfy $\hat{\mathbf{y}} = \mathbf{K}^T\hat{\mathbf{u}}$, where $\hat{\mathbf{y}}, \hat{\mathbf{u}}$ are the overall outputs and inputs of the new system:*

Figure 7.2: Summer, Connection Point and Block

i. Reverse direction of arrows everywhere, in particular, the overall outputs become overall inputs and vice versa.

ii. Replace summers by connection points and vice versa (since the arrows are reversed outputs become inputs and inputs, outputs).

iii. Each block of the form '$\mathbf{z}_y = \mathbf{K}\mathbf{z}_u$*' is replaced by one of the form* $\hat{\mathbf{z}}_y = \mathbf{K}^T\hat{\mathbf{z}}_u$ *(since the arrows are reversed outputs becomes inputs and vice versa).*

Before proving the result we adopt some convenient conventions and notation. We assume that
i. overall inputs do not touch a block directly but always go through summers. If necessary a summer with a single input and single output can be introduced to satisfy this condition.
ii. Similarly, overall outputs come only out of connection points.
iii. Output of a block always enters a summer or a connection point.
iv. Distinct summers, connection points, do not have common variables. Output of a summer (connection point) does not directly become input of a connection point (summer). If necessary a dummy 'identity' block could be introduced in between.

The reader may verify for himself that these assumptions cause no loss of generality.
Overall inputs are denoted by $\mathbf{u}$, overall outputs by $\mathbf{y}$, block inputs by $\mathbf{z}_u$ and block outputs by $\mathbf{z}_y$.

A homogeneous system of linear equations would be denoted by

$$\mathbf{F}(\mathbf{x},\mathbf{y},\mathbf{z},\cdots) = \mathbf{0}$$

$$\textit{or} \quad \mathbf{G}(\mathbf{x},\mathbf{y},\mathbf{z},\cdots) = \mathbf{0}$$

where $\mathbf{x},\mathbf{y},\mathbf{z}$ are vectors and together constitute the unknowns of the equations. $\mathbf{F}^{\perp}(\mathbf{u},\mathbf{v},\mathbf{w},\cdots) = \mathbf{0}$ would be the system of equations where solution space is complementary orthogonal to that of $\mathbf{F}(\mathbf{x},\mathbf{y},\mathbf{z},\cdots) = \mathbf{0}$ with $\mathbf{u}$ corresponding to $\mathbf{x}, \mathbf{v}$ to $\mathbf{y}, \mathbf{w}$ to $\mathbf{z}$ and so on. Thus, if $\mathbf{F}(\mathbf{x},\mathbf{y},\mathbf{z}) = \mathbf{0}$ denotes

$$\left[\begin{array}{cc:cc:cc} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{array}\right] \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \\ \mathbf{z} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}, \tag{7.8}$$

then, $\mathbf{F}^{\perp}(\mathbf{u},\mathbf{v},\mathbf{w}) = 0$ denotes

$$\left[\begin{array}{cc:cc:cc} -1 & -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & -1 & 0 & 1 & 0 \\ -1 & 0 & -1 & 0 & 0 & 1 \end{array}\right] \begin{bmatrix} \mathbf{u} \\ \mathbf{v} \\ \mathbf{w} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}. \tag{7.9}$$

We need the following lemma

Lemma 7.3.1 *i. If* $\mathbf{F}_s(\mathbf{x}_1,\cdots,\mathbf{x}_k,\mathbf{v}) = \mathbf{0}$, *where* $\mathbf{x}_1,\cdots,\mathbf{x}_k,\mathbf{v}$ *are vectors, denotes the summer*

$$\mathbf{v} = \mathbf{x}_1 + \cdots + \mathbf{x}_k$$

then $\mathbf{F}_s{}^{\perp}(-\beta_1,\cdots,-\beta_k,\alpha) = \mathbf{0}$ *(***equivalently** $\mathbf{F}_s{}^{\perp}(+\beta_1,\cdots,+\beta_k,-\alpha) = \mathbf{0}$*) is the connection point*

$$\alpha = \beta_1 = \cdots = \beta_k$$

denoted by

$$\mathbf{F}_c(\beta_1,\cdots,\beta_k,\alpha) = \mathbf{0}.$$

ii. If $\mathbf{F}_c(\beta_1,\cdots,\beta_k,\alpha) = \mathbf{0}$ *denotes the connection point as above then,*

$$\mathbf{F}_c{}^{\perp}(-\mathbf{x}_1,\cdots,-\mathbf{x}_k,\mathbf{v}) = \mathbf{0}$$

*(***equivalently** $\mathbf{F}_c{}^{\perp}(\mathbf{x}_1,\cdots,\mathbf{x}_k,-\mathbf{v}) = \mathbf{0}$*) denotes the summer*

$$\mathbf{F}_s(\mathbf{x}_1,\cdots,\mathbf{x}_k,\mathbf{v}) = \mathbf{0}.$$

iii. *If* $\mathbf{G}(\mathbf{x},\mathbf{v}) = 0$ *denotes the block constraint* $\mathbf{v} = \mathbf{K}\mathbf{x}$ *then,*
$\mathbf{G}^{\perp}(-\beta,\alpha) = 0$ *is the block constraint* $\beta = \mathbf{K}^T\alpha$.

The routine proof is omitted.

Proof of Theorem 7.3.1: (We remind the reader that constraints $\mathbf{f}(\mathbf{x},\mathbf{y})$ and $\mathbf{g}(\mathbf{x},\mathbf{z})$ are equivalent in $\mathbf{x}$ iff whenever $(\mathbf{x},\mathbf{y})$ satisfies the former (whenever $(\mathbf{x},\mathbf{z})$ satisfies the latter) there exists a solution $(\mathbf{x},\mathbf{z})$ of the latter (there exists a solution $(\mathbf{x},\mathbf{y})$ of the former)).

The linear system defined through the flow diagram can be denoted by
$\mathbf{F}(\mathbf{u},\mathbf{z}_u,\mathbf{z}_y,\mathbf{y}) = \mathbf{0}$ (Connection constraints:summer and connection points)
$\mathbf{G}(\mathbf{z}_u,\mathbf{z}_y) = \mathbf{0}$ (block constraints).
Let these constraints together be equivalent to $\mathbf{F}_r(\mathbf{u},\mathbf{y}) = \mathbf{0}$ as far as the variables $\mathbf{u},\mathbf{y}$ are concerned. Then the Implicit Duality Theorem assures us that the constraints

$$\mathbf{F}^{\perp}(-\hat{\mathbf{y}},\hat{\mathbf{z}}_y,-\hat{\mathbf{z}}_u,\hat{\mathbf{u}}) = \mathbf{0} \qquad (*)$$
$$\mathbf{G}^{\perp}(\hat{\mathbf{z}}_y,-\hat{\mathbf{z}}_u) = \mathbf{0} \qquad (**)$$

together are equivalent to

$$\mathbf{F}_r^{\perp}(-\hat{\mathbf{y}},\hat{\mathbf{u}}) = \mathbf{0}$$

as far as $(\hat{\mathbf{y}},\hat{\mathbf{u}})$ are concerned.
So if $\mathbf{F}_r(\mathbf{u},\mathbf{y}) = \mathbf{0}$ is equivalent to $\mathbf{y} = \mathbf{K}\mathbf{u}$ we have $\hat{\mathbf{y}} = \mathbf{K}^T\hat{\mathbf{u}}$.
We now show that the above constraints $(*)$ and $(**)$ are obtained by making the already delineated changes in the original flow diagram.
By our convention, the summers, connection points etc. do not have common variables. Similarly distinct block constraints also do not have common variables. Now the connection constraint $\mathbf{F}(\mathbf{u},\mathbf{z}_u,\mathbf{z}_y,\mathbf{y}) = \mathbf{0}$ is a direct sum of a number of summers and connection points (i.e., distinct summers, connection points have no common variables). Thus we can rewrite these constraints as

$$\bigoplus_i \mathbf{F}_{si}(\mathbf{u}_1^i,\cdots,\mathbf{u}_k^i,\mathbf{z}_{y1}^i,\cdots,\mathbf{z}_{ym}^i,\mathbf{z}_u^i) \quad + \bigoplus_j \mathbf{F}_{cj}(\mathbf{y}_1^j,\cdots,\mathbf{y}_t^j,\mathbf{z}_{u1}^j,\cdots,\mathbf{z}_{un}^j,\mathbf{z}_y^j) = \mathbf{0}.$$

To paraphrase, the i^{th} summer has k overall vector inputs, m other vector inputs which are outputs of blocks and a single vector output $\mathbf{z}_u^i$; the j^{th} connection point has t overall vector outputs, n vector outputs which are inputs to blocks and a single vector input $\mathbf{z}_y^j$.
A complementary orthogonal system to the above is

$$\mathbf{F}^{\perp}(-\hat{\mathbf{y}},\hat{\mathbf{z}}_y,-\hat{\mathbf{z}}_u,\hat{\mathbf{u}}) = \mathbf{0}$$

which is equivalent to

$$\bigoplus_i \mathbf{F}_{si}^{\perp}(-\hat{\mathbf{y}}_1^i,\cdots,-\hat{\mathbf{y}}_k^i,-\hat{\mathbf{z}}_{u1}^i,\cdots,-\hat{\mathbf{z}}_{um}^i,\hat{\mathbf{z}}_y^i) \quad +$$
$$\bigoplus_j \mathbf{F}_{cj}^{\perp}(\hat{\mathbf{u}}_1^j,\cdots,\hat{\mathbf{u}}_t^j,\hat{\mathbf{z}}_{y1}^j,\cdots,\hat{\mathbf{z}}_{yn}^j,-\hat{\mathbf{z}}_u^j) \qquad = \mathbf{0}.$$

But by Lemma 7.3.1 the above reduces to

$$\bigoplus_i \mathbf{F}_{ci}(\hat{\mathbf{y}}_1^i, \cdots, \hat{\mathbf{y}}_k^i, \hat{\mathbf{z}}_{u1}^i, \cdots, \hat{\mathbf{z}}_{um}^i, \hat{\mathbf{z}}_y^i) + \bigoplus_j \mathbf{F}_{sj}(\hat{\mathbf{u}}_1^j, \cdots \hat{\mathbf{u}}_t^j, \hat{\mathbf{z}}_{y1}^j, \cdots \hat{\mathbf{z}}_{yn}^j, \hat{\mathbf{z}}_u^j) = \mathbf{0}.$$

Observe that the inputs (outputs) of the i^{th} summer of the original have become the outputs (inputs) of the (corresponding) i^{th} connection point of the transformed system. Similarly, the inputs (outputs) of the j^{th} connection point have become the outputs (inputs) of the corresponding j^{th} summer of the transformed system. Thus, summers and connection points have been interchanged and the arrows reversed.
Next consider the block constraint

$$\mathbf{G}(\mathbf{z}_u, \mathbf{z}_y) = \mathbf{0}.$$

This can be written as

$$\bigoplus_j \mathbf{G}_j(\mathbf{z}_u^{bj}, \mathbf{z}_y^{bj}) = \mathbf{0}.$$

A complementary orthogonal system would be

$$\bigoplus_j \mathbf{G}_j^{\perp}(-\hat{\mathbf{z}}_y^{bj}, \hat{\mathbf{z}}_u^{bj}) = \mathbf{0}.$$

We know by Lemma 7.3.1 that if

$$\mathbf{G}_j(\mathbf{z}_u^{bj}, \mathbf{z}_y^{bj}) = \mathbf{0}$$

is equivalent to

$$\mathbf{z}_y^{bj} = \mathbf{K}\mathbf{z}_u^{bj}$$

then

$$\mathbf{G}_j^{\perp}(-\hat{\mathbf{z}}_y^{bj}, \hat{\mathbf{z}}_u^{bj}) = \mathbf{0}$$

is equivalent to

$$\hat{\mathbf{z}}_y^{bj} = \mathbf{K}^T\hat{\mathbf{z}}_u^{bj}.$$

Thus, the j^{th} block with the transfer matrix $\mathbf{K}$ of the original has become the j^{th} block with the transfer matrix $\mathbf{K}^T$ for the transformed system and the arrows have been reversed. This completes the proof of the theorem.

□

7.3.5 Rank, Nullity and the Hybrid rank

We show here how some standard notions associated with graphs and vector spaces can be viewed in terms of generalized minor and extension.
Let $\mathcal{V}_S$ be a vector space on S. Let $\mathcal{V}_S^{\circ}$ denote the zero space on S and $\mathcal{V}_S^1$, the space on S with rank $\mid S \mid$. An extension $\mathcal{V}_{SP}$ of spaces $\mathcal{V}_S$,$\hat{\mathcal{V}}_S$ is said to be their **minimal extension** iff whenever $\mathcal{V}_{SP'}$ is any other such extension we have $\mid P \mid \leq \mid P' \mid$. Define the **distance** between $\mathcal{V}_S$ and a space $\hat{\mathcal{V}}_S$, denoted by

$d(\mathcal{V}_S, \hat{\mathcal{V}}_S)$, to be $\mid P \mid$, where $\mathcal{V}_{SP}$, a space on $S \uplus P$, is a minimal extension of $\mathcal{V}_S$ and $\hat{\mathcal{V}}_S$. Using the results of Exercises 7.11 and 7.14 it may be verified that $d(\mathcal{V}, \mathcal{V}')$ as defined above equals $r(\mathcal{V} + \mathcal{V}') - r(\mathcal{V} \cap \mathcal{V}')$ and that $d(.,.)$ is a metric on the collection of vector spaces on S. Thus the rank of $\mathcal{V}_S = d(\mathcal{V}_S, \mathcal{V}_S^{\circ})$. The nullity of $\mathcal{V}_S = d(\mathcal{V}_S, \mathcal{V}_S^{1})$.

The hybrid rank of $\mathcal{V}_S \equiv \min_{S_1 \subseteq S}(r(\mathcal{V}_S \cdot S_1) + \nu(\mathcal{V}_S \times (S - S_1))$
$= \min_{S_1 \subseteq S} d(\mathcal{V}_S, (\mathcal{V}_{S_1}^{\circ} \oplus \mathcal{V}_{S-S_1}^{1}))$
= minimum distance between $\mathcal{V}_S$ and a space on S which has every element as a separator.

The hybrid rank can thus be seen to be the minimum distance between the given space and another where each element is decoupled from every other. More generally the hybrid rank of $\mathcal{V}_S$ relative to a partition Π of S can be defined to be the minimum distance between the given space and another space in which blocks of Π are separators. It is thus a measure of the difficulty in decoupling the blocks of Π in the space $\mathcal{V}_S$ or equivalently a measure of the strength of coupling between blocks of Π in $\mathcal{V}_S$. The efficient determination of this generalized hybrid rank appears to be a fundamental problem in 'Network Analysis by Decomposition'. For details see Chapter 15.

7.4 *Linear Inequality Systems

Inequality systems provide us with a good example of q-orthogonality, namely, **polarity**. In case of linear inequality systems the natural collections of vectors are **polyhedral cones** which are solution sets of inequality constraints of the form $\mathbf{Ax} \leq \mathbf{0}$. The collection of polyhedral cones is closed under sum, intersection and polarity. We will show that an analogue of the Implicit Duality Theorem holds also for inequality systems in terms of polyhedral cones.
We know that a vector space on a finite set can be thought of in two ways:

i. collection of vectors orthogonal to a finite set of vectors,

ii. collection of vectors generated by a finite basis through linear combination.

Similarly, a polyhedral cone can be defined in two ways:

i. collection of vectors polar to a given set of vectors (i.e., solution set of $\mathbf{A}\ \mathbf{x}\ \leq\ \mathbf{0}$),

ii. collection of real or rational vectors generated by a finite set of vectors through nonnegative linear combination, i.e., in the form $\mathbf{x} = \mathbf{By}, \mathbf{y} \geq \mathbf{0}$.

In the present case the $<,>$ operation is the usual dot product and q-orthogonality is polarity. Vectors $\mathbf{f}, \mathbf{g}$ on S are **polar** iff $< \mathbf{f}, \mathbf{g} > \leq 0$. Let $\mathcal{K}_S$ be a collection of vectors on S. The collection of all vectors on S which are polar to every vector in $\mathcal{K}_S$ is denoted by $\mathcal{K}_S^p$ and is called the **polar** of $\mathcal{K}_S$.

Theorem 7.4.1 *(Implicit Polarity Theorem) Let $\mathcal{K}_{SP}, \mathcal{K}_P$ be polyhedral cones whose members are vectors respectively on $S \uplus P, P$ respectively. Then*

$$(\mathcal{K}_{SP} \leftrightarrow \mathcal{K}_P)^p = \mathcal{K}_{SP}^p \leftrightarrow (-\mathcal{K}_P^p).$$

We need the following lemma for the proof of this theorem.

Lemma 7.4.1 *i. A cone $\mathcal{C}$ is polyhedral (i.e., defined by $\mathbf{A}\mathbf{x} \leq 0$) iff it is a nonnegative linear combination of a finite set of vectors (i.e., finitely generated). Hence, if $\mathcal{C}$ is a polyhedral cone so is $\mathcal{C}^p$.*

ii. (Farkas Lemma) If $\mathcal{C}$ is a finitely generated cone $\mathcal{C}^{pp} = \mathcal{C}$.

iii. Let $\mathcal{C}$ be a polyhedral cone on $S \uplus P$. Then $\mathcal{C} \cdot S$ is finitely generated and therefore polyhedral.

iv. If $\mathcal{C}_1, \mathcal{C}_2$ are polyhedral cones of vectors on S then $\mathcal{C}_1 + \mathcal{C}_2, \mathcal{C}_1 \cap \mathcal{C}_2$ are also polyhedral cones.

For proof see, for instance, [Stoer+Witzgall70].

Proof of Theorem 7.4.1: Let $(\mathbf{A}_S\ \mathbf{A}_P), \hat{\mathbf{A}}_P$ be matrices whose rows generate $\mathcal{K}_{SP}, \mathcal{K}_P$ respectively by nonnegative linear combinations. A vector $\mathbf{x}_S \in \mathcal{K}_{SP} \leftrightarrow \mathcal{K}_P$ iff there exist nonnegative vectors λ_1, λ_2 s.t.

$$\begin{bmatrix} \mathbf{A}_S{}^T & \mathbf{0} \\ \mathbf{A}_P{}^T & -\hat{\mathbf{A}}_P^T \end{bmatrix} \begin{matrix} \lambda_1 \\ \lambda_2 \end{matrix} = \begin{bmatrix} \mathbf{x}_S \\ \mathbf{0} \end{bmatrix}.$$

Thus, $\mathbf{x}_S \in \mathcal{K}_{SP} \leftrightarrow \mathcal{K}_P$ iff $\begin{bmatrix} \mathbf{x}_S \\ \mathbf{0} \end{bmatrix}$ belongs to the finitely generated cone $\mathcal{K}_{SP} - \mathcal{K}_P$. By Lemma 7.4.1 part (ii), $\begin{bmatrix} \mathbf{x}_S \\ \mathbf{0} \end{bmatrix}$ belongs to $(\mathcal{K}_{SP} - \mathcal{K}_P)$ iff it belongs to $(\mathcal{K}_{SP} - \mathcal{K}_P)^{pp}$. The cone $(\mathcal{K}_{SP} - \mathcal{K}_P)^p$ is defined by

$$\begin{pmatrix} \mathbf{A}_S & \mathbf{A}_P \\ \mathbf{0} & -\hat{\mathbf{A}}_P \end{pmatrix} \begin{matrix} \mathbf{y}_S \\ \mathbf{y}_P \end{matrix} \leq \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \end{pmatrix}. \qquad (*)$$

Thus, $\mathbf{x}_S \in \mathcal{K}_{SP} \leftrightarrow \mathcal{K}_P$ iff

$$\left(\begin{pmatrix} \mathbf{A}_S & \mathbf{A}_P \\ \mathbf{0} & -\hat{\mathbf{A}}_P \end{pmatrix} \begin{matrix} \mathbf{y}_S \\ \mathbf{y}_P \end{matrix} \leq \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \end{pmatrix} \right) \Rightarrow (\mathbf{x}_S^T\ \mathbf{y}_S \leq 0),$$

i.e., iff $(\mathbf{y}_S \in \mathcal{K}_{SP}^p \leftrightarrow -\mathcal{K}_P^p) \Rightarrow (\mathbf{x}_S^T\ \mathbf{y}_S \leq \mathbf{0})$.

Hence, $\mathbf{x}_S \in \mathcal{K}_{SP} \leftrightarrow \mathcal{K}_P$ iff $\mathbf{x}_S \in (\mathcal{K}_{SP}^p \leftrightarrow -\mathcal{K}_P^p)^p$.

Thus, $\mathcal{K}_{SP} \leftrightarrow \mathcal{K}_P = (\mathcal{K}^p_{SP} \leftrightarrow -\mathcal{K}^p_P)^p$. $\quad (**)$

Now $\mathbf{y}_S \in \mathcal{K}^p_{SP} \leftrightarrow -\mathcal{K}^p_P$ iff it is the restriction of a vector $\begin{pmatrix} \mathbf{y}_S \\ \mathbf{y}_P \end{pmatrix}$, satisfying the inequality $(*)$, to S.
Equivalently $\mathcal{K}^p_{SP} \leftrightarrow -\mathcal{K}^p_P$ is the restriction, of the polyhedral cone $(\mathcal{K}_{SP} - \mathcal{K}_P)^p$ on $S \uplus P$, to the subset S.
Now $(\mathcal{K}_{SP} - \mathcal{K}_P)^p$ is finitely generated by Lemma 7.4.1 and hence $\mathcal{K}^p_{SP} \leftrightarrow -\mathcal{K}^p_P$, which is its restriction to S, is also finitely generated. By the second part of the same lemma we must have

$$(\mathcal{K}^p_{SP} \leftrightarrow -\mathcal{K}^p_P)^{pp} = \mathcal{K}^p_{SP} \leftrightarrow -\mathcal{K}^p_P.$$

Hence, by $(**)$

$$(\mathcal{K}_{SP} \leftrightarrow \mathcal{K}_P)^p = (\mathcal{K}^p_{SP} \leftrightarrow -\mathcal{K}^p_P)^{pp} = \mathcal{K}^p_{SP} \leftrightarrow -\mathcal{K}^p_P.$$

□

Remark: i. The reader would notice that the above proof is a translation (with additional explanations) of the proof of Theorem 7.1.1.

ii. We note that both the pairs of polyhedral cones $\mathcal{K}_{SP}, \mathcal{K}_P$ as well $\mathcal{K}^p_{SP}, \mathcal{K}^p_P$ may be defined through inequalities:

$\mathcal{K}_{SP}$ be the solution set of $\begin{pmatrix} \mathbf{B}_S & \mathbf{B}_P \end{pmatrix} \begin{matrix} \mathbf{x}_S \\ \mathbf{x}_P \end{matrix} \leq \mathbf{0}$.

$\mathcal{K}_P$ be the solution set of $\hat{\mathbf{B}}_P \mathbf{x}_P \leq \mathbf{0}$

$\mathcal{K}^p_{SP}$ be the solution set of $\begin{pmatrix} \mathbf{A}_S & \mathbf{A}_P \end{pmatrix} \begin{matrix} \mathbf{y}_S \\ \mathbf{y}_P \end{matrix} \leq \mathbf{0}$

$\mathcal{K}^p_P$ be the solution set of $\hat{\mathbf{A}}_P \mathbf{y}_P \leq \mathbf{0}$.

Let $\mathcal{C}_S$ be the collection of all vectors $\mathbf{x}_S$ s.t. for some $\mathbf{x}_P$ we have

$$\begin{pmatrix} \mathbf{B}_S & \mathbf{B}_P \\ \mathbf{0} & \hat{\mathbf{B}}_P \end{pmatrix} \begin{matrix} \mathbf{x}_S \\ \mathbf{x}_P \end{matrix} \leq \mathbf{0}$$

Then $\mathcal{C}^p_S$ is the collection of all vectors $\mathbf{y}_S$ s.t. for some $\mathbf{y}_P$ we have

$$\begin{pmatrix} \mathbf{A}_S & \mathbf{A}_P \\ \mathbf{0} & -\hat{\mathbf{A}}_P \end{pmatrix} \begin{matrix} \mathbf{y}_S \\ \mathbf{y}_P \end{matrix} \leq \mathbf{0}.$$

iii. Since by Lemma 7.2.1, the collection of all vectors q-orthogonal to a vector space is its complementary orthogonal space, and since the q-bilinear operation used here is the dot product and further a vector space is a special case of a cone, it follows that the Implicit Duality Theorem is a special case of the Implicit Polarity Theorem.

The technique of the proof of Corollary 7.1.1 will work in the case of the present instance of q-orthogonality, namely, polarity. In this case also we would be working with a vector space $\mathcal{V}_{12}$ as defined in the proof of the above mentioned corollary. By Lemma 7.2.1, $\mathcal{V}_{12}^p = \mathcal{V}_{12}^\perp$ and therefore $\mathcal{V}_{12}^p$ would have the representative matrix

$$\begin{array}{cc} P_1 & P_2 \\ \multicolumn{2}{c}{\left[\begin{array}{cc} \mathbf{I} & -\mathbf{I} \end{array}\right].} \end{array}$$

Since $(\mathcal{K}_{S_1}{}' \oplus \mathcal{K}_{S_2}{}')^p = (\mathcal{K}_{S_1}{}')^p \oplus (\mathcal{K}_{S_2}{}')^p$, we must have

$$(\mathcal{K}_{S_1}{}' \oplus \mathcal{K}_{S_2}{}')^p \leftrightarrow \mathcal{V}_{12}^p = ((\mathcal{K}_{S_1}{}')^p \oplus (\mathcal{K}_{S_2}{}')^p) \leftrightarrow \mathcal{V}_{12}^p$$

The term in the RHS can now be seen to be equal to $\mathcal{K}_{S_1}{}^p \rightleftharpoons \mathcal{K}_{S_2}{}^p$. We thus have,

Corollary 7.4.1 *Let $\mathcal{K}_{S_1}, \mathcal{K}_{S_2}$ be polyhedral cones of vectors on S_1, S_2 respectively. Let '$<,>$' be the usual dot product operation. Then $(\mathcal{K}_{S_1} \leftrightarrow \mathcal{K}_{S_2})^p = (\mathcal{K}_{S_1}{}^p \rightleftharpoons \mathcal{K}_{S_2}{}^p)$, where the superscript 'p' denotes polarity.*

Exercise 7.16 *Redo the Exercises on implicit duality - vector space case, for the case of polyhedral collections.*

Exercise 7.17 *Using the Implicit Duality Theorem prove*
i. $(\mathcal{K}_{SP} \cdot S)^* = \mathcal{K}_{SP}^* \times S$
ii. $(\mathcal{K}_{SP} \times S)^* = \mathcal{K}_{SP}^* \cdot S$
where $\mathcal{K}_{SP}$ is a polyhedral cone of vectors on $S \cup P$.

Exercise 7.18 Proof of Implicit Duality from duality of contraction and restriction:
Let S, P be disjoint and let $P = P_1 \uplus P_2$.
i. *Show that*

$$\begin{aligned}(\mathcal{K}_{SP} \leftrightarrow \mathcal{K}_{P_1}) \leftrightarrow \mathcal{K}_{P_2} &= \mathcal{K}_{SP} \leftrightarrow (\mathcal{K}_{P_1} \oplus \mathcal{K}_{P_2}) \\ &= (\mathcal{K}_{SP} \leftrightarrow \mathcal{K}_{P_2}) \leftrightarrow \mathcal{K}_{P_1}\end{aligned}$$

ii. *Let $\mathcal{X}_P$ be the collection of all vectors on P and $\mathbf{T} : \mathcal{X}_P \rightarrow \mathcal{X}_P$, be a nonsingular transformation. Let $\mathbf{T}(\mathcal{K}_P) = \mathcal{K}'_P$. Let $\mathbf{T}_{SP}(\mathbf{f}_S \oplus \mathbf{f}_P) \equiv \mathbf{f}_S \oplus \mathbf{T}(\mathbf{f}_P)$ and let $\mathcal{K}'_{SP} \equiv \mathbf{T}_{SP}(\mathcal{K}_{SP})$. Show that*

$$\mathcal{K}'_{SP} \leftrightarrow \mathcal{K}'_P = \mathcal{K}_{SP} \leftrightarrow \mathcal{K}_P.$$

iii. *Assuming it is true for all $\mathcal{K}_{SP}$ and all finite sets S, P_1, P_2 s.t. $P = P_1 \uplus P_2$ that*

$$(\mathcal{K}_{SP} \leftrightarrow (\mathcal{X}_{P_1} \oplus \mathbf{0}_{P_2}))^* = \mathcal{K}_{SP}^* \leftrightarrow (\mathbf{0}_{P_1} \oplus \mathcal{X}_{P_2}),$$

show that

$$(\mathcal{K}_{SP} \leftrightarrow \mathcal{K}_P)^* = \mathcal{K}_{SP}^* \leftrightarrow (-\mathcal{K}_P^*),$$

for the following two cases:

- *$\mathcal{K}_P$ is a vector space on P, and $\mathcal{K}_{SP}$, a collection of vectors on $S \uplus P$*
- *$\mathcal{K}_P, \mathcal{K}_{SP}$ are collections of vectors on $P, S \uplus P$ respectively.*

7.4.1 Applications of the Polar Form

Both the polar version and the integrality version of the implicit duality theorem are presented in this chapter essentially for completeness [Narayanan85a]. However, we are able to cite atleast one reference in the literature on polyhedral combinatorics where a result, that could be regarded as an instance of the Implicit Polarity Theorem, is derived and applied [Balas+Pulleyblank87]. We state this Projection Theorem of Balas and Pulleyblank below but prove it using the Implicit Duality Theorem.

Theorem 7.4.2 *Let*

$$\mathcal{Z} \equiv \{(\mathbf{u},\mathbf{x}) \quad : \quad \begin{array}{l} \mathbf{Au}+\mathbf{Bx}=\mathbf{b}_1 \\ \mathbf{Du}+\mathbf{Ex}\le \mathbf{b}_2 \\ \mathbf{u}\ge \mathbf{0}\ \} \end{array}$$

Let $\mathcal{W} \equiv \{(\mathbf{y},\mathbf{z}) : \mathbf{y}^T\mathbf{A}+\mathbf{z}^T\mathbf{D} \ge \mathbf{0}, \mathbf{z}\ge\mathbf{0}\}$
Then, $\mathcal{X} \equiv \{\mathbf{x} : \exists \mathbf{u}\ s.t.\ (\mathbf{u},\mathbf{x}) \in \mathcal{Z}\}$
$= \{\mathbf{x} : (\mathbf{y}^T\mathbf{B}+\mathbf{z}^T\mathbf{E})\mathbf{x} \le \mathbf{y}^T\mathbf{b}_1 + \mathbf{z}^T\mathbf{b}_2 \quad \forall (\mathbf{y},\mathbf{z}) \in \mathcal{W}\}$.

Proof: For notational convenience we take the 'primal' cone to be made up of column vectors and the polar cone to be made up of row vectors. We regard $\mathbf{b}_1, \mathbf{b}_2$ also as variables initially.

Let $\mathcal{C} \equiv \mathcal{C}(\mathbf{u},\mathbf{x},\mathbf{b}_1,\mathbf{b}_2)$ denote the cone

$$\mathbf{Au}+\mathbf{Bx}-\mathbf{I}_{b_1}\mathbf{b}_1=\mathbf{0}$$

$$\mathbf{Du}+\mathbf{Ex}-\mathbf{I}_{b_2}\mathbf{b}_2\le\mathbf{0}.$$

and let $\mathcal{C}_u \equiv \mathcal{C}_u(\mathbf{u})$ denote the cone

$$(-\mathbf{I}_u)\mathbf{u} \le \mathbf{0}.$$

Let $\hat{\mathcal{C}} \equiv \hat{\mathcal{C}}(\mathbf{x},\mathbf{b}_1,\mathbf{b}_2) = \mathcal{C} \leftrightarrow \mathcal{C}_u$. The restriction of $\hat{\mathcal{C}}$ to the components corresponding to $\mathbf{x}$ for fixed $\mathbf{b}_1,\mathbf{b}_2$ is the set $\mathcal{X}$.
Then $(\hat{\mathcal{C}})^p = \mathcal{C}^p \leftrightarrow -\mathcal{C}_u^p$, by the Implicit Polarity Theorem. Now,

$$(\mathcal{C})^p = \{(\mathbf{y}^T\mathbf{A}+\mathbf{z}^T\mathbf{D}) \oplus (\mathbf{y}^T\mathbf{B}+\mathbf{z}^T\mathbf{E}) \oplus -\mathbf{y}^T \oplus -\mathbf{z}^T, \mathbf{z}\ge\mathbf{0}\}$$

and

$$-\mathcal{C}_u^p = \{\mathbf{p}^T\mathbf{I}_u, \mathbf{p}\ge\mathbf{0}\}$$

by Farkas Lemma (Theorem 2.3.2, also part (ii) of Lemma 7.4.1). Hence,

$$\begin{aligned} (\hat{\mathcal{C}})^p &= \{(\mathbf{y}^T\mathbf{B}+\mathbf{z}^T\mathbf{E}) \oplus -\mathbf{y}^T \oplus -\mathbf{z}^T), \mathbf{z}\ge\mathbf{0} \\ &\qquad\qquad \textit{and}\ \mathbf{y}^T\mathbf{A}+\mathbf{z}^T\mathbf{D} = \mathbf{p}^T \ge \mathbf{0}\} \\ &= \{(\mathbf{y}^T\mathbf{B}+\mathbf{z}^T\mathbf{E}) \oplus -\mathbf{y}^T \oplus -\mathbf{z}^T \quad \forall (\mathbf{y},\mathbf{z}) \in \mathcal{W}\}. \end{aligned}$$

Now,

$$\hat{\mathcal{C}} = (\hat{\mathcal{C}})^{pp} = \{(\mathbf{x}\oplus\mathbf{b}_1\oplus\mathbf{b}_2) :\ < (\mathbf{x}\oplus\mathbf{b}_1\oplus\mathbf{b}_2), (\alpha\oplus\beta\oplus\gamma) > \le 0 \quad \forall (\alpha\oplus\beta\oplus\gamma) \in \hat{\mathcal{C}}^p\}.$$

Thus,

$$\hat{\mathcal{C}} = \{(\mathbf{x} \oplus \mathbf{b}_1 \oplus \mathbf{b}_2) : (\mathbf{y}^T\mathbf{B} + \mathbf{z}^T\mathbf{E})\mathbf{x} - \mathbf{y}^T\mathbf{b}_1 - \mathbf{z}^T\mathbf{b}_2 \leq 0 \ \ \forall(\mathbf{y}, \mathbf{z}) \in \mathcal{W}\}.$$

To get $\mathcal{X}$ we restrict the above set to the components corresponding to $\mathbf{x}$ fixing $\mathbf{b}_1, \mathbf{b}_2$. Hence, $\mathcal{X} = \{\mathbf{x} : (\mathbf{y}^T\mathbf{B} + \mathbf{z}^T\mathbf{E})\mathbf{x} \leq \mathbf{y}^T\mathbf{b}_1 + \mathbf{z}^T\mathbf{b}_2 \ \ \forall(\mathbf{y}, \mathbf{z}) \in \mathcal{W}\}$.

□

The authors ([Balas+Pulleyblank87]) use this result to study the perfectly matchable subgraph polytope of an arbitrary graph. The variable of interest $\mathbf{x}$ is found in the naturally obtained constraints along with other variables. Direct elimination of these other variables would destroy the structure of the problem. The authors therefore, use the above 'implicit projection'.

7.5 *Integrality Systems

Another good example of q-orthogonality is integral duality. We say two vectors on a set S over the rational field are **integrally dual** iff their dot product is an integer. As in the case of polyhedral cones and polarity, here too we have a good family of (regularly generated) collections of vectors which is closed under sum, intersection and integral duality. We can therefore prove an implicit integral duality theorem for such systems which we do in the present section.
Throughout this section we will be dealing only with rational vectors. In this case also '$< .,. >$' denotes the usual dot product.
Let $\mathcal{K}_S$ be a collection of rational vectors. Then, the **integral dual** of $\mathcal{K}_S$ is denoted by ${\mathcal{K}_S}^d$ and is defined by
${\mathcal{K}_S}^d \equiv \{\mathbf{y}, < \mathbf{y}, \mathbf{x} > \text{ is an integer } \ \forall \mathbf{x} \in \mathcal{K}_S\ \}$
The analogue of vector space and polyhedral cone in the present case is a 'regularly generated' collection of vectors. We define this notion below.
Let $(\mathbf{A}, \mathbf{B})$ be an ordered pair of matrices whose rows are vectors over the rational field defined on a set S. The collection of all vectors of the form $\lambda_1^T\mathbf{A} + \lambda_2^T\mathbf{B}$, where λ_1 is an integral vector and λ_2, any rational vector, is said to be **regularly generated by the rows of** $(\mathbf{A}, \mathbf{B})$ or **regularly generated** for short.
For each row $\mathbf{A}_j$ of $\mathbf{A}$ let $\mathbf{A}_{jn}$ denote $\mathbf{A}_j - \mathbf{A}_{jp}$ where $\mathbf{A}_{jp}$ is the projection of $\mathbf{A}_j$ on the space spanned by rows of $\mathbf{B}$. (Note that $\mathbf{A}_{jp}$ is rational). Let $\mathbf{A}_n$ denote the matrix obtained from $\mathbf{A}$ by replacing each row $\mathbf{A}_j$ by $\mathbf{A}_{jn}$. Let $\hat{\mathbf{B}}$ denote a maximal linearly independent subset of rows of $\mathbf{B}$. It is then clear that the collection of vectors regularly generated by the rows of $(\mathbf{A}, \mathbf{B})$ is identical to that regularly generated by $(\mathbf{A}_n, \hat{\mathbf{B}})$. We show next that $\mathbf{A}_n$ can be replaced by an appropriate matrix $\hat{\mathbf{A}}$ with the same row space as $\mathbf{A}_n$ but with linearly independent rows.

Definition 7.5.1 *An integral matrix of full column rank is said to be in the* **Hermite Normal Form** *if it has the form* $\begin{bmatrix} \mathbf{B} \\ \mathbf{0} \end{bmatrix}$ *where* $\mathbf{B}$ *satisfies the following:*
i. *it is an upper triangular, integral, nonnegative matrix;*

ii. *its diagonal entries are positive and have the unique highest magnitude in their columns.*

We now have the following well known result.

Theorem 7.5.1 *By using the elementary integral row operations:*

i. interchanging two rows

ii. adding an integer multiple of one row to another

iii. multiplying a row by -1

any integral matrix $\tilde{\mathbf{A}}$ *can be transformed after column permutation to an integral matrix of the form* $\begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$,
where $\mathbf{A}_{11}$ *is a non singular matrix in the Hermite Normal form.*

Proof : After column permutations, if necessary, we can partition $\tilde{\mathbf{A}}$ as $(\tilde{\mathbf{A}}_{11} \vdots \tilde{\mathbf{A}}_{12})$, where $\tilde{\mathbf{A}}_{11}$ is composed of a maximal linearly independent set of columns of $\tilde{\mathbf{A}}$. With elementary integral row operations on the entire matrix modify the first column of $\tilde{\mathbf{A}}_{11}$ to a nonnegative vector with the least sum possible. Only one of these entries can be nonzero as otherwise the least nonzero entry can be subtracted from the others to reduce the sum. Bring this entry to the top left hand corner. The matrix $\tilde{\mathbf{A}}_{11}$ has now been converted into a matrix of the form $\begin{bmatrix} a_{11} & \mathbf{P} & \mathbf{Q} \\ \mathbf{0} & \mathbf{A}"_{11} & \mathbf{A}"_{12} \end{bmatrix}$. Repeat the procedure with the first column of $\mathbf{A}"_{11}$ and so on. At the end of this procedure we would have the matrix in the form $\begin{pmatrix} \mathbf{A}'_{11} & \mathbf{A}_{12} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}$ where $\mathbf{A}'_{11}$ would be in the upper triangular form. Now use the diagonal entries of $\mathbf{A}'_{11}$ to convert all entries above, by elementary integral row operations, to nonnegative numbers of value less than the diagonal entries. The resulting matrix $\mathbf{A}_{11}$, by definition, is in the Hermite Normal Form.

□

Suppose a collection of vectors $\mathcal{K}$ is regularly generated by rows of $(\mathbf{A}_n,\ \hat{\mathbf{B}})$ where rows of $\mathbf{A}_n$ are orthogonal to rows of $\hat{\mathbf{B}}$. Now $\mathbf{A}_n = \frac{1}{k}(\mathbf{A}_n{}')$, where $(\mathbf{A}_n{}')$ is an integral matrix and k is an integer. By elementary integral row operations as in Theorem 7.5.1, we can reduce $\mathbf{A}_n{}'$ to a row equivalent matrix $\mathbf{A}'$ which has linearly independent rows. Since it is clear that the inverse of each elementary integral row operation is another such operation, a vector is an integral linear combination of the rows of $\mathbf{A}_n{}'$ iff it is an integral linear combination of the rows of $\mathbf{A}'$. Let $\hat{\mathbf{A}} \equiv \frac{1}{k}(\mathbf{A}')$. Then rows of $\hat{\mathbf{A}}$ are linearly independent and further, rows of $\hat{\mathbf{A}}$ and $\mathbf{A}_n$ can be generated from each other by integral linear combinations. Thus, the collection of vectors $\mathcal{K}$ is regularly generated by rows of $(\hat{\mathbf{A}}, \hat{\mathbf{B}})$ where rows of $\hat{\mathbf{A}}$ and rows of $\hat{\mathbf{B}}$ are independent and are mutually orthogonal. We say that such an ordered pair $(\hat{\mathbf{A}}, \hat{\mathbf{B}})$ is in the **standard form**.

The integral dual operation is sufficiently well behaved, as we show below, for us to apply the implicit duality technique.

Theorem 7.5.2 *Let $\mathcal{K}$ be a collection of vectors over the rational field regularly generated by the rows of* $(\mathbf{A}, \mathbf{B})$ *in standard form. Then*

i. $\mathcal{K}^d$ *is regularly generated by rows of* $(\mathbf{C}, \mathbf{D})$ *in standard form where* $\mathbf{C}$ *is row equivalent to* $\mathbf{A}$, $\mathbf{AC}^T = \mathbf{I}$, *and* $\mathbf{D}$ *is a representative matrix for* $\mathcal{K}^{\perp}$.

ii. $\mathcal{K}^{dd} = \mathcal{K}$.

iii. If $\mathcal{K}$ *is a regularly generated collection of vectors on* $S \uplus P$ *then* $\mathcal{K} \cdot S$ *is regularly generated.*

iv. If $\mathcal{K}'$ *is another regularly generated collection then* $\mathcal{K} + \mathcal{K}', \mathcal{K} \cap \mathcal{K}'$ *are regularly generated.*

Proof :

i. We note that rows of $\mathbf{A}, \mathbf{B}, \mathbf{D}$ are mutually orthogonal. Also $\begin{pmatrix} \mathbf{A} \\ \mathbf{B} \\ \mathbf{D} \end{pmatrix}$ is a full rank matrix. This is also true of $\mathbf{C}, \mathbf{B}, \mathbf{D}$. Let $\mathcal{K}'$ be regularly generated by rows of $(\mathbf{C}, \mathbf{D})$, where $\mathbf{C}, \mathbf{D}$ are defined as in the statement of the theorem. Let $\mathbf{x} \in \mathcal{K}$ and $y \in \mathcal{K}'$. Then,

$$< \mathbf{x}, \mathbf{y} >= ({\lambda_1}^T {\lambda_2}^T) \begin{pmatrix} \mathbf{A} \\ \mathbf{B} \end{pmatrix} (\mathbf{C}^T \vdots \mathbf{D}^T) \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix}$$

for suitable vectors $\lambda_1, \lambda_2, \sigma_1, \sigma_2$ where λ_1, σ_1 are integral. Thus,

$$\begin{aligned} < \mathbf{x}, \mathbf{y} > &= ({\lambda_1}^T {\lambda_2}^T) \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \begin{matrix} \sigma_1 \\ \sigma_2 \end{matrix} \\ &= {\lambda_1}^T \sigma_1, \ \textit{which is an integer.} \end{aligned}$$

We see therefore, that $\mathcal{K}' \subseteq \mathcal{K}^d$. On the other hand, suppose $\mathbf{y}$ is integrally dual to all vectors in $\mathcal{K}$. We have, ${\lambda_2}^T \mathbf{By}$ is an integer for arbitrary λ_2. This can happen only if $\mathbf{By} = \mathbf{0}$, i.e., $\mathbf{y}^T$ belongs to the space spanned by rows of $\begin{pmatrix} \mathbf{C} \\ \mathbf{D} \end{pmatrix}$. Let $\mathbf{y}^T = \sigma_1^T \sigma_2^T \begin{pmatrix} \mathbf{C} \\ \mathbf{D} \end{pmatrix}$. Suppose σ_1 is not integral. We know that $\mathbf{AC}^T = \mathbf{I}$. Hence, for some integral value of λ_1 we would have ${\lambda_1}^T \mathbf{AC}^T \sigma_1$ nonintegral, which contradicts the fact that $\mathbf{y} \in \mathcal{K}^d$. We conclude that σ_1 must be integral and that $\mathcal{K}' \supseteq \mathcal{K}^d$. This proves the first part.

ii. From the above proof it is clear that if $\mathcal{K}^d$ is regularly generated by rows of $(\mathbf{C}, \mathbf{D})$ then $(\mathcal{K}^d)^d$ is regularly generated by rows of $(\mathbf{A}, \mathbf{B})$, i.e., $(\mathcal{K}^d)^d = \mathcal{K}$.

iii. This is immediate by the definition of regularly generated collections.

iv. If $\mathcal{K}, \mathcal{K}'$ are regularly generated by rows of $(\mathbf{A}, \mathbf{B})$, $(\mathbf{A}', \mathbf{B}')$ respectively then the rows of $\left(\begin{pmatrix} \mathbf{A} \\ \mathbf{A}' \end{pmatrix}, \begin{pmatrix} \mathbf{B} \\ \mathbf{B}' \end{pmatrix}\right)$ regularly generate $\mathcal{K} + \mathcal{K}'$.
Now $\mathcal{K}^d, (\mathcal{K}')^d$ are regularly generated and so is their sum.
By Lemma 7.2.2

$$(\mathcal{K}^d + (\mathcal{K}')^d)^d = \mathcal{K}^{dd} \cap \mathcal{K}'^{dd} = \mathcal{K} \cap \mathcal{K}'.$$

Hence, by the first part of the present theorem, $\mathcal{K} \cap \mathcal{K}'$ is regularly generated.

□

Remark: An easy way of constructing $\mathbf{C}$, from $(\mathbf{A}, \mathbf{B})$ in the standard form, is to first build a representative matrix $\mathbf{D}$ for $\mathcal{K}^{\perp}$. Let the matrix $\begin{pmatrix} \mathbf{A} \\ \mathbf{B} \\ \mathbf{D} \end{pmatrix}$ have the inverse $(\mathbf{P}^T \vdots \mathbf{Q}^T \vdots \mathbf{R}^T)$ with $\mathbf{A}\mathbf{P}^T = \mathbf{I}$. Then, $\mathbf{P}$ can be taken to be $\mathbf{C}$.

Theorem 7.5.3 (**Implicit Integral Duality Theorem**): *Let $\mathcal{K}_{SP}, \mathcal{K}_P$ be regularly generated collections of vectors on $S \uplus P, P$ respectively. Then,*

$$(\mathcal{K}_{SP} \leftrightarrow \mathcal{K}_P)^d = \mathcal{K}_{SP}^d \leftrightarrow \mathcal{K}_P^d.$$

Proof of Theorem 7.5.3: Let $\mathcal{K}_{SP}, \mathcal{K}_P$ be regularly generated by the rows of $((\mathbf{A}_S\ \mathbf{A}_P), (\mathbf{B}_S\ \mathbf{B}_P))$ and $(\hat{\mathbf{A}}_P,\ \hat{\mathbf{B}}_P)$ respectively. A vector $\mathbf{x}_S$ belongs to $\mathcal{K}_{SP} \leftrightarrow \mathcal{K}_P$ iff there exist vectors λ_1, μ_1 and λ_2, μ_2, where λ_1, λ_2 are integral s.t.

$$\begin{bmatrix} \mathbf{A}_S{}^T & \mathbf{B}_S{}^T & \mathbf{0} & \mathbf{0} \\ \mathbf{A}_P{}^T & \mathbf{B}_P{}^T & \hat{\mathbf{A}}_P^T & \hat{\mathbf{B}}_P^T \end{bmatrix} \begin{matrix} \lambda_1 \\ \mu_1 \\ \lambda_2 \\ \mu_2 \end{matrix} = \begin{bmatrix} \mathbf{x}_S \\ \mathbf{0} \end{bmatrix}$$

Thus, $\mathbf{x}_S \in \mathcal{K}_{SP} \leftrightarrow \mathcal{K}_P$ iff $\begin{bmatrix} \mathbf{x}_S \\ \mathbf{0} \end{bmatrix}$ belongs to the regularly generated collection $\mathcal{K}_{SP} + \mathcal{K}_P$. By part (ii) of Theorem 7.5.2, $\begin{bmatrix} \mathbf{x}_S \\ \mathbf{0} \end{bmatrix}$ belongs to $\mathcal{K}_{SP} + \mathcal{K}_P$ iff it belongs to $(\mathcal{K}_{SP} + \mathcal{K}_P)^{dd}$. The collection $(\mathcal{K}_{SP} + \mathcal{K}_P)^d$ is defined by

$$\left\{ \begin{pmatrix} \mathbf{y}_S \\ \mathbf{y}_P \end{pmatrix} : \begin{pmatrix} \mathbf{A}_S & \mathbf{A}_P \\ \mathbf{B}_S & \mathbf{B}_P \\ \mathbf{0} & \hat{\mathbf{A}}_P \\ \mathbf{0} & \hat{\mathbf{B}}_P \end{pmatrix} \begin{matrix} \mathbf{y}_S \\ \mathbf{y}_P \end{matrix} \quad \textit{is integral.} \right\} \qquad (*)$$

Thus, $\mathbf{x}_S \in \mathcal{K}_{SP} \leftrightarrow \mathcal{K}_P$ iff

$$\left(\begin{pmatrix} \mathbf{A}_S & \mathbf{A}_P \\ \mathbf{B}_S & \mathbf{B}_P \\ \mathbf{0} & \hat{\mathbf{A}}_P \\ \mathbf{0} & \hat{\mathbf{B}}_P \end{pmatrix} \begin{matrix} \mathbf{y}_S \\ \\ \mathbf{y}_P \end{matrix} \quad \textit{is integral.} \right) \Rightarrow (\mathbf{x}_S^T \mathbf{y}_S \textit{ is integral}),$$

i.e., iff

$$(\mathbf{y}_S \in \mathcal{K}^d_{SP} \leftrightarrow \mathcal{K}^d_P) \Rightarrow (\mathbf{x}^T_S \mathbf{y}_S \textit{ is integral}).$$

Hence, $\mathbf{x}_S \in (\mathcal{K}_{SP} \leftrightarrow \mathcal{K}_P)$ iff $\mathbf{x}_S \in (\mathcal{K}^d_{SP} \leftrightarrow \mathcal{K}^d_P)^d$. Thus,
$\mathcal{K}_{SP} \leftrightarrow \mathcal{K}_P = (\mathcal{K}^d_{SP} \leftrightarrow \mathcal{K}^d_P)^d$. (**)

Now, $\mathbf{y}_S \in \mathcal{K}^d_{SP} \leftrightarrow \mathcal{K}^d_P$ iff it is the restriction of a vector $\begin{pmatrix} \mathbf{y}_S \\ \mathbf{y}_P \end{pmatrix}$ satisfying the condition (*).

Equivalently $\mathcal{K}^d_{SP} \leftrightarrow \mathcal{K}^d_P$ is the restriction of $(\mathcal{K}_{SP} + \mathcal{K}_P)^d$ to S. Since $\mathcal{K}_{SP}, \mathcal{K}_P$ are regularly generated we must have $(\mathcal{K}_{SP} + \mathcal{K}_P)^d$ also regularly generated by Theorem 7.5.2. Hence, this must be true also of $(\mathcal{K}_{SP}+\mathcal{K}_P)^d \cdot S$, i.e., of $\mathcal{K}^d_{SP} \leftrightarrow \mathcal{K}^d_P$. By the same theorem we must have

$$(\mathcal{K}^d_{SP} \leftrightarrow \mathcal{K}^d_P)^{dd} = \mathcal{K}^d_{SP} \leftrightarrow \mathcal{K}^d_P$$

Hence, by (**)

$$(\mathcal{K}_{SP} \leftrightarrow \mathcal{K}_P)^d = (\mathcal{K}^d_{SP} \leftrightarrow \mathcal{K}^d_P)^{dd} = \mathcal{K}^d_{SP} \leftrightarrow \mathcal{K}^d_P.$$

□

The technique of the proof of Corollary 7.1.1 will work in the case of the present instance of q-orthogonality, namely, integral duality. In this case also we would be working with a vector space $\mathcal{V}_{12}$ as defined in the proof of the above mentioned corollary. By Lemma 7.2.1, $\mathcal{V}^d_{12} = \mathcal{V}^\perp_{12}$ and therefore $\mathcal{V}^d_{12}$ would have the representative matrix

$$\begin{array}{cc} P_1 & P_2 \end{array}$$
$$\begin{bmatrix} \mathbf{I} & -\mathbf{I} \end{bmatrix}.$$

Since $(\mathcal{K}_{S_1}{}' \oplus \mathcal{K}_{S_2}{}')^d = \mathcal{K}_{S_1}{}'^d \oplus \mathcal{K}_{S_2}{}'^d$, we must have

$$(\mathcal{K}_{S_1}{}' \oplus \mathcal{K}_{S_2}{}')^d \leftrightarrow \mathcal{V}^d_{12} = (\mathcal{K}_{S_1}{}'^d \oplus \mathcal{K}_{S_2}{}'^d) \leftrightarrow \mathcal{V}^d_{12}$$

The term in the RHS can now be seen to be equal to $\mathcal{K}_{S_1}{}^d \rightleftharpoons \mathcal{K}_{S_2}{}^d$. We thus have,

Corollary 7.5.1 *Let* $\mathcal{K}_{S_1}, \mathcal{K}_{S_2}$ *be regularly generated collections of vectors on* S_1, S_2 *respectively. Let '$<,>$' be the usual dot product operation. Then*
$(\mathcal{K}_{S_1} \leftrightarrow \mathcal{K}_{S_2})^d = \mathcal{K}_{S_1}{}^d \rightleftharpoons \mathcal{K}_{S_2}{}^d$.

Remark: As in the case of polarity, here too it is easy to see that the Implicit Duality Theorem is a special case of the Implicit Integral Duality Theorem.

Exercise 7.19 *Examine if implicit duality would work for the following instances of q-orthogonality.*
i. $\mathbf{f}, \mathbf{g}$ *are q-orthogonal iff* $<\mathbf{f}, \mathbf{g}>$ *is a nonnegative integer.*
ii. $\mathbf{f}, \mathbf{g}$ *are q-orthogonal iff* $<\mathbf{f}, \mathbf{g}>$ *is an integral multiple of a given integer.*

7.6 Problems

Problem 7.1
i. Algorithm for building representative matrix of generalized minor: *Given representative matrices for $\mathcal{V}_{SP}, \mathcal{V}_P$ show how to build a representative matrix for $\mathcal{V}_{SP} \leftrightarrow \mathcal{V}_P$.*
ii. Rank formula for generalized minor; another proof of the Implicit Duality Theorem: *Complete the details of the following alternative proof of the Implicit Duality Theorem.*
(a) $r(\mathcal{V}_{SP} \leftrightarrow \mathcal{V}_P) = r(\mathcal{V}_{SP} \times S) + r((\mathcal{V}_{SP} \cdot P) \cap \mathcal{V}_P)$ - $r((\mathcal{V}_{SP} \times P) \cap \mathcal{V}_P)$
(b) $(\mathcal{V}_{SP} \leftrightarrow \mathcal{V}_P)$ *is orthogonal to* $(\mathcal{V}_{SP}^{\perp} \leftrightarrow \mathcal{V}_P^{\perp})$.
(c) $r(\mathcal{V}_{SP} \leftrightarrow \mathcal{V}_P) + r(\mathcal{V}_{SP}^{\perp} \leftrightarrow \mathcal{V}_P^{\perp}) = \mid S \mid$.
Hence, $(\mathcal{V}_{SP} \leftrightarrow \mathcal{V}_P)^{\perp} = \mathcal{V}_{SP}^{\perp} \leftrightarrow \mathcal{V}_P^{\perp}$.

Problem 7.2 $(*)$ *Let $\mathcal{V}_{SP}, \mathcal{V}_{SP_1}$ be called* **equivalent** *in S iff $\mathcal{V}_{SP} \cdot S = \mathcal{V}_{SP_1} \cdot S$. We say $(\mathcal{V}_{SP}, \mathcal{V}_{SP}^{\perp})$, $(\mathcal{V}_{SP_1}, \mathcal{V}_{SP_1}^{\perp})$ are* **equivalent in** *S iff $\mathcal{V}_{SP}, \mathcal{V}_{SP_1}$ and $\mathcal{V}_{SP}^{\perp}, \mathcal{V}_{SP_1}^{\perp}$ are equivalent in S. We say $(\mathcal{V}_{SP}, \mathcal{V}_{SP}^{\perp})$ are* **minimal in** *P iff whenever $(\mathcal{V}_{SP_1}, \mathcal{V}_{SP_1}^{\perp})$ is equivalent to $(\mathcal{V}_{SP}, \mathcal{V}_{SP}^{\perp})$, $\mid P_1 \mid \geq \mid P \mid$.*
Show that $(\mathcal{V}_{SP}, \mathcal{V}_{SP}^{\perp})$ is minimal in P iff either of the following equivalent conditions hold
i. $\mid P \mid = r(\mathcal{V}_{SP} \cdot S) - r(\mathcal{V}_{SP} \times S) = r(\mathcal{V}_{SP}^{\perp} \cdot S) - r(\mathcal{V}_{SP}^{\perp} \times S)$
ii. $r(\mathcal{V}_{SP} \times P) = r(\mathcal{V}_{SP}^{\perp} \times P) = 0.$

Problem 7.3 Minor of generalized minors in terms of minor of $\mathcal{V}_{SP}$: *Let $S_2 \subseteq S_1 \subseteq S$ and let $\mathcal{V}_{S_1}$ be a vector space on S_1. Show that*

$$(\mathcal{V}_{SP} \leftrightarrow \mathcal{V}_P) \cdot S_1 \times S_2 = (\mathcal{V}_{SP} \cdot (S_1 \cup P) \times (S_2 \cup P)) \leftrightarrow \mathcal{V}_P$$

Problem 7.4 $(*)$ *Let $(\mathcal{V}_{SP}, \mathcal{V}_P)$ be compatible, i.e., $\mathcal{V}_{SP} \cdot P \supseteq \mathcal{V}_P$ and $\mathcal{V}_{SP} \times P \subseteq \mathcal{V}_P$.*
i. *Let $r(\mathcal{V}_{SP} \times P) \neq 0$, let $\hat{\mathbf{f}}_P \in \mathcal{V}_{SP} \times P$, and let e belong to the support of $\hat{\mathbf{f}}_P$. Let $P_1 \equiv P - e, \mathcal{V}_{P_1} \equiv \mathcal{V}_P \times P_1$, and let $\mathcal{V}_{SP_1} \equiv \mathcal{V}_{SP} \times (S \cup P_1)$. Show that*
(a) $\mathcal{V}_{SP} \leftrightarrow \mathcal{V}_{P_1} = \mathcal{V}_{SP} \leftrightarrow \mathcal{V}_P$
(b) $\mathcal{V}_{SP_1} \cdot P_1 \supseteq \mathcal{V}_{P_1}$ *and* $\mathcal{V}_{SP_1} \times P_1 \subseteq \mathcal{V}_{P_1}$.
(c) $\mathcal{V}_{SP_1} \cdot S = \mathcal{V}_{SP} \cdot S$ *and* $\mathcal{V}_{SP_1} \times S = \mathcal{V}_{SP} \times S$.

ii. *Let $r(\mathcal{V}_{SP}^{\perp} \times P) \neq 0$, let $\hat{\mathbf{f}}_P \in \mathcal{V}_{SP}^{\perp} \times P$ and let e belong to the support of $\hat{\mathbf{f}}_P$. Let $P_2 \equiv P - e, \mathcal{V}_{P_2} \equiv \mathcal{V}_P \cdot P_2$ and let $\mathcal{V}_{SP_2} \equiv \mathcal{V}_{SP} \cdot (S \cup P_2)$. Show that*
(a) $\mathcal{V}_{SP_2} \leftrightarrow \mathcal{V}_{P_2} = \mathcal{V}_{SP} \leftrightarrow \mathcal{V}_P$,
(b) $\mathcal{V}_{SP_2} \cdot P_2 \supseteq \mathcal{V}_{P_2}$ *and* $\mathcal{V}_{SP_2} \times P_2 \subseteq \mathcal{V}_{P_2}$,
(c) $\mathcal{V}_{SP_2} \cdot S = \mathcal{V}_{SP} \cdot S$ *and* $\mathcal{V}_{SP_2} \times S = \mathcal{V}_{SP} \times S$.

iii. *Repeat the above problem when 'compatibility' is replaced by 'strong compatibility' (i.e., $\mathcal{V}_{SP} \cdot P = \mathcal{V}_P$ and $\mathcal{V}_{SP} \times P = \mathcal{V}_P$).*

Problem 7.5 Compatibility permits recovery of $\mathcal{V}_P$ from $\mathcal{V}_{SP}$, $\mathcal{V}_{SP} \leftrightarrow \mathcal{V}_P$: *Prove*

Theorem 7.6.1 . *Let $\mathcal{V}_S = \mathcal{V}_{SP} \leftrightarrow \mathcal{V}_P$. Then, $\mathcal{V}_P = \mathcal{V}_{SP} \leftrightarrow \mathcal{V}_S$ iff $\mathcal{V}_{SP} \cdot P \supseteq \mathcal{V}_P$ and $\mathcal{V}_{SP} \times P \subseteq \mathcal{V}_P$, (i.e., iff $(\mathcal{V}_{SP}, \mathcal{V}_P)$ are compatible).*

Problem 7.6 (*) *Let $\mathcal{V}_1, \mathcal{V}_2$ be vector spaces on S_1, S_2 respectively. Find a vector space $\mathcal{V}_{12P}$ on $(S_1 \cup S_2) \uplus P$ that is an extension of both $\mathcal{V}_1$ and $\mathcal{V}_2$ such that $\mid P \mid$ is the minimum possible. Show that the minimum value of $\mid P \mid$ is $r(\mathcal{V}_1 + \mathcal{V}_2) + r((\mathcal{V}_1 \times (S_1 \cap S_2)) \cap (\mathcal{V}_2 \times (S_1 \cap S_2)))$.*

Problem 7.7 Minimal common extension and algorithms for construction of the spaces:
i. *Given vector spaces $\mathcal{V}_S^1, \cdots, \mathcal{V}_S^k$ show that if $\mathcal{V}_{SP}$ is to be a common extension of $\mathcal{V}_S^i$ through spaces $\mathcal{V}_P^i$ then*

$$\mid P \mid \geq r(\sum_i \mathcal{V}_S^i) - r(\bigcap_i \mathcal{V}_S^i).$$

ii. *Give a procedure for building $\mathcal{V}_{SP}, \mathcal{V}_P^i$ s.t. $\mid P \mid$ is equal to the RHS of the above inequality.*

Problem 7.8 *Let $\mathcal{V}_{SP}$ be a common extension of $\mathcal{V}_S^1, \mathcal{V}_S^2$. Then*
i. *$\mathcal{V}_{SP}$ is a common extension of $\mathcal{V}_{SP} \leftrightarrow \mathcal{V}_S^1$ and $\mathcal{V}_{SP} \leftrightarrow \mathcal{V}_S^2$.*
ii. Distance between spaces does not change when we take generalized minor with respect to an extension: *If $\mathcal{V}_{SP}$ is an extension of $\mathcal{V}_S^1, \mathcal{V}_S^2, \cdots, \mathcal{V}_S^k$*

$$d(\mathcal{V}_S^1, \mathcal{V}_S^2, \cdots, \mathcal{V}_S^k) = d((\mathcal{V}_{SP} \leftrightarrow \mathcal{V}_S^1), (\mathcal{V}_{SP} \leftrightarrow \mathcal{V}_S^2), \cdots, (\mathcal{V}_{SP} \leftrightarrow \mathcal{V}_S^k)).$$

Here, $d(\cdots, \mathcal{V}_S^j, \cdots)$ denotes $r(\sum \mathcal{V}_S^j)$ - $r(\cap \mathcal{V}_S^j)$.
iii. *Suppose $\mathcal{V}_{SP}$ is a common extension of $\mathcal{V}_S^1, \mathcal{V}_S^2, \cdots, \mathcal{V}_S^k$. Show that, to find a minimal extension, the following procedure is valid . Let*
$\mathcal{V}_P^1 = \mathcal{V}_{SP} \leftrightarrow \mathcal{V}_S^1$
$\mathcal{V}_P^2 = \mathcal{V}_{SP} \leftrightarrow \mathcal{V}_S^2$
$\vdots$
$\mathcal{V}_P^k = \mathcal{V}_{SP} \leftrightarrow \mathcal{V}_S^k$.
Let $\mathcal{V}_{PQ}$ be a minimal extension of $\mathcal{V}_P^1, \mathcal{V}_P^2, \cdots, \mathcal{V}_P^k$. Then, $\mathcal{V}_{SP} \leftrightarrow \mathcal{V}_{PQ}$ is a minimal extension of $\mathcal{V}_S^1, \mathcal{V}_S^2, \cdots, \mathcal{V}_S^k$.

Problem 7.9 How to build $\mathbf{R}_1, \mathbf{R}_{P_2}$ efficiently in special situations:
The procedure described in subsection 7.3.3 is useful only if the matrices $\mathbf{R}_1, \mathbf{R}_{P_2}$ can be built efficiently. Brute force Gaussian elimination should be avoided. Linear time algorithms, if available, would obviously be the best.
Let $\{E_1, \cdots, E_k\}$ be a partition of $E(\mathcal{G})$. For the following cases build $\mathbf{R}_1, \mathbf{R}_{P_2}$ efficiently: Let $\mathcal{G}'$ be
i. $\bigoplus_i \mathcal{G} \cdot E_i$
ii. $\bigoplus_i \mathcal{G} \times E_i$
iii. *obtained from $\mathcal{G}$ by fusing nodes,*
iv. *obtained from $\mathcal{G}$ by node splitting,*
v. *obtained from $\mathcal{G}$ by first fusing certain nodes and then splitting some nodes of the resulting graph.*

Problem 7.10 Nodal analysis of $\mathcal{N}$ by bordering nodal matrix of $\mathcal{N}'$: *In subsection 7.3.3 let* $\mathbf{N} = \hat{\mathbf{G}}$ *and* $\mathbf{M} = -\mathbf{I}$. *For this case derive a nodal analysis-like procedure in which the nodal matrix of graph* $\mathcal{G}'$ *appears as the core of the overall coefficient matrix. Specialise this derivation to the case where* $\mathcal{G}'$ *is made up only of self loops and show that the usual nodal analysis equations result.*

Problem 7.11
i. Using circuit matrix of $\mathcal{G}'$ instead of reduced incidence matrix: *In subsection 7.3.3 the reduced incidence matrix of* $\mathcal{G}'$ *is used while writing KCE and KVL has been applied in terms of node potentials of* $\mathcal{G}'$. *Instead derive a dual set of equations using a representative matrix of the current space of* $\mathcal{G}'$ *for writing KVE and applying KVL in terms of loop currents of* $\mathcal{G}'$.

ii. Loop analysis of $\mathcal{N}$ by bordering loop matrix of $\mathcal{N}'$: *Let* $\mathbf{M} = \hat{\mathbf{R}}$ *and* $\mathbf{N} = -\mathbf{I}$. *For this case derive a loop analysis-like procedure in which the loop analysis coefficient matrix of graph* $\mathcal{G}'$ *appears as the core of the overall coefficient matrix. Specialise this derivation to the case where* $\mathcal{G}'$ *is made up only of coloops and show that the usual loop analysis equations result.*

Problem 7.12 *(Tellegen)* **Reciprocity:** *Let* $\mathcal{N}$ *be an electrical network with graph* $\mathcal{G}$. *Let* $E(\mathcal{G})$ *be partitioned into* E_P, E_R, *where the devices in* E_P *are norators and the devices in* E_R *have the following characteristic:*
Let $\mathbf{x}_R$ *denote the vector* $(\mathbf{v}_R, \mathbf{i}_R)$, $\mathbf{y}_R$ *denote the vector* $(-\mathbf{i}_R, \mathbf{v}_R)$,
where $(\mathbf{v}_R, \mathbf{i}_R)$ *belongs to the device characteristic of* E_R. *Then the collection* $\mathcal{V}_{xR}$ *of all the* $\mathbf{x}_R$*'s is a vector space and is complementary orthogonal to the collection* $\mathcal{V}_{yR}$ *of all the* $\mathbf{y}_R$*'s.*
e.g. $\mathbf{v}_R = (\mathbf{R})\mathbf{i}_R$ *where* $\mathbf{R}$ *is a symmetric matrix.*
Let $\mathbf{x}_P$ *denote the vector* $(\mathbf{v}_P, \mathbf{i}_P)$,
$\mathbf{y}_P$ *denote the vector* $(-\mathbf{i}_P, \mathbf{v}_P)$,
where $((\mathbf{v}_P, \mathbf{v}_R), (\mathbf{i}_P, \mathbf{i}_R))$ *is a solution of the network.*
Show that the collection $\mathcal{V}_{xP}$ *of all the* $\mathbf{x}_P$*'s is complementary orthogonal to the collection* $\mathcal{V}_{yP}$ *of all the* $\mathbf{y}_P$*'s.*

Problem 7.13 Adjoint Networks: *Let* $\mathcal{N}, \mathcal{N}'$ *be electrical networks with graph* $\mathcal{G}$. *Let* $E(\mathcal{G})$ *be partitioned into* $E_D \uplus E_{yv} \uplus E_{yi} \uplus E_{uv} \uplus E_{ui}$, *where the characteristic*

i. *of the devices in* E_D *is*

$$\begin{pmatrix} \mathbf{M}_D & \mathbf{N}_D \end{pmatrix} \begin{bmatrix} \mathbf{i}_D \\ \mathbf{v}_D \end{bmatrix} = \mathbf{0} \textit{ in } \mathcal{N}.$$

$$\begin{pmatrix} \mathbf{M}_D^{\perp} & \mathbf{N}_D^{\perp} \end{pmatrix} \begin{bmatrix} \mathbf{v}'_D \\ \mathbf{i}'_D \end{bmatrix} = \mathbf{0} \textit{ in } \mathcal{N}',$$

ii. *of the devices in* E_{yv} *is*

$$\mathbf{i}_{yv} = \mathbf{0} \textit{ in } \mathcal{N}$$

unconstrained (norators) in $\mathcal{N}'$
(in particular no constraints on $\mathbf{v}'_{yv}$*)*

iii. *of the devices in* E_{yi} *is*

$$\mathbf{v}_{yi} = \mathbf{0} \ in \ \mathcal{N}$$

unconstrained (norators) in $\mathcal{N}'$
(in particular no constraints on $\mathbf{i'}_{yi}$*)*

iv. *of the devices in* E_{uv} *is*
unconstrained in $\mathcal{N}$
(in particular $\mathbf{i}_{uv}$ *has no constraints)*

$$\mathbf{v'}_{uv} = \mathbf{0} \ in \ \mathcal{N}'.$$

v. *of the devices in* E_{ui} *is*
unconstrained in $\mathcal{N}$
(in particular $\mathbf{v}_{ui}$ *has no constraints)*

$$\mathbf{i'}_{ui} = \mathbf{0} \ in \ \mathcal{N}'.$$

The symbols u, y *indicate input, output in* $\mathcal{N}$ *and output, input in* $\mathcal{N}'$. *If*

$$\begin{bmatrix} \mathbf{v}_{yv} \\ \mathbf{i}_{yi} \end{bmatrix} = \begin{bmatrix} \mathbf{K}_{11} & \mathbf{K}_{12} \\ \mathbf{K}_{21} & \mathbf{K}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{v}_{uv} \\ \mathbf{i}_{ui} \end{bmatrix} \ in \ \mathcal{N}$$

show that

$$\begin{bmatrix} \mathbf{i'}_{uv} \\ \mathbf{v'}_{ui} \end{bmatrix} = - \begin{bmatrix} \mathbf{K}_{11}^T & \mathbf{K}_{21}^T \\ \mathbf{K}_{12}^T & \mathbf{K}_{22}^T \end{bmatrix} \begin{bmatrix} \mathbf{i'}_{yv} \\ \mathbf{v'}_{yi} \end{bmatrix} \ in \ \mathcal{N}'.$$

Remark:

i. $\left(\mathbf{M}_D \quad \mathbf{N}_D \right), \left(\mathbf{M}_D^{\perp} \quad \mathbf{N}_D^{\perp} \right)$ *are representative matrices of complementary orthogonal spaces.*

ii. The usual electrical network adjoint $\mathcal{N}$" *is obtained by replacing* $\mathbf{i'}$ *by* $-\mathbf{i}$", $\mathbf{v'}$ *by* $\mathbf{v}$" *in the device characteristic.*

iii. u, y *indicate 'input','output' respectively in* $\mathcal{N}$ *and 'output','input' respectively in* $\mathcal{N}'$.

Problem 7.14 *Proof of the Implicit Duality Theorem without using finiteness assumption:
Let $< \cdot, \cdot >$ *be a q-bilinear operation as in Section 7.2. Further whenever* S, P *are disjoint let* $< \mathbf{f}_S \oplus \mathbf{f}_P, \mathbf{g}_S \oplus \mathbf{g}_P > = < \mathbf{f}_S, \mathbf{g}_S > + < \mathbf{f}_P, \mathbf{g}_P >$. *Let '*$*$*' denote the q-orthogonality operation with the set* $\mathcal{A}$ *being closed with respect to both* **addition** *and* **subtraction**. *We say a collection* $\mathcal{K}$ *of vectors on a set* T *is* **closed** *iff* $(\mathcal{K}^*)^* = \mathcal{K}$. *Prove the following:*

i.

Lemma 7.6.1 *Let $\mathcal{V}_{SP}, \mathcal{V}_P$, be vector spaces.*
Let $\mathcal{U}_P \equiv \mathcal{V}_P \cap (\mathcal{V}_{SP} \cdot P)$. Then
(a) $\mathcal{V}_{SP} \leftrightarrow \mathcal{V}_P = \mathcal{V}_{SP} \leftrightarrow \mathcal{U}_P$
(b) *if $\mathcal{V}_P, \mathcal{V}_{SP}, \mathcal{V}_{SP} \cdot P, \mathcal{V}_P^* + \mathcal{V}_{SP}^* \times P$ are closed, then $\mathcal{V}_{SP}^* \leftrightarrow \mathcal{U}_P^* = \mathcal{V}_{SP}^* \leftrightarrow \mathcal{V}_P^*$.*

ii.

Theorem 7.6.2 *Let $\mathcal{U}_P \subseteq \mathcal{V}_{SP} \cdot P$. If $\mathcal{V}_{SP}, \mathcal{V}_{SP}^* \cdot S$ are closed, then $(\mathcal{V}_{SP} \leftrightarrow \mathcal{U}_P)^* = \mathcal{V}_{SP}^* \leftrightarrow \mathcal{U}_P^*$.*

iii. *Hence, if $\mathcal{V}_P, \mathcal{V}_{SP}, \mathcal{V}_{SP} \cdot P, \mathcal{V}_{SP}^* \cdot S, \mathcal{V}_{SP}^* + \mathcal{V}_{SP}^* \times P$ are closed, then*

$$(\mathcal{V}_{SP} \leftrightarrow \mathcal{V}_P)^* = \mathcal{V}_{SP}^* \leftrightarrow \mathcal{V}_P^*.$$

7.7 Notes

It seems very dificult to trace the origins of the Implicit Duality Theorem. The first publication, that the author could trace, which refers to the ideal transformer version of the result is [Belevitch68]. However in that reference the proof of the result only deals with orthogonality and omits the crucial aspect of the ranks of the orthogonal spaces adding up to the full rank. Kron's use of the 'power invariance postulate', reminds us of this theorem [Kron39]. Unfortunately he never makes it clear under what conditions the 'postulate' can be used. Some of the proofs of the theorem presented in this chapter may be found in [Narayanan86a], [Narayanan86b], [Narayanan87].

7.8 Solutions of Exercises

E 7.1: We remind the reader that the dot product of $\mathbf{f}_1$ on S_1 with $\mathbf{f}_2$ on S_2 is equal to

$$\sum_{e \in S_1 \cap S_2} \mathbf{f}_1(e)\mathbf{f}_2(e).$$

If $S_1 \cap S_2 = \emptyset$, then, by definition, the dot product is zero.
Wherever possible we state and solve a more general version. However, throughout, the q-bilinear operation is a dot product. The reader is referred to Section 7.2 for the definition of q-orthogonality and of the set $\mathcal{A}$.
i. If $S_1 \cap S_2 = \emptyset$, then

$$(\mathcal{K}_1 \oplus \mathcal{K}_2)^* = \mathcal{K}_1{}^* \oplus \mathcal{K}_2{}^*,$$

where $\mathcal{K}_1$ ($\mathcal{K}_2$) is a collection of vectors on $S_1 (S_2)$ with the zero vector as a member.
Proof : Any vector in the RHS is of the form $\mathbf{f}_1 + \mathbf{f}_2$ where $\mathbf{f}_1 \in \mathcal{K}_1{}^*$ and $\mathbf{f}_2 \in \mathcal{K}_2{}^*$. Consider $\mathbf{g}_1 \in \mathcal{K}_1, \mathbf{g}_2 \in \mathcal{K}_2$. We have

$$< \mathbf{g}_1 + \mathbf{g}_2, \mathbf{f}_1 + \mathbf{f}_2 > = < \mathbf{g}_1, \mathbf{f}_1 > + < \mathbf{g}_2, \mathbf{f}_2 > .$$

Now $< \mathbf{g}_1, \mathbf{f}_1 >, < \mathbf{g}_2, \mathbf{f}_2 >$, belong to $\mathcal{A}$. Therefore, so does their sum (taking $\mathbf{g}, \mathbf{f}$ to be q-orthogonal iff $< \mathbf{g}, \mathbf{f} > \in \mathcal{A}$) i.e., $\mathbf{f}_1 + \mathbf{f}_2$ is q-orthogonal to every vector in $\mathcal{K}_1 \oplus \mathcal{K}_2$.
Thus, RHS $\subseteq$ LHS.
On the other hand if $\mathbf{f} \in$ LHS, then $\mathbf{f} = \mathbf{f}/S_1 \oplus \mathbf{f}/S_2 = \mathbf{f}_1 \oplus \mathbf{f}_2$, say.
Now for every $\mathbf{g}_1 \in \mathcal{K}_1, \mathbf{g}_2 \in \mathcal{K}_2$, we have

$$\mathcal{A} \ni < \mathbf{f}_1 \oplus \mathbf{f}_2, \mathbf{g}_1 \oplus \mathbf{g}_2 > = < \mathbf{f}_1, \mathbf{g}_1 > + < \mathbf{f}_2, \mathbf{g}_2 > .$$

Setting $\mathbf{g}_1$ to $\mathbf{0}$, we see that $< \mathbf{f}_2, \mathbf{g}_2 > \in \mathcal{A}$. Similarly, $< \mathbf{f}_1, \mathbf{g}_1 > \in \mathcal{A}$. Hence $\mathbf{f}_1 \in \mathcal{K}_1{}^*$ and $\mathbf{f}_2 \in \mathcal{K}_2{}^*$ and $\mathbf{f} \in$ RHS.
Thus, RHS $\supseteq$ LHS.

ii. $(\mathcal{K}_1 \oplus \mathcal{K}_2) \leftrightarrow \mathcal{K}_2 = \mathcal{K}_1$ with the usual dot product as the q-bilinear operation. Proof straight forward.

E 7.2: $\mathcal{K}_P, \mathcal{K}_P^2$ are collections of vectors on P and $\mathcal{K}_1, \mathcal{K}_2$ are collections on $S \uplus P$. We assume the collections are all closed under addition.
i. $(\mathcal{K}_1 + \mathcal{K}_2) \leftrightarrow \mathcal{K}_P \supseteq (\mathcal{K}_1 \leftrightarrow \mathcal{K}_P) + (\mathcal{K}_2 \leftrightarrow \mathcal{K}_P)$.
Proof : Let $\mathbf{f}_S \in$ RHS. Then $\mathbf{f}_S = \mathbf{f}_S{}^1 + \mathbf{f}_S{}^2$ where $\mathbf{f}_S{}^1 \in \mathcal{K}_1 \leftrightarrow \mathcal{K}_P$ and $\mathbf{f}_S{}^2 \in \mathcal{K}_2 \leftrightarrow \mathcal{K}_P$. There exist vectors $\mathbf{f}_S{}^1 \oplus \mathbf{f}_P{}^1 \in \mathcal{K}_1, \mathbf{f}_P{}^1 \in \mathcal{K}_P, \mathbf{f}_S{}^2 \oplus \mathbf{f}_P{}^2 \in \mathcal{K}_2$ and $\mathbf{f}_P{}^2 \in \mathcal{K}_P$. Then, $\mathbf{f}_S{}^1 + \mathbf{f}_S{}^2 \oplus \mathbf{f}_P{}^1 + \mathbf{f}_P{}^2 \in \mathcal{K}_1 + \mathcal{K}_2$ and $\mathbf{f}_P{}^1 + \mathbf{f}_P{}^2 \in \mathcal{K}_P$.
Hence, $\mathbf{f}_S \in$ L.H.S.

ii. $(\mathcal{K}_1 \cap \mathcal{K}_2) \leftrightarrow \mathcal{K}_P \subseteq (\mathcal{K}_1 \leftrightarrow \mathcal{K}_P) \cap (\mathcal{K}_2 \leftrightarrow \mathcal{K}_P)$.
This is immediate.

iii. $\mathcal{K}_1 \leftrightarrow (\mathcal{K}_P + \mathcal{K}_P^2) \supseteq (\mathcal{K}_1 \leftrightarrow \mathcal{K}_P) + (\mathcal{K}_1 \leftrightarrow \mathcal{K}_P^2)$.
Proof is similar to that of part (a) above. We use the fact that $\mathcal{K}_1$ is closed under addition.

iv. $\mathcal{K}_1 \leftrightarrow (\mathcal{K}_P \cap \mathcal{K}_P^2) \subseteq (\mathcal{K}_1 \leftrightarrow \mathcal{K}_P) \cap (\mathcal{K}_1 \leftrightarrow \mathcal{K}_P^2)$.
This is immediate.

v. $(\mathcal{K}_1 \leftrightarrow \mathcal{K}_P)^* = (\mathcal{K}_1 - \mathcal{K}_P)^* \cdot S$. (Note that $\mathcal{K}_1 \leftrightarrow \mathcal{K}_P = (\mathcal{K}_1 - \mathcal{K}_P) \times S$).
This holds if $((\mathcal{K}_1 - \mathcal{K}_P) \times S)^* = (\mathcal{K}_1 - \mathcal{K}_P)^* \cdot S$.

E 7.3: For (iii) below, we assume $\mathcal{K}_{SP}$ is closed under addition and
$\mathcal{K}_{SP} - \mathcal{K}_{SP} \times P \subseteq \mathcal{K}_{SP}$.
i. $\mathcal{K}_{SP} \leftrightarrow \mathcal{K}_P = \mathcal{K}_{SP} \leftrightarrow (\mathcal{K}_P \cap (\mathcal{K}_{SP} \cdot P))$

ii. $\mathcal{K}_{SP} \leftrightarrow \mathcal{K}_P = (\mathcal{K}_{SP} \cap (\mathcal{K}_{SP} \cdot S \oplus \mathcal{K}_P)) \leftrightarrow \mathcal{K}_P$.

iii. Let $\hat{\mathcal{K}}_P - \mathcal{K}_{SP} \times P = \mathcal{K}_P - \mathcal{K}_{SP} \times P$.
Then $\mathcal{K}_{SP} \leftrightarrow \hat{\mathcal{K}}_P = \mathcal{K}_{SP} \leftrightarrow \mathcal{K}_P$.
Proof :
i. If $\mathbf{f}_S \in \mathcal{K}_{SP} \leftrightarrow \mathcal{K}_P$, then there exists $\mathbf{f}_P \in \mathcal{K}_P$ s.t. $\mathbf{f}_S \oplus \mathbf{f}_P \in \mathcal{K}_{SP}$. Clearly $\mathbf{f}_P \in \mathcal{K}_P \cap (\mathcal{K}_{SP} \cdot P)$. Hence, $\mathbf{f}_S \in \mathcal{K}_{SP} \leftrightarrow (\mathcal{K}_P \cap (\mathcal{K}_{SP} \cdot P))$. The reverse containment is clear since
$\mathcal{K}_{SP} \leftrightarrow \mathcal{K}_P^1 \supseteq \mathcal{K}_{SP} \leftrightarrow \mathcal{K}_P^2$
whenever $\mathcal{K}_P^1 \supseteq \mathcal{K}_P^2$.

ii. We have $\mathcal{K}_{SP}^1 \leftrightarrow \mathcal{K}_P \supseteq \mathcal{K}_{SP}^2 \leftrightarrow \mathcal{K}_P$ whenever $\mathcal{K}_{SP}^1 \supseteq \mathcal{K}_{SP}^2$. Hence, LHS $\supseteq$ RHS.

Let $\mathbf{f}_S \in \mathcal{K}_{SP} \leftrightarrow \mathcal{K}_P$.
Then there exists $\mathbf{f}_P \in \mathcal{K}_P$ s.t. $\mathbf{f}_S \oplus \mathbf{f}_P \in \mathcal{K}_{SP}$.
Clearly, $\mathbf{f}_S \oplus \mathbf{f}_P \in (\mathcal{K}_{SP} \cdot S \oplus \mathcal{K}_P)$.
Hence, $\mathbf{f}_S \oplus \mathbf{f}_P \in \mathcal{K}_{SP} \cap (\mathcal{K}_{SP} \cdot S \oplus \mathcal{K}_P)$.
Thus, LHS $\subseteq$ RHS.

iii. Let $\mathbf{f}_S \in$ L.H.S. Then $\exists \mathbf{f}_P \in \hat{\mathcal{K}}_P$ s.t. $\mathbf{f}_S \oplus \mathbf{f}_P \in \mathcal{K}_{SP}$. By the given condition on $\mathcal{K}_P, \hat{\mathcal{K}}_P$ there exist vectors ${\mathbf{f}_P}^1, {\mathbf{f}_P}^2$ in $\mathcal{K}_{SP} \times P$, s.t. $\mathbf{f}_P - {\mathbf{f}_P}^1 + {\mathbf{f}_P}^2 \in \mathcal{K}_P$. Denote this last vector by ${\mathbf{f}_P}^3$. Clearly, $\mathbf{f}_S \oplus {\mathbf{f}_P}^3 \in \mathcal{K}_{SP}$ since $\mathcal{K}_{SP} - \mathcal{K}_{SP} \times P \subseteq \mathcal{K}_{SP}$. Hence, $\mathbf{f}_S \in \mathcal{K}_{SP} \leftrightarrow \mathcal{K}_P$. Thus LHS $\subseteq$ RHS.
The reverse containment is proved identically, interchanging the roles of $\mathcal{K}_P$ and $\hat{\mathcal{K}}_P$.

E 7.4: Let $Q \subseteq T \subseteq S$. If Implicit q-orthogonality holds:
i. $(\mathcal{K} \cdot T)^* = \mathcal{K}^* \times T$.
ii. $(\mathcal{K} \times T)^* = \mathcal{K}^* \cdot T$.
iii. $(\mathcal{K} \times T \cdot Q)^* = \mathcal{K}^* \cdot T \times Q$.
Proof :
i. Take $\mathcal{K}_{S-T} \equiv \mathcal{V}_{S-T}$, where $\mathcal{V}_{S-T}$ is the space $\mathcal{X}_{S-T}$ of all vectors on $S-T$. Then by Lemma 7.2.1
$\mathcal{V}^*_{S-T} = V^{\perp}_{S-T} = \{\mathbf{0}_{S-T}\}$.
We have

$$(\mathcal{K} \cdot T)^* = (\mathcal{K} \leftrightarrow \mathcal{K}_{S-T})^* = \mathcal{K}^* \leftrightarrow \mathcal{K}^*_{S-T} = \mathcal{K}^* \times T.$$

ii. In this case $\mathcal{K}_{S-T} \equiv \{\mathbf{0}_{S-T}\}$ and $\mathcal{K}^*_{S-T}$ is the vector space $\mathcal{X}_{S-T}$ on $S-T$ having full rank.

iii. Letting $\mathcal{X}_P, P \subseteq S$, denote the space on P with full rank, we have
$\mathcal{K} \times T \cdot Q = \mathcal{K} \leftrightarrow \mathcal{K}_{S-Q}$,
where $\mathcal{K}_{S-Q} \equiv \mathbf{0}_{S-T} \oplus \mathcal{X}_{T-Q}$.
Now $\mathcal{K}^*_{S-Q} = \mathcal{X}_{S-T} \oplus \mathbf{0}_{T-Q}$.
So, $\mathcal{K}^* \leftrightarrow \mathcal{K}^*_{S-Q} = \mathcal{K}^* \cdot T \times Q$.

E 7.5: Let $\mathcal{K}_{SP}, \mathcal{K}_S$ be collections of vectors on $S \uplus P, S$ respectively. Then there exists a collection of vectors $\mathcal{K}_P$ on P s.t. $\mathbf{0} \in \mathcal{K}_P$ and $\mathcal{K}_{SP} \leftrightarrow \mathcal{K}_P = \mathcal{K}_S$ only if $\mathcal{K}_{SP} \times S \subseteq \mathcal{K}_S \subseteq \mathcal{K}_{SP} \cdot S$. The latter condition is sufficient for the existence of $\mathcal{K}_P$ with $\mathbf{0} \in \mathcal{K}_P$, provided $\mathcal{K}_{SP}$ is a vector space and $\mathcal{K}_S$ is closed under addition.
Proof : Suppose $\mathcal{K}_{SP} \leftrightarrow \mathcal{K}_P = \mathcal{K}_S$ and $\mathbf{0} \in \mathcal{K}_P$. It is clear from the definition of the generalized minor operation that $\mathcal{K}_{SP} \cdot S \supseteq \mathcal{K}_S$. Since $\mathbf{0} \in \mathcal{K}_P$ every vector $\mathbf{f}_S$ s.t. $\mathbf{f}_S \oplus \mathbf{0}_P \in \mathcal{K}_{SP}$ must belong to $\mathcal{K}_S$. Thus, $\mathcal{K}_{SP} \times S \subseteq \mathcal{K}_S$.
On the other hand suppose $\mathcal{K}_{SP} \cdot S \supseteq \mathcal{K}_S \supseteq \mathcal{K}_{SP} \times S$.
Let $\mathcal{K}_P$ be the collection of all vectors $\mathbf{f}_P$ s.t. for some vector $\mathbf{f}_S \in \mathcal{K}_S$, $\mathbf{f}_S \oplus \mathbf{f}_P \in \mathcal{K}_{SP}$. Clearly $\mathcal{K}_{SP} \leftrightarrow \mathcal{K}_P \supseteq \mathcal{K}_S$. If ${\mathbf{f}_S}' \in \mathcal{K}_{SP} \times S$ then ${\mathbf{f}_S}' \in \mathcal{K}_S$ and ${\mathbf{f}_S}' \oplus \mathbf{0}_P \in \mathcal{K}_{SP}$. Hence, by definition of $\mathcal{K}_P, \mathbf{0} \in \mathcal{K}_P$.
Let $\mathbf{f}_S$ be s.t. $\mathbf{f}_S \oplus \mathbf{f}_P \in \mathcal{K}_{SP}$ for some $\mathbf{f}_P \in \mathcal{K}_P$. We know that there exists ${\mathbf{f}_S}' \in \mathcal{K}_S$ s.t. ${\mathbf{f}_S}' \oplus \mathbf{f}_P \in \mathcal{K}_{SP}$. Since $\mathcal{K}_{SP}$ is a vector space, we must have, $(\mathbf{f}_S - {\mathbf{f}_S}') \oplus \mathbf{0}_P \in \mathcal{K}_{SP}$. Hence, $\mathbf{f}_S - {\mathbf{f}_S}' \in \mathcal{K}_{SP} \times S \subseteq \mathcal{K}_S$. Since $\mathcal{K}_S$ is closed under addition and ${\mathbf{f}_S}' \in \mathcal{K}_S$,

it follows that $(\mathbf{f}_S - \mathbf{f}_S{}') + \mathbf{f}_S{}' = \mathbf{f}_S$ also belong to $\mathcal{K}_S$. Thus, $\mathcal{K}_{SP} \leftrightarrow \mathcal{K}_P \subseteq \mathcal{K}_S$. □

E 7.6:
i. A vector $\mathbf{f}_S \oplus \mathbf{f}_Q$ belongs to each of the spaces (whose equality is to be proved) iff there exist vectors $\mathbf{f}_S \oplus \mathbf{f}_T$, $\mathbf{f}_T \oplus \mathbf{f}_P$, $\mathbf{f}_P \oplus \mathbf{f}_Q$, belonging respectively to $\mathcal{V}_{ST}, \mathcal{V}_{TP}, \mathcal{V}_{PQ}$.
The skewed sum case is similar.

ii. For each of these spaces a vector $\mathbf{f}_{S1} \oplus \mathbf{f}_{P1} \oplus \mathbf{f}_{S2} \oplus \mathbf{f}_{P2}$ is a member iff there exist vectors $\mathbf{f}_{S1} \oplus \mathbf{f}_{T1}$, $\mathbf{f}_{T1} \oplus \mathbf{f}_{P1}$, $\mathbf{f}_{S2} \oplus \mathbf{f}_{T2}$, $\mathbf{f}_{T2} \oplus \mathbf{f}_{P2}$ in vector spaces $\mathcal{V}_{S_1T_1}$, $\mathcal{V}_{T_1P_1}$, $\mathcal{V}_{S_2T_2}$, $\mathcal{V}_{T_2P_2}$, respectively. The skewed sum case is similar.

iii. A vector $\mathbf{f}_{S_1} \oplus \cdots \oplus \mathbf{f}_{S_n}$ belongs to LHS (as well as RHS) iff there exist vectors

$$\mathbf{f}_{S_1} \oplus \mathbf{f}_{T_1}, \mathbf{f}_{S_2} \oplus \mathbf{f}_{T_2}, \cdots, \mathbf{f}_{S_n} \oplus \mathbf{f}_{T_n}$$

belonging respectively to $\mathcal{V}_{S_1T_1} \oplus \cdots \oplus \mathcal{V}_{S_nT_n}$ and a vector $\mathbf{f}_{T_1} \oplus \cdots \oplus \mathbf{f}_{T_n}$ belonging to $\mathcal{V}_{T_1T_2\cdots T_n}$. The skewed sum case is similar.

E 7.7:
i. This is clear since

$$(\mathcal{V}_{S_1T} \cdot T)^{\perp} = \mathcal{V}_{S_1T}^{\perp} \times T,$$

$$(\mathcal{V}_{S_1T} \times T)^{\perp} = \mathcal{V}_{S_1T}^{\perp} \cdot T, \quad \textit{and}$$

$$\mathcal{V}_1 \supseteq \mathcal{V}_2 \text{ iff } \mathcal{V}_1^{\perp} \subseteq \mathcal{V}_2^{\perp}.$$

ii. Let $\mathcal{V}_{S_1S_2} \equiv (\mathcal{V}_{S_1T} \leftrightarrow \mathcal{V}_{S_2T})$.
A vector $\mathbf{f}_{S2} \oplus \mathbf{f}_T \in \mathcal{V}_{S_1T} \leftrightarrow \mathcal{V}_{S_1S_2}$
iff there exist vectors
$\mathbf{f}_{S1} \oplus \mathbf{f}_T \in \mathcal{V}_{S_1T}$, $\mathbf{f}_{S1} \oplus \hat{\mathbf{f}}_T \in \mathcal{V}_{S_1T}$, and $\mathbf{f}_{S2} \oplus \hat{\mathbf{f}}_T \in \mathcal{V}_{S_2T}$.
Now, $\mathbf{0}_{S_1} \oplus (\mathbf{f}_T - \hat{\mathbf{f}}_T) \in \mathcal{V}_{S_1T}$, i.e., $\mathbf{f}_T - \hat{\mathbf{f}}_T \in \mathcal{V}_{S_1T} \times T \subseteq \mathcal{V}_{S_2T} \times T$.
Thus, $\mathbf{f}_{S2} \oplus \hat{\mathbf{f}}_T + (\mathbf{f}_T - \hat{\mathbf{f}}_T) \in \mathcal{V}_{S_2T}$, i.e., $\mathbf{f}_{S2} \oplus \mathbf{f}_T \in \mathcal{V}_{S_2T}$.
Next let $\mathbf{f}_{S2} \oplus \mathbf{f}_T \in \mathcal{V}_{S_2T}$. Since $\mathcal{V}_{S_1T} \cdot T \supseteq \mathcal{V}_{S_2T} \cdot T$, there exists a vector $\mathbf{f}_{S1} \oplus \mathbf{f}_T \in \mathcal{V}_{S_1T}$. Hence, $\mathbf{f}_{S1} \oplus \mathbf{f}_{S_2} \in \mathcal{V}_{S_1S_2}$. Therefore, $\mathbf{f}_{S2} \oplus \mathbf{f}_T \in \mathcal{V}_{S_1T} \leftrightarrow \mathcal{V}_{S_1S_2}$.
This proves the result.

iii. We have $(\mathcal{V}_{S_1T}^{\perp}, \mathcal{V}_{S_2T}^{\perp})$ compatible. Therefore,

$$(\mathcal{V}_{S_1T})^{\perp} \leftrightarrow (\mathcal{V}_{S_1T}^{\perp} \leftrightarrow \mathcal{V}_{S_2T}^{\perp}) = \mathcal{V}_{S_2T}^{\perp}$$

Taking orthogonal complements on both sides we get,

$$\mathcal{V}_{S_1T} \rightleftharpoons (\mathcal{V}_{S_1T} \rightleftharpoons \mathcal{V}_{S_2T}) = \mathcal{V}_{S_2T}.$$

E 7.8: Let $\mathbf{B}_f = [\mathbf{I} \vdots \mathbf{B}_{12}]$ be the f-circuit matrix of $\mathcal{G}$, where the identity matrix columns correspond to a coforest $\bar{f}$ and the columns of $\mathbf{B}_{12}$ correspond to the forest f. Associate with each edge of f an ordered pair of terminals with no two pairs corresponding to different edges having common terminals.

(We describe a procedure which would work for any rational matrix $[\mathbf{I}\vdots\mathbf{B}_{12}]$ with the provision that we use $1:b_i$ transformers instead of $1:1$ transformers). Suppose a cotree voltage $v_e = \mathbf{B}_e\mathbf{v}_f = b_1e_1 + \cdots + b_ke_k$, where $\mathbf{B}_e = (b_1, \cdots, b_k)$ is the appropriate row of $\mathbf{B}_{12}$ and $v_{e_1}, \cdots, v_{e_k}$ are the voltages associated with the forest branches. We will use one 2-port ideal transformer for each nonzero entry of $\mathbf{B}_{12}$. Corresponding to $b_i, i = 1, \cdots, k$, we would have a $1 : b_i$ transformer. The primary and secondary of this transformer have reference arrows. So the nodes of the primary (secondary) can be called tail node and head node corresponding to tail of the arrow and head of the arrow associated with the primary (secondary). Attach the tail (head) node of the primary of the $1 : b_i$ transformer to the first (second) node of the ordered pair of nodes associated with edge e_i.
Put the secondaries of the $1 : b_i$ transformers in series. Assume for simplicity that $b_1, \cdots, b_k$ are non zero.
The sum $b_1v_1 + \cdots + b_kv_k$ would now be associated with an ordered pair of nodes (n_1, n_2),where n_1 is the tail node of the secondary of the $1 : b_1$ transformer and n_2 is the head node of the secondary of the $1 : b_k$ transformer. The ordered pair (n_2, n_1) of nodes will now be associated with the directed cotree edge e. When this procedure is completed each directed edge of the graph would be associated with an ordered pair of nodes. The voltage constraint at these pairs of terminals is given by

$$\begin{bmatrix} \mathbf{I} & \mathbf{B_{12}} \end{bmatrix} \begin{bmatrix} \mathbf{v}_{\bar{f}} \\ \mathbf{v}_f \end{bmatrix} = \mathbf{0}.$$

By the Implicit Duality Theorem the current constraints must be

$$\begin{bmatrix} -\mathbf{B}_{12}^T & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{i}_{\bar{f}} \\ \mathbf{i}_f \end{bmatrix} = \mathbf{0}.$$

These are precisely the voltage and current constraints of the graph.

E 7.9: Ideal transformers cannot be connected inconsistently since the zero solution would always work for linear homogeneous equations. When two ideal transformers of different turns ratio are connected in parallel they would still permit zero voltage across the ports.

E 7.10: The voltage constraints are $\mathcal{V}_v(\mathcal{G}) \leftrightarrow \mathcal{V}_P$ and the current constraints, $\mathcal{V}_i(\mathcal{G}) \leftrightarrow \mathcal{V}_P^{\perp}$. Thus the ideal transformer that the remaining devices of the network 'see', is

$$(\mathcal{V}_v(\mathcal{G}) \leftrightarrow \mathcal{V}_P, \;\; \mathcal{V}_i(\mathcal{G}) \leftrightarrow \mathcal{V}_P^{\perp})$$

E 7.11: [Narayanan87] If $\mathcal{V}_{SP} \leftrightarrow \mathcal{V}_P = \mathcal{V}$, then clearly $\mathcal{V}_{SP} \cdot S \supseteq \mathcal{V}$. Further, $\mathcal{V}_{SP}^{\perp} \leftrightarrow \mathcal{V}_P^{\perp} = \mathcal{V}^{\perp}$. So $\mathcal{V}_{SP}^{\perp} \cdot S \supseteq \mathcal{V}^{\perp}$, i.e., $(\mathcal{V}_{SP}^{\perp} \cdot S)^{\perp} \subseteq \mathcal{V}$, i.e., $\mathcal{V}_{SP} \times S \subseteq \mathcal{V}$. In the present problem it is therefore clear that $\mathcal{V}_{SP} \cdot S \supseteq \mathcal{V} + \mathcal{V}'$ and $\mathcal{V}_{SP} \times S \subseteq \mathcal{V} \cap \mathcal{V}'$. Hence,

$$r(\mathcal{V}_{SP} \cdot S) - r(\mathcal{V}_{SP} \times S) \geq r(\mathcal{V} + \mathcal{V}') - r(\mathcal{V} \cap \mathcal{V}').$$

But

$$r(\mathcal{V}_{SP} \cdot S) - r(\mathcal{V}_{SP} \times S) = r(\mathcal{V}_{SP} \cdot P) - r(\mathcal{V}_{SP} \times P) \leq \mid P \mid.$$

Thus, $\mid P \mid \geq r(\mathcal{V}+\mathcal{V}') - r(\mathcal{V}\cap\mathcal{V}')$.
However, the construction given in the subsection 7.3.3 actually achieves equality. Thus, for $\mathcal{V}_{SP}$ to be a minimum extension of $\mathcal{V}$ and $\mathcal{V}'$ it is necessary and sufficient that

$$\mid P \mid = \ r(\mathcal{V}+\mathcal{V}') - r(\mathcal{V}\cap\mathcal{V}').$$

E 7.12: Let $\mathcal{G}$ be composed of two edges e_1, e_2 in parallel directed in the same direction and let $\mathcal{G}'$ have e_1, e_2 in parallel directed oppositely.
To make both $\mathcal{G}$ and $\mathcal{G}'$ the minors of a graph $\mathcal{G}_{EP}$, we may think of P as composed of $P_1 \uplus P_2$. When edges of P_1 are shorted and those of P_2 opened, we should get $\mathcal{G}$ and when edges of P_2 are shorted and those of P_1 opened, $\mathcal{G}'$. It can be seen that this requires four edges. Starting from $\mathcal{G}$ we first introduce two edges in series with e_2 one at its tail end and the other at the head end. These would be P_1. Now add an edge from the tail of e_2 to the head of e_1 and an edge from the head of e_2 to the tail of e_1. These would be P_2. The reason we cannot do with less number of edges in P is that to reverse e_2 both its endpoints have to be detached and reattached. However, our construction clearly requires

$$r(\mathcal{V}_v(\mathcal{G})+\mathcal{V}_v(\mathcal{G}')) - r(\mathcal{V}_v(\mathcal{G})\cap\mathcal{V}_v(\mathcal{G}')) = 2$$

elements in P. So $\mathcal{V}_{EP}$ with $\mid P \mid = 2$ cannot be the voltage space of a graph.
Remark: Node pair fusion and node fission operations (to be discussed in the chapter on hybrid rank) are more powerful than graph minor operations. In the present case to move from $\mathcal{G}$ to $\mathcal{G}'$ we require only two such operations.

E 7.13: If $\mathcal{G}'$ is made up only of coloops, then $\mathcal{V}_v(\mathcal{G}') \supseteq \mathcal{V}_v(\mathcal{G})$. Hence, $\mathbf{R}_1$ would have zero rows, i.e., would not exist. As described in Subsection 7.3.3 KVE of $\mathcal{G}$ can be written as the two constraints

$$\mathbf{v}_E = (\mathbf{A}_r{}')^T\mathbf{v}_n{}' \qquad and \qquad \mathbf{R}_{P_2}^T\mathbf{v}_n{}' = \mathbf{0},$$

where $\mathbf{A}_r{}'$ is the reduced incidence matrix of $\mathcal{G}'$. But $\mathbf{A}_r{}'$ is the unit matrix since $\mathcal{G}'$ is made up of coloops. So the KVE of $\mathcal{G}$ is equivalent to $\mathbf{R}_{P_2}^T\mathbf{v}_E = \mathbf{0}$. Further, the columns of $\mathbf{R}_{P_2}$ are linearly independent. Thus, $\mathbf{R}_{P_2}^T$ must be a representative matrix of $\mathcal{V}_i(\mathcal{G})$.
If $\mathcal{G}'$ is made up entirely of self loops, $\mathbf{A}_r{}'$ will have no rows, i.e., would not exist. Neither would $\mathbf{R}_{P_2}$. The matrix $\mathbf{R}_1$ would be the same as the reduced incidence matrix $\mathbf{A}_r$ of $\mathcal{G}$.

E 7.14:
i. Let $\mathcal{V}, \mathcal{V}', \mathcal{V}"$ denote $\mathcal{V}_v(\mathcal{G}), \mathcal{V}_v(\mathcal{G}'), \mathcal{V}_v(\mathcal{G}")$ respectively. We will only verify the triangle inequality. We have,

$$\begin{aligned} d(\mathcal{G},\mathcal{G}') + d(\mathcal{G}',\mathcal{G}") &= r(\mathcal{V}+\mathcal{V}') + r(\mathcal{V}'+\mathcal{V}") - r(\mathcal{V}\cap\mathcal{V}') - r(\mathcal{V}'\cap\mathcal{V}") \\ d(\mathcal{G},\mathcal{G}") &= r(\mathcal{V}+\mathcal{V}") - r(\mathcal{V}\cap\mathcal{V}"). \end{aligned}$$

We use the identity,

$$r(\mathcal{V}_1) + r(\mathcal{V}_2) = r(\mathcal{V}_1\cap\mathcal{V}_2) + r(\mathcal{V}_1+\mathcal{V}_2).$$

Hence,

$$d(\mathcal{G},\mathcal{G}') + d(\mathcal{G}',\mathcal{G}") - d(\mathcal{G},\mathcal{G}") = 2[(r(\mathcal{V}') - r(\mathcal{V}\cap\mathcal{V}' + \mathcal{V}'\cap\mathcal{V}"))$$
$$+(r(\mathcal{V}\cap\mathcal{V}") - r(\mathcal{V}\cap\mathcal{V}'\cap\mathcal{V}"))] \geq 0.$$

ii. The first part is similar to the above.
The second part follows from the fact that
$(\mathcal{V}+\mathcal{V}')^{\perp} = \mathcal{V}^{\perp}\cap(\mathcal{V}')^{\perp}$
and $\quad r(\mathcal{V})^{\perp} = \mid E \mid - r(\mathcal{V})$ if $\mathcal{V}$ is on E.

E 7.15: Let us rewrite Equation 7.7 as follows:

$$\begin{bmatrix} \mathbf{C}_{11} & \vdots & \mathbf{C}_{12} \\ \cdots & \cdots & \cdots \\ \mathbf{C}_{21} & \vdots & \mathbf{0} \end{bmatrix} \begin{matrix} \mathbf{x} \\ \mathbf{y} \end{matrix} = \begin{bmatrix} \alpha \\ \mathbf{0} \end{bmatrix} \tag{7.10}$$

where $\mathbf{x} = \begin{bmatrix} \mathbf{i}_E \\ \mathbf{v'}_n \\ \mathbf{v}_E \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} \mathbf{i}_{P_2} \\ \mathbf{v}_{P_1} \end{bmatrix}$. We will describe the plan of our method first algebraically and later give it a network interpretation.
Assume that the overall matrix is invertible. We have, from the above equations, if $\mathbf{C}_{11}$ is invertible,

$$\mathbf{x} = {\mathbf{C}_{11}}^{-1}[\alpha - \mathbf{C}_{12}\ \ \mathbf{y}] \tag{7.11}$$
$$\mathbf{C}_{21}\mathbf{x} = \mathbf{0}. \tag{7.12}$$

Hence,

$$\mathbf{C}_{21}{\mathbf{C}_{11}}^{-1}(\alpha - \mathbf{C}_{12}\ \ \mathbf{y}) = \mathbf{0}, \tag{7.13}$$
$$i.e., \qquad \mathbf{C}_{21}{\mathbf{C}_{11}}^{-1}\mathbf{C}_{12}\mathbf{y} = \mathbf{C}_{21}{\mathbf{C}_{11}}^{-1}\ \alpha. \tag{7.14}$$

Since we assumed that the equation 7.10 is uniquely solvable the above equation in $\mathbf{y}$ must also be uniquely solvable. Substituting this value of $\mathbf{y}$ in equation 7.11 we get the value of $\mathbf{x}$.
Now the network interpretation:
We first show that our assumptions above follow from assumptions about unique solvability of $\mathcal{N}$ and $\mathcal{N}'$.

Equation 7.7 is equivalent to the constraints of the network $\mathcal{N}$ as far as the variables $\mathbf{v}_E, \mathbf{i}_E$ are concerned. Hence the solution of this equation is unique as far as $\mathbf{v}_E, \mathbf{i}_E$ are concerned. The overall solution would be unique if we can show that the columns corresponding to $\mathbf{v}_n{}', \mathbf{i}_{P_2}, \mathbf{v}_{P_1}$ are linearly independent. This is so because (a) columns of $\begin{pmatrix} (\mathbf{A}_r{}')^T & \mathbf{R}_1^T \end{pmatrix}$ are linearly independent - indeed they form a basis for $\mathcal{V}+\mathcal{V}'$. The matrix $\mathbf{R}_{P_2}$ is row equivalent to an identity matrix and therefore, has independent columns. Thus, $\begin{pmatrix} (\mathbf{A}_r{}')^T & \mathbf{0} & \mathbf{R}_1^T \\ \mathbf{0} & \mathbf{R}_{P_2} & \mathbf{0} \end{pmatrix}$ has

linearly independent columns as desired.
We assumed $\mathcal{N}'$ is uniquely solvable. The matrix $\mathbf{C}_{11}$ would be the coefficient matrix, if for $\mathcal{N}'$, we write KCE, KVL (in the potential difference form) and device characteristic equations. Hence, $\mathbf{C}_{11}$ is invertible.

Next we interpret the steps of the solution when we solve

$$\mathbf{C}_{11}\ \mathbf{x} = \alpha - \mathbf{C}_{12}\ \ \mathbf{y}$$

We are solving network $\mathcal{N}'$, $(1+ \mid P \mid)$ times. First solve with α in place but $\mathbf{y} = 0$. Next set $\alpha = 0$ and keep one component of $\mathbf{y}$ at a time equal to 1 and all the rest zero. The solutions we get are respectively equivalent to ${\mathbf{C}_{11}}^{-1}\ \alpha$ and columns of ${\mathbf{C}_{11}}^{-1}\ \mathbf{C}_{12}$. Referring to Equation 7.7 we see that each column of $\mathbf{C}_{12}$ is effectively either a column of $\mathbf{R}_{P_2}$ or a column of $-\mathbf{R}_1^T$. When we set one of the components of $\mathbf{i}_{P_2}$ equal to 1 the corresponding column $\mathbf{R}_{P_2 j}$ comes into the equation since we have

$$\mathbf{A}_r{}'\ \mathbf{i}_E = -\mathbf{R}_{P_2 j}.$$

This means that in the network $\mathcal{N}'$ with sources $\mathbf{j}, \mathbf{e}$ set to zero we have the current source vector $-\mathbf{R}_{P_2 j}$ entering the nodes. When we set one of the components of $\mathbf{v}_{P_1}$ equal to 1 the corresponding column $-\mathbf{R}_{1j}^T$ comes into the picture since we have

$$\mathbf{v}_E - \mathbf{R}_{1j}^T = (\mathbf{A}_r{}')^T \mathbf{v}_n{}'.$$

Here, $-\mathbf{R}_{1j}^T$ may be thought of as a branch voltage source vector, each entry being the value of a source voltage in series with the corresponding branch.

Because $\mid P \mid \ll r(\mathcal{G}')$ we may neglect the computational effort needed to construct Equation 7.14. (The matrix $({\mathbf{C}_{11}}^{-1}\ \ \mathbf{C}_{12})$ has already been constructed. It has $\mid E \mid$ rows and $\mid P \mid$ columns. The matrix $\mathbf{C}_{21}$ has $\mid P \mid$ rows and $\mid E \mid$ columns. So the effort to build the coefficient matrix is $O(\mid P \mid^2 \mid E \mid))$. The solution of this equation is also easy since the coefficient matrix is $\mid P \mid \times \mid P \mid$. Once the value of $\mathbf{y}$ is found, substituting it in Equation 7.11 would give us $\mathbf{x}$ which contains $(\mathbf{v}_E, \mathbf{i}_E)$, the solution of $\mathcal{N}$.

E 7.16: Already done earlier (see for instance solution of Exercises 7.1, 7.2 etc.).

E 7.17: See Solution of Exercise 7.4.

E 7.18: Let $P = P_1 \uplus P_2$.
i. We need to show that

$$(\mathcal{K}_{SP} \leftrightarrow \mathcal{K}_{P_1}) \leftrightarrow \mathcal{K}_{P_2} = \mathcal{K}_{SP} \leftrightarrow (\mathcal{K}_{P_1} \oplus \mathcal{K}_{P_2}).$$

Both LHS and RHS are composed of vectors $\mathbf{f}_S$ s.t. $\mathbf{f}_S \oplus \mathbf{f}_{P_1} \oplus \mathbf{f}_{P_2} \in \mathcal{K}_{SP}$ for some $\mathbf{f}_{P_1}, \mathbf{f}_{P_2}$ in $\mathcal{K}_{P_1}, \mathcal{K}_{P_2}$ respectively. So the equality holds.

ii. Let $\mathbf{f}_S \in$ RHS. Then there exists $\mathbf{f}_P \in \mathcal{K}_P$ s.t. $\mathbf{f}_S \oplus \mathbf{f}_P \in \mathcal{K}_{SP}$. But then $\mathbf{T}_P(\mathbf{f}_P) \in \mathcal{K}'_P$ and $\mathbf{f}_S \oplus \mathbf{T}_P(\mathbf{f}_P) \in \mathcal{K}'_{SP}$. Hence, $\mathcal{K}_{SP} \leftrightarrow \mathcal{K}_P \subseteq \mathcal{K}'_{SP} \leftrightarrow \mathcal{K}'_P$. The reverse containment follows similarly by working with ${\mathbf{T}_P}^{-1}$.

iii. We will first consider the case where $\mathcal{K}'_P$ is a vector space. Let $\mathcal{K}'_P$ have the representative matrix $[\mathbf{I}_{P_1}\ \mathbf{R}_{P_2}]$ and let $\mathcal{K}_P$ have the representative matrix $[\mathbf{I}_{P_1}\ \mathbf{0}_{P_2}]$. Let $\mathbf{T}_P$ be the nonsingular transformation defined by

$$\mathbf{T}_P(\mathbf{f}_{P_1} \oplus \mathbf{f}_{P_2}) = \begin{pmatrix} \mathbf{f}^T_{P_1} & \mathbf{f}^T_{P_2} \end{pmatrix} \begin{bmatrix} \mathbf{I}_{P_1} & \mathbf{R}_{P_2} \\ \mathbf{0} & \mathbf{I}_{P_2} \end{bmatrix}$$

Clearly, $\mathbf{T}_P(\mathbf{f}_{P_1} \oplus \mathbf{0}_{P_2}) = \mathbf{f}^T_{P_1} + (\mathbf{f}^T_{P_1}(\mathbf{R}_{P_2}))$
So, $\mathcal{K}'_P = \mathbf{T}_P(\mathcal{K}_P)$.
Let $\mathcal{K}'_{SP} = \mathbf{T}_{SP}(\mathcal{K}_{SP})$, where $\mathbf{T}_{SP}$ is defined in terms of $\mathbf{T}_P$ as in the previous section of this problem. We have

$$\mathcal{K}_{SP} \leftrightarrow \mathcal{K}_P = \mathcal{K}'_{SP} \leftrightarrow \mathcal{K}'_P.$$

Now since $\mathcal{K}'_P = \mathbf{T}_P(\mathcal{K}_P)$, we claim that
$(\mathcal{K}'_P)^* = ({\mathbf{T}_P}^T)^{-1}(\mathcal{K}^*_P)$ $\qquad (*)$
To prove this we first check that vectors in $\mathcal{K}'_P$ and $({\mathbf{T}_P}^T)^{-1}(\mathcal{K}^*_P)$ are q-orthogonal (i.e., that

$$(\mathcal{K}'_P)^* \supseteq ({\mathbf{T}_P}^T)^{-1}(\mathcal{K}^*_P)).$$

We have, if ${\mathbf{f}_P}' \in \mathcal{K}'_P$ and ${\mathbf{g}_P}' \in ({\mathbf{T}_P}^T)^{-1}(\mathcal{K}^*_P)$, that there must exist $\mathbf{f}_P \in \mathcal{K}_P$ and $\mathbf{g}_P \in \mathcal{K}^*_P$, s.t ${\mathbf{f}_P}' = \mathbf{T}_P(\mathbf{f}_P)$ and ${\mathbf{g}_P}' = ({\mathbf{T}_P}^T)^{-1}(\mathbf{g}_P)$. Hence,

$$({\mathbf{f}_P}')^T({\mathbf{g}_P}') = (\mathbf{T}_P(\mathbf{f}_P))^T(({\mathbf{T}_P}^T)^{-1}(\mathbf{g}_P)) = {\mathbf{f}_P}^T\mathbf{g}_P \in \mathcal{A}.$$

(Here we have treated $\mathbf{T}_P(\mathbf{f}_P)$ as a matrix product). On the other hand if $\mathbf{f}_P \in \mathcal{K}_P$ and $\mathbf{g}_P \in ({\mathbf{T}_P}^T)((\mathcal{K}'_P)^*)$, then there must exist ${\mathbf{f}_P}' \in \mathcal{K}'_P$ and ${\mathbf{g}_P}' \in (\mathcal{K}'_P)^*$, s.t $\mathbf{f}_P = {\mathbf{T}_P}^{-1}({\mathbf{f}_P}')$ and $\mathbf{g}_P = ({\mathbf{T}_P}^T)({\mathbf{g}_P}')$.Hence,

$$(\mathbf{f}_P)^T(\mathbf{g}_P) =< {\mathbf{T}_P}^{-1}{\mathbf{f}_P}',\ {\mathbf{T}_P}^T{\mathbf{g}_P}' >=< {\mathbf{f}_P}',\ {\mathbf{g}_P}' > \in \mathcal{A},$$

i.e., ${\mathbf{T}_P}^T(\mathcal{K}'_P)^* \subseteq \mathcal{K}^*_P$, equivalently, $(\mathcal{K}'_P)^* \subseteq ({\mathbf{T}_P}^T)^{-1}\mathcal{K}^*_P$. This proves $(*)$.
Since

$$\mathbf{T}_{SP}(\mathbf{f}_S \oplus \mathbf{f}_P) = \mathbf{f}_S \oplus \mathbf{T}_P(\mathbf{f}_P),$$

it follows that

$$(\mathbf{T}^T_{SP})^{-1}(\mathbf{f}_S \oplus \mathbf{f}_P) = \mathbf{f}_S \oplus ({\mathbf{T}_P}^T)^{-1}(\mathbf{f}_P).$$

As in the case of $\mathcal{K}_P, \mathcal{K}'_P$ we can verify in the case of $\mathcal{K}_{SP}, \mathcal{K}'_{SP}(\equiv \mathbf{T}_{SP}(\mathcal{K}_{SP}))$ also that $(\mathcal{K}'_{SP})^* = (\mathbf{T}^T_{SP})^{-1}(\mathcal{K}^*_{SP})$. By the previous section of the present problem we must have

$$(\mathcal{K}'_{SP})^* \leftrightarrow (\mathcal{K}'_P)^* = \mathcal{K}^*_{SP} \leftrightarrow \mathcal{K}^*_P.$$

But it is given that

$$(\mathcal{K}_{SP} \leftrightarrow \mathcal{K}_P)^* = \mathcal{K}^*_{SP} \leftrightarrow \mathcal{K}^*_P$$

and we have already seen that

$$\mathcal{K}_{SP} \leftrightarrow \mathcal{K}_P = \mathcal{K}'_{SP} \leftrightarrow \mathcal{K}'_P.$$

It follows therefore that

$$(\mathcal{K}'_{SP} \leftrightarrow \mathcal{K}'_P)^* = (\mathcal{K}'_{SP})^* \leftrightarrow (\mathcal{K}'_P)^*$$

Next, we handle the case where $\mathcal{K}_P$ is not a vector space.
Let $\mathcal{K}_{P'}$ be a copy of $\mathcal{K}_P$ on set P' which itself is a copy of P disjoint from $S \uplus P$.
Let $\mathcal{K}_{PP'}$ be the vector space with representative matrix

$$\begin{array}{cc} P & P' \\ \end{array}$$
$$\left[\begin{array}{cc} \mathbf{I} & \mathbf{I} \end{array}\right],$$

where the rows have 1's on columns corresponding to elements which are copies of each other. We then have, by the definition of generalized minor,

$$(\mathcal{K}_{SP} \oplus \mathcal{K}'_P) \leftrightarrow \mathcal{K}_{PP'} = \mathcal{K}_{SP} \leftrightarrow \mathcal{K}_P.$$

Since $\mathcal{K}_{PP'}$ is a vector space we must have,

$$\begin{aligned} ((\mathcal{K}_{SP} \oplus \mathcal{K}_{P'}) \leftrightarrow \mathcal{K}_{PP'})^* &= (\mathcal{K}_{SP} \oplus \mathcal{K}_{P'})^* \leftrightarrow \mathcal{K}^*_{PP'} \\ &= (\mathcal{K}^*_{SP} \oplus \mathcal{K}^*_{P'}) \leftrightarrow \mathcal{K}^*_{PP'} \end{aligned}$$

(see the solution of Exercise 7.1).
Now $\mathcal{K}^*_{PP'} (= \mathcal{K}^{\perp}_{PP'})$ has the representative matrix

$$\begin{array}{cc} P & P' \\ \end{array}$$
$$\left[\begin{array}{cc} \mathbf{I} & -\mathbf{I} \end{array}\right].$$

Hence,

$$(\mathcal{K}^*_{SP} \oplus \mathcal{K}^*_{P'}) \leftrightarrow \mathcal{K}^*_{PP'} = \mathcal{K}^*_{SP} \leftrightarrow (-\mathcal{K}^*_P),$$

which proves the required result.

E 7.19: We need to check if there is an appropriate class of collections of vectors s.t. if $\mathcal{K}$ belongs to this class $(\mathcal{K}^*)^* = \mathcal{K}$.
i. $\mathbf{f}, \mathbf{g}$ are q-orthogonal iff $< \mathbf{f}, \mathbf{g} >$ is a nonnegative integer:
Consider the collection of integral solutions to

$$\mathbf{Ax} \leq \mathbf{0}.$$

It can be shown that this is not closed under q-orthogonality.So implicit duality would not not work in this case if we take the above mentioned collection as basic.

ii. $\mathbf{f}, \mathbf{g}$ are q-orthogonal iff $< \mathbf{f}, \mathbf{g} >$ is an integral multiple of a given integer n:
The Implicit Duality Theorem should go through in this case. We sketch the argument below:
In this case, we could define 'regularly generated by $(\mathbf{A}, \mathbf{B})$' to be the collection of vectors of the form $\lambda^{\mathbf{T}}\mathbf{A} + \sigma^{\mathbf{T}}\mathbf{B}$ where λ is an integral vector, and σ, an arbitrary rational vector. If $\mathcal{K}$ is regularly generated by $(\mathbf{A}, \mathbf{B})$ we can, without loss of generality, take $\begin{pmatrix} \mathbf{A} \\ \mathbf{B} \end{pmatrix}$ to have linearly independent rows - the argument as in Theorem 7.5.1. Then, using the arguments of Theorem 7.5.2, we conclude that $\mathcal{K}^*$ would be regularly generated by $(\mathbf{C}, \mathbf{D})$ where $\mathbf{C}$ satisfies $\mathbf{AC}^T = n(\mathbf{I})$ and rows of $\mathbf{D}$ span the orthogonal complement of the space spanned by rows of $\begin{pmatrix} \mathbf{A} \\ \mathbf{B} \end{pmatrix}$.
The rest of Theorem 7.5.2 also goes through in this case. Hence, we can mimic the proof of Theorem 7.5.3 and conclude

$$(\mathcal{K}_{SP} \leftrightarrow \mathcal{K}_P)^* = \mathcal{K}^*_{SP} \leftrightarrow \mathcal{K}^*_P.$$

7.9 Solutions of Problems

P 7.1:

i. Let $\mathcal{V}'_{SP}$ be the subspace consisting of all vectors of $\mathcal{V}_{SP}$ whose restrictions to P belong to $\mathcal{V}_P$. Choose a representative matrix for $\mathcal{V}'_{SP}$ of the form

$$\begin{array}{c} \begin{array}{cc} S & P \end{array} \\ \left[\begin{array}{cc} \mathbf{R}_{1S} & \mathbf{0} \\ \mathbf{0} & \mathbf{R}_{2P} \\ \mathbf{R}_{3S} & \mathbf{R}_{3P} \end{array}\right] \end{array} \qquad (*)$$

where rows of $\begin{pmatrix} \mathbf{R}_{1S} \\ \mathbf{R}_{3S} \end{pmatrix}$ as well as rows of $\begin{pmatrix} \mathbf{R}_{2P} \\ \mathbf{R}_{3P} \end{pmatrix}$ are linearly independent. The number of rows (as well as rank) of this representative matrix is clearly $r(\mathcal{V}_{SP} \times S) + r((\mathcal{V}_{SP} \cdot P) \cap \mathcal{V}_P)$. Now $\begin{pmatrix} \mathbf{R}_{1S} \\ \mathbf{R}_{3S} \end{pmatrix}$ is a representative matrix for $\mathcal{V}_{SP} \leftrightarrow \mathcal{V}_P$.

To determine the space $\mathcal{V}'_{SP}$: Construct representative matrices $\mathbf{R} = (\mathbf{R}_S \vdots \mathbf{R}_2)$ for $\mathcal{V}_{SP}$ and $\mathbf{R}_P$ for $\mathcal{V}_P$. Find the solution space of the equation

$$\begin{pmatrix} \mathbf{R}_2^T & \mathbf{R}_P^T \end{pmatrix} \begin{matrix} \lambda_1 \\ \lambda_2 \end{matrix} = \mathbf{0}$$

Let $(\mathbf{Q}_{11}\ \mathbf{Q}_{12})$ be a representative matrix of the solution space. (Note $\begin{pmatrix} \mathbf{R}_2^T & \mathbf{R}_P^T \end{pmatrix} \begin{pmatrix} \mathbf{Q}_{11}^T \\ \mathbf{Q}_{12}^T \end{pmatrix} = 0$). The rows of $\mathbf{Q}_{11}\mathbf{R}_2$ generate $(\mathcal{V}_{SP} \cdot P) \cap \mathcal{V}_P$ and the rows of $\mathbf{Q}_{11}(\mathbf{R}_S \vdots \mathbf{R}_2)$ generate $\mathcal{V}'_{SP}$.

ii. The rank of $\mathcal{V}_{SP} \leftrightarrow \mathcal{V}_P$:

(a) The rank is the number of rows of $\begin{pmatrix} \mathbf{R}_{1S} \\ \mathbf{R}_{3S} \end{pmatrix}$. Noting that the number of rows of $\mathbf{R}_{2P}$ (in $(*)$) equals $r((\mathcal{V}_{SP} \times P) \cap \mathcal{V}_P)$ we see that

$$r(\mathcal{V}_{SP} \leftrightarrow \mathcal{V}_P) = r(\mathcal{V}_{SP} \times S) + r((\mathcal{V}_{SP} \cdot P) \cap \mathcal{V}_P) - r((\mathcal{V}_{SP} \times P) \cap \mathcal{V}_P).$$

(b) If $\mathbf{f}_S, \mathbf{g}_S$ belong respectively to $\mathcal{V}_{SP} \leftrightarrow \mathcal{V}_P$ and $(\mathcal{V}_{SP}^\perp \leftrightarrow \mathcal{V}_P^\perp)$, there exist vectors $\mathbf{f}_P \in \mathcal{V}_P$ and $\mathbf{g}_P \in \mathcal{V}_P^\perp$ s.t. $\mathbf{f}_S \oplus \mathbf{f}_P \in \mathcal{V}_{SP}$ and $\mathbf{g}_S \oplus \mathbf{g}_P \in \mathcal{V}_{SP}^\perp$. Hence, $< \mathbf{f}_S, \mathbf{g}_S > = < \mathbf{f}_S \oplus \mathbf{f}_P,\ \mathbf{g}_S \oplus \mathbf{g}_P >$ - $< \mathbf{f}_P,\ \mathbf{g}_P > = 0$.

(c)

$$\begin{aligned} r(\mathcal{V}_{SP}^\perp \leftrightarrow \mathcal{V}_P^\perp) + r(\mathcal{V}_{SP} \leftrightarrow \mathcal{V}_P) \quad = \quad & [r(\mathcal{V}_{SP}^\perp \times S) + r((\mathcal{V}_{SP}^\perp \cdot P) \cap \mathcal{V}_P^\perp) - \\ & r((\mathcal{V}_{SP}^\perp \times P) \cap \mathcal{V}_P^\perp) + r(\mathcal{V}_{SP} \times S) + \\ & r((\mathcal{V}_{SP} \cdot P) \cap \mathcal{V}_P) - r((\mathcal{V}_{SP} \times P) \cap \mathcal{V}_P)], (**) \end{aligned}$$

where we have used the results of a previous section of the present problem. Now we make use of the following facts:

$$
\begin{gathered}
r(\mathcal{V}_{SP}^{\perp} \times S) = \mid S \mid - r(\mathcal{V}_{SP} \cdot S), \\
r((\mathcal{V}_{SP}^{\perp} \cdot P) \cap \mathcal{V}_P^{\perp}) = \mid P \mid - r((\mathcal{V}_{SP} \times P) + \mathcal{V}_P), \\
r((\mathcal{V}_{SP}^{\perp} \times P) \cap \mathcal{V}_P^{\perp}) = \mid P \mid - r((\mathcal{V}_{SP} \cdot P) + \mathcal{V}_P), \\
r((\mathcal{V}_{SP} \times P) + \mathcal{V}_P) = r(\mathcal{V}_{SP} \times P) + r(\mathcal{V}_P) - r((\mathcal{V}_{SP} \times P) \cap \mathcal{V}_P), \\
r((\mathcal{V}_{SP} \cdot P) + \mathcal{V}_P) = r(\mathcal{V}_{SP} \cdot P) + r(\mathcal{V}_P) - r((\mathcal{V}_{SP} \cdot P) \cap \mathcal{V}_P).
\end{gathered}
$$

The RHS of $(**)$ then simplifies to

$$\mid S \mid - (r(\mathcal{V}_{SP} \cdot S) - r(\mathcal{V}_{SP} \times S)) + (r(\mathcal{V}_{SP} \cdot P) - r(\mathcal{V}_{SP} \times P)) = \mid S \mid,$$

since $r(\mathcal{V}_{SP} \cdot S) + r(\mathcal{V}_{SP} \times P) = r(\mathcal{V}_{SP} \cdot P) + r(\mathcal{V}_{SP} \times S) = r(\mathcal{V}_{SP})$.

P 7.2: We have

$$\mid P \mid \geq r(\mathcal{V}_{SP} \cdot P) - r(\mathcal{V}_{SP} \times P).$$

The RHS equals $(r(\mathcal{V}_{SP} \cdot S) - r(\mathcal{V}_{SP} \times S))$ by Corollary 3.4.2.
Select $\mathcal{V}_{SP}$ so that it has the following representative matrix

$$
\begin{array}{c}
\begin{array}{cc} S & P \end{array} \\
\left[\begin{array}{cc} \mathbf{R}_{1S} & \mathbf{I}_P \\ \mathbf{R}_{2S} & \mathbf{0} \end{array} \right],
\end{array}
\tag{7.15}
$$

where $\mathbf{R}_{2S}$ is a representative matrix of $\mathcal{V}_{SP} \times S$ and rows of $\mathbf{R}_{1S}$ are linearly independent. It is clear that

$$\mid P \mid = r(\mathcal{V}_{SP} \cdot P) - r(\mathcal{V}_{SP} \times P) = r(\mathcal{V}_{SP} \cdot S) - r(\mathcal{V}_{SP} \times S).$$

Now

$$
\begin{aligned}
r(\mathcal{V}_{SP}^{\perp} \cdot P) - r(\mathcal{V}_{SP}^{\perp} \times P) &= r(\mathcal{V}_{SP}^{\perp}) - r(\mathcal{V}_{SP}^{\perp} \times S) - r(\mathcal{V}_{SP}^{\perp}) + r(\mathcal{V}_{SP}^{\perp} \cdot S) \\
&= r(\mathcal{V}_{SP}^{\perp} \cdot S) - r(\mathcal{V}_{SP}^{\perp} \times S) \\
&= \mid S \mid - r(\mathcal{V}_{SP} \times S) - \mid S \mid + r(\mathcal{V}_{SP} \cdot S) \\
&= r(\mathcal{V}_{SP} \cdot S) - r(\mathcal{V}_{SP} \times S) \\
&= \mid P \mid .
\end{aligned}
$$

Since we have already seen that, in general, $\mid P \mid$ cannot be less than $r(\mathcal{V}_{SP} \cdot S) - r(\mathcal{V}_{SP} \times S)$, the pair $(\mathcal{V}_{SP}, \mathcal{V}_{SP}^{\perp})$ is minimal in P.
To see the validity of the second condition we first observe that

$$\mid P \mid = r(\mathcal{V}_{SP} \cdot P) - r(\mathcal{V}_{SP} \times P) (= r(\mathcal{V}_{SP} \cdot S) - r(\mathcal{V}_{SP} \times S))$$

only if $r(\mathcal{V}_{SP} \times P) = 0$ and

$$\mid P \mid = r(\mathcal{V}_{SP}^{\perp} \cdot P) - r(\mathcal{V}_{SP}^{\perp} \times P) (= r(\mathcal{V}_{SP}^{\perp} \cdot S) - r(\mathcal{V}_{SP}^{\perp} \times S))$$

only if $r(\mathcal{V}_{SP}^{\perp} \times P) = 0$.
On the other hand since

$$r(\mathcal{V}_{SP}^{\perp} \cdot P) = \mid P \mid - r(\mathcal{V}_{SP} \times P)$$

and

$$r(\mathcal{V}_{SP} \cdot P) = \mid P \mid - r(\mathcal{V}_{SP}^{\perp} \times P),$$

we must have, if $r(\mathcal{V}_{SP} \times P) = r(\mathcal{V}_{SP}^{\perp} \times P) = 0$, then

$$\begin{aligned} \mid P \mid &= r(\mathcal{V}_{SP} \cdot P) - r(\mathcal{V}_{SP} \times P) \\ &= r(\mathcal{V}_{SP}^{\perp} \cdot P) - r(\mathcal{V}_{SP}^{\perp} \times P). \end{aligned}$$

So the result follows.

P 7.3: Let $\mathcal{V}_{S-S_2}$ have the representative matrix

$$\begin{array}{cc} S - S_1 & S_1 - S_2 \\ \end{array}$$
$$\begin{bmatrix} \mathbf{I} & \mathbf{0} \end{bmatrix},$$

where $S_2 \subseteq S_1 \subseteq S$.
In Exercise 7.18 we saw that if $P_1 \uplus P_2 = P$ that

$$(\mathcal{K}_{SP} \leftrightarrow \mathcal{K}_{P_1}) \leftrightarrow \mathcal{K}_{P_2} = (\mathcal{K}_{SP} \leftrightarrow \mathcal{K}_{P_2}) \leftrightarrow \mathcal{K}_{P_1}$$

Hence,

$$(\mathcal{V}_{SP} \leftrightarrow \mathcal{V}_P) \leftrightarrow \mathcal{V}_{(S_1 - S_2)} = (\mathcal{V}_{SP} \leftrightarrow \mathcal{V}_{(S_1 - S_2)}) \leftrightarrow \mathcal{V}_P.$$

But the LHS is $(\mathcal{V}_{SP} \leftrightarrow \mathcal{V}_P) \cdot S_1 \times S_2$ while the RHS is $(\mathcal{V}_{SP} \cdot (S_1 \cup P) \times (S_2 \cup P)) \leftrightarrow \mathcal{V}_P$. Thus the desired result follows.

P 7.4:
i(a) Let $\mathbf{f}_S \in \mathcal{V}_{SP_1} \leftrightarrow \mathcal{V}_{P_1}$. Then there exists $\mathbf{f}_{P_1} \in \mathcal{V}_{P_1}$ s.t. $\mathbf{f}_S \oplus \mathbf{f}_{P1} \in \mathcal{V}_{SP_1}$. Since $\mathcal{V}_{P_1} = \mathcal{V}_P \times (P - e)$, and $\mathcal{V}_{SP_1} = \mathcal{V}_{SP} \times (S \cup (P - e))$, we must have $\mathbf{f}_{P1} \oplus \mathbf{0}_e \in \mathcal{V}_P$ and $\mathbf{f}_S \oplus \mathbf{f}_{P1} \oplus \mathbf{0}_e \in \mathcal{V}_{SP}$. Hence, $\mathbf{f}_S \in \mathcal{V}_{SP} \leftrightarrow \mathcal{V}_P$.
On the other hand let $\mathbf{f}_S \in \mathcal{V}_{SP} \leftrightarrow \mathcal{V}_P$. Then there exists $\mathbf{f}_P \in \mathcal{V}_P$ s.t. $\mathbf{f}_S \oplus \mathbf{f}_P \in \mathcal{V}_{SP}$. It is given that there exists $\hat{\mathbf{f}_P} \in \mathcal{V}_{SP} \times P \subseteq \mathcal{V}_P$ with e in the support of $\hat{\mathbf{f}_P}$. For some λ we must have $(\mathbf{f}_P + \lambda \hat{\mathbf{f}_P})(e) = 0$. It is clear that $\mathbf{f}_S \oplus (\mathbf{f}_P + \lambda \hat{\mathbf{f}}_P) \in \mathcal{V}_{SP}$ and $\mathbf{f}_P + \lambda \hat{\mathbf{f}}_P \in \mathcal{V}_P$. Let $\mathbf{f}_{P_1}$ be the restriction of $(\mathbf{f}_P + \lambda \hat{\mathbf{f}_P})$ to P_1. Clearly $\mathbf{f}_S \oplus \mathbf{f}_{P_1} \in \mathcal{V}_{SP_1}$ and $\mathbf{f}_{P_1} \in \mathcal{V}_{P_1}$. Hence, $\mathbf{f}_S \in \mathcal{V}_{SP_1} \leftrightarrow \mathcal{V}_{P_1}$.

i(b) We have that $\mathcal{V}_{SP} \cdot P \supseteq \mathcal{V}_P$. Let $\mathbf{f}_{P_1} \in \mathcal{V}_{P_1}$. Then $\mathbf{f}_{P_1} \oplus \mathbf{0}_e \in \mathcal{V}_P \subseteq \mathcal{V}_{SP} \cdot P$. Hence, there exists $\mathbf{f}_S \oplus \mathbf{f}_{P_1} \oplus \mathbf{0}_e \in \mathcal{V}_{SP}$. Hence, $\mathbf{f}_S \oplus \mathbf{f}_{P_1} \in \mathcal{V}_{SP} \times (S \cup P_1) = \mathcal{V}_{SP_1}$. Hence $\mathbf{f}_{P_1} \in \mathcal{V}_{SP_1} \cdot P_1$. Thus, $\mathcal{V}_{P_1} \subseteq \mathcal{V}_{SP_1} \cdot P$.
Next, let $\mathbf{f}_{P_1} \in \mathcal{V}_{SP_1} \times P_1$. Then, there exists $\mathbf{f}_S \oplus \mathbf{f}_{P_1} \oplus \mathbf{0}_e \in \mathcal{V}_{SP}$. Hence, $\mathbf{f}_{P_1} \oplus \mathbf{0}_e \in \mathcal{V}_{SP} \times P \subseteq \mathcal{V}_P$. Hence, $\mathbf{f}_{P_1} \in \mathcal{V}_P \times P_1$. Thus, $\mathcal{V}_{SP_1} \cdot P_1 \supseteq \mathcal{V}_{P_1} \supseteq \mathcal{V}_{SP_1} \times P_1$.

i(c) It is clear that $\mathcal{V}_{SP} \cdot S \supseteq \mathcal{V}_{SP} \times (S \cup P_1) \cdot S = \mathcal{V}_{SP_1} \cdot S$. To see the reverse containment let $\mathbf{f}_S \in \mathcal{V}_{SP} \cdot S$. Then there exists $\mathbf{f}_P$ s.t. $\mathbf{f}_S \oplus \mathbf{f}_P \in \mathcal{V}_{SP}$. We have a vector $\hat{\mathbf{f}_P}$ in $\mathcal{V}_{SP} \times P$ with e in its support. Hence, for a suitable λ, $(\mathbf{f}_P + \lambda \hat{\mathbf{f}_P})(e) = \mathbf{0}$. Now $\mathbf{f}_S \oplus (\mathbf{f}_P + \lambda \hat{\mathbf{f}}_P) \in \mathcal{V}_{SP}$ and this vector takes zero value on e. Let $\mathbf{f}_{P_1}$ be the restriction of $(\mathbf{f}_P + \lambda\hat{\mathbf{f}_P})$ to P_1. Then $\mathbf{f}_S \oplus \mathbf{f}_{P_1} \in \mathcal{V}_{SP_1}$. Thus, $\mathbf{f}_S \in \mathcal{V}_{SP_1} \cdot S$. This proves that $\mathcal{V}_{SP} \cdot S \subseteq \mathcal{V}_{SP_1} \cdot S$, and since the reverse containment is clear we have $\mathcal{V}_{SP} \cdot S = \mathcal{V}_{SP_1} \cdot S$. Next, to prove that $\mathcal{V}_{SP} \times S = \mathcal{V}_{SP_1} \times S$, we merely note that $\mathcal{V}_{SP_1} = \mathcal{V}_{SP} \times (S \cup P_1)$.

ii(a) From the previous section it is clear that

$$\mathcal{V}_{SP}^{\perp} \leftrightarrow \mathcal{V}_{P}^{\perp} = \mathcal{V}_{SP_2}^{\perp} \leftrightarrow \mathcal{V}_{P_2}^{\perp}$$

since $\mathcal{V}_{SP_2}^{\perp} = (\mathcal{V}_{SP} \cdot (S \cup P_2))^{\perp} = \mathcal{V}_{SP}^{\perp} \times (S \cup P_2)$
and $\mathcal{V}_{P_2}^{\perp} = (\mathcal{V}_P \cdot P_2)^{\perp} = \mathcal{V}_P^{\perp} \times P_2$. Hence,

$$\mathcal{V}_{SP} \leftrightarrow \mathcal{V}_P = (\mathcal{V}_{SP}^{\perp} \leftrightarrow \mathcal{V}_P^{\perp})^{\perp} = (\mathcal{V}_{SP_2}^{\perp} \leftrightarrow \mathcal{V}_{P_2}^{\perp})^{\perp} = \mathcal{V}_{SP_2} \leftrightarrow \mathcal{V}_{P_2}$$

ii(b) Since $\mathcal{V}_{SP} \cdot P \supseteq \mathcal{V}_P \supseteq \mathcal{V}_{SP} \times P$, it follows that $(\mathcal{V}_{SP} \cdot P)^{\perp} \subseteq \mathcal{V}_P^{\perp} \subseteq (\mathcal{V}_{SP} \times P)^{\perp}$. Hence, $\mathcal{V}_{SP}^{\perp} \cdot P \supseteq \mathcal{V}_P^{\perp} \supseteq \mathcal{V}_{SP}^{\perp} \times P$. Now by the argument of the previous section of the present problem

$$\mathcal{V}_{SP_2}^{\perp} \cdot P_2 \supseteq \mathcal{V}_{P_2}^{\perp} \supseteq \mathcal{V}_{SP_2}^{\perp} \times P_2.$$

Hence, $\mathcal{V}_{SP_2} \times P_2 \subseteq \mathcal{V}_{P_2} \subseteq \mathcal{V}_{SP_2} \cdot P_2$.

ii(c) We have, $\mathcal{V}_{SP}^{\perp} \cdot S = \mathcal{V}_{SP_2}^{\perp} \cdot S$ by the arguments of the previous section. Hence, $\mathcal{V}_{SP} \times S = \mathcal{V}_{SP_2} \times S$. Similarly, $\mathcal{V}_{SP}^{\perp} \times S = \mathcal{V}_{SP_2}^{\perp} \times S$ by the arguments of the previous section. Hence, by taking orthogonal complements of both sides, $\mathcal{V}_{SP} \cdot S = \mathcal{V}_{SP_2} \cdot S$.

iii. The proof is similar to the 'compatibility' case.

P 7.5: If $\mathcal{V}_P = \mathcal{V}_{SP} \leftrightarrow \mathcal{V}_S'$ for some $\mathcal{V}_S'$, then by Exercise 7.5,

$$\mathcal{V}_{SP} \times P \subseteq \mathcal{V}_P \subseteq \mathcal{V}_{SP} \cdot P.$$

This takes care of the necessity of the condition.
Suppose

$$\mathcal{V}_{SP} \times P \subseteq \mathcal{V}_P \subseteq \mathcal{V}_{SP} \cdot P.$$

Let $\mathbf{f}_P \in \mathcal{V}_{SP} \leftrightarrow \mathcal{V}_S$. Then there exists $\mathbf{f}_S \in \mathcal{V}_S$ s.t. $\mathbf{f}_S \oplus \mathbf{f}_P \in \mathcal{V}_{SP}$. But $\mathcal{V}_S = \mathcal{V}_{SP} \leftrightarrow \mathcal{V}_P$. Hence, there exists $\mathbf{f}_P{}' \in \mathcal{V}_P$ s.t. $\mathbf{f}_S \oplus \mathbf{f}_P{}' \in \mathcal{V}_{SP}$. It follows that $\mathbf{f}_P{}' - \mathbf{f}_P \in \mathcal{V}_{SP} \times P \subseteq \mathcal{V}_P$. Hence, $\mathbf{f}_P \in \mathcal{V}_P$.
Next let $\mathbf{f}_P \in \mathcal{V}_P$. Then there exists $\mathbf{f}_S$ s.t. $\mathbf{f}_S \oplus \mathbf{f}_P \in \mathcal{V}_{SP}$ since $\mathcal{V}_{SP} \cdot P \supseteq \mathcal{V}_P$.
Now $\mathbf{f}_S \in \mathcal{V}_{SP} \leftrightarrow \mathcal{V}_P = \mathcal{V}_S$, by the definition of the generalized minor operation. Hence, $\mathbf{f}_P \in \mathcal{V}_{SP} \leftrightarrow \mathcal{V}_S$, once again using the definition of generalized minor.

P 7.6: Extend each vector in $\mathcal{V}_1, \mathcal{V}_2$ to $S_1 \cup S_2$ by padding with zeros. Let us call the resulting vector spaces $\mathcal{V}_1', \mathcal{V}_2'$. Using the result in Problem 7.7, if $\mathcal{V}_{SP}$ is a minimum extension of $\mathcal{V}_1', \mathcal{V}_2'$, then

$$\mid P \mid = r(\mathcal{V}_1' + \mathcal{V}_2') - r(\mathcal{V}_1' \cap \mathcal{V}_2').$$

It is clear that $\mathcal{V}_{SP}$ is a minimum common extension of $\mathcal{V}_1', \mathcal{V}_2'$ iff it is a minimum common extension of $\mathcal{V}_1, \mathcal{V}_2$.

Now $r(\mathcal{V}_1' + \mathcal{V}_2') = r(\mathcal{V}_1 + \mathcal{V}_2)$ and $\mathcal{V}_1' \cap \mathcal{V}_2'$ is the collection of vectors which are zero outside $S_1 \cap S_2$ and belong to both $\mathcal{V}_1'$ and $\mathcal{V}_2'$. Hence,

$$r(\mathcal{V}_1' \cap \mathcal{V}_2') = r((\mathcal{V}_1 \times (S_1 \cap S_2)) \cap (\mathcal{V}_2 \times (S_1 \cap S_2))).$$

P 7.7: This is a generalization of Exercise 7.11 and can be solved similarly. The essential difference is that the minors may have to be generalized while in the case of two spaces ordinary minors were adequate. For details see [Narayanan87].

P 7.8:
i. is obvious.

ii. Let $\mathcal{V}_{SP}$ be a common extension of $\mathcal{V}_S^1, \cdots, \mathcal{V}_S^k$.
Let $\mathcal{V}_P^i = \mathcal{V}_{SP} \leftrightarrow \mathcal{V}_S^i, i = 1, ...k$.
By the result in Problem 7.5 it is clear that $\mathcal{V}_{SP} \leftrightarrow \mathcal{V}_P^i = \mathcal{V}_S^i, i = 1, ..k$.
Grow a basis for $\sum \mathcal{V}_S^i$ starting with vectors in $\bigcap_i \mathcal{V}_S^i$ and using vectors which belong to some $\mathcal{V}_S^i$. Let $\mathbf{f}_S{}^1, \cdots, \mathbf{f}_S{}^r$ be the vectors of this basis which are outside $\bigcap_i \mathcal{V}_S^i$ and let them belong to spaces $\mathcal{V}_{S1}, \cdots, \mathcal{V}_{Sr}$ respectively where the $\mathcal{V}_{Si}$ are not necessarily distinct and are each equal to one of $\mathcal{V}_S^j$. Then there exist $\mathbf{f}_P{}^1, \cdots, \mathbf{f}_P{}^r$ in $\mathcal{V}_{P1}, \cdots, \mathcal{V}_{Pr}$ respectively such that $\mathbf{f}_S{}^1 \oplus \mathbf{f}_P{}^1, \cdots, \mathbf{f}_S{}^r \oplus \mathbf{f}_P{}^r$ belong to $\mathcal{V}_{SP}$. (Here $\mathcal{V}_{Pi}$ are spaces not necessarily distinct each equal to one of the $\mathcal{V}_P^j$). Suppose $\mathbf{f}_P{}^1, \cdots, \mathbf{f}_P{}^r$ are dependent modulo $(\bigcap_i \mathcal{V}_P^i)$. Then there is a linear combination $\mathbf{f}_S = \lambda_1 \mathbf{f}_S{}^1 + \cdots + \lambda_r \mathbf{f}_S{}^r$ which does not belong to $\bigcap_i \mathcal{V}_S^i$ but $\mathbf{f}_P = \lambda_1 \mathbf{f}_P{}^1 + \cdots + \lambda_r \mathbf{f}_P{}^r \in \bigcap_i \mathcal{V}_P^i$. Now the vector $\mathbf{f}_S \in \bigcap_i (\mathcal{V}_{SP} \leftrightarrow \mathcal{V}_P^i)$ but $\mathbf{f}_S \notin \bigcap \mathcal{V}_S^i$. This is a contradiction. We conclude therefore, that

$$r(\sum_i \mathcal{V}_P^i) - r(\bigcap_i \mathcal{V}_P^i) \geq r(\sum_i \mathcal{V}_S^i) - r(\bigcap_i \mathcal{V}_S^i)$$

The reverse inequality follows by repeating the argument interchanging S and P.

iii. We first show that $\mathcal{V}_{SP} \leftrightarrow \mathcal{V}_{PQ}$ is an extension of $\mathcal{V}_S^1, \cdots, \mathcal{V}_S^k$. Let $\mathbf{f}_S \in \mathcal{V}_S^i$. Then there exists $\mathbf{f}_P$ s.t. $\mathbf{f}_S \oplus \mathbf{f}_P \in \mathcal{V}_{SP}$. Hence, $\mathbf{f}_P \in \mathcal{V}_{SP} \leftrightarrow \mathcal{V}_S^i = \mathcal{V}_P^i$. Since $\mathcal{V}_{PQ}$ is an extension of $\mathcal{V}_P^i$ there exists $\mathbf{f}_Q$ s.t. $\mathbf{f}_P \oplus \mathbf{f}_Q \in \mathcal{V}_{PQ}$. Hence, $\mathbf{f}_S \oplus \mathbf{f}_Q \in \mathcal{V}_{SP} \leftrightarrow \mathcal{V}_{PQ}$. Then $\mathcal{V}_S^i \subseteq (\mathcal{V}_{SP} \leftrightarrow \mathcal{V}_{PQ}) \cdot S$. Similarly we can show that $(\mathcal{V}_S^i)^\perp \subseteq (\mathcal{V}_{SP}^\perp \leftrightarrow \mathcal{V}_{PQ}^\perp) \cdot S = (\mathcal{V}_{SP} \leftrightarrow \mathcal{V}_{PQ})^\perp \cdot S$. Hence, $\mathcal{V}_S^i \supseteq (\mathcal{V}_{SP} \leftrightarrow \mathcal{V}_{PQ}) \times S$. By Exercise 7.5 this proves that $(\mathcal{V}_{SP} \leftrightarrow \mathcal{V}_{PQ})$ is an extension of $\mathcal{V}_S^i$).
To see that it is a minimal extension we note that

$$\begin{aligned} \mid Q \mid &= r(\sum_i \mathcal{V}_P^i) - r(\bigcap_i \mathcal{V}_P^i) \\ &= r(\sum_i \mathcal{V}_S^i) - r(\bigcap_i \mathcal{V}_S^i), \end{aligned}$$

using the previous section of the present problem and Problem 7.7.
Thus, once again using the result in the abovementioned problem, $\mathcal{V}_{SQ} \equiv \mathcal{V}_{SP} \leftrightarrow \mathcal{V}_{PQ}$ is a minimal extension of $\mathcal{V}_S^1, \cdots, \mathcal{V}_k^k$.

P 7.9:
i. This is a special case of part (iv) of the present problem, whose solution is given below.

ii. $\mathcal{G}' \equiv \bigoplus_i \mathcal{G} \times E_i$ where $E_1, \cdots, E_k$ is a partition of E. Now,

$$\begin{aligned}\mathcal{V}_v(\bigoplus_i \mathcal{G} \times E_i) &= \bigoplus_i \mathcal{V}_v(\mathcal{G} \times E_i) \\ &= \bigoplus_i (\mathcal{V}_v(\mathcal{G})) \times E_i.\end{aligned}$$

Any vector of the form $\mathbf{f}_{E_i} \oplus \mathbf{0}_{E-E_i}$ where $\mathbf{f}_{E_i} \in (\mathcal{V}_v(\mathcal{G})) \times E_i$ is also in $\mathcal{V}_v(\mathcal{G})$ by the definition of contraction.
So $\mathcal{V}_v(\bigoplus_i \mathcal{G} \times E_i) \subseteq \mathcal{V}_v(\mathcal{G})$. Clearly $\mathbf{R}_{P_2}$ cannot exist in this case. We show how to build $\mathbf{R}_1$. Select a forest T of $\mathcal{G}' \equiv \bigoplus_i \mathcal{G} \times E_i$. Build the graph $\mathcal{G} \times (E-T)$. Let $\hat{\mathbf{A}}_{1r}$ be the reduced incidence matrix of $\mathcal{G} \times (E-T)$. Let $\mathbf{A}_{1r} \equiv (\hat{\mathbf{A}}_{1r}\mathbf{0})$, where the zero submatrix corresponds to the set T. We claim $\mathbf{A}_{1r}$ can be chosen as the matrix $\mathbf{R}_1$.
To prove this statement we first observe that if $\begin{pmatrix} \mathbf{A}_{1r} \\ \mathbf{A}_r{}' \end{pmatrix}$ is a representative matrix of $\mathcal{G}$ with $\mathbf{A}'_r$, a representative matrix of $\mathcal{G}'$, then $\mathbf{R}_1$ can be taken to be $\mathbf{A}_{1r}$. The rows of $\mathbf{A}_{1r}$ are voltage vectors of $\mathcal{G}$ since $\mathcal{G} \times (E-T)$ is a contraction of $\mathcal{G}$. The columns T are independent in the reduced incidence matrix $\mathbf{A}_r{}'$ of $\mathcal{G}'$ whereas they are zero in $\mathbf{A}_{1r}$. Hence, the matrix $\begin{pmatrix} \mathbf{A}_{1r} \\ \mathbf{A}_r{}' \end{pmatrix}$ has linearly independent rows.
Next $r(\mathcal{G} \times (E-T)) + r(\mathcal{G} \ . \ T) = r(\mathcal{G})$.
Hence, $r(\mathcal{G} \times (E-T)) + \mid T \mid \geq r(\mathcal{G})$.
Thus, the above matrix must have $r(\mathcal{G})$ rows and is therefore a representative matrix of $\mathcal{G}$.
This completes the proof that $\mathbf{R}_1$ can be taken to be $\mathbf{A}_{1r}$. The labour involved is to build $\mathcal{G} \times (E-T)$ and its reduced incidence matrix. So the algorithm is $O(\mid E \mid)$.

iii. Let $\mathcal{G}'$ be obtained from $\mathcal{G}$ by fusing nodes. Then every voltage vector of $\mathcal{G}'$ can be derived from a node potential vector of $\mathcal{G}$ by assigning the same potential to each group of nodes of $\mathcal{G}$ which make up a node of $\mathcal{G}'$. Hence,

$$\mathcal{V}_v(\mathcal{G}') \subseteq \mathcal{V}_v(\mathcal{G}).$$

Thus, the method of the previous section of this problem can be used to show the following: Let T be a forest of $\mathcal{G}'$. Let $\hat{\mathbf{A}}_{1r}$ be the reduced incidence matrix of $\mathcal{G} \times (E-T)$. Then $\mathbf{R}_1$ can be taken to be $(\hat{\mathbf{A}}_{1r} \ \mathbf{0})$, with zero submatrix corresponding to T.

iv. Since $\mathcal{G}'$ is obtained from $\mathcal{G}$ by node splitting, $\mathcal{V}_v(\mathcal{G}') \supseteq \mathcal{V}_v(\mathcal{G})$. The matrix $\mathbf{R}_1$ is composed of row vectors which belong to $\mathcal{V}_v(\mathcal{G})$ and not to $\mathcal{V}_v(\mathcal{G}')$. Hence, in this case $\mathbf{R}_1$ would not exist. We have to construct $\mathbf{R}_{P_2}$ through a fast algorithm.

From the discussion in Subsection 7.3.3 it is clear that imposing KVL for $\mathcal{G}$ is equivalent to

$$\begin{aligned}\mathbf{v}_E - (\mathbf{A}_r{}')^T \mathbf{v}_n{}' &= \mathbf{0} \qquad (*) \\ \mathbf{R}_{P_2}{}^T \mathbf{v}_n{}' &= \mathbf{0} \qquad (**)\end{aligned}$$

The equation $(*)$ represents the KVL conditions for $\mathcal{G}'$. It expresses the branch voltages of $\mathcal{G}'$ in terms of the node voltages. Suppose $\mathcal{G}'$ is made up of m connected components. For each connected component we choose a pseudo datum node. The node voltage vector $\mathbf{v}_n{}'$ represents the voltages of the nodes in each component with respect to the pseudo datum node voltage. We could draw additional edges between each node and the corresponding pseudo datum node with the arrow directed towards the latter. The voltages of these additional branches E_a would be given by $\mathbf{v}_n{}'$. Let us call the graphs obtained by adding E_a to $\mathcal{G}'$ and that by adding E_a to $\mathcal{G}$ respectively $\mathcal{G}'_a, \mathcal{G}_a$. ($\mathcal{G}_a$ is obtained by performing those node fusions on $\mathcal{G}'_a$ through which $\mathcal{G}'$ becomes $\mathcal{G}$. In $\mathcal{G}'$ there are no voltage constraints on E_a. The voltage constraints on E_a in the graph $\mathcal{G}_a$ would be equivalent to $\mathbf{R}_{P_2}{}^T\mathbf{v}_n{}' = \mathbf{0}$. But these constraints are precisely the voltage constraints of $\mathcal{G}_a \cdot E_a$. Thus, $\mathbf{R}_{P_2}{}^T$ may be taken to be an f-circuit matrix of $\mathcal{G}_a \cdot E_a$. The complexity of this construction is O(number of nonzero entries of $\mathbf{R}_{P_2}$). This is bounded above by $(\nu(\mathcal{G}\ .\ E_a))(\mid E_a \mid -\nu(\mathcal{G}\ .\ E_a)) + \nu(\mathcal{G}\ .\ E_a)$. Usually however the effort required would be much less particularly if we choose a variation of the mesh matrix.

v. Let $\mathcal{G}$" be obtained from $\mathcal{G}$ by fusing some nodes and $\mathcal{G}'$ from $\mathcal{G}$" by splitting some nodes of $\mathcal{G}$". Let $\mathcal{G}$ have KCE $\quad \mathbf{A}_r\mathbf{i}_E = 0$
and KVL constraints $\quad \mathbf{A}_r{}^T\mathbf{v}_n = \mathbf{v}_E$.
Then, KCE of $\mathcal{G}$ can be written as

$$
\begin{aligned}
\mathbf{A}_r"\mathbf{i}_E" &= \mathbf{0} \qquad (\surd)\\
\mathbf{i}_E" - \mathbf{i}_E &= \mathbf{0} \qquad (\surd)\\
\mathbf{R}_1\mathbf{i}_E &= \mathbf{0} \qquad (\surd)
\end{aligned}
$$

where $\mathbf{A}_r$" is the reduced incidence matrix of $\mathcal{G}$" and $\begin{pmatrix} \mathbf{R}_1 \\ \mathbf{A}_r" \end{pmatrix}$ is a representative matrix of $\mathcal{V}_v(\mathcal{G})$. Observe that the set of equations $(\surd)$ are the KCE of $\mathcal{G}$". Since $\mathcal{G}$" is obtained by fusing vertices of $\mathcal{G}$, we can use the procedure outlined in the solution of parts (ii) and (iii) of the present problem to construct the matrix $\mathbf{R}_1$. The KVL of $\mathcal{G}$ is imposed by

$$
\begin{pmatrix} (\mathbf{A}_r")^T & \mathbf{R}_1{}^T \end{pmatrix} \begin{pmatrix} \mathbf{v}_n" \\ \mathbf{v}_{P_1} \end{pmatrix} - \mathbf{v}_E = \mathbf{0}.
$$

Let us define $\mathbf{v}"_E \equiv (\mathbf{A}_r")^T\mathbf{v}_n"$. So the KVL of $\mathcal{G}$ can be written as

$$
\begin{aligned}
(\mathbf{A}_r")^T\mathbf{v}_n" - \mathbf{v}"_E &= \mathbf{0} \qquad (\surd\surd)\\
\mathbf{v}"_E + \mathbf{R}_1{}^T\mathbf{v}_{P_1} - \mathbf{v}_E &= \mathbf{0}. \qquad (\surd\surd)
\end{aligned}
$$

Observe that the first set of equations of $(\surd\surd)$ above are the KVE of $\mathcal{G}$".
Now let $\mathcal{G}'$ be obtained from $\mathcal{G}$" by splitting nodes. Then KCE of $\mathcal{G}$" can be written as

$$
\mathbf{A}_r{}'\mathbf{i}_E" + \mathbf{R}_{P_2}\mathbf{i}_{P_2} = \mathbf{0}
$$

and the KVE of $\mathcal{G}$" can be written as

$$
\begin{aligned}
(\mathbf{A}_r{}')^T\mathbf{v}_n{}' - \mathbf{v}"_E &= \mathbf{0}\\
\mathbf{R}_{P_2}{}^T\mathbf{v}_n{}' &= \mathbf{0},
\end{aligned}
$$

where $\mathbf{A}_r{}'$ is the reduced incidence matrix of $\mathcal{G}'$. The matrix $\mathbf{R}_{P_2}$ can be constructed therefore by using the procedure outlined in the solution of parts (i) and (iv) above. Observe that the result of the above procedure is that KCL and KVL constraints of $\mathcal{G}$ are written equivalently (as far as $\mathbf{i}_E, \mathbf{v}_E$ are concerned) as

$$\begin{aligned} \mathbf{A}_r{}'\mathbf{i}_E + \mathbf{R}_{P_2}\mathbf{i}_{P_2} &= \mathbf{0} \\ \mathbf{R}_1\mathbf{i}_E &= \mathbf{0} \\ (\mathbf{A}_r{}')^T\mathbf{v}_n{}' + \mathbf{R}_1{}^T\mathbf{v}_{P_1} - \mathbf{v}_E &= \mathbf{0} \\ \mathbf{R}_{P_2}{}^T\mathbf{v}_n{}' &= \mathbf{0}. \end{aligned}$$

P 7.10: Our starting point is Equation 7.7. We will assume that the device characteristic can be written in the form

$$-\mathbf{G}(\mathbf{v}_E - \mathbf{e}) + (\mathbf{i}_E - \mathbf{j}) = \mathbf{0}.$$

We start with KCE in the modified form

$$\begin{aligned} \mathbf{A}_r{}'\mathbf{i}_E + \mathbf{R}_{P_2}\mathbf{i}_{P_2} &= \mathbf{0} \\ \mathbf{R}_1\mathbf{i}_E &= \mathbf{0}. \end{aligned}$$

We now use the device characteristic and get

$$\begin{aligned} \mathbf{A}_r{}'\mathbf{G}\mathbf{v}_E + \mathbf{R}_{P_2}\mathbf{i}_{P_2} &= -\mathbf{A}_r{}'\mathbf{j} + \mathbf{A}_r{}'\mathbf{G}\mathbf{e} \\ \mathbf{R}_1\mathbf{G}\mathbf{v}_E &= -\mathbf{R}_1\mathbf{j} + \mathbf{R}_1\mathbf{G}\mathbf{e} \end{aligned}$$

Next we use KVL constraints in the altered form:

$$\begin{aligned} \mathbf{A}_r{}'\mathbf{G}((\mathbf{A}_r{}')^T\mathbf{v}_n{}' + \mathbf{R}_1{}^T\mathbf{v}_{P_1}) + \mathbf{R}_{P_2}\mathbf{i}_{P_2} &= -\mathbf{A}_r{}'\mathbf{j} + \mathbf{A}_r{}'\mathbf{G}\mathbf{e} \\ \mathbf{R}_1\mathbf{G}((\mathbf{A}_r{}')^T\mathbf{v}_n{}' + \mathbf{R}_1{}^T\mathbf{v}_{P_1}) &= -\mathbf{R}_1\mathbf{j} + \mathbf{R}_1\mathbf{G}\mathbf{e} \\ \mathbf{R}_{P_2}{}^T\mathbf{v}_n{}' &= \mathbf{0} \end{aligned}$$

The final transformed nodal equations are as follows:

$$\begin{bmatrix} \mathbf{A}_r{}'\mathbf{G}(\mathbf{A}_r{}')^T & \mathbf{A}_r{}'\mathbf{G}(\mathbf{R}_1)^T & \mathbf{R}_{P_2} \\ \mathbf{R}_1\mathbf{G}(\mathbf{A}_r{}')^T & \mathbf{R}_1\mathbf{G}(\mathbf{R}_1)^T & \mathbf{0} \\ \mathbf{R}_{P_2}{}^T & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{matrix} \mathbf{v}_n{}' \\ \mathbf{v}_{P_1} \\ \mathbf{i}_{P_2} \end{matrix} = \begin{bmatrix} -\mathbf{A}_r{}'\mathbf{j} + \mathbf{A}_r{}'\mathbf{G}\mathbf{e} \\ -\mathbf{R}_1\mathbf{j} + \mathbf{R}_1\mathbf{G}\mathbf{e} \\ \mathbf{0} \end{bmatrix}$$

P 7.11:
i. Let the KVE of $\mathcal{G}$ be

$$\mathbf{B}\mathbf{v}_E = \mathbf{0}.$$

We write this as

$$\begin{pmatrix} \mathbf{B}_1 \\ \mathbf{B}_\cap \end{pmatrix} \mathbf{v}_E = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \end{pmatrix},$$

where $\mathbf{B}_\cap$ is a representative matrix of $(\mathcal{V}_i(\mathcal{G})) \cap (\mathcal{V}_i(\mathcal{G}'))$.
Similarly, let the KVE of $\mathcal{G}'$ be

$$\begin{pmatrix} \mathbf{B}_2 \\ \mathbf{B}_\cap \end{pmatrix} \mathbf{v}_E = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \end{pmatrix}.$$

Let $\mathcal{V}_E^1, \mathcal{V}_E^2$ be the spaces $\mathcal{V}_i(\mathcal{G})$ and $\mathcal{V}_i(\mathcal{G}')$. Then a minimal common extension $\mathcal{V}_{EP}$ will have the following representative matrix:

$$\begin{array}{ccc} E & P_1 & P_2 \end{array}$$
$$\left[\begin{array}{ccc} \mathbf{B}_1 & \mathbf{I} & \mathbf{0} \\ \mathbf{B}_\cap & \mathbf{0} & \mathbf{0} \\ \mathbf{B}_2 & \mathbf{0} & \mathbf{I} \end{array}\right],$$

or equivalently the matrix

$$\left[\begin{array}{ccc} \mathbf{B}_1 & \mathbf{I} & \mathbf{0} \\ \mathbf{B}_2' & \mathbf{0} & \mathbf{B}_{P_2} \end{array}\right],$$

where $\mathbf{B}_2'$ is a representative matrix of $\mathcal{V}_E^2$. Let $\mathcal{V}_{EP} \leftrightarrow \mathcal{V}_P^1 = \mathcal{V}_E^1$ and let $\mathcal{V}_{EP} \leftrightarrow \mathcal{V}_P^2 = \mathcal{V}_E^2$.

Then it is clear that $\mathcal{V}_P^1$ can be taken to have the representative matrix $\overset{\begin{array}{cc} P_1 & P_2 \end{array}}{[\begin{array}{cc}\mathbf{I} & \mathbf{0}\end{array}]}$ and $\mathcal{V}_P^2$ the representative matrix $\overset{\begin{array}{cc} P_1 & P_2 \end{array}}{[\begin{array}{cc}\mathbf{0} & \mathbf{I}\end{array}]}$. Thus the KVL, KCL constraints of $\mathcal{G}$ can be rewritten (using a procedure analogous to that followed in Subsection 7.3.3) as follows:

$$\begin{aligned} \mathbf{B}_2'\mathbf{v}_E + \mathbf{B}_{P_2}\mathbf{v}_{P_2} &= \mathbf{0} \\ \mathbf{B}_1\mathbf{v}_E &= \mathbf{0} \\ \mathbf{i}_E - (\mathbf{B}_2')^T\mathbf{i}'_l - \mathbf{B}_1^T\mathbf{i}_{P_1} &= \mathbf{0} \\ \mathbf{B}_{P_2}^T i_l' &= \mathbf{0} \end{aligned}$$

ii. We give the final transformed loop equations below:

$$\left[\begin{array}{ccc} \mathbf{B}_2'\mathbf{R}(\mathbf{B}_2')^T & \mathbf{B}_2'\mathbf{R}\mathbf{B}_1^T & \mathbf{B}_{P_2} \\ \mathbf{B}_1\mathbf{R}(\mathbf{B}_2')^T & \mathbf{B}_1\mathbf{R}\mathbf{B}_1^T & \mathbf{0} \\ \mathbf{B}_{P_2}^T & \mathbf{0} & \mathbf{0} \end{array}\right] \begin{array}{c} \mathbf{i}'_l \\ \mathbf{i}_{P_1} \\ \mathbf{v}_{P_2} \end{array} = \left[\begin{array}{c} -\mathbf{B}_2'\mathbf{e} + \mathbf{B}_2'\mathbf{R}\mathbf{j} \\ -\mathbf{B}_1\mathbf{e} + \mathbf{B}_1\mathbf{R}\mathbf{j} \\ \mathbf{0} \end{array}\right]$$

P 7.12: Consider the constraints

$$\left(\begin{array}{cc} \mathbf{A}_R & \mathbf{A}_P \end{array}\right) \left[\begin{array}{c} \mathbf{i}_R \\ \mathbf{i}_P \end{array}\right] = \mathbf{0} \qquad (*)$$

$$\left(\begin{array}{cc} \mathbf{B}_R & \mathbf{B}_P \end{array}\right) \left[\begin{array}{c} \mathbf{v}_R \\ \mathbf{v}_P \end{array}\right] = \mathbf{0} \qquad (**)$$

$$\left(\begin{array}{cc} \mathbf{v}_R & \mathbf{i}_R \end{array}\right) \in \mathcal{V}_{xR} \qquad (***)$$

Let us denote $(*)$ and $(**)$ by

$$\mathbf{F}_{top}(\mathbf{i}_R, \mathbf{i}_P, \mathbf{v}_R, \mathbf{v}_P) = \mathbf{0} \qquad (\surd)$$

and $(***)$ by

$$\mathbf{F}_d(\mathbf{v}_R, \mathbf{i}_R) = \mathbf{0}. \qquad (\surd\surd)$$

It is clear that $(*)$, $(**)$ can also be denoted by

$$\mathbf{F}^{\perp}_{top}(\mathbf{v}_R, \mathbf{v}_P, -\mathbf{i}_R, -\mathbf{i}_P) = \mathbf{0}.$$

It is given that $\mathbf{F}_d(\mathbf{v}_R, \mathbf{i}_R) = \mathbf{0}$ can also be written as $\mathbf{F}^{\perp}_d(-\mathbf{i}_R, \mathbf{v}_R) = \mathbf{0}$. As far as the variables $(\mathbf{v}_P, \mathbf{i}_P)$ are concerned $(*), (**), (\surd\surd)$ are equivalent to

$$(\mathbf{v}_P, \mathbf{i}_P) \in \mathcal{V}_{xP}.$$

By the Implicit Duality Theorem it follows that

$$\begin{aligned} &[(\text{Solution space of } (\surd)) \leftrightarrow (\text{solution space of } (\surd\surd))]^{\perp} \\ &= (\text{Solution space of } (\surd))^{\perp} \leftrightarrow (\text{solution space of } (\surd\surd))^{\perp} \end{aligned}$$

But, LHS $= \mathcal{V}_{xP}$,
and RHS $= \mathcal{V}_{yP}$.
The result follows.

Remark: The above may be used to prove reciprocity for certain kinds of networks.
Example $\mathbf{v}_R = \mathbf{R}\mathbf{i}_R$ and $\mathbf{R}$ is a symmetric matrix.
Suppose $\mathcal{V}_{xP}$ is defined by $\begin{bmatrix} \mathbf{I} & \mathbf{K} \end{bmatrix} \begin{bmatrix} \mathbf{v}_P \\ \mathbf{i}_P \end{bmatrix} = \mathbf{0}$. The $\mathcal{V}_{yP}$ would be defined by $\begin{bmatrix} -\mathbf{K}^T & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{i}_P \\ \mathbf{v}_P \end{bmatrix} = \mathbf{0}$. This is possible only if $\mathbf{K}$ is a symmetric matrix.

Let us now subject the network to two different port excitations. Let us call the $(\mathbf{v}_P, \mathbf{i}_P)$ vector corresponding to the first situation, $(\mathbf{v}'_P, \mathbf{i}'_P)$ and that corresponding to the second situation, $(\mathbf{v}''_P, \mathbf{i}''_P)$. Then

$$(\mathbf{v}'_P, \mathbf{i}'_P)(-\mathbf{i}''_P, \mathbf{v}''_P)^T = 0$$

Hence, $\mathbf{v}'_P \mathbf{i}''_P = \mathbf{v}''_P \mathbf{i}'_P$.

P 7.13: We use the notation of Subsection 7.3.4. The constraints of $\mathcal{N}$ are

$$\begin{aligned} \mathbf{F}_i(\mathbf{i}_D, \mathbf{i}_{yv}, \mathbf{i}_{yi}, \mathbf{i}_{uv}, \mathbf{i}_{ui}) &= \mathbf{0} \qquad \cdots KCE \\ \mathbf{F}_v(\mathbf{v}_D, \mathbf{v}_{yv}, \mathbf{v}_{yi}, \mathbf{v}_{uv}, \mathbf{v}_{ui}) &= \mathbf{0} \qquad \cdots KVE \end{aligned}$$

The device characteristic constraints are

$$\left.\begin{array}{l} \begin{bmatrix} \mathbf{M} & \mathbf{N} \end{bmatrix} \begin{bmatrix} \mathbf{i}_D \\ \mathbf{v}_D \end{bmatrix} = \mathbf{0} \\ (\mathbf{I})\mathbf{i}_{yv} = \mathbf{0} \\ (\mathbf{I})\mathbf{v}_{yi} = \mathbf{0} \\ \textit{no constraints on } \mathbf{v}_{ui} \\ \textit{no constraints on } \mathbf{i}_{uv} \end{array}\right\} \mathbf{F}_d(\mathbf{v}_D, \mathbf{v}_{yi}, \mathbf{v}_{ui}, \mathbf{i}_D, \mathbf{i}_{yv}, \mathbf{i}_{uv}) = \mathbf{0}.$$

As far as the variables $\mathbf{v}_{yv}, \mathbf{i}_{yi}, \mathbf{v}_{uv}, \mathbf{i}_{ui}$ are concerned, these constraints are equivalent to, say,

$$\mathbf{F}_{uy}(\mathbf{v}_{yv}, \mathbf{i}_{yi}, \mathbf{v}_{uv}, \mathbf{i}_{ui}) = \mathbf{0}.$$

By the Implicit Duality Theorem, the constraints

$$\mathbf{F}_i^{\perp}(\mathbf{v}'_D, \mathbf{v}'_{yv}, \mathbf{v}'_{yi}, \mathbf{v}'_{uv}, \mathbf{v}'_{ui}) = \mathbf{0} \qquad \cdots KVE$$
$$\mathbf{F}_v^{\perp}(\mathbf{i}'_D, \mathbf{i}'_{yv}, \mathbf{i}'_{yi}, \mathbf{i}'_{uv}, \mathbf{i}'_{ui}) = \mathbf{0} \qquad \cdots KCE$$
$$\mathbf{F}_d^{\perp}(\mathbf{i}'_D, \mathbf{i}'_{yi}, \mathbf{i}'_{ui}, \mathbf{v}'_D, \mathbf{v}'_{yv}, \mathbf{v}'_{uv}) = \mathbf{0} \qquad \cdots \textit{device characteristic}$$

are together equivalent, as far as the variables

$$\mathbf{i}'_{yv}, \mathbf{v}'_{yi}, \mathbf{i}'_{uv}, \mathbf{v}'_{ui}$$

are concerned, to

$$\mathbf{F}_{uy}^{\perp}(\mathbf{i}'_{yv}, \mathbf{v}'_{yi}, \mathbf{i}'_{uv}, \mathbf{v}'_{ui}) = \mathbf{0}.$$

But the constraint

$$\mathbf{F}_d^{\perp}(\mathbf{i}'_D, \mathbf{i}'_{yi}, \mathbf{i}'_{ui}, \mathbf{v}'_D, \mathbf{v}'_{yv}, \mathbf{v}'_{uv}) = \mathbf{0}$$

is equivalent to the following:

$$\begin{pmatrix} \mathbf{M}_D^{\perp} & \mathbf{N}_D^{\perp} \end{pmatrix} \begin{bmatrix} \mathbf{v}'_D \\ \mathbf{i}'_D \end{bmatrix} = \mathbf{0},$$
$$\textit{no constraints on } \mathbf{v}'_{yv},$$
$$\textit{no constraints on } \mathbf{i}'_{yi},$$
$$(\mathbf{I})\mathbf{i}'_{ui} = \mathbf{0},$$
$$(\mathbf{I})\mathbf{v}'_{uv} = \mathbf{0}.$$

The result follows.

P 7.14:
Proof of Lemma 7.6.1: (a) is immediate.
(b). Since $\mathcal{U}_P \subseteq \mathcal{V}_P$ it is clear that $\mathcal{U}_P^* \supseteq \mathcal{V}_P^*$. Hence,

$$\mathcal{V}_{SP}^* \leftrightarrow \mathcal{V}_P^* \subseteq \mathcal{V}_{SP}^* \leftrightarrow \mathcal{U}_P^*$$

To prove the reverse containment, we first observe that, by Lemma 7.2.4,

$$(\mathcal{V}_{SP} \cdot P)^* = \mathcal{V}_{SP}^* \times P.$$

Next, since $\mathcal{V}_P^* + \mathcal{V}_{SP}^* \times P$ is given to be closed, we must have

$$\mathcal{U}_P^* = (\mathcal{V}_P \cap \mathcal{V}_{SP} \cdot P)^* = \mathcal{V}_P^* + \mathcal{V}_{SP}^* \times P$$

(by Lemma 7.2.2).
Suppose $\mathbf{g}_S \in \mathcal{V}_{SP}^* \leftrightarrow \mathcal{U}_P^*$.
Then there exists $\mathbf{g}_P \in \mathcal{U}_P^*$ s.t. $\mathbf{g}_S \oplus \mathbf{g}_P \in \mathcal{V}_{SP}^*$. Now $\mathcal{U}_P^* = \mathcal{V}_P^* + \mathcal{V}_{SP}^* \times P$. Thus, there exists $\mathbf{g}_P{}' \in \mathcal{V}_{SP}^* \times P$ s.t. $\mathbf{g}_P - \mathbf{g}_P{}' \in \mathcal{V}_P^*$. Since $\mathbf{g}_P{}' \in \mathcal{V}_{SP}^* \times P$, we must have $\mathbf{0}_S \oplus \mathbf{g}_P{}' \in \mathcal{V}_{SP}^*$. Hence,(($\mathbf{g}_S \oplus \mathbf{g}_P) - (\mathbf{0}_S \oplus \mathbf{g}_P{}')) \in \mathcal{V}_{SP}^*$. ($\mathcal{V}_{SP}^*$ is a vector space since $\mathcal{V}_{SP}$ is one - by Lemma 7.2.1). Hence, $\mathbf{g}_S \oplus (\mathbf{g}_P - \mathbf{g}_P{}') \in \mathcal{V}_{SP}^*$.

It follows that $\mathbf{g}_S \in \mathcal{V}^*_{SP} \leftrightarrow \mathcal{V}^*_P$.

□

Proof of Theorem 7.6.2: We have $\mathcal{U}_P \subseteq \mathcal{V}_{SP} \cdot P$. It is easily seen that vectors in $\mathcal{V}_{SP} \leftrightarrow \mathcal{U}_P$ and $\mathcal{V}^*_{SP} \leftrightarrow \mathcal{U}^*_P$ are q-orthogonal. For, if $\mathbf{f}_S \in \mathcal{V}_{SP} \leftrightarrow \mathcal{U}_P$, then there exists $\mathbf{f}_P \in \mathcal{U}_P$ s.t. $\mathbf{f}_S \oplus \mathbf{f}_P \in \mathcal{V}_{SP}$. If $\mathbf{g}_S \in \mathcal{V}^*_{SP} \leftrightarrow \mathcal{U}^*_P$, then there exists $\mathbf{g}_P \in \mathcal{U}^*_P$ s.t. $\mathbf{g}_S \oplus \mathbf{g}_P \in \mathcal{V}^*_{SP}$.
Now $< \mathbf{f}_S \oplus \mathbf{f}_P,\ \mathbf{g}_S \oplus \mathbf{g}_P > \in \mathcal{A}$ and $< \mathbf{f}_P,\ \mathbf{g}_P > \in \mathcal{A}$. Since $\mathcal{A}$ is closed under subtraction, it follows that

$$< \mathbf{f}_S \oplus \mathbf{f}_P,\ \mathbf{g}_S \oplus \mathbf{g}_P > - < \mathbf{f}_P,\ \mathbf{g}_P > \in \mathcal{A},$$

i.e., $< \mathbf{f}_S,\ \mathbf{g}_S > \in \mathcal{A}$. So $\mathbf{f}_S, \mathbf{g}_S$ are q-orthogonal.
Thus $(\mathcal{V}_{SP} \leftrightarrow \mathcal{U}_P)^* \supseteq (\mathcal{V}^*_{SP} \leftrightarrow \mathcal{U}^*_P)$.

Next let $\mathbf{g}_S \in (\mathcal{V}_{SP} \leftrightarrow \mathcal{U}_P)^*$. Let $\mathbf{f}_S \in \mathcal{V}_{SP} \times S$. Since $\mathbf{0}_P \in \mathcal{U}_P$ and $\mathbf{f}_S \oplus \mathbf{0}_P \in \mathcal{V}_{SP}$, it is clear that $\mathbf{f}_S \in \mathcal{V}_{SP} \leftrightarrow \mathcal{U}_P$ and $< \mathbf{f}_S,\ \mathbf{g}_S > \in \mathcal{A}$. Hence, $\mathbf{g}_S \in (\mathcal{V}_{SP} \times S)^*$. Since $\mathcal{V}_{SP}, \mathcal{V}^*_{SP} \cdot S$ are closed, we have, using Lemma 7.2.4,

$$\begin{aligned}\mathcal{V}^*_{SP} \cdot S &= (\mathcal{V}^*_{SP} \cdot S)^{**} \\ &= (\mathcal{V}^{**}_{SP} \times S)^* = (\mathcal{V}_{SP} \times S)^*.\end{aligned}$$

Thus, $\mathbf{g}_S \in \mathcal{V}^*_{SP} \cdot S$. Hence, there exists $\mathbf{g}_P$ on P s.t. $\mathbf{g}_S \oplus \mathbf{g}_P \in \mathcal{V}^*_{SP}$. We will show that $\mathbf{g}_P \in \mathcal{U}^*_P$. Let $\mathbf{f}_P \in \mathcal{U}_P \subseteq \mathcal{V}_{SP} \cdot P$. Then there exists $\mathbf{f}_S$ on S s.t. $\mathbf{f}_S \oplus \mathbf{f}_P \in \mathcal{V}_{SP}$. Since $\mathbf{g}_S \in (\mathcal{V}_{SP} \leftrightarrow \mathcal{U}_P)^*$, we must have $< \mathbf{f}_S,\ \mathbf{g}_S > \in \mathcal{A}$. We also have

$$< \mathbf{f}_S \oplus \mathbf{f}_P,\ \mathbf{g}_S \oplus \mathbf{g}_P > = < \mathbf{f}_S,\ \mathbf{g}_S > + < \mathbf{f}_P,\ \mathbf{g}_P > \in \mathcal{A}.$$

Since $\mathcal{A}$ is closed under subtraction it follows that $< \mathbf{f}_P, \mathbf{g}_P > \in \mathcal{A}$, i.e., $\mathbf{g}_P \in \mathcal{U}^*_P$ and therefore, $\mathbf{g}_S \in \mathcal{V}^*_{SP} \leftrightarrow \mathcal{U}^*_P$. Thus $(\mathcal{V}_{SP} \leftrightarrow \mathcal{U}_P)^* \subseteq (\mathcal{V}^*_{SP} \leftrightarrow \mathcal{U}^*_P)$.

□

Chapter 8

Multiport Decomposition

8.1 Introduction

An informal discussion of the role of multiports in electrical networks may be found in Section 5.8 of Chapter 5. Its relation to the Implicit Duality Theorem is brought out briefly in Subsection 7.3.2 of Chapter 7. In this chapter, we give a formal description of multiport decomposition using the notion of **generalized minor** and the **Implicit Duality Theorem**. The primary application we have in mind is to network analysis by decomposition. Relevant to this application is the port minimization of component multiports. These topics we deal with in detail. The port connection diagram can be viewed as the graph of a reduced network which keeps invariant the interrelationship between the subnetworks which go into the making of the different multiports. We give some instances of this idea in Section 8.5.

Remark: The word multiport has been used in two different senses in this chapter. An **electrical multiport** is an electrical network with some devices, which are norators, specified as **ports**. A **component multiport** is a vector space $\mathcal{V}_{EP}$ on $E \uplus P$ with the subset P specified as **ports**. Formally, in both cases no further property of ports needs to be specified.

8.2 Multiport Decomposition of Vector Spaces

Definition 8.2.1 *Let $\mathcal{V}_E$ be a vector space on E and let E be partitioned into $E_1, E_2, \cdots, E_k$. Let $\mathcal{V}_{E_1P_1}, \cdots, \mathcal{V}_{E_kP_k}, \mathcal{V}_P$, where $P \equiv P_1 \uplus \cdots \uplus P_k$ is a set disjoint from E, be vector spaces on $E_1 \cup P_1, \cdots, E_k \cup P_k, P$ respectively. We say $(\mathcal{V}_{E_1P_1}, \cdots, \mathcal{V}_{E_kP_k}; \mathcal{V}_P)$ is a* **k-multiport decomposition** *(***k-decomposition** *or*

decomposition *for short) of* $\mathcal{V}_E$ *iff*

$$(\bigoplus_i \mathcal{V}_{E_iP_i}) \leftrightarrow \mathcal{V}_P = \mathcal{V}_E.$$

The set P *is called the set of* **ports**, $\mathcal{V}_{E_iP_i}, i = 1, \cdots, k$ *are called the* **components or component multiports** *of the decomposition while* $\mathcal{V}_P$ *is called the* **coupler**. *A multiport decomposition* $(\mathcal{V}_{E_1P_1}, \cdots, \mathcal{V}_{E_kP_k}; \mathcal{V}_P)$ *of* $\mathcal{V}_E$ *is said to be* **minimal** *iff whenever* $(\mathcal{V}_{E_1P_1'}, \cdots, \mathcal{V}_{E_kP_k'}; \mathcal{V}_{P'})$ *is a multiport decomposition of* $\mathcal{V}_E$*, we have* $\mid P \mid \leq \mid P' \mid$.

Remark: Instead of $(\mathcal{V}_{E_1P_1}, \cdots, \mathcal{V}_{E_kP_k}; \mathcal{V}_P)$ we would usually write $((\mathcal{V}_{E_jP_j})_k; \mathcal{V}_P)$. Instead of $(\cdots \mathcal{V}_{E_jP_j} \cdots; \mathcal{V}_{P_I})$ where $j \in I \subseteq \{1, \cdots, k\}$ and $P_I = \cup_{j \in I} P_j$ we write $((\mathcal{V}_{E_jP_j})_{j\in I}; \mathcal{V}_{P_I})$.
As an immediate consequence of the Implicit Duality Theorem, we have

Theorem 8.2.1 $(\mathcal{V}_{E_1P_1}, \cdots, \mathcal{V}_{E_kP_k}; \mathcal{V}_P)$ *is a multiport decomposition of* $\mathcal{V}_E$ *iff*

$$(\mathcal{V}^{\perp}_{E_1P_1}, \cdots, \mathcal{V}^{\perp}_{E_kP_k}; \mathcal{V}^{\perp}_P)$$

is a multiport decomposition of $\mathcal{V}^{\perp}_E$.

Proof : We have

$$(\bigoplus_i \mathcal{V}_{E_iP_i}) \leftrightarrow \mathcal{V}_P = \mathcal{V}_E$$

iff

$$(\bigoplus_i \mathcal{V}_{E_iP_i})^{\perp} \leftrightarrow \mathcal{V}^{\perp}_P = \mathcal{V}^{\perp}_E$$

by the Implicit Duality Theorem (Theorem 7.1.1)
The LHS of the second condition $= (\bigoplus_i \mathcal{V}^{\perp}_{E_iP_i}) \leftrightarrow \mathcal{V}^{\perp}_P$,
and the result follows.

□

We can now define multiport decomposition for graphs in a natural manner.

Multiport Decomposition of a Graph

Let $\mathcal{G}$ be a directed graph on edge set E. Let E be partitioned into $E_1, \cdots, E_k$. Let $P_1 \uplus \cdots \uplus P_k = P$ be a set disjoint from E. Let $\mathcal{G}_{E_1P_1}, \cdots, \mathcal{G}_{E_kP_k}, \mathcal{G}_P$ be graphs on sets of edges $E_1 \uplus P_1, \cdots, E_k \uplus P_k, P$ respectively, such that

$$\mathcal{V}_v(\mathcal{G}) = (\bigoplus_j \mathcal{V}_v(\mathcal{G}_{E_jP_j})) \leftrightarrow \mathcal{V}_v(\mathcal{G}_P),$$

equivalently, such that

$$\mathcal{V}_i(\mathcal{G}) = (\bigoplus_i \mathcal{V}_i(\mathcal{G}_{E_jP_j})) \leftrightarrow \mathcal{V}_i(\mathcal{G}_p).$$

Then we say that $(\mathcal{G}_{E_1P_1}, \cdots, \mathcal{G}_{E_kP_k}; \mathcal{G}_P)$ is a **k-multiport decomposition** of $\mathcal{G}$. The edges in P are called **ports**. The graphs $\mathcal{G}_{E_iP_i}, i = 1, \cdots, k$ are called the **components** or **component multiports** in the decomposition while $\mathcal{G}_P$ is called the **port connection diagram**.

We would usually write $((\mathcal{G}_{E_iP_i})_k; \mathcal{G}_P)$ instead of $(\mathcal{G}_{E_1P_1}, \cdots, \mathcal{G}_{E_kP_k}; \mathcal{G}_P)$. If the components and coupler of a multiport decomposition of $\mathcal{V}_v(G)$ are voltage spaces of graphs we say that the decomposition is **graphic**. In general, a multiport decomposition of $\mathcal{V}_v(\mathcal{G})$ would not be graphic. Also the procedure for minimization of number of ports that we describe in this section yields a multiport decomposition of $\mathcal{V}_v(\mathcal{G})$ that is not always graphic. Computationally this is not a great hindrance, as we shall show.

Coupling of given components to yield a given vector space

We describe below necessary and sufficient conditions under which given components $\mathcal{V}_{E_jP_j}, j = 1, \cdots, k$ can be coupled to yield a given vector space $\mathcal{V}_E$. First we state a simple lemma which characterizes an extension of a vector space.

Lemma 8.2.1 *Let $\mathcal{V}_{EP}, \mathcal{V}_E$ be vector spaces on $E \uplus P, E$. Then, there exists a vector space $\mathcal{V}_P$ on P s.t. $\mathcal{V}_{EP} \leftrightarrow \mathcal{V}_P = \mathcal{V}_E$ iff*

$$\mathcal{V}_{EP} \cdot E \supseteq \mathcal{V}_E \supseteq \mathcal{V}_{EP} \times E.$$

For proof see the solution of Exercise 7.5.

Theorem 8.2.2 *Let $\mathcal{V}_{E_1P_1}, \cdots, \mathcal{V}_{E_kP_k}$ be vector spaces on $E_1 \uplus P_1, \cdots, E_k \uplus P_k$. Let $E \equiv \uplus E_i, P \equiv \uplus P_i$ and let $\mathcal{V}_E$ be a vector space on E. Then there exists a vector space $\mathcal{V}_P$ on P such that $(\mathcal{V}_{E_1P_1}, \cdots, \mathcal{V}_{E_kP_k}; \mathcal{V}_P)$ is a k-multiport decomposition of $\mathcal{V}_E$ iff the following two equivalent conditions are satisfied:*

i. $\bigoplus_j (\mathcal{V}_{E_jP_j} \cdot E_j) \supseteq \mathcal{V}_E \supseteq \bigoplus_j (\mathcal{V}_{E_jP_j} \times E_j), j = 1, \cdots, k.$

ii. $\mathcal{V}_{E_jP_j} \cdot E_j \supseteq \mathcal{V}_E \cdot E_j$ *and*
$\mathcal{V}_{E_jP_j} \times E_j \subseteq \mathcal{V}_E \times E_j, j = 1, \cdots, k$

Proof : i. From Lemma 8.2.1 we have that there exists $\mathcal{V}_P$ s.t. $(\bigoplus_j \mathcal{V}_{E_jP_j}) \leftrightarrow \mathcal{V}_P = \mathcal{V}_E$ iff

$$(\bigoplus_j \mathcal{V}_{E_jP_j}) \cdot E \supseteq \mathcal{V}_E \supseteq (\bigoplus_j \mathcal{V}_{E_jP_j}) \times E.$$

We observe that

$$(\bigoplus_j \mathcal{V}_{E_jP_j}) \cdot E = \bigoplus_j (\mathcal{V}_{E_jP_j} \cdot E_j)$$

and

$$(\bigoplus_j \mathcal{V}_{E_jP_j}) \times E = \bigoplus_j (\mathcal{V}_{E_jP_j} \times E_j).$$

The result follows.

ii. We observe that $(\bigoplus_j \mathcal{V}_{E_jP_j}) \cdot E \supseteq \mathcal{V}_E$ iff $(\bigoplus_j \mathcal{V}_{E_jP_j}) \cdot E \cdot E_j \supseteq \mathcal{V}_E \cdot E_j, j = 1, \cdots, k$,
i.e., iff $\mathcal{V}_{E_jP_j} \cdot E_j \supseteq \mathcal{V}_E \cdot E_j, j = 1, \cdots, k$.
Next $\mathcal{V}_E \supseteq (\bigoplus_j \mathcal{V}_{E_jP_j}) \times E$ iff $(\bigoplus_j \mathcal{V}_{E_jP_j}) \times E \times E_j \subseteq \mathcal{V}_E \times E_j, j = 1, \cdots, k$,
i.e., iff $\mathcal{V}_{E_jP_j} \times E_j \subseteq \mathcal{V}_E \times E_j, j = 1, \cdots, k$.

□

Compatibility of a decomposition

In general one cannot recover the coupler $\mathcal{V}_P$ of a decomposition given the components $\mathcal{V}_{E_jP_j}$ and the decomposed space $\mathcal{V}_E$. This is possible (as we show below) precisely when
$\mathcal{V}_{E_jP_j} \cdot P_j \supseteq \mathcal{V}_P \cdot P_j, j = 1, \cdots, k$
and $\mathcal{V}_{E_jP_j} \times P_j \subseteq \mathcal{V}_P \times P_j, j = 1, \cdots, k$.
When these conditions are satisfied we say that the components and the coupler of the decomposition are **compatible**, or more briefly, that the decomposition is **compatible**.

Theorem 8.2.3 *Let $((\mathcal{V}_{E_jP_j})_k, \mathcal{V}_P)$ be a decomposition of $\mathcal{V}_E$. Then*

$$(\bigoplus_j \mathcal{V}_{E_jP_j}) \leftrightarrow \mathcal{V}_E = \mathcal{V}_P$$

iff the decomposition is compatible.

Proof : Given spaces $\mathcal{V}_{EP}, \mathcal{V}_P, \mathcal{V}_E$ on $E \uplus P, P, E$ respectively such that
$\mathcal{V}_{EP} \leftrightarrow \mathcal{V}_P = \mathcal{V}_E$, we have
$\mathcal{V}_P = \mathcal{V}_{EP} \leftrightarrow \mathcal{V}_E$ iff $\mathcal{V}_{EP} \cdot P \supseteq \mathcal{V}_P$ *and* $\mathcal{V}_{EP} \times P \subseteq \mathcal{V}_P$ (by Theorem 7.6.1 of Problem 7.5).
The result now follows from the fact that the two sets of conditions of Theorem 8.2.2 are equivalent.

□

Further Decomposition of Components

Let $(\mathcal{V}_{E_1P_1}, \cdots, \mathcal{V}_{E_kP_k}; \mathcal{V}_P)$ be a decomposition of $\mathcal{V}_E$. It would often be convenient to further decompose the components $\mathcal{V}_{E_jP_j}$. There are two ways in which this could be done:

- We could perform an m_j-multiport decomposition of $\mathcal{V}_{E_jP_j}, j = 1, \cdots, k$. In this case while decomposing $\mathcal{V}_{E_jP_j}$ we would treat E_j and P_j the same way, i.e., not distinguish between them.
- We could try another kind of decomposition in which the final ports P_j do not appear in the individual components $\mathcal{V}_{E_{jt}Q_t}$ but only in the port connection diagram V_{P_jQ} (see Figure 8.1.)

Figure 8.1: Decomposition of a Multiport

The latter is encountered more often in network theory when electrical multiports are decomposed. For instance, in network theory, there are procedures for 2-port synthesis where simpler electrical 2-ports are first built and their ports then connected together to form the final electrical 2-port. In Figure 8.1, two component multiports on $\mathcal{G}_{E_1Q_1}$ and $\mathcal{G}_{E_2Q_2}$ are connected acccording to the port connection diagram $\mathcal{G}_{QP}$ to yield the final multiport $\mathcal{G}_{EP}$. The reader would notice that this corresponds to the series connection of the two 2-ports.

We formally define the 'decomposition of component multiports', as opposed to 'decomposition of vector spaces', below. The reader may, if he so wishes, identify $\mathcal{V}_{EP}$ with one of the $\mathcal{V}_{E_iP_i}$ in the decomposition of a vector space $\mathcal{V}_E$.

Let $\mathcal{V}_{EP}$ be a vector space on $E \uplus P$. The ordered pair $(\mathcal{V}_{EP}, P)$ is called a **vector space on $E \uplus P$ with ports P.** More briefly we might say $\mathcal{V}_{EP}$ is a vector space on $E \uplus P$ with ports P. We say $(\mathcal{V}_{E_1Q_1}, \cdots, \mathcal{V}_{E_kQ_k}; \mathcal{V}_{QP})$, where $Q \cap (E \uplus P) = \emptyset$, $\uplus Q_j = Q$, is a **matched k-multiport decomposition** of $(\mathcal{V}_{EP}, P)$ iff

$$(\bigoplus_j \mathcal{V}_{E_jQ_j}) \leftrightarrow \mathcal{V}_{QP} = \mathcal{V}_{EP}.$$

We say it is a **skewed k-multiport decomposition** of $(\mathcal{V}_{EP}, P)$ iff

$$(\bigoplus_j \mathcal{V}_{E_jQ_j}) \rightleftharpoons \mathcal{V}_{QP} = \mathcal{V}_{EP}.$$

Thus the notion of decomposition of component multiports may be used to decompose a vector space hierarchically. Suppose we have $\mathcal{V}_E \equiv (\bigoplus \mathcal{V}_{E_iP_i}) \leftrightarrow \mathcal{V}_P$. We could then further decompose the components $\mathcal{V}_{E_iP_i}$ as $(\mathcal{V}_{E_{i1}Q_{i1}}, \cdots, \mathcal{V}_{E_{ik}Q_{ik}}; \mathcal{V}_{Q_iP_i})$, i.e., $\mathcal{V}_{E_iP_i} = (\bigoplus_j \mathcal{V}_{E_{ij}Q_{ij}}) \leftrightarrow \mathcal{V}_{Q_iP_i}$.

Remark: As in the case of decomposition of vector spaces, we usually write $((\mathcal{V}_{E_iQ_i})_k; \mathcal{V}_{QP})$ instead of $(\mathcal{V}_{E_1Q_1}, \cdots, \mathcal{V}_{E_kQ_k}; \mathcal{V}_{QP})$. Again as in the case of decomposition of vector spaces, $\mathcal{V}_{E_jQ_j}$ would be called **components** and $\mathcal{V}_{QP}$ the **coupler**. We say a k-multiport decomposition of $(\mathcal{V}_{EP}, P)$ is graphic if the components and the coupler space are voltage spaces of graphs.
We now have

Theorem 8.2.4 *Let $\mathcal{V}_{EP}$ be a vector space with ports P. Then $(\mathcal{V}_{E_1Q_1}, \cdots, \mathcal{V}_{E_kQ_k}; \mathcal{V}_{QP})$ is a matched k-multiport decomposition of $\mathcal{V}_{EP}$ iff $(\mathcal{V}^{\perp}_{E_1Q_1}, \cdots, \mathcal{V}^{\perp}_{E_kQ_k}; \mathcal{V}^{\perp}_{QP})$ is a skewed k-multiport decomposition $\mathcal{V}^{\perp}_{EP}$.*

Proof : We use Corollary 7.1.1. We have in general,

$$(\mathcal{V}_{EP} \leftrightarrow \mathcal{V}_{QP})^{\perp} = (\mathcal{V}^{\perp}_{EP} \rightleftharpoons \mathcal{V}^{\perp}_{QP})$$

Hence,

$$((\bigoplus_j \mathcal{V}_{E_jQ_j}) \leftrightarrow \mathcal{V}_{QP})^{\perp} = ((\bigoplus_j \mathcal{V}^{\perp}_{E_jQ_j}) \rightleftharpoons \mathcal{V}^{\perp}_{QP})$$

and the theorem follows.

□

Exercise 8.1 Number of ports needed to access a graph: *Let $\mathcal{G}$ be a connected graph on edgeset E_1. Suppose $\mathcal{G}$ were a subgraph of another graph $\mathcal{G}'$ on the set of edges E'.*

i. If nothing is known about the structure of $\mathcal{G}'$, show that the minimum number of ports P_1 we require, in a multiport decomposition of $\mathcal{V}_v(\mathcal{G}')$ with respect to a partition $\{E_1, E_2, \cdots, E_k\}$ of E', is equal to $r(\mathcal{G})$ and that these ports can in general be arranged as the copy of a tree of $\mathcal{G}$.

ii. What if $\mathcal{G}$ had p connected components?

iii. Repeat for a multiport decomposition of $\mathcal{V}_i(\mathcal{G}')$.

Exercise 8.2 Essential information about one part of a solution for the remaining part: *Consider the complementary orthogonal linear equations*

$$(\mathbf{A}_1 \vdots \mathbf{A}_2)\binom{\mathbf{x}_1}{\mathbf{x}_2} = \mathbf{0} \qquad (*)$$

$$(\mathbf{B}_1 \vdots \mathbf{B}_2)\binom{\mathbf{z}_1}{\mathbf{z}_2} = \mathbf{0} \qquad (\perp)$$

Suppose $\mathbf{x}_1 = \hat{\mathbf{x}}_1$. *Let* $S_2(\hat{\mathbf{x}}_1)$ *be the collection of all* $\mathbf{x}_2$ *s.t.* $(\hat{\mathbf{x}}_1, \mathbf{x}_2)$ *is a solution of Equations* $(*)$. *To determine* $S_2(\hat{\mathbf{x}}_1)$ *we usually would not have to know* **individual entries** *of* $\hat{\mathbf{x}}_1$.

i. *Show that it is sufficient to know the image* $\mathbf{p}_1$ *of* $\mathbf{x}_1$ *through an appropriate linear transformation* $\mathbf{Q}$*, where the number of entries of the vector* $\mathbf{p}_1 = r(\mathcal{V} \cdot E_1) - r(\mathcal{V} \times E_1)$, $\mathcal{V}$ *denoting the row space of* $\mathbf{A}$ *and* E_1 *the columns corresponding to* $\mathbf{x}_1$.
Also show that if the vector $\mathbf{p}_1$ *is obtained by a linear transformation of* $\mathbf{x}_1$*, then it cannot have less entries than the above.*

ii. *Repeat for the complementary orthogonal set of equations (Equations* $(\perp)$*).*

iii. *How is this notion related to multiport decomposition?*

Exercise 8.3 Counter-intuitive behaviour of decomposition: *Give an example of a graph of* $\mathcal{G}$ *and a multiport decomposition* $(\mathcal{G}_{E_1P_1}, \mathcal{G}_{E_2P_2}, \cdots; \mathcal{G}_P)$ *such that if the graphs* $\mathcal{G}_{E_1P_1}, \mathcal{G}_{E_2P_2}, \cdots$ *are connected along their ports according to the port connection diagram* $\mathcal{G}_P$*, we do not get back a graph with the same voltage space as* $\mathcal{G}$.

Exercise 8.4 Violation of port conditions after connection: *In some practical synthesis procedures, one first synthesizes 'component' multiports, which are put together later to construct the desired larger multiport. It is however necessary to check, after connection, whether the component multiports continue to satisfy the port conditions they were originally assumed to satisfy. In Figure 8.2,* $\mathcal{G}_{E_1Q_1}, \mathcal{G}_{E_2Q_2}$ *are put together according to* $\mathcal{G}_{QP}$ *to yield* $\mathcal{G}_{EP}$.

i. *Check if the port conditions of* $\mathcal{G}_{E_1Q_1}, \mathcal{G}_{E_2Q_2}$ *are satisfied in* $\mathcal{G}_{EP}$.

ii. *[Narayanan85b] Give a general procedure for testing whether port conditions of component multiports are satisfied in the combined multiport.*

Exercise 8.5 The decomposition of generalised minors of $\mathcal{V}_E$**:**
Let $(\mathcal{V}_{E_1P_1}, \cdots, \mathcal{V}_{E_kP_k}; \mathcal{V}_P)$ *be a k-multiport decomposition of* $\mathcal{V}_E$. *Let* $Q \subseteq E$ *and let* $\mathcal{V}_Q = \mathcal{V}_{Q_1} \oplus \cdots \oplus \mathcal{V}_{Q_k}$*, where*

$$Q_j = Q \cap E_j, \quad j = 1, \cdots, k.$$

Then, $\mathcal{V}_E \leftrightarrow \mathcal{V}_Q$ *has the decomposition* $((\mathcal{V}_{E_1P_1} \leftrightarrow \mathcal{V}_{Q_1}), \cdots, (\mathcal{V}_{E_kP_k} \leftrightarrow \mathcal{V}_{Q_k}); \mathcal{V}_P)$.
Hence, if $T \subseteq S \subseteq E$, $\mathcal{V}_E \times S \cdot T$ *has the decomposition*
$(\mathcal{V}_{E_jP_j} \times (S_j \cup P_j) \cdot (T_j \cup P_j)_k; \mathcal{V}_P)$ *where* $S_j \equiv S \cap E_j$*, and* $T_j \equiv T \cap E_j$.

Exercise 8.6 If P_i **are separators then** E_i **are separators:** *Let* $\mathcal{V}_P$ *have* $P_1, \cdots, P_k$ *as separators. Then*

$$(\bigoplus_j \mathcal{V}_{E_jP_j}) \leftrightarrow \mathcal{V}_P$$

Figure 8.2: Testing Port Conditions of Component Multiports

has $E_1, \cdots, E_k$ *as separators.*

Exercise 8.7 Compatible decomposition - Minors of $\mathcal{V}_E$ that can be obtained through minors of $\mathcal{V}_P$: *Let* $((\mathcal{V}_{E_jP_j})_k; \mathcal{V}_P)$ *be a compatible decomposition of* $\mathcal{V}_E$. *Let* $I_1 \subseteq \{1, \cdots, k\}$. *Let* $P_{I_1} \equiv \bigcup_{j \in I_1} P_j$ *and* $E_{I_1} \equiv \bigcup_{j \in I_1} E_j$. *Then*

i. $((\bigoplus_{j \in I_1} \mathcal{V}_{E_jP_j})_{j \in I_1} \leftrightarrow \mathcal{V}_P \cdot P_{I_1}) = \mathcal{V}_E \cdot E_{I_1}$.

ii. $((\bigoplus_{j \in I_1} \mathcal{V}_{E_jP_j})_{j \in I_1} \leftrightarrow \mathcal{V}_P \times P_{I_1}) = \mathcal{V}_E \times E_{I_1}$.

iii. $((\bigoplus_{j \in I_2} \mathcal{V}_{E_jP_j})_{j \in I_2} \leftrightarrow \mathcal{V}_P \times P_{I_1} \cdot P_{I_2}) = \mathcal{V}_E \times E_{I_1} \cdot E_{I_2}$.
where $I_2 \subseteq I_1 \subseteq \{1, \cdots, k\}$

iv. *In each of the above cases the derived decomposition is also compatible.*

Exercise 8.8 Counterintuitive behaviour of decomposition of a multiport: *Give an example of the graphic decomposition of a vector space* $\mathcal{V}_{EP}$ *with ports* P *such that when the graphs* $\mathcal{G}_{E_jQ_j}$ *are connected according to* $\mathcal{G}_{QP}$ *the resulting graph* $\mathcal{G}_{EP}$ *does not have* $\mathcal{V}_{EP}$ *as its voltage space.*

Exercise 8.9 Flattening a hierarchical multiport decomposition:
Let $\mathcal{M} \equiv (\mathcal{V}_{E_1P_1}, \cdots, \mathcal{V}_{E_kP_k}; \mathcal{V}_{PQ})$ *be a matched k-multiport decomposition of the*

vector space $\mathcal{V}_{EQ}$ with ports Q.
Let $\mathcal{V}_{E_jP_j}$ with ports P_j have a matched m_j-multiport decomposition $((\mathcal{V}_{E_{jt}T_{jt}})_{m_j};\mathcal{V}_{T_jP_j})$. Show that $\mathcal{V}_{EQ}$ has the matched Σm_j multiport decomposition $(\cdots,\mathcal{V}_{E_{ji}T_{ji}},\cdots;\mathcal{V}_{TQ})$ where $\mathcal{V}_{TQ} = (\bigoplus_j \mathcal{V}_{T_jP_j}) \leftrightarrow \mathcal{V}_{PQ}$.

8.3 Analysis through Multiport Decomposition

Let $\mathcal{N}$ be an electrical network on the directed graph $\mathcal{G}$. Let $E(\mathcal{G})$ be partitioned into subsets $E_1,\cdots,E_k$ which are mutually decoupled in the device characteristic of the network. Let $(\mathcal{V}_{E_1P_1},\cdots\mathcal{V}_{E_kP_k};\mathcal{V}_P)$ be a multiport decomposition of $\mathcal{V}_v(\mathcal{G})$. We now describe a scheme for analyzing the network $\mathcal{N}$ using the above multiport decomposition. The procedure is valid for a network with arbitrary devices but we illustrate our ideas through a linear network since computationally this is the most important case.

8.3.1 Rewriting Network Constraints in the Multiport Form

Let the device characteristic of the network $\mathcal{N}$ be

$$\mathbf{M}_j(\mathbf{i}_{E_j} - \mathbf{j}_j) + \mathbf{N}_j(\mathbf{v}_{E_j} - \mathbf{e}_j) = \mathbf{0}, j = 1,\cdots,k, \tag{8.1}$$

where $\mathbf{i}_{E_j},\mathbf{v}_{E_j}$ denote the current and voltage vectors respectively associated with the edge subset E_j. Let $\mathcal{V}_E$ be the voltage space of the graph $\mathcal{G}$. Let the representative matrix of the space $\mathcal{V}_{E_jP_j}$ be $(\mathbf{R}_j\ \mathbf{R}_{P_j}), j = 1,\cdots,k$ and of the space $\mathcal{V}_P$ be $\tilde{\mathbf{R}}_p$. Thus, the KVL constraints can be written equivalently as far as the variables $\mathbf{v}_E(= \mathbf{v}_{E_1}\oplus\cdots\oplus\mathbf{v}_{E_k})$ are concerned, as follows:

$$\begin{bmatrix} \mathbf{v}_{E_j} \\ \mathbf{v}_{P_j} \end{bmatrix} - \begin{bmatrix} \mathbf{R}_j^T \\ \mathbf{R}_{P_j}^T \end{bmatrix} \mathbf{v}_{n_j} = \mathbf{0}, j = 1,\cdots,k \tag{8.2}$$

$$\mathbf{v}_P - \tilde{\mathbf{R}}_P^T\,\tilde{\mathbf{v}}_{n_P} = \mathbf{0}. \tag{8.3}$$

We note that $\mathbf{v}_P = (\mathbf{v}_{P_1}\oplus\cdots\oplus\mathbf{v}_{P_k})$
and $\tilde{\mathbf{R}}_P = (\tilde{\mathbf{R}}_{P_1}\cdots\tilde{\mathbf{R}}_{P_k})$. We know by Theorem 8.2.1 that $\mathcal{V}_i(\mathcal{G})$ has the multiport decomposition $(\mathcal{V}_{E_1P_1}^{\perp}\cdots\mathcal{V}_{E_kP_k}^{\perp};\mathcal{V}_P^{\perp})$. Hence, the KCE of $\mathcal{N}$ may be written equivalently as far as the variable $\mathbf{i}_E = (\mathbf{i}_{E_1}\oplus\cdots\oplus\mathbf{i}_{E_k})$ are concerned as follows:

$$\begin{pmatrix} \mathbf{R}_j & \mathbf{R}_{P_j} \end{pmatrix} \begin{pmatrix} \mathbf{i}_{E_j} \\ \mathbf{i}_{P_j} \end{pmatrix} = \mathbf{0}, j = 1,\cdots,k \tag{8.4}$$

$$\tilde{\mathbf{R}}_P\mathbf{i}_P = \mathbf{0}. \tag{8.5}$$

We note that $\mathbf{i}_P = (\mathbf{i}_{P_1}\oplus\cdots\oplus\mathbf{i}_{P_k})$.
Equations 8.1, 8.2, 8.3, 8.4 and 8.5 are together equivalent to the constraints of the network $\mathcal{N}$ as far as the variables $\mathbf{i}_E,\mathbf{v}_E$ are concerned.

For convenience we rearrange these equations according to multiports and the coupler as follows:

$$\begin{aligned}
\begin{pmatrix} \mathbf{R}_j & \mathbf{R}_{P_j} \end{pmatrix} \begin{matrix} \mathbf{i}_{E_j} \\ \mathbf{i}_{P_j} \end{matrix} &= \mathbf{0} && (8.6)\\
\begin{bmatrix} \mathbf{v}_j \\ \mathbf{v}_{P_j} \end{bmatrix} - \begin{bmatrix} \mathbf{R}_j^T \\ \mathbf{R}_{P_j}^T \end{bmatrix} \mathbf{v}_{n_j} &= \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \end{pmatrix}. && (8.7)\\
\mathbf{M}_j(\mathbf{i}_{E_j} - \mathbf{j}_j) + \mathbf{N}_j(\mathbf{v}_{E_j} - \mathbf{e}_j) &= \mathbf{0}, j = 1, \cdots, k && (8.8)\\
\tilde{\mathbf{R}}_P \mathbf{i}_P &= \mathbf{0} && (8.9)\\
\mathbf{v}_P - \tilde{\mathbf{R}}_P^T \, \tilde{\mathbf{v}}_{n_P} &= \mathbf{0}. && (8.10)
\end{aligned}$$

We may regard the equations 8.6 through 8.8 as consisting of the KCL, KVL and device characteristic constraints of the electrical multiport networks $\mathcal{N}_{E_j P_j}, j = 1, \cdots, k$. We remind the reader that informally an electrical multiport (multiport for short) is a network with some devices which are norators, specified as ports. It is in this sense that we use this word when we talk of solution of multiports henceforth.

8.3.2 An Intuitive Procedure for Solution through Multiports

Let us assume that each of the electrical multiports $\mathcal{N}_{E_j P_j}$ can be uniquely solved for arbitrary values of $(\mathbf{i}_{P_{j1}}, \mathbf{v}_{P_{j2}})$ for some partitions $\{P_{j1}, P_{j2}\}$ of P_j. The natural way of solving equations 8.6 to 8.10 is as follows: .

STEP 1: Solve $\mathcal{N}_{E_j P_j}, j = 1, \cdots, k$
(a) setting all entries of $(\mathbf{i}_{P_{j1}}, \mathbf{v}_{P_{j2}})$ equal to zero; let $(\mathbf{v}^{\circ}_{E_j P_j}, \mathbf{i}^{\circ}_{E_j P_j})$ be the corresponding solution.
(b) setting the source values $\mathbf{e}_j, \mathbf{j}_j$ equal to zero and setting one entry of $(\mathbf{i}_{P_{j1}}, \mathbf{v}_{P_{j2}})$ equal to one in turn and the rest to zero. If P_j has k_j elements we would have k_j solutions $(\mathbf{v}^t_{E_j P_j}, \mathbf{i}^t_{E_j P_j})$, $t = 1, \cdots, k_j$.
(c) write the general solution (denoting k_j by q)

$$\begin{bmatrix} \mathbf{v}_{E_j} \\ \mathbf{i}_{E_j} \end{bmatrix} = \begin{bmatrix} \mathbf{v}^{\circ}_{E_j} \\ \mathbf{i}^{\circ}_{E_j} \end{bmatrix} + \begin{bmatrix} \mathbf{v}^1_{E_j} & \cdots & \mathbf{v}^q_{E_j} \\ \mathbf{i}^1_{E_j} & \cdots & \mathbf{i}^q_{E_j} \end{bmatrix} \begin{bmatrix} x^1 \\ \vdots \\ x^q \end{bmatrix} \tag{8.11}$$

$$\begin{bmatrix} \mathbf{v}_{P_j} \\ \mathbf{i}_{P_j} \end{bmatrix} = \begin{bmatrix} \mathbf{v}^{\circ}_{P_j} \\ \mathbf{i}^{\circ}_{P_j} \end{bmatrix} + \begin{bmatrix} \mathbf{v}^1_{P_j} & \cdots & \mathbf{v}^q_{P_j} \\ \mathbf{i}^1_{P_j} & \cdots & \mathbf{i}^q_{P_j} \end{bmatrix} \begin{bmatrix} x^1 \\ \vdots \\ x^q \end{bmatrix} \tag{8.12}$$

(Here $x^1, \cdots, x^q$ are the entries of $\mathbf{i}_{P_{j1}}, \mathbf{v}_{P_{j2}})$). Thus the x variables, appearing in the RHS of Equation 8.12, also appear as some of the variables on the left. So the equation can be rewritten involving only the variables in $\mathbf{v}_{P_j}$ and in $\mathbf{i}_{P_j}$. Thus the Equation 8.12, has the form of a device characteristic on the set of edges P_j. We will call this equation the equivalent device characteristic of P_j.

STEP 2: Combine all the equivalent device characteristics of $P_j, j = 1, \cdots k$ (called $\mathcal{D}_p$ say) with the KCL and KVL constraints of the coupler given in equation 8.9 and 8.10 and solve the resultant 'generalized network' $\mathcal{N}_P$ on P.
Let $(\mathbf{v}_P^f, \mathbf{i}_p^f)$ be the solution of this network.

STEP 3: Substitute appropriate entries of $(\mathbf{v}_P^f, \mathbf{i}_P^f)$ in $(\mathbf{i}_{P_{j1}}, \mathbf{v}_{P_{j2}})$ (i.e., in the x variables) of Equation 8.11.

STOP

Remark: We define a **generalized network** $\mathcal{N}_P$ to be a pair $(\mathcal{V}, \mathcal{D})$ where $\mathcal{V}$ is a real vector space on P and $\mathcal{D}$ is a device characteristic on P as in Definition 6.2.1. The space $\mathcal{V}$ takes the place of $\mathcal{V}_v(\mathcal{G})$ of ordinary network. A solution is a pair of vectors $(\mathbf{v}(\cdot), \mathbf{i}(\cdot))$ s.t. $\mathbf{v}(t) \in \mathcal{V}\ \forall t \in \Re$, $\mathbf{i}(t) \in \mathcal{V}^{\perp}\quad \forall\ t \in \Re$ and $(\mathbf{v}(\cdot), \mathbf{i}(\cdot)) \in \mathcal{D}$.

Detailed description of STEPS 1(a) and 1(b)

In subsequent sections we describe a procedure for port minimization which allows the electrical multiport $\mathcal{N}_{E_jP_j}$ to have a graph structure (i.e., $\mathcal{V}_{E_jP_j}$ can be chosen to be the voltage space of a graph $\mathcal{G}_{E_jP_j}$). We now go into details of steps 1(a) and 1(b) assuming this. (It should be noted however that graph structure is not as important as the sparsity that is its consequence.) We also simplify the notation as follows: $S \equiv E_j, T \equiv P_j$, $T_1 \equiv P_{j1}, T_2 \equiv P_{j2}$, $\hat{\mathcal{G}} \equiv \mathcal{G}_{E_jP_j}$, $\mathcal{V}_{ST} \equiv \mathcal{V}_{E_jP_j}$, $\hat{\mathcal{G}}_S \equiv \hat{\mathcal{G}} \times (S \cup T_1) \cdot S$.
Select a reduced incidence matrix $\hat{\mathbf{A}}_1$ of $\hat{\mathcal{G}} \times (S \cup T_1)$. When port minimization is done it would follow that in the graph $\mathcal{G}_{E_jP_j}$, the edges P_j would have no circuit or cutset. So we may assume that in the graph $\hat{\mathcal{G}} \times (S \cup T_1)$ there is a forest t which does not intersect T_1. Let $\hat{\mathbf{A}}_2$ be an f-cutset matrix with respect to the forest T_2 of $\hat{\mathcal{G}} \times ((S - t) \cup T)$. Let $\mathbf{A}_1, \mathbf{A}_2$ be the matrices obtained from $\hat{\mathbf{A}}_1, \hat{\mathbf{A}}_2$ by lengthening the rows padding them with zeros so that they become vectors on $S \uplus T$. Let us partition the matrix $\begin{pmatrix} \mathbf{A}_1 \\ \mathbf{A}_2 \end{pmatrix}$ as $\begin{pmatrix} \mathbf{A}_{1S} & \mathbf{A}_{1T_1} & \mathbf{0} \\ \mathbf{A}_{2S} & \mathbf{A}_{2T_1} & \mathbf{I}_{T_2} \end{pmatrix}$. This matrix is a representative matrix of $\mathcal{V}_v(\hat{\mathcal{G}})$ (see Problem 7.9). We can rewrite the

constraints of the electrical multiport $\mathcal{N}_{E_j P_j}$ as

$$\left[\begin{array}{ccc|ccccc} \mathbf{0} & \mathbf{A}_{1S} & \mathbf{0} & \mathbf{0} & \mathbf{A}_{1T_1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -\mathbf{A}_{1S}^T & \mathbf{0} & \mathbf{I} & -\mathbf{A}_{2S}^T & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{M} & \mathbf{N} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\ \mathbf{0} & \mathbf{A}_{2S} & \mathbf{0} & \mathbf{0} & \mathbf{A}_{2T_1} & \mathbf{I} & \mathbf{0} & \mathbf{0} \\ -\mathbf{A}_{1T_1}^T & \mathbf{0} & \mathbf{0} & -\mathbf{A}_{2T_1}^T & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{I}_{T_2} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} \end{array}\right] \begin{array}{c} \mathbf{v}_{n_1} \\ \mathbf{i}_S \\ \mathbf{v}_S \\ \mathbf{v}_{n_2} \\ \mathbf{i}_{T_1} \\ \mathbf{i}_{T_2} \\ \mathbf{v}_{T_1} \\ \mathbf{v}_{T_2} \end{array} = \begin{array}{c} \mathbf{0} \\ \mathbf{0} \\ \mathbf{M}\mathbf{j}_j + \mathbf{N}\mathbf{e}_j \\ \cdots \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{array} \tag{8.13}$$

The submatrix $\mathbf{A}_{1S}$ is the reduced incidence matrix of $\hat{\mathcal{G}}_S (\equiv \mathcal{G}_{E_j P_j} \times (E_j \cup P_{j1}) \cdot E_j)$. When the independent sources $\mathbf{j}_j$ and $\mathbf{e}_j$ are active and the variables $\mathbf{i}_{T_1}, \mathbf{v}_{T_2} (= \mathbf{v}_{n_2})$ are set to zero, we are left with the inner core of the equations with unknowns $\mathbf{v}_{n_1}, \mathbf{i}_S, \mathbf{v}_S$. After these have been solved $\mathbf{v}_{T_1}^{\circ}$ is computed in terms of $\mathbf{v}_{n_1}$ and $\mathbf{i}_{T_2}^{\circ}$ in terms of $\mathbf{i}_S^{\circ}$ using the fifth and fourth sets of rows respectively. Clearly the network structure and therefore, the constraint structure corresponds to $\hat{\mathcal{G}}_S$ (as we have seen it must in Section 6.3).

Next let us set the sources $\mathbf{j}_j$ and $\mathbf{e}_j$ to zero and say the r^{th} entry of $\mathbf{i}_{T_1}$ equal to 1, remaining entries of $\mathbf{i}_{T_1}$ equal to zero and all of $\mathbf{v}_{T_2}$ equal to zero. This corresponds to current sources **leaving the nodes** of $\hat{\mathcal{G}}_S$ whose value is given by the r^{th} column of $\mathbf{A}_{1T_1}$, i.e., if the p^{th} entry of this column of $\mathbf{A}_{1T_1}$ is β then a current source of value β leaves the p^{th} node of $\hat{\mathcal{G}}_S$ (and enters the datum node). So in this case again we are solving a network with graph $\hat{\mathcal{G}}_S$ with the source free device characteristic the same as before but a different source distribution.
Finally let us set the sources $\mathbf{j}_j, \mathbf{e}_j$ to zero and say the r^{th} entry of $\mathbf{v}_{T_2}$ equal to value 1, remaining entries of $\mathbf{v}_{T_2}$ equal to zero and all of $\mathbf{i}_{T_1}$ equal to zero. Since $\mathbf{v}_{n_2} = \mathbf{v}_{T_2}$, this is clearly equivalent to **impressing a voltage source distribution on the branches** of the network given by the j^{th} column of $\mathbf{A}_{2S}^T$. (Suppose the p^{th} entry of this column is α, then the p^{th} branch will acquire the voltage source of value α as shown in the Figure 8.3.

In each of the three cases described above $\mathbf{v}_{T_1}$ and $\mathbf{i}_{T_2}$ are computed after completing the solution of the rest of the variables, i.e., $\mathbf{i}_S$, $\mathbf{v}_S$ and $\mathbf{v}_{n_1}$.

Computational Effort for the Procedure

Let us examine the computational effort required in steps 1, 2 and 3 of the above procedure.
Solving the network $\mathcal{N}_{E_j P_j}$ in STEP 1 entails the solution of a linear network on E_j, $(\mid P_j \mid +1)$ times for appropriate source distributions corresponding to (a) the actual source vectors $\mathbf{j}_j, \mathbf{e}_j$ and (b) setting one term at a time of $\mathbf{i}_{P_{j1}}, \mathbf{v}_{P_{j2}}$ to value 1 and the rest as well as source terms to zero. We expect this step to be the most expensive computationally because of the size of the E_j's. In practice the partition $\{E_1, \cdots, E_k\}$ of E can usually be chosen (using heuristics) such that $\mid P \mid$ is less than about 5% of $\mid E \mid$ (assuming $\mid E \mid > 10000$). Hence, the effort involved in

Figure 8.3: Voltage Source Distribution when $\mathbf{v}_{T_1}$ is Active

Figure 8.4: Graph Of A Multiport

STEP 2, i.e., in computing the solution of $\mathcal{N}_P$, should be regarded as negligible in comparison with the effort involved in solving the $\mathcal{N}_{E_jP_j}$ repeatedly.

Exercise 8.10 Structure of constraints of electrical multiports during solution: *The method of multiport decomposition gives us the additional freedom of imposing appropriate structure on the equations corresponding to $\mathcal{N}_{E_jP_j}$ (on $\mathcal{V}_{E_jP_j}$) by solving in terms of $\mathbf{i}_{P_{j1}}, \mathbf{v}_{P_{j2}}$.*

i. *What is this structure?*

ii. *What is the best partition (P_{j1}, P_{j2}) for the multiport in Figure 8.4. Assume that all the edges are decoupled in the device characteristic.*

iii. *For a given partition $\{E_1, \cdots, E_k\}$ what is the complete range of structures possible for the electrical multiport constraint equations through variation of the port sets (through nonsingular transformation) as well as the partition of ports P_j into P_{j1}, P_{j2}.*

Remark: Let us suppose that we use LU factorisation to solve the linear equations. When we solve the network $\mathcal{N}_{E_jP_j}$ for various source distributions, the coefficient matrix, i.e., the core submatrix composed of the first three sets of rows and the first three sets of columns of Equation 8.13, remains the same. So the **LU factorisation of this submatrix has to be done only once**. We have to solve an equation say $(\mathbf{L}_j\mathbf{U}_j)\mathbf{x}_j = \mathbf{b}$ for $(| P_j | +1)$ values of $\mathbf{b}$. For each of these values of $\mathbf{b}$ we have to premultiply it by $\mathbf{L}_j^{-1}$ and then premultiply the result of the multiplication by $\mathbf{U}_j^{-1}$. Thus when we say solve $N_{E_jP_j}, (| P_j | +1)$ times we are really speaking of an upper bound of the effort involved.

8.4 Port Minimization

The discussion in the previous section suggests that port minimization (see the definition at the beginning of Section 8.2) is useful since

i. the number of times each of the electrical multiports $N_{E_jP_j}$ has to be solved equals $(| P_j | +1)$, and

ii. if the ports are not minimized, they may contain circuits or cutsets so that the imposed port conditions cannot be treated as independent. Our procedure of keeping one entry of $(\mathbf{i}_{P_{j1}}, \mathbf{v}_{P_{j2}})$ equal to one and the rest to zero would not be feasible.

However, it would be relevant to network analysis only if it is near-linear time. In the present section we give a few minimization algorithms. Two of these algorithms are very fast and can be used during the preprocessing stage of network analysis.

8.4.1 An Algorithm for Port Minimization

We begin with a general algorithm based on vector spaces [Narayanan86a], [Narayanan87]. The reader might like to review Section 3.4, particularly Subsections 3.4.2 and 3.4.5 and Exercises 3.53 and 3.54.

ALGORITHM 8.1 PORT MINIMIZATION 1

INPUT *Representative matrix* $\hat{\mathbf{R}}$ *of vector space* $\mathcal{V}_E$ *on* E, *a partition* $\{E_1, \cdots E_k\}$ *of* E.

OUTPUT *Representative matrices* $\begin{pmatrix} \mathbf{R}_j & \mathbf{R}_{P_j} \end{pmatrix}$ *of space* $\mathcal{V}_{E_jP_j}, j = 1, \cdots, k$ *and representative matrix* $\tilde{\mathbf{R}}_P$ *of space* $\mathcal{V}_P$, *where* $(\mathcal{V}_{E_1P_1}, \cdots, \mathcal{V}_{E_kP_k}; \mathcal{V}_P)$ *is a k-multiport decomposition of* $\mathcal{V}_E$ *such that* $|P|$ *is a minimum.*

STEP 1 *Let* $\hat{\mathbf{R}} \equiv (\hat{\mathbf{R}}_{E_1} \vdots \cdots \vdots \hat{\mathbf{R}}_{E_k})$. *Do reversible row transformations on* $\hat{\mathbf{R}}$ *so that we get a row equivalent matrix*

$$\mathbf{R} \equiv \begin{bmatrix} \mathbf{R}_{11} & & & \\ & \mathbf{R}_{22} & & \\ & & \ddots & \\ & & & \mathbf{R}_{kk} \\ \tilde{\mathbf{R}}_{(k+1)1} & \tilde{\mathbf{R}}_{(k+1)2} & \cdots & \tilde{\mathbf{R}}_{(k+1)k} \end{bmatrix},$$

where the submatrices $\mathbf{R}_{jj}$ *are representative matrices of* $\mathcal{V} \times E_j, j = 1, \cdots, k$ *and the submatrices* **other than** $\mathbf{R}_{jj}$ *and* $\tilde{\mathbf{R}}_{(k+1)j}, j = 1, \cdots, k$ *are* **zero** *submatrices.*
(Note that some of the R_{jj} *may not exist).*

STEP 2 *Let* $\mathbf{R}_{(k+1)j}$ *be such that*

$$\begin{pmatrix} \mathbf{R}_{jj} \\ \mathbf{R}_{(k+1)j} \end{pmatrix}$$

is composed of a maximal linearly independent set of rows of

$$\begin{pmatrix} \mathbf{R}_{jj} \\ \tilde{\mathbf{R}}_{(k+1)j} \end{pmatrix}$$

Let T_j *be a maximal linearly independent set of columns of* $\mathbf{R}_{(k+1)j}$. *Let* $\mathbf{R}_{(k+1)T_j}$ *be the full row submatrix of* $\mathbf{R}_{(k+1)j}$ *corresponding to* T_j. *Take the representative matrix* $(\mathbf{R}_j \ \mathbf{R}_{P_j})$ *of* $\mathcal{V}_{E_jP_j}, j = 1, \cdots k$ *to be*

$$\begin{pmatrix} \mathbf{R}_j & \vdots & \mathbf{R}_{P_j} \end{pmatrix} \equiv \begin{pmatrix} \mathbf{R}_{jj} & \vdots & \mathbf{0} \\ \mathbf{R}_{(k+1)j} & \vdots & \mathbf{R}_{(k+1)T_j} \end{pmatrix}$$

(Observe that $\mathbf{R}_{(k+1)T_j}$ *is a square submatrix of size* $r(\mathcal{V}_E \cdot E_j) - r(\mathcal{V}_E \times E_j)$*).*

STEP 3 *Let* $\tilde{\mathbf{R}}_{(k+1)T_j}$ *be the full row submatrix of* $\tilde{\mathbf{R}}_{(k+1)j}$ *corresponding to* T_j. *Let* $\tilde{\mathbf{R}}_{P_j} \equiv \tilde{\mathbf{R}}_{(k+1)T_j}, j = 1, \cdots, k$. *Take the representative matrix* $\tilde{\mathbf{R}}_P$ *to be*

$$\tilde{\mathbf{R}}_P \equiv (\tilde{\mathbf{R}}_{P_1} \vdots \cdots \vdots \tilde{\mathbf{R}}_{P_k}).$$

STOP

Complexity of Algorithm (Port Minimization 1)

In STEP 1 to compute the matrix $\mathbf{R}$ for the case where k = 2, we could proceed as follows. For the submatrix of $\hat{\mathbf{R}}$ composed of columns E_1, and all rows, build the RRE but extend the row operations to all columns. This takes $O(r_1 r \mid E \mid)$

operations where $r_1 \equiv r(\mathcal{V}\cdot E_1), r \equiv r(\mathcal{V}_E)$ The result is a matrix of the form shown below:

$$\begin{pmatrix} \mathbf{A}_1 \\ \mathbf{A}_2 \end{pmatrix} = \begin{bmatrix} \mathbf{A}_{11} & \vdots & \mathbf{A}_{12} \\ \cdots & \vdots & \cdots \\ \mathbf{0} & \vdots & \mathbf{A}_{22} \end{bmatrix}$$

For the matrix $\mathbf{A}_{12}$ compute the RRE but extend the row operations to all columns. This converts $\mathbf{A}_1$ to $\mathbf{A}'_1$ where

$$\mathbf{A}'_1 = \begin{bmatrix} \mathbf{A}'_{a1} & \mathbf{0} \\ \mathbf{A}'_{b1} & \mathbf{A}'_{b2} \end{bmatrix}$$

This takes $O(r_1^2 \mid E \mid)$ steps.
At the end of these steps we have the matrix $\mathbf{A}'$ shown below:

$$\mathbf{A}' = \begin{bmatrix} \mathbf{A}'_{a1} & \mathbf{0} \\ \mathbf{A}'_{b1} & \mathbf{A}'_{b2} \\ \mathbf{0} & \mathbf{A}_{22} \end{bmatrix}$$

To compute $\mathbf{A}'$ from $\hat{\mathbf{R}}$, as we have shown, takes $O(r_1 r \mid E \mid)$ steps.
Now if $k > 2$ we proceed as above but the columns of $\mathbf{A}_{22}$ correspond to $E - E_1$. Also we have to repeat the procedure with $\mathbf{A}_{22}$ recursively. To break up $\mathbf{A}_{22}$ as above takes $O(r_2 r \mid E - E_1 \mid)$ steps where $r_2 = r(\mathcal{V} \times (E - E_1) \cdot E_2)$. Repeating this procedure we see that computation of $\mathbf{R}$ takes
$O(r_1 r \mid E \mid + r_2 r \mid E - E_1 \mid + \cdots + r_{k-1} r \mid E_{k-1} \cup E_k \mid)$ steps, where
$r_j = r(\mathcal{V} \times (E - (E_1 \cup \cdots \cup E_{j-1})) \cdot E_j)$. By using Corollary 3.4.2 we see that
$r_1 + \cdots + r_k = r(\mathcal{V})$. Hence, complexity of the above computation is $O(r^2 \mid E \mid)$, where $r \equiv r(\mathcal{V}_E)$.
In STEP 2 we need to compute the RRE of

$$\begin{pmatrix} \mathbf{R}_{jj} \\ \tilde{\mathbf{R}}_{(k+1)j} \end{pmatrix} j = 1, \cdots, k.$$

The columns T_j are merely those columns corresponding to the unit matrix of appropriate order appearing in the second set of rows of the RRE (corresponding to the rows of $\tilde{\mathbf{R}}_{(k+1)j}$). Thus, this computation, for all $j = 1, \cdots, k$, is $O(r^2 \mid E \mid)$. **In STEP 3**, we merely put together the matrices $\tilde{\mathbf{R}}_{(k+1)T_j}$ that are already computed. Thus, the **overall complexity of Algorithm 8.1 is** $O(r^2 \mid E \mid)$, where $r \equiv r(\mathcal{V}_E)$.

Remark: Algorithm 8.1 can be easily adapted to the case where $\mathcal{V}_E$ is the voltage space of a graph so that it is near linear time and all the matrices generated are sparse. Further, for this case it produces a coupler space $\mathcal{V}_P$ which is voltage space of a graph. The component spaces $\mathcal{V}_{E_j P_j}$ are the sum of the voltage spaces of two graphs. Algorithm 8.2 described later produces component spaces which are voltage spaces of graphs but a coupler space which is not the voltage space of a graph.

Justification of Algorithm (Port Minimization 1)

We now proceed to show that the output of the Algorithm 8.1 is a minimal k-multiport decomposition of $\mathcal{V}_E$. We first show that

$$\mathcal{V}_E = (\bigoplus_j \mathcal{V}_{E_j P_j}) \leftrightarrow \mathcal{V}_P.$$

Let $\mathbf{f}_E \in \mathcal{V}_E$. We denote $\mathbf{f}_E/E_j$ by $\mathbf{f}_{E_j}$. Then

$$\mathbf{f}_{E_j}{}^T = \begin{pmatrix} \lambda_j^T & \vdots & \lambda_{k+1}^T \end{pmatrix} \begin{bmatrix} \mathbf{R}_{jj} \\ \tilde{\mathbf{R}}_{(k+1)j} \end{bmatrix}, \ j = 1, \cdots, k.$$

Observe that by the structure of the representative matrix $\mathbf{R}$, when j varies, λ_j would vary but λ_{k+1} would remain fixed.
Let

$$\mathbf{f}_{E_j P_j}{}^T = \lambda_j^T (\mathbf{R}_{jj} \vdots \mathbf{0}_{P_j}) + (\sigma^j_{(k+1)})^T (\mathbf{R}_{(k+1)j} \vdots \mathbf{R}_{(k+1)T_j}), \ j = 1, \cdots, k,$$

where $(\sigma^j_{(k+1)})^T (\mathbf{R}_{(k+1)j}) = \lambda^T_{(k+1)} (\tilde{\mathbf{R}}_{(k+1)j})$. Such a $\sigma^j_{(k+1)}$ must exist since rows of $\mathbf{R}_{(k+1)j}$ span rows of $\tilde{\mathbf{R}}_{(k+1)j}$. Observe that

$$(\sigma^j_{(k+1)})^T (\mathbf{R}_{(k+1)T_j}) = \lambda^T_{(k+1)} (\tilde{\mathbf{R}}_{(k+1)T_j}) = \lambda^T_{(k+1)} \tilde{\mathbf{R}}_{P_j}.$$

Hence,

$$\mathbf{f}_{E_j P_j}{}^T = \begin{pmatrix} \lambda_j^T & \vdots & \lambda_{(k+1)}^T \end{pmatrix} \begin{pmatrix} \mathbf{R}_{jj} \\ \tilde{\mathbf{R}}_{(k+1)j} \end{pmatrix} \oplus \lambda^T_{(k+1)} \tilde{\mathbf{R}}_{P_j}, j = 1, \cdots, k.$$

Let, $\mathbf{f}_P \equiv \lambda^T_{(k+1)} \tilde{\mathbf{R}}_P$,
i.e., $\mathbf{f}_P/P_j = \lambda^T_{(k+1)} \tilde{\mathbf{R}}_{P_j}, j = 1, \cdots, k$.
It is now clear that

$$\bigoplus_j \mathbf{f}_{E_j P_j} - \mathbf{f}_P = \mathbf{f}_E \oplus \mathbf{0}_P.$$

Hence,

$$\mathbf{f}_E \in ((\bigoplus_j \mathcal{V}_{E_j P_j}) \leftrightarrow \mathcal{V}_P)$$

On the other hand, let

$$\mathbf{f}_E \in ((\bigoplus_j \mathcal{V}_{E_j P_j}) \leftrightarrow \mathcal{V}_P).$$

Then there exist $\mathbf{f}_{E_j P_j}, j = 1, \cdots, k$ and $\mathbf{f}_P$ such that

$$\bigoplus_j \mathbf{f}_{E_j P_j} - \mathbf{f}_P = \mathbf{f}_E \oplus \mathbf{0}_P$$

If $\mathbf{f}_{E_jP_j} = \lambda_j^T(\mathbf{R}_{jj} \vdots \mathbf{0}_{P_j}) + (\sigma^j_{(k+1)})^T(\mathbf{R}_{(k+1)j} \vdots \mathbf{R}_{(k+1)T_j})$
then $\mathbf{f}_{E_jP_j}/P_j = \mathbf{f}_P/P_j = (\sigma^j_{(k+1)})^T(\mathbf{R}_{(k+1)T_j})$

But $\mathbf{f}_P = \lambda^T_{(k+1)}(\tilde{\mathbf{R}}_{P_1} \vdots \cdots \vdots \tilde{\mathbf{R}}_{P_k})$ for some $\lambda_{(k+1)}$. It follows that

$$(\sigma^j_{(k+1)})^T(\mathbf{R}_{(k+1)T_j}) = \lambda^T_{(k+1)}(\tilde{\mathbf{R}}_{P_j}),\ j = 1, \cdots, k.$$

and since the columns corresponding to T_j span the columns of $\tilde{\mathbf{R}}_{(k+1)j}$ as well as $\mathbf{R}_{(k+1)j}$, we must have

$$(\sigma^j_{(k+1)})^T)(\mathbf{R}_{(k+1)j}) = \lambda^T_{(k+1)}(\tilde{\mathbf{R}}_{(k+1)j}),\ j = 1, \cdots, k.$$

Thus,

$$\mathbf{f}_{E_jP_j}/E_j = \lambda_j^T(\mathbf{R}_{jj}) + \lambda^T_{(k+1)}(\tilde{\mathbf{R}}_{(k+1)j}),\ j = 1, \cdots, k.$$

Hence,

$$\bigoplus_j(\mathbf{f}_{E_jP_j}/E_j) = \bigoplus_j(\lambda_j^T(\mathbf{R}_{jj}) + \lambda^T_{(k+1)})(\tilde{\mathbf{R}}_{(k+1)j})).$$

But the RHS of the above equation is clearly a linear combination of the rows of $\mathbf{R}$ and therefore belongs to $\mathcal{V}_E$.
Next we need to show that the k-multiport decomposition that is generated by the algorithm is minimal. It is clear that in the decomposition generated by the algorithm

$$\mid P_j \mid = r(\mathcal{V}_E \cdot E_j) - r(\mathcal{V}_E \times E_j),\ j = 1, \cdots, k.$$

Lemma 8.4.1 proved below assures us that for every k-multiport decomposition $\mid P_j \mid$ cannot be less than the above RHS. The minimality of the decomposition follows.

Lemma 8.4.1 *Let* $(\mathcal{V}_{E_1P_1}, \cdots, \mathcal{V}_{E_kP_k}; \mathcal{V}_P)$ *be a k-multiport decomposition of* $\mathcal{V}_E$. *Then,* $\mid P_j \mid \geq r(\mathcal{V}_E \cdot E_j) - r(\mathcal{V}_E \times E_j),\ j = 1, \cdots, k.$

Proof : By Corollary 3.4.2

$$\begin{aligned} r(\mathcal{V}_{E_jP_j}) &= r(\mathcal{V}_{E_jP_j} \cdot E_j) + r(\mathcal{V}_{E_jP_j} \times P_j) \\ &= r(\mathcal{V}_{E_jP_j} \times E_j) + r(\mathcal{V}_{E_jP_j} \cdot P_j). \end{aligned}$$

Thus, $r(\mathcal{V}_{E_jP_j} \cdot E_j) - r(\mathcal{V}_{E_jP_j} \times E_j) = r(\mathcal{V}_{E_jP_j} \cdot P_j) - r(\mathcal{V}_{E_jP_j} \times P_j) \leq \mid P_j \mid$.
Next, by Theorem 8.2.2, we know that $\mathcal{V}_{E_jP_j} \cdot E_j \supseteq \mathcal{V}_E \cdot E_j$, and $\mathcal{V}_{E_jP_j} \times E_j \subseteq \mathcal{V}_E \times E_j$. We conclude therefore, that

$$\mid P_j \mid \geq r(\mathcal{V}_E \cdot E_j) - r(\mathcal{V}_E \times E_j),\ j = 1, \cdots, k.$$

□

8.4.2 Characterization of Minimal Decomposition

In Theorem 8.4.1, below, we give a number of equivalent conditions for the minimality of a k-multiport decomposition. A preliminary lemma is needed for the proof of this theorem.

Lemma 8.4.2 *Let $(\mathcal{V}_{E_1P_1}, \cdots, \mathcal{V}_{E_kP_k}; \mathcal{V}_P)$ be a k-multiport decomposition of $\mathcal{V}_E$. Then*

$$r(\mathcal{V}_E \cdot E_j) - r(\mathcal{V}_E \times E_j) \leq r(\mathcal{V}_P \cdot P_j) - r(\mathcal{V}_P \times P_j), \ j = 1, \cdots, k.$$

Proof : Let $\mathbf{f}_{E_j}{}^1, \cdots, \mathbf{f}_{E_j}{}^r$ be a set of vectors which together with a basis of $\mathcal{V}_E \times E_j$ form a basis for $\mathcal{V}_E \cdot E_j$. Then there exist vectors $\mathbf{f}_{P_j}{}^1, \cdots, \mathbf{f}_{P_j}{}^r$ in $\mathcal{V}_P \cdot P_j$ s.t. $\mathbf{f}_{E_j}{}^1 \oplus \mathbf{f}_{P_j}{}^1, \cdots, \mathbf{f}_{E_j}{}^r \oplus \mathbf{f}_{P_j}{}^r \in \mathcal{V}_{E_jP_j}$. Suppose a nontrivial linear combination $\mathbf{f}_{P_j}$ of $\mathbf{f}_{P_j}{}^1, \cdots, \mathbf{f}_{P_j}{}^r$ belongs to $\mathcal{V}_P \times P_j$. Let the same linear combination of $\mathbf{f}_{E_j}{}^1, \cdots, \mathbf{f}_{E_j}{}^r$ yield the vector $\mathbf{f}_{E_j}$. Then $\mathbf{f}_{E_j} \oplus \mathbf{f}_{P_j} \in \mathcal{V}_{E_jP_j}$ with $\mathbf{f}_{E_j} \in \mathcal{V}_E \cdot E_j - \mathcal{V}_E \times E_j$ and $\mathbf{f}_{P_j} \in \mathcal{V}_P \times P_j$. Let $\mathbf{f}_{EP}$ be the vector, on $E \uplus P$, whose restriction to $E_j \uplus P_j$ is $\mathbf{f}_{E_j} \oplus \mathbf{f}_{P_j}$ and whose value outside this set is zero. Let $\mathbf{f}_P$ be the vector, on P, whose restriction to P_j is $\mathbf{f}_{P_j}$ and whose value outside this set is zero. Since $\mathcal{V}_E = (\oplus_j \mathcal{V}_{E_jP_j}) \leftrightarrow \mathcal{V}_P$, it follows that $\mathbf{f}_{E_j} \oplus \mathbf{0}_{(E-E_j)}$ belongs to $\mathcal{V}_E$. Hence, $\mathbf{f}_{E_j} \in \mathcal{V}_E \times E_j$, which is a contradiction. We conclude that $\mathbf{f}_{P_j}{}^1, \cdots, \mathbf{f}_{P_j}{}^r$ together with a basis of $\mathcal{V}_P \times P_j$ forms an independent set. The result follows immediately.

□

Theorem 8.4.1 *Let $(\mathcal{V}_{E_1P_1}, \cdots, \mathcal{V}_{E_kP_k}; \mathcal{V}_P)$ be a k-multiport decomposition of $\mathcal{V}_E$. Then it is minimal iff the following equivalent conditions are satisfied.*

i. $| P_j |= r(\mathcal{V}_E \cdot E_j) - r(\mathcal{V}_E \times E_j), \ j = 1, \cdots, k.$

ii. $r(\mathcal{V}_{E_jP_j} \cdot E_j) = r(\mathcal{V}_E \cdot E_j)$
$r(\mathcal{V}_{E_jP_j} \times E_j) = r(\mathcal{V}_E \times E_j)$ *and*
$r(\mathcal{V}_{E_jP_j} \times P_j) = r(\mathcal{V}^{\perp}_{E_jP_j} \times P_j) = 0, j = 1, \cdots, k.$

iii. $r(\mathcal{V}_{E_jP_j} \cdot P_j) = r(\mathcal{V}_P \cdot P_j) =| P_j |, j = 1, \cdots, k.$
$r(\mathcal{V}_{E_jP_j} \times P_j) = r(\mathcal{V}_P \times P_j) = 0, j = 1, \cdots, k.$

iv. If $(\mathbf{R}_j \vdots \mathbf{R}_{P_j}{}')$, $(\mathbf{B}_j \vdots \mathbf{B}_{P_j}{}')$ are representative matrices of $\mathcal{V}_{E_jP_j}, \mathcal{V}^{\perp}_{E_jP_j}$ respectively and $(\mathbf{R}_{P_1} \vdots \cdots \vdots \mathbf{R}_{P_k})$, $(\mathbf{B}_{P_1} \vdots \cdots \vdots \mathbf{B}_{P_k})$ are representative matrices of $\mathcal{V}_P, \mathcal{V}_P^{\perp}$ respectively then the matrices $\mathbf{R}_{P_j}, \mathbf{R}_{P_j}{}', \mathbf{B}_{P_j}, \mathbf{B}_{P_j}{}'$ all have independent columns, $j = 1, \cdots, k$.

Proof : By Lemma 8.4.1 it follows that $| P_j |\geq r(\mathcal{V}_E \cdot E_j)$ - $r(\mathcal{V}_E \times E_j), \ j = 1, \cdots k$ for every k-multiport decomposition. But in Algorithm 8.1 we have constructed a k-multiport decomposition which satisfies the above condition with equality. Thus,

$(\mathcal{V}_{E_1P_1}, \cdots, \mathcal{V}_{E_kP_k}; \mathcal{V}_P)$ is a minimal k-multiport decomposition iff Condition (i) is satisfied.

Conditions (i) and (ii) are equivalent: Let Condition (i) hold. We have

$$| P_j | \geq r(\mathcal{V}_{E_jP_j} \cdot P_j) - r(\mathcal{V}_{E_jP_j} \times P_j) = r(\mathcal{V}_{E_jP_j} \cdot E_j) - r(\mathcal{V}_{E_jP_j} \times E_j). \qquad (*)$$

But $r(\mathcal{V}_{E_jP_j} \cdot E_j) - r(\mathcal{V}_{E_jP_j} \times E_j) \geq r(\mathcal{V}_E \cdot E_j) - r(\mathcal{V}_E \times E_j)$ by Theorem 8.2.2. It follows that Condition (i) can hold only if we have equality in place of the above inequality in Equation $(*)$.
But $| P_j | \geq r(\mathcal{V}_{E_jP_j} \cdot P_j)$, $r(\mathcal{V}_{E_jP_j} \times P_j) \geq 0$, $r(\mathcal{V}_{E_jP_j} \cdot E_j) \geq r(\mathcal{V}_E \cdot E_j)$ and $r(\mathcal{V}_{E_jP_j} \times E_j) \leq r(\mathcal{V}_E \times E_j)$.
Hence, equality holds in $(*)$ only if
$| P_j | = r(\mathcal{V}_{E_jP_j} \cdot P_j)$, $r(\mathcal{V}_{E_jP_j} \times P_j) = 0$, $r(\mathcal{V}_{E_jP_j} \cdot E_j) = r(\mathcal{V}_E \cdot E_j)$ and $r(\mathcal{V}_{E_jP_j} \times E_j) = r(\mathcal{V}_E \times E_j)$.
Now $| P_j | = r(\mathcal{V}_{E_jP_j} \cdot P_j) + r(\mathcal{V}^{\perp}_{E_jP_j} \times P_j)$. So $| P_j | = r(\mathcal{V}_{E_jP_j} \cdot E_j)$ iff $r(\mathcal{V}^{\perp}_{E_jP_j} \times P_j) = 0$.
This proves that Condition (i) implies Conditions (ii).
Next let Condition (ii) hold. We have (by Theorem 3.4.3)

$$\begin{aligned} | P_j | &= r(\mathcal{V}_{E_jP_j} \cdot P_j) + r(\mathcal{V}^{\perp}_{E_jP_j} \times P_j) \\ &= r(\mathcal{V}^{\perp}_{E_jP_j} \cdot P_j) + r(\mathcal{V}_{E_jP_j} \times P_j) \end{aligned}$$

So we must have, using Condition (ii),

$$| P_j | = r(\mathcal{V}_{E_jP_j} \cdot P_j) - r(\mathcal{V}_{E_jP_j} \times P_j)$$

The RHS equals $r(\mathcal{V}_{E_jP_j} \cdot E_j) - r(\mathcal{V}_{E_jP_j} \times E_j)$ by Corollary 3.4.2. But this expression equals $r(\mathcal{V}_E \cdot E_j) - r(\mathcal{V}_E \times E_j)$. Hence, Condition (i) holds.

Conditions (i) and (iii) are equivalent:
Let $| P_j | = r(\mathcal{V}_E \cdot E_j) - r(\mathcal{V}_E \times E_j)$.
Now $| P_j | = r(\mathcal{V}_{E_jP_j} \cdot P_j) + r(\mathcal{V}^{\perp}_{E_jP_j} \times P_j)$.
Hence, by Condition (ii), $| P_j | = r(\mathcal{V}_{E_jP_j} \cdot P_j)$ and $r(\mathcal{V}_{E_jP_j} \times P_j) = 0$. Further by Lemma 8.4.2

$$r(\mathcal{V}_E \cdot E_j) - r(\mathcal{V}_E \times E_j) \leq r(\mathcal{V}_P \cdot P_j) - r(\mathcal{V}_P \times P_j), \ j = 1, \cdots, k,$$

i.e., $\qquad | P_j | \leq r(\mathcal{V}_P \cdot P_j) - r(\mathcal{V}_P \times P_j), \ j = 1, \cdots, k.$
The only way this inequality can be satisfied is to have $| P_j | = r(\mathcal{V}_P \cdot P_j)$ and $r(\mathcal{V}_P \times P_j) = 0$. Thus Condition (i) implies Condition (iii).
Next let us assume that

$$r(\mathcal{V}_{E_jP_j} \cdot P_j) = r(\mathcal{V}_P \cdot P_j) = | P_j |, j = 1, \cdots, k,$$

$$r(\mathcal{V}_{E_jP_j} \times P_j) = r(\mathcal{V}_P \times P_j) = 0, j = 1, \cdots, k.$$

By Lemma 8.4.1 we already have

$$| P_j | \geq r(\mathcal{V}_E \cdot E_j) - r(\mathcal{V}_E \times E_j), \ j = 1, \cdots, k.$$

Suppose $| P_j | > r(\mathcal{V}_E \cdot E_j) - r(\mathcal{V}_E \times E_j)$ for some j.
So we have $r(\mathcal{V}_{E_j P_j} \cdot P_j) - r(\mathcal{V}_{E_j P_j} \times P_j) > r(\mathcal{V}_E \cdot E_j) - r(\mathcal{V}_E \times E_j)$. So by Theorem 8.2.2 there exists
(a) $\mathbf{f}_{E_j} \in (\mathcal{V}_{E_j P_j} \cdot E_j - \mathcal{V}_E \cdot E_j)$, or
(b) $\mathbf{g}_{E_j} \in (\mathcal{V}_E \times E_j - \mathcal{V}_{E_j P_j} \times E_j)$.
Case (a): $\mathbf{f}_{E_j} \in (\mathcal{V}_{E_j P_j} \cdot E_j - \mathcal{V}_E \cdot E_j)$.
In this case there exists a vector $\mathbf{f}_{E_j P_j} \in \mathcal{V}_{E_j P_j}$ s.t. $\mathbf{f}_{E_j P_j}/E_j = \mathbf{f}_{E_j}$. Since $\mathcal{V}_P \cdot P_j$ has full rank there must exist a vector $\mathbf{f}_P \in \mathcal{V}_P$ such that $\mathbf{f}_P/P_j = \mathbf{f}_{E_j P_j}/P_j$. Since $\mathcal{V}_{E_i P_i} \cdot P_i$ has full rank for all i, it follows that there exist vectors $\mathbf{f}_{E_i P_i}$ for all $i \neq j$ also s.t. $\mathbf{f}_{E_i P_i}/P_i = \mathbf{f}_P/P_i$. But this means that $\bigoplus_i \mathbf{f}_{E_i P_i}/E$ belongs to $\mathcal{V}_E$. But then $\mathbf{f}_{E_j} \in \mathcal{V}_E \cdot E_j$, which is a contradiction.
Case (b): $\mathbf{g}_{E_j} \in (\mathcal{V}_E \times E_j - \mathcal{V}_{E_j P_j} \times E_j)$.
In this case there exists a vector $\mathbf{g}_E \in \mathcal{V}_E$ s.t. $\mathbf{g}_E/E_j = \mathbf{g}_{E_j}$ and $\mathbf{g}_E/(E - E_j) = \mathbf{0}$. By the definition of k-multiport decomposition it follows that there must exist vectors $\mathbf{g}_{E_i P_i} \in \mathcal{V}_{E_i P_i}, i = 1, \cdots, k$ and $\mathbf{g}_P \in \mathcal{V}_P$ s.t.
$\mathbf{g}_{E_i P_i}/E_i = \mathbf{g}_E/E_i, i = 1, \cdots, k$ and $\mathbf{g}_P/P_i = \mathbf{g}_{E_i P_i}/P_i, i = 1, \cdots, k$.
But then it follows that $\mathbf{g}_{E_i P_i}/E_i = \mathbf{0}, i \neq j$. Hence, $\mathbf{g}_{E_i P_i}/P_i \in \mathcal{V}_{E_i P_i} \times P_i, i \neq j$. But $r(\mathcal{V}_{E_i P_i} \times P_i) = 0, i = 1, \cdots, k$. Thus $\mathbf{g}_{E_i P_i}/P_i = 0, i \neq j$. It follows that $\mathbf{g}_P/P_i = 0, i \neq j$. Hence, $\mathbf{g}_P/P_j \in \mathcal{V}_P \times P_j$. But this latter vector space also has zero rank. We conclude that $\mathbf{g}_P/P_j = \mathbf{0}$. But this means that $\mathbf{g}_{E_j P_j}/P_j = \mathbf{0}$, i.e., $\mathbf{g}_{E_j P_j}/E_j (= \mathbf{g}_{E_j})$ belongs to $\mathcal{V}_{E_j P_j} \times E_j$, a contradiction.

Thus, in both cases we arrive at contradictions. We therefore must have

$$| P_j | = r(\mathcal{V}_E \cdot E_j) - r(\mathcal{V}_E \times E_j) \ \ \forall j.$$

Thus, Conditions (i) and (iii) are equivalent.

Conditions (iii) and (iv) are equivalent:
We observe that since

$$\begin{aligned}
| P_j | &= r(\mathcal{V}_{E_j P_j} \times P_j) + r(\mathcal{V}^{\perp}_{E_j P_j} \cdot P_j) \\
&= r(\mathcal{V}_P \times P_j) + r(\mathcal{V}^{\perp}_P \cdot P_j) \\
| P_j | &= r(\mathcal{V}^{\perp}_{E_j P_j} \cdot P_j) \qquad iff \quad r(\mathcal{V}_{E_j P_j} \times P_j) = 0 \\
and \ | P_j | &= r(\mathcal{V}^{\perp}_P \cdot P_j) \qquad iff \quad r(\mathcal{V}_P \times P_j) = 0
\end{aligned}$$

Now Condition (iv) merely states that

$$\begin{aligned}
| P_j | &= r(\mathcal{V}_{E_j P_j} \cdot P_j) = r(\mathcal{V}^{\perp}_{E_j P_j} \cdot P_j) \\
&= r(\mathcal{V}_P \cdot P_j) = r(\mathcal{V}^{\perp}_P \cdot P_j)
\end{aligned}$$

It follows that Condition (iii) is equivalent to Condition (iv).

□

Exercise 8.11 (Strongly) Compatible Decomposition and Minimization Starting from a Strongly Compatible Decomposition: *We remind the reader that a k-multiport decomposition* $(\mathcal{V}_{E_1P_1}, \cdots, \mathcal{V}_{E_kP_k}; \mathcal{V}_P)$ *of* $\mathcal{V}_E$ *is* **compatible** *iff*

$$\mathcal{V}_{E_iP_i} \cdot P_i \supseteq \mathcal{V}_P \cdot P_i, \quad i = 1, \cdots, k,$$

$$\mathcal{V}_{E_iP_i} \times P_i \subseteq \mathcal{V}_P \times P_i, \quad i = 1, \cdots, k.$$

Let us define a k-multiport decomposition $(\mathcal{V}_{E_1P_1}, \cdots, \mathcal{V}_{E_kP_k}; \mathcal{V}_P)$ *of* $\mathcal{V}_E$ *to be* **strongly compatible** *iff*

$$\mathcal{V}_{E_iP_i} \cdot P_i = \mathcal{V}_P \cdot P_i, \quad i = 1, \cdots, k,$$

$$\mathcal{V}_{E_iP_i} \times P_i = \mathcal{V}_P \times P_i, \quad i = 1, \cdots, k.$$

Prove the following:

i. $((\mathcal{V}_{E_jP_j})_k; \mathcal{V}_P)$ *is a (strongly) compatible decomposition of* $\mathcal{V}_E$ *iff*

$$(\bigoplus_j \mathcal{V}_{E_jP_j}) \cdot P \supseteq \mathcal{V}_P((\bigoplus_j \mathcal{V}_{E_jP_j}) \cdot P = \mathcal{V}_P)$$

and

$$(\bigoplus_j \mathcal{V}_{E_jP_j}) \times P \subseteq \mathcal{V}_P((\bigoplus_j \mathcal{V}_{E_jP_j}) \times P = \mathcal{V}_P).$$

ii. $((\mathcal{V}_{E_jP_j})_k; \mathcal{V}_P)$ *is a compatible k-multiport decomposition of* $\mathcal{V}_E$

(a) iff $((\mathcal{V}_{E_jP_j})_k; \mathcal{V}_E)$ *is a compatible k-multiport decomposition of* $\mathcal{V}_P$;

(b) iff $((\mathcal{V}_{E_jP_j})_k^{\perp}; \mathcal{V}_P^{\perp})$ *is a compatible k-multiport decomposition of* $\mathcal{V}_E^{\perp}$.

iii. $((\mathcal{V}_{E_jP_j})_k; \mathcal{V}_P)$ *is a strongly compatible k-multiport decomposition of* $\mathcal{V}_E$ *iff* $((\mathcal{V}_{E_jP_j})_k^{\perp}; \mathcal{V}_P^{\perp})$ *is a strongly compatible k-multiport decomposition of* $\mathcal{V}_E^{\perp}$.

iv. (∗) *Let* $r(\mathcal{V}_{E_iP_i} \times P_i) > 0$*, let* $\hat{\mathbf{f}}_{P_i}$ *be a nonzero vector in* $\mathcal{V}_{E_iP_i} \times P_i$ *and let* e *belong to its support. If* $(\mathcal{V}_{E_1P_1}, \cdots, \mathcal{V}_{E_kP_k}; \mathcal{V}_P)$ *is a strongly compatible k-multiport decomposition of* $\mathcal{V}_E$*, then* $(\mathcal{V}_{E_1P_1}, \cdots, \mathcal{V}_{E_iP_i'}, \cdots, \mathcal{V}_{E_kP_k}; \mathcal{V}_{P'})$*, where* $P_i' = P_i - e, P' = P - e, \mathcal{V}_{E_iP_i'} = \mathcal{V}_{E_iP_i} \times (E_i \cup (P_i - e))$ *and* $\mathcal{V}_{P'} = \mathcal{V}_P \times (P - e)$,
is also a strongly compatible k-multiport decomposition of $\mathcal{V}_E$.

v. (∗) *Let* $r((\mathcal{V}_{E_iP_i})^{\perp} \times P_i) > 0$*, let* $\mathbf{g}_{P_i}$ *be a nonzero vector in* $(\mathcal{V}_{E_iP_i})^{\perp} \times P_i$ *and let* e *belong to the support of* $\mathbf{g}_{P_i}$*. Let* $Q_j \equiv P_j \quad \forall j \neq i$, $Q_i \equiv P_i - e$, $Q \equiv P - e, \mathcal{V}_{E_jQ_j} \equiv \mathcal{V}_{E_jP_j} \quad \forall j \neq i$, $\mathcal{V}_{E_iQ_i} \equiv \mathcal{V}_{E_iP_i} \cdot (E_i \cup Q_i)$ *and let* $\mathcal{V}_Q \equiv \mathcal{V}_P \cdot Q$.
Then $((\mathcal{V}_{E_jQ_j})_k; \mathcal{V}_Q)$ *is a strongly compatible k-multiport decomposition of* $\mathcal{V}_E$ *if* $((\mathcal{V}_{E_jP_j})_k; \mathcal{V}_P)$ *is strongly compatible.*

vi. (*) *The preceding two sections give us an algorithm for constructing a minimal decomposition starting from a strongly compatible decomposition by successively contracting and deleting suitable elements. Show that this algorithm terminates in a minimal decomposition. Show further that every minimal decomposition is a strongly compatible decomposition.*

Exercise 8.12 Natural transformation from $\mathcal{V}_E$ to $\mathcal{V}_P$ for minimal decompositions: *Let $((\mathcal{V}_{E_jP_j})_k, \mathcal{V}_P)$ be a minimal decomposition of $\mathcal{V}_E$. Show that*

i. *If $\bigoplus_j \mathbf{f}_{E_j} \in \mathcal{V}_E$ there is a* **unique** $\bigoplus_j \mathbf{f}_{P_j} \in \mathcal{V}_P$ *s.t.* $\mathbf{f}_{E_j} \oplus \mathbf{f}_{P_j} \in \mathcal{V}_{E_jP_j}$, $j = 1, \cdots, k$, *and hence , there is a linear transformation $T : \mathcal{V}_E \rightarrow \mathcal{V}_P$ s.t. $T(\bigoplus_j \mathbf{f}_{E_j}) = \bigoplus_j \mathbf{f}_{P_j}$ and $\mathbf{f}_{E_j} + \mathbf{f}_{P_j} \in \mathcal{V}_{E_jP_j}, j = 1, \cdots, k$*

ii. *if $\bigoplus_j \mathbf{f}_{P_j} \in \mathcal{V}_P$,then there exist $\bigoplus_j \mathbf{f}_{E_j} \in \mathcal{V}_E$ s.t $\mathbf{f}_{E_j} \oplus \mathbf{f}_{P_j} \in \mathcal{V}_{E_jP_j}, j = 1, \cdots, k$. If $\bigoplus_j {\mathbf{f}_{E_j}}^1$ and $\bigoplus_j {\mathbf{f}_{E_j}}^2$ correspond in this manner to $\bigoplus_j \mathbf{f}_{P_j}$ then*

$$ {\mathbf{f}_{E_j}}^1 - {\mathbf{f}_{E_j}}^2 \in \mathcal{V}_{E_jP_j} \times E_j = \mathcal{V}_E \times E_j. $$

Exercise 8.13 Uniqueness of $\mathcal{V}_P$ for minimal decompositions:
Let $((\mathcal{V}_{E_jP_j})_k, \mathcal{V}_P), ((\mathcal{V}_{E_jP_j})_k, \hat{\mathcal{V}}_P)$ both be miinimal decompositions of $\mathcal{V}_E$. Then $\mathcal{V}_P = \hat{\mathcal{V}}_P$.

Exercise 8.14 Nonsingular transformation of port variables:

i. *Let $(\mathcal{V}_{E_1P_1}, \cdots, \mathcal{V}_{E_kP_k}; \mathcal{V}_P)$ be a minimal k-decomposition of $\mathcal{V}_E$.*
Let $(\mathcal{V}_{E_1P'_1}, \cdots, \mathcal{V}_{E_kP'_k}, \mathcal{V}_{P'})$ be obtained as follows:

$$ \mathbf{f}_{E_j} \oplus \mathbf{f}_{P_j} \in \mathcal{V}_{E_jP_j} \quad iff \quad \mathbf{f}_{E_j} \oplus \mathbf{T}_j(\mathbf{f}_{P_j}) \in \mathcal{V}_{E_jP'_j}, j = 1, \cdots, k $$

$$ \bigoplus_j \mathbf{f}_{P_j} \in \mathcal{V}_P \quad iff \quad \bigoplus_j \mathbf{T}_j(\mathbf{f}_{P_j}) \in \mathcal{V}_{P'}, $$

where $\mathbf{T}_j$ is a nonsingular linear transformation acting on vectors defined on P_j.
Show that the latter decomposition is also minimal.

ii. *Given two minimal k-decompositions of $\mathcal{V}_E$ show that one can be obtained from the other by nonsingular linear transformations acting on vectors on port sets (as in the previous section of this problem).*

Exercise 8.15 Structure of the columns P_j for a minimal decomposition
Let $(\mathcal{V}_{E_1P_1}, \cdots, \mathcal{V}_{E_kP_k}; \mathcal{V}_P)$ be a minimal k-decomposition of $\mathcal{V}_E$.

i. *Show that in any representative matrix of $\mathcal{V}_{E_jP_j}, \mathcal{V}^{\perp}_{E_jP_j}, \mathcal{V}_P or \mathcal{V}^{\perp}_P$ the columns P_j would be linearly independent.*

ii. *The hybrid rank of a vector space $\mathcal{V}_P$ is defined to be*

$$ min_{K_1 \subseteq P}(r(\mathcal{V}_P \cdot K_1) + r(\mathcal{V}_P^{\perp} \cdot (P - K_1))). $$

Show that hybrid rank of $\mathcal{V}_P \geq \mid P_j \mid j = 1, \cdots, k$.

iii. We say $\mathcal{V}_{P'}$ is obtained from $\mathcal{V}_P$ by nonsingular transformation of the P_j if

$$\mathbf{f}_{P_1} \oplus \cdots \oplus \mathbf{f}_{P_K} \in \mathcal{V}_P \Leftrightarrow \mathbf{T}_1(\mathbf{f}_{P_1}) \oplus \cdots \oplus \mathbf{T}_k(\mathbf{f}_{P_K}) \in \mathcal{V}_{P'},$$

where $\mathbf{f}_{P'_j} \equiv \mathbf{T}_j(\mathbf{f}_{P_j})$.
Show that $\mid P_j \mid \leq$ hybrid rank of $\mathcal{V}_{P'}$, if $\mathcal{V}_{P'}$ is obtained by nonsingular transformation of the P_j.

8.4.3 Complexity of Algorithm (Port minimization 1) for Graphic Spaces and Sparsity of the Output Matrices

We have proposed multiport decomposition as a prelude to network analysis. It is necessary therefore that the algorithm for decomposition be near linear time in the size of the set E and that the matrices generated be sparse. However, this can be hoped for, and is essential, only in the case where $\mathcal{V}_E$ is the voltage or current space of a graph. Algorithm 8.1 is intended to work on an arbitrary vector space. While it is polynomial time ($O(r^2 \mid E \mid)$ as shown earlier), it needs to be shown that it is acceptably fast when $\mathcal{V}_E$ is the voltage space of a graph. The same holds for sparsity. In general we can say little about sparsity of the matrices generated but with voltage spaces of graphs one can hope to do better. Below, first we show that the algorithm is near linear time for voltage spaces of graphs. From the discussion it would also be clear that the matrices involved are quite sparse.

Adaptation of Algorithm (Port minimization 1) to graphic spaces

Let $\mathbf{R}$ be a reduced incidence matrix of a graph $\mathcal{G}$.
i. The matrix $\mathbf{R}_{jj}$ can be taken to be the reduced incidence matrix for $\mathcal{G} \times E_j, j = 1, \cdots, k$. Building all the $\mathcal{G} \times E_j$ takes $O(k \mid E \mid)$ time (in fact this can be shown to take $O(k|V| + |E|)$ time) and then all the $\mathbf{R}_{jj}$ takes an additional $O(\mid E \mid)$ time).
ii. To build the matrix $(\tilde{\mathbf{R}}_{(k+1)1} : \cdots : \tilde{\mathbf{R}}_{(k+1)k})$, we could first select a forest t of the graph $\bigoplus_j \mathcal{G} \times E_j$. This would be a disjoint union of forests t_j of $\mathcal{G} \times E_j, j = 1, \cdots, k$. Construct a reduced incidence matrix of the graph $\mathcal{G} \times (E - t)$. This takes $O(\mid E \mid)$ time. Adjoin a zero submatrix corresponding to the set t. This would be the desired matrix $(\tilde{\mathbf{R}}_{(k+1)1} : \cdots : \tilde{\mathbf{R}}_{(k+1)k})$ since the rows of this matrix along with the rows of the reduced incidence matrix of $\bigoplus_j \mathcal{G} \times E_j$ form a basis for $\mathcal{V}_v(\mathcal{G})$ (see Problem 7.9).
Building $(\tilde{\mathbf{R}}_{(k+1)1} : \cdots : \tilde{\mathbf{R}}_{(k+1)k})$ takes $O(\mid E \mid)$ time.
iii. The representative matrix $(\mathbf{R}_j : \mathbf{R}_{P_j})$ of $\mathcal{V}_{E_j P_j}$ is

$$\begin{pmatrix} \mathbf{R}_{jj} & \vdots & \mathbf{0} \\ \mathbf{R}_{(k+1)j} & \vdots & \mathbf{R}_{(k+1)T_j} \end{pmatrix}$$

where $\mathbf{R}_{(k+1)T_j}$ is a nonsingular submatrix of $\mathbf{R}_{(k+1)j}$ of full rank. We obtain the rows of $(\mathbf{R}_{(k+1)j} : \mathbf{R}_{(k+1)T_j})$ as follows:

Observe that the matrix $(\tilde{\mathbf{R}}_{(k+1)1} : \cdots : \tilde{\mathbf{R}}_{(k+1)k})$ is the reduced incidence matrix of the graph $\mathcal{G}'$ obtained by adding self loops t to the graph $\mathcal{G} \times (E-t)$. Let $(\mathbf{R}'_{(k+1)1} : \cdots : \mathbf{R}'_{(k+1)k})$ be the incidence matrix of $\mathcal{G}'$. Then, $\mathbf{R}'_{(k+1)j}$ is the incidence matrix of $\mathcal{G}' \cdot E_j$. Thus, $\mathbf{R}_{(k+1)j}$ can be taken to be the reduced incidence matrix of this graph. Let T_j be a forest of this graph. The columns corresponding to this set form a maximal linearly independent subset of columns of $\mathbf{R}_{(k+1)j}$. Let this submatrix of $\mathbf{R}_{(k+1)j}$ be denoted by $\mathbf{R}_{(k+1)T_j}$. The matrix $(\mathbf{R}_{(k+1)j} : \mathbf{R}_{(k+1)T_j})$ is the reduced incidence matrix of the graph $\mathcal{G}'_{E_jP_j}$ obtained from $\mathcal{G}' \cdot E_j$ by adding a forest P_j that is a copy of T_j to $\mathcal{G}' \cdot E_j$.
Building $\mathbf{R}_{(k+1)j}, j = 1, \cdots, k$ takes $O(\mid E \mid)$ time overall. Building the forests T_j and $\mathbf{R}_{(k+1)T_j},\ j = 1, \cdots, k$ takes $O(\mid E \mid)$ time overall.
iv. Let $\tilde{\mathbf{R}}_{(k+1)T_j}$ be the submatrix of $\tilde{\mathbf{R}}_{(k+1)j}$ corresponding to the columns T_j. Select $\tilde{\mathbf{R}}_{P_j} \equiv \tilde{\mathbf{R}}_{(k+1)T_j}$.
Building $\tilde{\mathbf{R}}_{P_j}, j = 1, \cdots, k$, clearly takes $O(\mid E \mid)$ time overall.
Thus, the **Algorithm (Port minimization 1) takes** $O(k \mid E \mid)$ **time**.
We now briefly speak of the sparsity of the above matrices. We saw in the above discussion that the matrices $\mathbf{R}_{jj}, \mathbf{R}_{(k+1)j}, \mathbf{R}_{(k+1)T_j}$ are reduced incidence matrices of appropriate graphs. Thus, the matrix

$$\left(\mathbf{R}_j \ \vdots \ \mathbf{R}_{P_j} \right) = \begin{pmatrix} \mathbf{R}_{jj} & \vdots & \mathbf{0} \\ \mathbf{R}_{(k+1)j} & \vdots & \mathbf{R}_{(k+1)T_j} \end{pmatrix}$$

has at most four nonzero entries per column in $\mathbf{R}_j$ and atmost two nonzero entries per column in $\mathbf{R}_{P_j}$. The matrix $\tilde{\mathbf{R}}_P$ is the reduced incidence matrix of a graph. So it has atmost two nonzero entries per column.

8.4.4 *Minimal Decomposition of Graphic Vector Spaces to make Component Spaces Graphic

We saw in the previous subsection that Algorithm (Port minimization 1) permits us to minimally decompose the voltage space of a graph in such a way that the coupler space is graphic. For such a vector space, it is not clear whether minimal multiport decomposition is possible with both component spaces and the coupler, graphic. However, in this subsection we give an algorithm which makes the component spaces graphic while losing control over the coupler space. In the interest of brevity we only sketch the justification.

ALGORITHM 8.2 (PORT MINIMIZATION 2)

INPUT *A connected directed graph $\mathcal{G}$ with a partition $\{E_1, \cdots, E_k\}$ of $E \equiv E(\mathcal{G})$. The space $\mathcal{V}_E$ to be decomposed is $\mathcal{V}_v(\mathcal{G})$.*

OUTPUT *Graphs $\mathcal{G}_{E_jP'_j}, j = 1, \cdots, k$ and a representative matrix $\tilde{\mathbf{R}}_{P'}$ of space $\mathcal{V}_{P'}$ s.t. $(\mathcal{V}_{E_1P'_1}, \cdots, \mathcal{V}_{E_kP'_k}; \mathcal{V}_{P'})$ is a minimal k-multiport decomposition of $\mathcal{V}_E$, where $\mathcal{V}_{E_jP'_j} \equiv \mathcal{V}_v(\mathcal{G}_{E_jP'_j})$.*

STEP 1 *Construct a reduced incidence matrix* $\hat{\mathbf{R}} \equiv (\hat{\mathbf{R}}_{E_1} \vdots \cdots \vdots \hat{\mathbf{R}}_{E_k})$ *of* $\mathcal{G}$*. This is a representative matrix for* $\mathcal{V}_E$*.*

STEP 2 *For j=1 to k, do the following:*
Construct a forest f^j *of* $\mathcal{G} \,.\, E_j$ *extending* f^j *to a forest* f *of* $\mathcal{G}$*.*
Extend $f - f^j$ *to a forest* f^{-j} *of* $\mathcal{G} \,.\, (E - E_j)$*.*
Let ${P_j}' \equiv f^{-j} - (f - f^j)$*. Contract* $(f - f^j)$ *in* $\mathcal{G}$ *and delete* $E - E_j - (f^{-j} - (f - f^j))$*.*
The resulting graph is on $E_j \cup {P_j}'$ *and will be denoted by* $\mathcal{G}_{E_j P_j'}$*.*
Take $\mathcal{V}_{E_j P_j'}$ *to be the voltage space of* $\mathcal{G}_{E_j P_j'}$*.*
A reduced incidence matrix of $\mathcal{G}_{E_j P_j'}$ *would be a representative matrix of* $\mathcal{V}_{E_j P_j'}$*.*

STEP 3 *Let* t_j *be a forest of* $\mathcal{G} \times E_j, j = 1, \cdots, k$*. Let* $t \equiv \bigcup t_j$*. Construct the graph* $\mathcal{G} \times (E - t)$*. Add branches of* t *as self loops to this graph. Call the resulting graph* $\mathcal{G}'$*. Select a forest* T_j *for* $\mathcal{G}' \cdot E_j, j = 1, \cdots, k$*.*
Let $T \equiv \bigcup T_j$*. Let* $\mathcal{G}_T \equiv \mathcal{G}' \cdot T$*. Let* $(\tilde{\mathbf{R}}_{T_1} \vdots \cdots \vdots \tilde{\mathbf{R}}_{T_k})$ *be a reduced incidence matrix of* $\mathcal{G}_T$*. In the graph* $\mathcal{G}_{E_j P_j'}$ *contract* t_j *and delete* $E_j \cup {P_j}' - (T_j \cup t_j)$*. The resulting graph is on* $T_j \cup {P_j}'$ *and is denoted by* $\mathcal{G}'_{T_j P_j'}$*. Let* $(\mathbf{I} \vdots\, \mathbf{Q}_{T_j P_j'})$ *be an f-cutset matrix of* $\mathcal{G}_{T_j P_j'}$ *with respect to the forest* T_j*. Let* $\tilde{\mathbf{R}}_{P_j'} \equiv (\tilde{\mathbf{R}}_{T_j})(\mathbf{Q}_{T_j P_j'}), j = 1, \cdots, k$ *and let* $\tilde{\mathbf{R}}_{P'} \equiv (\tilde{\mathbf{R}}_{P_1'} \vdots \cdots \vdots\, \tilde{\mathbf{R}}_{P_k'})$*.*
Output $\tilde{\mathbf{R}}_{P'}$ *as the representative matrix of* $\mathcal{V}_{P'}$*.*

STOP.

Justification of Algorithm (Port minimization 2)

We confine ourselves to a statement of the main steps in the justification, omitting details, in the interest of brevity. We need the following elementary lemma. This essentially states that if we apply the same nonsingular transformation on the 'P_j part' of vectors in $\mathcal{V}_{E_j P_j}$ and $\mathcal{V}_P$ the resulting spaces would still constitute a k-multiport decomposition of $\mathcal{V}_E$.

Lemma 8.4.3 *Let* $(\mathcal{V}_{E_1 P_1}, \cdots, \mathcal{V}_{E_k P_k}; \mathcal{V}_P)$ *be a k-multiport decomposition of* $\mathcal{V}_E$*. Let* $(\mathbf{R}_j \vdots \mathbf{R}_{P_j})$ *be a representative matrix of* $\mathcal{V}_{E_j P_j}, j = 1, \cdots, k$ *and let* $(\tilde{\mathbf{R}}_{P_1} \vdots \cdots \vdots \tilde{\mathbf{R}}_{P_k})$ *be a representative matrix of* $\mathcal{V}_P$*. Let* $\mathbf{K}_j$ *be a square nonsingular submatrix of size* $\mid P_j \mid$*. Let* $(\mathbf{R}_j \vdots \mathbf{R}_{P_j'})$ *be a representative matrix of* $\mathcal{V}_{E_j P_j'}$*,* $j = 1, \cdots, k$ *and let* $(\tilde{\mathbf{R}}_{P_1'} \vdots \cdots \vdots \tilde{\mathbf{R}}_{P_k'})$ *be a representative matrix of* $\mathcal{V}_{P'}$*,*
where $\mathbf{R}_{P_j'} = \mathbf{R}_{P_j}(\mathbf{K}_j)$ *and* $\tilde{\mathbf{R}}_{P_j'} = \tilde{\mathbf{R}}_{P_j}(\mathbf{K}_j)$
Then, $(\mathcal{V}_{E_1 P_1'}, \cdots, \mathcal{V}_{E_k P_k'}; \mathcal{V}_{P'})$ *is a k-multiport decomposition of* $\mathcal{V}_E$*.*

We omit the routine proof. It is essentially the same as that of Exercise 7.18.

We indicate below how the k-multiport decomposition output by Algorithm 8.2 (Port minimization 2) is related to that output by the adaptation of Algorithm 8.1 (Port minimization 1) (hereinafter called 'Modified algorithm (Port minimization 1)') given in Subsection 8.4.3, through nonsingular transformation of the P_j part of vectors in $\mathcal{V}_{E_jP_j}$ and $\mathcal{V}_P$. This would justify Algorithm (Port minimization 2). Since $\mid P' \mid = \mid P \mid$, it would also follow that the latter algorithm is minimal.

i. The graph $\mathcal{G}_j \equiv \mathcal{G}_{E_jP'_j} \cdot E_j$ of Algorithm (Port minimization 2) is 2-isomorphic to $\mathcal{G} \,.\, E_j$. This is because in the graph $\mathcal{G} \,.\, (E_j \cup (f - f^j))$, $(f - f^j)$ is a separator. Thus, contraction or deletion of $(f - f^j)$ will result in 2-isomorphic graphs. It is also easily seen that

$$\mathcal{G}_{E_jP'_j} \times E_j = \mathcal{G} \times E_j$$

ii. Since the graph $\mathcal{G}_{E_jP'_j}$ in the same algorithm is built so that

$$\mathcal{G}_{E_jP'_j} \cdot E_j = \mathcal{G} \,.\, E_j$$

$$\mathcal{G}_{E_jP_j} \times E_j = \mathcal{G} \times E_j,$$

it follows that,

$$\mathcal{V}_{E_jP'_j} \cdot E_j = \mathcal{V}_E \cdot E_j.$$

$$\mathcal{V}_{E_jP'_j} \times E_j = \mathcal{V}_E \times E_j$$

iii. In the Modified Algorithm (Port minimization 1)

$$\mathcal{V}_{E_jP_j} \cdot E_j = \mathcal{V}_E \cdot E_j$$

$$\mathcal{V}_{E_jP_j} \times E_j = \mathcal{V}_E \times E_j.$$

Now $\mid P_j \mid = \mid P'_j \mid$. Hence, a representative matrix of $\mathcal{V}_{E_jP'_j}$ can be obtained from that of $\mathcal{V}_{E_jP_j}$ by post multiplying columns P_j by a nonsingular matrix $\mathbf{K}_j$.
By Lemma 8.4.3, if

$$\tilde{\mathbf{R}}_{P'_j} = (\tilde{\mathbf{R}}_{P_j})\mathbf{K}_j, j = 1, \cdots, k,$$

and $\mathcal{V}_{P'}$ has the representative matrix $(\tilde{\mathbf{R}}_{P'_1} \vdots \cdots \vdots \tilde{\mathbf{R}}_{P'_k})$, then $(\mathcal{V}_{E_1P'_1}, \cdots, \mathcal{V}_{E_kP'_k}; \mathcal{V}_{P'})$ is a k-multiport decomposition of $\mathcal{V}_E$.

iv. We need to show that our computation of $\mathbf{K}_j$ is correct.
The graph $\mathcal{G}'$ of the Modified Algorithm (Port minimization 1) and the Algorithm (Port minimization 2) are identical. In Algorithm (Port minimization 2) let $\mathcal{G}'_{E_jP'_j}$ denote the graph obtained from $\mathcal{G}_{E_jP'_j}$ by contracting the forest t_j of $\mathcal{G}_{E_jP'_j} \times E_j$ ($\cong \mathcal{G} \times E_j$) and adding it as self loops. The graph $\mathcal{G}'_{E_jP'_j}$ is related to $\mathcal{G}'_{E_jP_j}$ of Modified Algorithm (Port minimization 1) as follows: In $\mathcal{G}'_{E_jP'_j}$, the set of edges ${P_j}'$

form a forest. If we delete this forest and replace it by a copy P_j of the forest T_j of $\mathcal{G}'_{E_j P'_j}$ we get $\mathcal{G}'_{E_j P_j}$. Let $\mathcal{G}'_{E_j P'_j P_j}$ denote the graph obtained from $\mathcal{G}'_{E_j P'_j}$ by adding P_j but not deleting $P_j{}'$. If $(\mathbf{A}_{rj}\ \mathbf{A}_{rP'_j}\ \mathbf{A}_{rP_j})$ is a reduced incidence matrix of this graph, there is no loss of generality in assuming that $(\mathbf{A}_{rj}\ \mathbf{A}_{rP_j})$ is the same as the matrix $(\mathbf{R}_{(k+1)j} \vdots \mathbf{R}_{(k+1)T_j})$ of Modified Algorithm (Port minimization 1). Further $(\mathbf{A}_{rj} \vdots \mathbf{A}_{rP'_j})$ is the reduced incidence matrix of the graph $\mathcal{G}'_{E_j P'_j}$. The matrix $\mathbf{K}_j$ is defined by $\mathbf{A}_{rP'_j} = (\mathbf{A}_{rP_j})\mathbf{K}_j$. Now $\mathbf{A}_{rP_j}$ is identical to the submatrix $\mathbf{A}_{rT_j}$ of $\mathbf{A}_{rj}$ corresponding to T_j. Thus, to compute K_j, we need only consider the matrix $(\mathbf{A}_{rT_j} \vdots \mathbf{A}_{rP'_j})$. This is the reduced incidence matrix of the graph $\mathcal{G}'_{T_j P'_j} \equiv \mathcal{G}'_{E_j P'_j} \cdot (T_j \cup P'_j)$. The column dependence structure of this matrix is identical to that of the f-cutset matrix $(\mathbf{I} \vdots \mathbf{Q}_{T_j P'_j})$ of this graph with respect to T_j. Hence,

$$(\mathbf{A}_{rT_j})\mathbf{Q}_{T_j P'_j} = \mathbf{A}_{rP'_j},\ i.e., \mathbf{K}_j = \mathbf{Q}_{T_j P'_j}.$$

Complexity of Algorithm (Port minimization 2)

Computation of $\mathbf{R}_{jj}, j = 1, \cdots, k$ is as in Modified Algorithm (Port minimization 1). This takes $O(k \mid E \mid)$ time overall. Computation of $(\mathbf{R}_j \vdots \mathbf{R}_{P'_j})$ (the representative matrix of $\mathcal{V}_{E_j P'_j}), j = 1, \cdots, k$ is $O(k \mid E \mid)$ since the graph operations to reach $\mathcal{G}_{E_j P'_j}, j = 1, \cdots, k$, are each $O(\mid E \mid)$. We need to examine more carefully the following:

(a) Construction of the matrices $\mathbf{K}_j$. This involves building an f-cutset matrix for $\mathcal{G}'_{T_j P'_j}$. This is $O(\Sigma \mid P_j{}' \mid^2)(= O(\Sigma \mid P_j \mid^2))$.

(b) Post multiplication of $\tilde{\mathbf{R}}_{P_j}$ by $\mathbf{K}_j,\ j = 1, \cdots, k$. Now $\tilde{\mathbf{R}}_{P_j}$ is a reduced incidence matrix. Multiplying $\mathbf{K}_j$ by a row of $\tilde{\mathbf{R}}_{P_j}$ (corresponding to a vertex v_i) is equivalent to adding some rows of $\mathbf{K}_j$ and subtracting some other rows. If q_i is the degree of v_i (the number of nonzero entries in the row), then the number of operations is $q_i(\mid P_j \mid)$. Thus, the product can be computed in $(\Sigma q_i) \mid P_j \mid$ steps. But $\Sigma q_i < 2 \mid P_j \mid$ since $\tilde{\mathbf{R}}_{P_j}$ is a reduced incidence matrix. Thus the product takes $O(\mid P_j \mid^2)$ steps and the overall multiplication takes $O(\Sigma \mid P_j \mid^2)$ steps.

Thus Algorithm (Port minimization 2) has time complexity $O(k \mid E \mid + \mid P_j \mid^2)$. It is slower than Algorithm (Port minimization 1) but would still be acceptable if $\mid P \mid \ll \mid E \mid$, which is often the case.

8.5 *Multiport Decomposition for Network Reduction

We saw in Subsection 8.3.2 that multiport decomposition can be used to solve a network $\mathcal{N}$ by
first solving electrical multiports in terms of suitable port variables, in the process obtaining constraints on port variables
next using these constraints as the device characteristic in another network $\mathcal{N}_P$ based on the coupler space, and
finally using the solution of the latter network to obtain the port variables and thence the solution of $\mathcal{N}$.
Thus, if the network $\mathcal{N}$ is on graph $\mathcal{G}$, $(\mathcal{V}_{E_1P_1}, \cdots \mathcal{V}_{E_kP_k}; \mathcal{V}_P)$ is a k-multiport decomposition of $\mathcal{V}_E \equiv \mathcal{V}_v(\mathcal{G})$, and if $E_1, \cdots E_k$ are decoupled in the device characteristic of $\mathcal{N}$, then we can define a reduced generalized network $\mathcal{N}_p \equiv (\mathcal{V}_P, \mathcal{D}_p)$ with respect to $\{E_1, \cdots, E_k\}$ that accurately mirrors the relationship between $E_1, \cdots, E_k$ in the network $\mathcal{N}$. We will consider an application of this idea informally.We assume the reader to be familiar with the nature of solutions of linear constant coefficient differential equations in terms of eigenvalues of the **A** matrix of the state equations.(For standard material on eigenvalues see [Hoffman+Kunze72]).

We begin with some preliminary definitions.
Let $\mathcal{N}$ be a network on graph $\mathcal{G}$ with device characteristic $\mathcal{D}$. Let $\{E_1, \cdots, E_k\}$ be a partition of $E(\mathcal{G})$ such that the E_i are decoupled in $\mathcal{D}$.
We say a solution $(\mathbf{v}, \mathbf{i})$ of $\mathcal{N}$ is **trapped** relative to E_j iff $\mathbf{v}(t)/E_j \in \mathcal{V}_v(\mathcal{G}) \times E_j$ and $\mathbf{i}(t)/E_j \in \mathcal{V}_i(\mathcal{G}) \times E_j$.
A solution that is not **trapped** for any of $E_1, \cdots, E_k$ is said to be **interactive** with respect to $\{E_1, \cdots, E_k\}$.
Network reduction by multiport decomposition can reveal information about interactive solutions in a compact manner. Indeed, a solution of the reduced network $\mathcal{N}_P$ will determine a solution of the original network uniquely modulo a trapped solution.

Trapped solutions in an RLMC network

In a source free RLMC network (i.e., a network with resistors, inductors with coupling and capacitors) when the RLMC matrices are positive definite, if initial conditions were specified, then the solution is unique. In the discussion that follows **we assume** that the **initial conditions are unknown** and then study the nature of the trapped solution.
Consider an RLMC network $\mathcal{N}$ on graph $\mathcal{G}$. Let $E(\mathcal{G})$ be partitioned into E_R, E_M, E_C corresponding to resistors, coupled inductors and capacitors. If the resistor matrix **R** is positive definite and a solution (**v**,**i**) of $\mathcal{N}$ is trapped relative to E_R then

$$\mathbf{v}_R(t) \equiv \mathbf{v}(t)/E_R = \mathbf{0} \textit{ and } \mathbf{i}_R(t) \equiv \mathbf{i}(t)/E_R = \mathbf{0}.$$

For, $< \mathbf{v}_R(t), \mathbf{i}_R(t) >= 0$ since spaces $\mathcal{V}_v(\mathcal{G}) \times E_R, \mathcal{V}_i(\mathcal{G}) \times E_R$ are orthogonal. On the other hand, since $\mathbf{v}_R(t) = \mathbf{R}\mathbf{i}_R(t)$ we must have

$$< \mathbf{v}_R(t), \mathbf{i}_R(t) >= (\mathbf{i}_R(t))^T(\mathbf{R})\mathbf{i}_R(t).$$

The RHS is zero only if $\mathbf{i}_R(t) = \mathbf{0}$ since $\mathbf{R}$ is positive definite. Thus $\mathbf{i}_R(t) = \mathbf{0}$ and $\mathbf{v}_R(t) = \mathbf{0}$.
If the capacitor matrix $\mathcal{C}$ is positive definite, then we can show that, solution $(\mathbf{v}, \mathbf{i})$ is trapped with respect to E_C iff $\mathbf{i}_C = \mathbf{0}$ and $\mathbf{v}_C(t) \in (\mathcal{V}_v(\mathcal{G})) \times E_C$ $(= \mathcal{V}_v(\mathcal{G} \times E_C))$, and further is constant in time. An example of this is where two capacitors are in series with nonzero initial voltages which are equal and opposite. The proof is similar to the resistor case discussed above. But here we work with $\dot{\mathbf{v}}_C$, $\mathbf{i}_C$ and show that $\dot{\mathbf{v}}_C(t) = \mathbf{0}$, $\mathbf{i}_C(t) = \mathbf{0}$ using the fact that $\mathcal{C}$ is a positive definite matrix.
If the mutual inductor matrix $\mathcal{L}$ is positive definite then solution $(\mathbf{v},\mathbf{i})$ is trapped with respect to E_L iff $\mathbf{v}_L = \mathbf{0}$ and $\mathbf{i}_L(t) \in (\mathcal{V}_i(\mathcal{G})) \times E_L = \mathcal{V}_i(\mathcal{G} \cdot E_L)$ and further is constant in time. An example of this is where inductors form a circuit and their initial currents circulate within the loop. To prove this fact we work with $\mathbf{v}_L, \dot{\mathbf{i}}_L$ and show that $\mathbf{v}_L(t) = \mathbf{0}$ and $\dot{\mathbf{i}}_l(t) = \mathbf{0}$, using the fact that $\mathcal{L}$ is a positive definite matrix and using arguments similar to the resistive case.

State equations for a linear network

We will now show how to use minimal multiport decomposition to write state equations for a circuit with capacitors, mutual inductors and nondynamic devices. A byproduct of our method is that we would get a reduced generalized network of the same kind whose solution mimics the solution of the original network except for the trapped solution corresponding to zero eigen values.
Let $\mathcal{N}$ be a linear network on graph $\mathcal{G}$. Let $E \equiv E(\mathcal{G})$ be partitioned into E_C, E_L, E_R. Let the voltage and current vectors associated with these edges be denoted by $\mathbf{v}_C, \mathbf{i}_C, \mathbf{v}_L, \mathbf{i}_L, \mathbf{v}_R, \mathbf{i}_R$. Let the device characteristic of $\mathcal{N}$ be given by

$$(\mathcal{C})\dot{\mathbf{v}}_C - \mathbf{i}_C = \mathbf{0}$$

$$(\mathcal{L})\dot{\mathbf{i}}_L - \mathbf{v}_L = \mathbf{0}$$

$$\mathbf{M}(\mathbf{v}_R - \mathbf{e}_R) + \mathbf{N}(\mathbf{i}_R - \mathbf{j}_R) = \mathbf{0}.$$

where $\mathcal{C}, \mathcal{L}$ are symmetric positive definite matrices.
We could use any algorithm for minimal 3-multiport decomposition of the space $\mathcal{V}_E \equiv \mathcal{V}_v(\mathcal{G})$. But for convenience we use the notation of Algorithm (Port minimization 1). Let $(\mathbf{R}_j \vdots \mathbf{R}_{P_j}), (\mathbf{B}_j \vdots \mathbf{B}_{P_j})$ be representative matrices of $\mathcal{V}_{E_j P_j}, \mathcal{V}^{\perp}_{E_j P_j}, j = C, L, R$. Let $(\tilde{\mathbf{R}}_{P_C} \vdots \tilde{\mathbf{R}}_{P_L} \vdots \tilde{\mathbf{R}}_{P_R})$ be the representative matrix of the space $\mathcal{V}_P$. The capacitor multiport equations are

$$\begin{pmatrix} \mathbf{R}_C & \vdots & \mathbf{R}_{P_C} \end{pmatrix} \begin{matrix} \mathbf{i}_C \\ \mathbf{i}_{P_C} \end{matrix} = \mathbf{0} \tag{8.14}$$

$$\mathcal{C}\dot{\mathbf{v}}_C - \mathbf{i}_C = \mathbf{0} \tag{8.15}$$

$$\begin{pmatrix} \mathbf{v}_C \\ \mathbf{v}_{P_C} \end{pmatrix} - \begin{pmatrix} \mathbf{R}_C^T \\ \mathbf{R}_{P_C}^T \end{pmatrix} \mathbf{v}_{n_C} = \mathbf{0} \tag{8.16}$$

These may be rewritten to obtain the relationship between $\mathbf{i}_{P_C}, \mathbf{v}_{P_C}$ and $\mathbf{v}_C$

$$\begin{aligned}
\mathbf{R}_{P_C}\ \mathbf{i}_{P_C} &= -\mathbf{R}_C\ \mathcal{C}\ \dot{\mathbf{v}}_C && (8.17)\\
&= -\mathbf{R}_C\ \mathcal{C}\ \mathbf{R}_C^T \dot{\mathbf{v}}_{n_C} && (8.18)\\
\mathbf{v}_{P_C} &= \mathbf{R}_{P_C}^T\ \mathbf{v}_{n_C} && (8.19)\\
\dot{\mathbf{v}}_{P_C} &= -\mathbf{R}_{P_C}^T (\mathbf{R}_C\ \mathcal{C}\ \mathbf{R}_C^T)^{-1}\ \mathbf{R}_{P_C}\ \mathbf{i}_{P_C} && (8.20)\\
&= -(\mathcal{C}_P)^{-1}\ \mathbf{i}_{P_C},\ \text{say.} && (8.21)
\end{aligned}$$

Since the decomposition is minimal, rows of $\mathbf{R}_C$ are linearly independent and so are columns of $\mathbf{R}_{P_C}$. It follows that since $\mathcal{C}$ is symmetric positive definite so are $(\mathbf{R}_C\ \mathcal{C}\ \mathbf{R}_C^T)^{-1}$ and $\mathbf{R}_{P_C}^T\ (\mathbf{R}_C\ \mathcal{C}\ \mathbf{R}_C^T)^{-1}\ \mathbf{R}_{P_C}$.
We call the inverse of this latter matrix $\mathcal{C}_P$.
The inductor multiport equations are

$$\begin{pmatrix} \mathbf{B}_L & \vdots & \mathbf{B}_{P_L} \end{pmatrix} \begin{matrix} \mathbf{v}_L \\ \mathbf{v}_{P_L} \end{matrix} = \mathbf{0} \tag{8.22}$$

$$\mathcal{L}\dot{\mathbf{i}}_L - \mathbf{v}_L = \mathbf{0} \tag{8.23}$$

$$\begin{pmatrix} \mathbf{i}_L \\ \mathbf{i}_{P_L} \end{pmatrix} - \begin{pmatrix} \mathbf{B}_L^T \\ \mathbf{B}_{P_L}^T \end{pmatrix} \mathbf{i}_{l_L} = \mathbf{0} \tag{8.24}$$

This may be rewritten to obtain the relationship between $\mathbf{i}_{P_L}, \mathbf{v}_{P_L}$ and $\mathbf{i}_L$

$$\begin{aligned}
\mathbf{B}_{P_L}\ \mathbf{v}_{P_L} &= -\mathbf{B}_L\ (\mathcal{L})\ \dot{\mathbf{i}}_L && (8.25)\\
&= -\mathbf{B}_L\ (\mathcal{L})\ \mathbf{B}_L^T \dot{\mathbf{i}}_{l_L} && (8.26)\\
\mathbf{i}_{P_L} &= \mathbf{B}_{P_L}^T\ \mathbf{i}_{l_L} && (8.27)\\
\dot{\mathbf{i}}_{P_L} &= -\mathbf{B}_{P_L}^T (\mathbf{B}_L\ \mathcal{L}\ \mathbf{B}_L^T)^{-1}\ \mathbf{B}_{P_L}\ \mathbf{v}_{P_L} && (8.28)\\
&= -(\mathcal{L}_P)^{-1}\ \mathbf{v}_{P_L}\ \text{say.} && (8.29)
\end{aligned}$$

Since the decomposition is minimal, rows of $\mathbf{B}_L$ are linearly independent and so are columns of $\mathbf{B}_{P_L}$. It follows that since $\mathcal{L}$ is symmetric positive definite so are $(\mathbf{B}_L\ \mathcal{L}\ \mathbf{B}_L^T)^{-1}$ and $\mathbf{B}_{P_L}^T\ (\mathbf{B}_L\ \mathcal{L}\ \mathbf{B}_L^T)^{-1}\ \mathbf{B}_{P_L}$.
We call the inverse of this latter matrix $\mathcal{L}_P$.
The resistor multiport equations are

$$\begin{pmatrix} \mathbf{R}_R & \vdots & \mathbf{R}_{P_R} \end{pmatrix} \begin{matrix} \mathbf{i}_R \\ \mathbf{i}_{P_R} \end{matrix} = \mathbf{0} \tag{8.30}$$

$$\mathbf{M}(\mathbf{v}_R - \mathbf{e}_R) + \mathbf{N}(\mathbf{i}_R - \mathbf{j}_R) = \mathbf{0} \tag{8.31}$$

$$\begin{pmatrix} \mathbf{v}_R \\ \mathbf{v}_{P_R} \end{pmatrix} - \begin{pmatrix} \mathbf{R}_R^T \\ \mathbf{R}_{P_R}^T \end{pmatrix} \mathbf{v}_{n_R} = \mathbf{0}. \tag{8.32}$$

Let us suppose that these latter can be equivalently written as

$$\begin{pmatrix} \mathbf{v}_R \\ \mathbf{i}_R \end{pmatrix} = \begin{pmatrix} \mathbf{K}_{vv} & \mathbf{K}_{vi} \\ \mathbf{K}_{iv} & \mathbf{K}_{ii} \end{pmatrix}^{\mathbf{v}_{P_R}}_{\mathbf{i}_{P_R}} + \begin{pmatrix} \mathbf{s}_{v_R} \\ \mathbf{s}_{i_R} \end{pmatrix}. \tag{8.33}$$

$$\mathbf{M}_P(\mathbf{v}_{P_R} - \mathbf{e}_{P_R}) + \mathbf{N}_P(\mathbf{i}_{P_R} - \mathbf{j}_{P_R}) = \mathbf{0}. \tag{8.34}$$

(We will now work with a generalized network as defined in page 283- the network $\mathcal{N}_P$ is defined in Subsection 8.3.2). Consider a network $\mathcal{N}'_p \equiv (\mathcal{V}_P, \mathcal{D}'_p)$ where $P = P_C \uplus P_L \uplus P_R$
and the device characteristic $\mathcal{D}'_p$ is defined by

$$\mathcal{C}_P \dot{\mathbf{v}}'_{P_C} = \mathbf{i}'_{P_C}$$

$$\mathcal{L}_P \dot{\mathbf{i}}'_{P_L} = \mathbf{v}'_{P_L}$$

$$-\mathbf{M}_P(\mathbf{v}'_{P_R} + \mathbf{e}_{P_R}) + \mathbf{N}_P(\mathbf{i}'_{P_R} - \mathbf{j}_{P_R}) = \mathbf{0}.$$

We take $\mathbf{v}'_{P_C} \equiv -\mathbf{v}_{P_C}$, $\mathbf{v}'_{P_R} \equiv -\mathbf{v}_{P_R}$, $\mathbf{v}'_{P_L} \equiv -\mathbf{v}_{P_L}$, and $\mathbf{i}'_P \equiv \mathbf{i}_P$. This network differs from $\mathcal{N}_P$ in that the voltages of $\mathcal{N}'_P$ are negatives of the voltages of $\mathcal{N}_P$ but the currents in both the networks are the same. We use $\mathcal{N}'_P$ instead of $\mathcal{N}_P$ since the former is of the same kind as $\mathcal{N}$.
Let the state equations for this network be written as in Section 5.7.
Let these equations be

$$\begin{bmatrix} \dot{\mathbf{v}}'_{P_C} \\ \dot{\mathbf{i}}'_{P_L} \end{bmatrix} = \begin{bmatrix} \mathbf{A}'_{CC} & \mathbf{A}'_{CL} \\ \mathbf{A}'_{LC} & \mathbf{A}_{LL} \end{bmatrix} \begin{matrix} \mathbf{v}'_{P_C} \\ \mathbf{i}'_{P_L} \end{matrix} + \begin{bmatrix} \mathbf{B}'_{C\mathcal{E}} & \mathbf{B}'_{C\mathcal{J}} \\ \mathbf{B}'_{L\mathcal{E}} & \mathbf{B}'_{L\mathcal{J}} \end{bmatrix} \begin{matrix} \mathbf{e}_R \\ \mathbf{j}_R \end{matrix} \tag{8.35}$$

Let the output equations be

$$[\mathbf{y}_P] = \begin{bmatrix} \mathbf{C}'_{P_C} & \mathbf{C}'_{P_L} \end{bmatrix} \begin{matrix} \mathbf{v}'_{P_C} \\ \mathbf{i}'_{P_L} \end{matrix} + \begin{bmatrix} \mathbf{D}'_{\mathcal{E}} & \mathbf{D}'_{\mathcal{J}} \end{bmatrix} \begin{matrix} \mathbf{e}_R \\ \mathbf{j}_R \end{matrix} \tag{8.36}$$

where $\mathbf{y}_p$ includes variables such as $\mathbf{i}_{P_C}, \mathbf{v}_{P_L}, \mathbf{i}_{P_R}, \mathbf{v}_{P_R}$. Here we have assumed that $\mathbf{v}'_{P_C}, \mathbf{i}'_{P_L}$ do not become dependent in the network $\mathcal{N}'_P$ when the resistive device characteristic alone is used (and P_C, P_L are treated as voltage sources and current sources respectively). This is done only for notational convenience.

$\mathcal{N}'_P$ contains the dynamics of $\mathcal{N}$

In the discussion to follow we relate the solutions of $\mathcal{N}_P$ and $\mathcal{N}$ without assuming that the initial conditions are specified.
We will show that Equation 8.35 contains a description of the 'essential dynamics' of the network $\mathcal{N}$. We have, from Equations 8.16 and 8.24

$$\begin{bmatrix} \mathbf{R}^T_C \\ \mathbf{R}^T_{P_C} \end{bmatrix} \mathbf{v}_{n_C} = \begin{bmatrix} \mathbf{v}_C \\ \mathbf{v}_{P_C} \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{B}^T_L \\ \mathbf{B}^T_{P_L} \end{bmatrix} \mathbf{i}_{l_L} = \begin{bmatrix} \mathbf{i}_L \\ \mathbf{i}_{P_L} \end{bmatrix}.$$

Note that knowledge of $\mathbf{v}_{n_C}(t)$ and $\mathbf{i}_{l_L}(t)$ gives us complete knowledge of the dynamics of the network.
Decompose $\mathbf{v}_{n_C}(t)$ into two orthogonal components $\mathbf{v}^1_{n_C}(t)$ and $\mathbf{v}^2_{n_C}(t)$, where $\mathbf{R}^T_{P_C}(\mathbf{v}^1_{n_C}(t)) = \mathbf{0}$ and $\mathbf{v}^2_{n_C}(t)$ is spanned by the columns of $\mathbf{R}_{P_C}$. Similarly decompose $\mathbf{i}_{l_L}(t)$ into two orthogonal components $\mathbf{i}^1_{l_L}(t)$ and $\mathbf{i}^2_{l_L}(t)$ where $\mathbf{B}^T_{P_L}(\mathbf{i}^1_{l_L}(t)) = \mathbf{0}$ and $\mathbf{i}^2_{l_L}(t)$ is spanned by columns of $\mathbf{B}_{P_L}$.
We will show that $\mathbf{v}^2_{n_C}(t), \mathbf{i}^2_{l_L}(t)$ are uniquely determinable from $\mathbf{v}_{P_C}(t)$, $\mathbf{i}_{P_L}(t)$. The ambiguity in obtaining $\mathbf{v}_{n_C}(t), \mathbf{i}_{l_L}(t)$ from the latter variables is contained in $\mathbf{v}^1_{n_C}(t)$ and $\mathbf{i}^1_{l_L}(t)$.
We now show that $\mathbf{v}^1_{n_C}(t)$ and $\mathbf{i}^1_{l_L}(t)$ correspond to trapped constant solutions. We prove only the $\mathbf{v}^1_{n_C}$ case. The other case is similar (dual).
Let us split $\mathbf{v}_C$ into $\mathbf{v}^1_C$ and $\mathbf{v}^2_C$ where $\mathbf{v}^1_C(t) = (\mathbf{R}^T_C)\mathbf{v}^1_{n_C}(t)$. We then have

$$\begin{bmatrix} \mathbf{v}^1_C(t) \\ \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathbf{R}^T_C \\ \mathbf{R}^T_{P_C} \end{bmatrix} \mathbf{v}^1_{n_C}(t).$$

So $\mathbf{v}^1_C(t)$ and therefore,$\dot{\mathbf{v}}^1_C(t)$, also belong to $\mathcal{V}_{E_C P_C} \times E_C$, i.e., to $\mathcal{V}_E \times E_C$ (using Theorem 8.4.1 since the multiport decomposition is minimal).Hence, $< \mathbf{i}_C(t), \dot{\mathbf{v}}^1_C(t) >$ $= 0$, i.e., $< \mathcal{C}\dot{\mathbf{v}^1}_C(t), \dot{\mathbf{v}}^1_C(t) >= 0$. The matrix $\mathcal{C}$ is positive definite. Hence, $\dot{\mathbf{v}}^1_C(t) = \mathbf{0}$ and therefore, $\mathbf{i}^1_C(t) = \mathbf{0}$. Thus, $\mathbf{v}^1_C(t)$ is a constant and $(\mathbf{v}^1_C \oplus \mathbf{0}_L \oplus \mathbf{0}_R$, $\mathbf{0}_C \oplus \mathbf{0}_L \oplus \mathbf{0}_R)$ is a trapped solution. Similarly one can show that ${\mathbf{i}_L}^1(t)$ is constant (where ${\mathbf{i}_L}^1(t) = \mathbf{B}^T_L \mathbf{i}^1_{l_L}$) and $(\mathbf{0}_C \oplus \mathbf{0}_L \oplus \mathbf{0}_R$, $\mathbf{0}_C \oplus \mathbf{i}_L \oplus \mathbf{0}_R)$ is a trapped solution.
To obtain $\mathbf{v}^2_{n_C}(t)$ from $\mathbf{v}_{P_C}(t)$ we proceed as follows:

$$\mathbf{v}^2_{n_C}(t) = (\mathbf{R}_{P_C})\mathbf{k}(t) \qquad \textit{say}.$$

Hence, $\mathbf{v}_{P_C}(t) = ({\mathbf{R}_{P_C}}^T \mathbf{R}_{P_C})\mathbf{k}(t)$. Now $\mathbf{R}_{P_C}$ has linearly independent columns, since by the minimality of the decomposition, $r(\mathcal{V}_{E_C P_C} \cdot P_C) =\mid P_C \mid$. Hence,

$$\mathbf{v}^2_{n_C}(t) = \mathbf{R}_{P_C}({\mathbf{R}_{P_C}}^T \mathbf{R}_{P_C})^{-1}\mathbf{v}_{P_C}(t).$$

Similarly one can show that

$$\mathbf{i}^2_{l_L}(t) = \mathbf{B}_{P_L}(\mathbf{B}^T_{P_L}\ \mathbf{B}_{P_L})^{-1}\mathbf{i}_{P_L}(t).$$

The trapped solutions corresponding to $\mathbf{v}^1_{n_C}, \mathbf{i}^1_{l_L}$ are constant solutions of linear constant coefficient differential equations which can exist in the absence of inputs. They are of the form ke^{ot} and therefore, correspond to zero eigen values.
For completeness we show how to obtain the overall state equations. We have

$$\dot{\mathbf{v}}_{n_C}(t) = \dot{\mathbf{v}}^2_{n_C}(t) = \mathbf{R}_{P_C}({\mathbf{R}_{P_C}}^T \mathbf{R}_{P_C})^{-1}\dot{\mathbf{v}}_{P_C}(t)$$

$$\dot{\mathbf{i}}_{l_L}(t) = \dot{\mathbf{i}}^2_{l_L}(t) = \mathbf{B}_{P_L}(\mathbf{B}^T_{P_L}\ \mathbf{B}_{P_L})^{-1}\dot{\mathbf{i}}_{P_L}(t)$$

$(\dot{\mathbf{v}}^1_{n_C}(t), \dot{\mathbf{i}}^1_{l_L}(t)$ are zero since $\dot{\mathbf{v}}^1_C(t), \dot{\mathbf{i}}^1_L(t)$ are zero, $\mathbf{v}^1_C(t) = (\mathbf{R}^T_C)\mathbf{v}^1_{n_C}(t)$, $\mathbf{i}^1_L(t) = (\mathbf{B}^T_L)\mathbf{i}^1_{l_L}(t)$, and the coefficient matrices $\mathbf{R}^T_C, \mathbf{B}^T_L$ have full column rank).

Now in Equation 8.35, $\dot{\mathbf{v}}'_{P_C}, \dot{\mathbf{i}}'_{P_L}$ have been expressed as time invariant linear functions of $\mathbf{v}'_{P_C}, \mathbf{i}'_{P_L}, \mathbf{e}_R$ and $\mathbf{j}_R$. Now since $\mathbf{v}_{P_C} = -\mathbf{v}'_{P_C}, \mathbf{i}_{P_L} = \mathbf{i}'_{P_L}$ we can express $\dot{\mathbf{v}}_{n_C}, \dot{\mathbf{i}}_{l_L}$ as time invariant linear functions of $\mathbf{v}_{P_C}, \mathbf{i}_{P_L}, \mathbf{e}_R$ and $\mathbf{j}_R$. But $\mathbf{v}_{P_C} = \mathbf{R}_{P_C}{}^T \mathbf{v}_{n_C}$ and $\mathbf{i}_{P_L} = \mathbf{B}^T_{P_L} \mathbf{i}_{l_L}$. Thus, $\dot{\mathbf{v}}_{n_C}, \dot{\mathbf{i}}_{l_L}$ are expressed as time invariant linear functions of $\mathbf{v}_{n_C}, \mathbf{i}_{l_L}, \mathbf{e}_R$ and $\mathbf{j}_R$. These would be the required state equations.

The number of state variables that we have obtained equals the number of entries in $\mathbf{v}_{n_C}$ and $\mathbf{i}_{l_L}$. Clearly this equals $r(\mathcal{V}_{E_C P_C}) + r(\mathcal{V}^{\perp}_{E_L P_L})$. Now, $r(\mathcal{V}_{E_C P_C}) = r(\mathcal{V}_{E_C P_C} \cdot E_C) = r(\mathcal{V}_E \cdot E_C)$ since the multiport decomposition is minimal (Theorem 8.4.1). But $r(\mathcal{V}_E \cdot E_C) = r(\mathcal{V}_v(\mathcal{G} \ . \ E_C)) = r(\mathcal{G} \ . \ E_C)$. Similarly,

$$\begin{aligned} r(\mathcal{V}^{\perp}_{E_L P_L}) &= r(\mathcal{V}^{\perp}_E \cdot E_L) \\ &= r(\mathcal{V}_i(\mathcal{G} \times E_L)) \\ &= \nu(\mathcal{G} \times E_L). \end{aligned}$$

If in the network $\mathcal{N}'_P$, when P_C, P_L are treated as voltage sources and current sources, the voltages in $\mathbf{v}'_{P_C}$ and currents in $\mathbf{i}'_{P_L}$ are independent, then it is clear from the above discussion on writing state equations in terms of the variables $\mathbf{v}_{n_C}, \mathbf{i}_{l_L}$, using equation 8.35, that $\mathbf{v}_{n_C}, \mathbf{i}_{l_L}$ can have their initial conditions arbitrary. Thus, $r(\mathcal{G} \ . \ E_C) + \nu(\mathcal{G} \times E_L)$ is the least number of state variables for this network.

If $\mathcal{N}$ has the device characteristic of E_R of the form

$$-\mathbf{G}(\mathbf{v}_R - \mathbf{e}_R) + (\mathbf{i}_R - \mathbf{j}_R) = \mathbf{0},$$

where $\mathbf{G}$ is positive definite, then one can show that $\mathcal{N}'_P$ has the device characteristic of P_R of the form

$$-\mathbf{G}_P(\mathbf{v}_{P_R} - \mathbf{e}_{P_R}) + (\mathbf{i}_{P_R} - \mathbf{j}_{P_R}) = \mathbf{0},$$

where $\mathbf{G}_P$ is positive definite. In this case, $\mathcal{N}'_P$ has no constant solution and therefore no zero eigen values (see Problem 8.7). Thus, all the zero eigen values of the network $\mathcal{N}$ are concentrated in the trapped solutions. For the above case, the number of zero eigen values of $\mathcal{N}$

$$\begin{aligned} &= r(\mathcal{V}_E \times E_C) + r(\mathcal{V}^{\perp}_E \times E_L) \\ &= r(\mathcal{G} \times E_C) + \nu(\mathcal{G} \ . \ E_L). \end{aligned}$$

8.6 Problems

Problem 8.1 Given two decompositions to get a decomposition of one of the couplers: *Let* $((\mathcal{V}_{E_j P_j})_k, \mathcal{V}_P), ((\mathcal{V}_{E_j Q_j})_k; \mathcal{V}_Q)$ *be decompositions of* $\mathcal{V}_E$ *and further let the former be compatible. Then,* $((\mathcal{V}_{Q_j P_j})_k; \mathcal{V}_Q)$, *where* $\mathcal{V}_{Q_j P_j} \equiv (\mathcal{V}_{E_j P_j} \leftrightarrow \mathcal{V}_{E_j Q_j}), j = 1, \cdots, k,$ *is a compatible decomposition of* $\mathcal{V}_P$.

Problem 8.2 In a compatible decomposition the generalized hybrid rank of $\mathcal{V}_E$ and $\mathcal{V}_P$ are the same: *The generalized hybrid rank of a vector space $\mathcal{V}_T$ relative to a partition $\{T_1, \cdots, T_k\}$, of T equals $min_{\mathcal{V}'_T}$ $\{d(\mathcal{V}_T, \mathcal{V}'_T)\}$, where $\mathcal{V}'_T$ is a vector space on T which has $T_j, j = 1, \cdots, k$ as separators. We remind the reader that*

$$d(\mathcal{V}_T, \mathcal{V}'_T) = r(\mathcal{V} + \mathcal{V}'_T) - r(\mathcal{V}_T \cap \mathcal{V}'_T).$$

i. *Let $\mathcal{V}_E$ be a vector space on E. Let E be partitioned into $\{E_1, \cdots, E_k\}$. Let $\hat{\mathcal{V}}_E$ have $E_j, j = 1, \cdots, k$, as separators and further be such that*

$$d(\mathcal{V}_E, \hat{\mathcal{V}}_E) \leq d(\mathcal{V}_E, \mathcal{V}'_E)$$

whenever $\mathcal{V}'_E$ has the E_j as separators. Then

$$\bigoplus_j (\mathcal{V}_E \cdot E_j) \supseteq \hat{\mathcal{V}}_E \supseteq (\bigoplus \mathcal{V}_E \times E_j).$$

ii. *Let $(\mathcal{V}_{E_1P_1}, \cdots, \mathcal{V}_{E_kP_k}; \mathcal{V}_P)$ be a compatible decomposition of $\mathcal{V}_E$. Then (the generalized hybrid rank of $\mathcal{V}_E$ relative to $\{E_1, \cdots, E_k\}$) equals (the generalized hybrid rank of $\mathcal{V}_P$ relative to $\{P_1, \cdots, P_k\}$).*

Problem 8.3 *Let $(\mathcal{G}_{E_1P_1}, \cdots, \mathcal{G}_{E_kP_k}; \mathcal{G}_P)$ be a* **strongly compatible** *multiport decomposition $\mathcal{G}$, i.e.,*
$(\mathcal{V}_{E_1P_1}, \cdots, \mathcal{V}_{E_kP_k}; \mathcal{V}_P)$, where $\mathcal{V}_{E_jP_j} \equiv \mathcal{V}_v(\mathcal{G}_{E_jP_j}), j = 1, \cdots, k$, $\mathcal{V}_P \equiv \mathcal{V}_v(\mathcal{G}_P)$, is a strongly compatible decomposition of $\mathcal{V}_E \equiv \mathcal{V}_v(\mathcal{G})$.
Justify the following algorithm for building a minimal k-multiport decomposition starting from the above decomposition.

ALGORITHM 8.3 *(***Port minimization 3***)*

STEP 1 *Construct graphs $\mathcal{G}_{E_jP_j} \times P_j$, $j = 1, \cdots, k$. Let $t_j, j = 1, \cdots, k$ respectively be forests of these graphs.*

STEP 2 *Construct graphs $\mathcal{G}_{E_jP_j} \cdot P_j$, $j = 1, \cdots, k$. Let $L_j, j = 1, \cdots, k$ respectively be the coforests of these graphs such that $L_j \cap t_j = \emptyset, j = 1, \cdots, k$.*
Let $Q_j = P_j - (t_j \cup L_j), Q = \bigcup Q_j$,
$\mathcal{G}_{E_jQ_j} = \mathcal{G}_{E_jP_j} \times (E_j \cup (P_j - t_j)) \cdot (E_j \cup (P_j - t_j - L_j))$
$\mathcal{G}_Q = \mathcal{G}_P \times (P - \bigcup t_j) \cdot (P - \bigcup t_j - \bigcup L_j)$
$(\mathcal{G}_{E_1Q_1}, \cdots, \mathcal{G}_{E_kQ_k}; \mathcal{G}_Q)$ is a minimal k-multiport decomposition of $\mathcal{G}$.

STOP

Problem 8.4 Fast algorithm for minimal graphic 2-decomposition: *Let $\mathcal{G}$ be a graph on E, and let $E = E_1 \uplus E_2$. Let P_1 be a copy of E_2, and P_2, a copy of E_1. Let $\mathcal{G}_{E_1E_2} \equiv \mathcal{G}$ and let $\mathcal{G}_{E_1P_1}, \mathcal{G}_{E_2P_2}, \mathcal{G}_{P_1P_2}$ all be copies of $\mathcal{G}$.*

i. *Show that $(\mathcal{G}_{E_1P_1}, \mathcal{G}_{E_2P_2}; \mathcal{G}_{P_1P_2})$ is a strongly compatible decomposition of $\mathcal{G}$..*

ii. *Use the algorithm of Problem 8.3 to obtain a minimal graphic 2-decomposition of $\mathcal{G}$.*

Problem 8.5 ***$\mathcal{N}_{AL} - \mathcal{N}_{BK}$ method and 2-decomposition:** *Show that the $\mathcal{N}_{AL} - \mathcal{N}_{BK}$ method (see Section 6.4) can be regarded as a special case of network analysis through decomposition into two multiports.*

Problem 8.6 ***Formal description of network reduction:** *Let $\mathcal{N} \equiv (\mathcal{G}, \mathcal{D})$ be a network. Let $E \equiv E(\mathcal{G})$ be partitioned into $\{E_1, \cdots, E_k\}$ such that the E_i are decoupled in the device characteristic, i.e., $\mathcal{D} = D_{E_1} \times \cdots \times \mathcal{D}_{E_k}$. Let $((\mathcal{V}_{E_jP_j})_k; \mathcal{V}_P)$ be a minimal k-multiport decomposition of $\mathcal{V}_E$.*
Let multiport networks $\mathcal{N}_{E_jP_j} \equiv (\mathcal{V}_{E_jP_j}, \mathcal{D}_{E_j} \times \delta_{P_j}),\ j = 1, \cdots, k$.
Define $\mathcal{D}'_{P_j},\ j = 1, \cdots, k$ as follows:
$(\mathbf{v}'_{P_j}, \mathbf{i}'_{P_j}) \in \mathcal{D}'_{P_j}$ iff there exist vectors $\mathbf{v}_{E_j}, \mathbf{i}_{E_j}$ s.t. $(\mathbf{v}_{E_j} \oplus (-\mathbf{v}'_{P_j}), \mathbf{i}_{E_j} \oplus \mathbf{i}'_{P_j})$ is a solution of $\mathcal{N}_{E_jP_j}$.
Let $\mathcal{N}'_P \equiv (\mathcal{V}_P, \mathcal{D}'_{P_1} \times \cdots \mathcal{D}'_{P_k})$

i. *Let $(\mathbf{v}, \mathbf{i})$ be a solution of $\mathcal{N}$. Then there is a unique solution $(\mathbf{v}'_P, \mathbf{i}'_P)$ of $\mathcal{N}'_P$ which corresponds to $(\mathbf{v}, \mathbf{i})$ s.t.*

$$(\mathbf{v}/E_j \oplus (-\mathbf{v}'_P/P_j), \mathbf{i}/E_j \oplus \mathbf{i}'_P/P_j)$$

is a solution of $\mathcal{N}_{E_jP_j},\ j = 1, \cdots, k$.

ii. **The reverse process is not quite unique:** *Given a solution $(\mathbf{v}'_P, \mathbf{i}'_P)$ of $\mathcal{N}'_P$, a corresponding solution in the above sense is* **unique** *within a pair $(\hat{\mathbf{v}}, \hat{\mathbf{i}})$ where $\hat{\mathbf{v}} \in \bigoplus_j \mathcal{V}_E \times E_j$ and $\hat{\mathbf{i}} \in \bigoplus_j \mathcal{V}_E^{\perp} \times E_j$.*

iii. *Let $((\mathcal{V}_{E_jQ_j})_k, \mathcal{V}_Q)$ be a compatible (not necessarily minimal) k-multiport decomposition of $\mathcal{V}_E$. Define $\mathcal{N}'_Q$ and relate its solutions to those of $\mathcal{N}'_P$.*

Problem 8.7 ***Minimal reduced network for an RLC network has no zero eigen values:** *Show that the network $\mathcal{N}'_P$ defined in Section 8.5 has*

i. *no nonzero trapped solution if the sources have zero values.*

ii. *all eigenvalues nonzero.*

Problem 8.8 ***If the decomposition is compatible $\mathcal{V}_E, \mathcal{V}_P$ have essentially the same polymatroid:** *Let $(\mathcal{V}_{E_jP_j})_k; \mathcal{V}_P)$ be a k-multiport compatible decomposition of $\mathcal{V}_E$, i.e.,*

$$\mathcal{V}_{E_jP_j} \cdot P_j \supseteq \mathcal{V}_P \cdot P_j \quad \textit{and}$$

$$\mathcal{V}_{E_jP_j} \times P_j \subseteq \mathcal{V}_P \times P_j, \quad j = 1, \cdots, k$$

Define the following set functions on subsets of $\{1, 2 \cdots k\}$

$$\begin{aligned}\rho_E(I) &= r(\mathcal{V}_E \cdot (\bigcup_{i \in I} E_i)) \\ \omega_E(I) &= \sum_{i \in I} r(\mathcal{V}_{E_iP_i} \times E_i)\end{aligned}$$

$$\rho_P(I) = r(\mathcal{V}_P \cdot (\bigcup_{i\in I} P_i))$$
$$\omega_P(I) = \sum_{i\in I} r(\mathcal{V}_{E_iP_i} \times P_i).$$

Show that

i. *ρ_E, ρ_P are polymatroid rank functions while ω_E, ω_P are modular functions.*

ii. *$\rho_E - \omega_E = \rho_P - \omega_P$*

Problem 8.9 *Is the generalized minor operation matroidal?:
Let $\mathcal{V}_{EP} \leftrightarrow \mathcal{V}_P = \mathcal{V}_E$. Show that

i. *if the vector spaces are over GF2 then $\mathcal{M}(\mathcal{V}_E)$ can be determined from $\mathcal{M}(\mathcal{V}_{EP}), \mathcal{M}(\mathcal{V}_P)$;*

ii. *in general, knowledge of $\mathcal{M}(\mathcal{V}_{EP}), \mathcal{M}(\mathcal{V}_P)$ would not be sufficient for us to determine $\mathcal{M}(\mathcal{V}_E)$.*

8.7 Solutions of Exercises

E 8.1:
i. We denote E_1 by E. By Theorem 8.2.2 we have

$$\mathcal{V}_{EP_1} \cdot E \supseteq \mathcal{V}_{E'} \cdot E$$

$$\mathcal{V}_{EP_1} \times E \subseteq \mathcal{V}_{E'} \times E$$

Thus, $r(\mathcal{V}_{EP_1} \cdot E) - r(\mathcal{V}_{EP_1} \times E) \geq r(\mathcal{V}_{E'} \cdot E) - r(\mathcal{V}_{E'} \times E)$
But the LHS $= r(\mathcal{V}_{EP_1} \cdot P_1) - r(\mathcal{V}_{EP_1} \times P_1) \leq\mid P_1 \mid$.
Hence, $\mid P_1 \mid\geq r(\mathcal{V}_{E'} \cdot E) - r(\mathcal{V}_{E'} \times E)$.
Next suppose the graph $\mathcal{G}'$ is obtained by adding a copy t' of a tree t of $\mathcal{G}$ again to $\mathcal{G}$. We have $r(\mathcal{V}_{E'} \cdot E) - r(\mathcal{V}_{E'} \times E) = r(\mathcal{G})$. So we cannot do with number of ports less than this number. On the other hand this number of ports is adequate. For, from these port voltages all voltages of the graph are uniquely determined.
More formally, use of Algorithm (Port minimisation 2) will allow us to have a multiport decomposition of $\mathcal{V}_v(\mathcal{G}')$ in which $\mathcal{V}_{EP_1}$ is the voltage space of a graph $\mathcal{G}_{EP_1}$ with P_1 containing **no circuits** or **cutsets**. Hence a copy of a tree of $\mathcal{G}$ is always sufficient to act as ports in $\mathcal{V}_{EP_1}$. The structure of the tree does not matter by Lemma 8.4.3.

ii. We require $r(\mathcal{G})$ ports in general. For each component we could add a tree of ports on the set of nodes of the component.

iii. If $(\mathcal{V}_{E_1P_1}, \cdots \mathcal{V}_{E_kP_k}; \mathcal{V}_P)$ is a multiport decomposition of $\mathcal{V}_v(\mathcal{G}')$ then we know that $(\mathcal{V}^{\perp}_{E_1P_1}, \cdots \mathcal{V}^{\perp}_{E_kP_k}; \mathcal{V}^{\perp}_P)$ is a multiport decomposition of $\mathcal{V}_i(\mathcal{G}')$. So the arguments of the previous sections of this problem are adequate to prove that a copy of a forest of $\mathcal{G}$ would be adequate as ports in general.

E 8.2:
i. Consider the equation

$$\begin{pmatrix} \overset{E_1}{\mathbf{A}_1} & \vdots & \overset{E-E_1}{\mathbf{A}_2} \end{pmatrix} \begin{matrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{matrix} = \mathbf{0}$$

By row transformation we can transform these into an equivalent form

$$\begin{pmatrix} \mathbf{A}_{11} & \mathbf{0} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \\ \mathbf{0} & \mathbf{A}_{32} \end{pmatrix} \begin{matrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{matrix} = \mathbf{0} \tag{8.37}$$

where $\mathbf{A}_{11}, \mathbf{A}_{32}$ are representative matrices of $\mathcal{V} \times E_1, \mathcal{V} \times (E - E_1)$, respectively. The rows of $\mathbf{A}_{21}, \mathbf{A}_{22}$ are linearly independent and in number equal to $r(\mathcal{V} \;.\; E_1) - r(\mathcal{V} \times E_1)$. If $\mathbf{A}_{11}\hat{\mathbf{x}}_1 \neq \mathbf{0}$, then $S_2(\hat{\mathbf{x}}_1) = \emptyset$. So let us assume that $\mathbf{A}_{11}\hat{\mathbf{x}}_1 = \mathbf{0}$. Now we can rewrite Equation 8.37 as shown below as far as $\hat{\mathbf{x}}_1, \mathbf{x}_2$ are concerned.

$$\begin{pmatrix} \mathbf{A}_{21} & \mathbf{A}_{22} \\ \mathbf{0} & \mathbf{A}_{32} \end{pmatrix} \begin{matrix} \hat{\mathbf{x}}_1 \\ \mathbf{x}_2 \end{matrix} = \mathbf{0}. \tag{8.38}$$

The affine space $S_2(\hat{\mathbf{x}}_1)$ is determined by $\mathbf{A}_{21}\hat{\mathbf{x}}_1$. So if $\mathbf{p}_1 \equiv \mathbf{A}_{21}\hat{\mathbf{x}}_1$ we do not require a vector of number of entries greater than that of $\mathbf{p}_1$ to determine $S_2(\hat{\mathbf{x}}_1)$.
One cannot do with a vector in place of $\mathbf{p}_1$ with less number of entries provided that vector is obtained from $\mathbf{x}_1$ by a linear transformation.To prove this we proceed as follows: The rows of $\mathbf{A}_{21}$ are linearly independent. So for each $\mathbf{p}_1$ there exists a corresponding $\hat{\mathbf{x}}_1$. Also for different values of $\mathbf{p}_1$ the spaces S_2 would be disjoint. Given a value of $\mathbf{x}_2, \mathbf{p}_1$ is determined uniquely. Suppose there is a matrix $\mathbf{M}$ s.t. $\mathbf{y} = \mathbf{M}\mathbf{x}_1$ determines $S_2(\mathbf{x}_1)$ uniquely. Since $S_2(\mathbf{x}_1)$ determines $\mathbf{A}_{21}(\mathbf{x}_1)$ uniquely $\mathbf{A}_{21}(\mathbf{x}_1) = f(\mathbf{y})$ for some function $f(.)$. It is easily seen that $f(.)$ must be a linear transformation. If $\mathcal{V}_p, \mathcal{V}_y$ represent the range spaces of $\mathbf{A}_{21}$ and $\mathbf{M}$ then we have an onto linear transformation $f : (\mathcal{V}_y) \to \mathcal{V}_p$. Thus $dim(\mathcal{V}_y) \geq dim(\mathcal{V}_P)$ as required.

ii. For $(\mathbf{B}_1 \vdots \mathbf{B}_2)^{\mathbf{z}_1}_{\mathbf{z}_2} = \mathbf{0}$ the minimum number of entries of a vector to determine the affine space of vectors $\mathbf{z}_2$ knowing $\mathbf{z}_1$ can be similarly shown to be

$$r(\mathcal{V}^{\perp} \cdot E_1) - r(\mathcal{V}^{\perp} \times E_1) = r(\mathcal{V} \;.\; E_1) - r(\mathcal{V} \times E_1).$$

iii. If we decompose $\mathcal{V}_E$ into $(\mathcal{V}_{E_1P_1}, \mathcal{V}_{E_2P_2}; \mathcal{V}_P)$ where $E_2 = E - E_1$ then the minimum size of P_2 equals $(r(\mathcal{V} \;.\; E_1) - r(\mathcal{V} \times E_1))$. P_2 represents E_1 as far as E_2 is concerned, in the space $\mathcal{V}_{E_2P_2}$. If $\mathbf{x} \in \mathcal{V}_E$ and we are given $\mathbf{x}/E_1$ then the range of possible values $\mathbf{x}/E_2$ can take can be determined as follows:
First find all possible vectors $\mathbf{x}/E_1 \oplus \mathbf{x}_{P_1}$ in $\mathcal{V}_{E_1P_1}$. If the decomposition is minimal there would be only one such vector. Let us for simplicity assume the decomposition is minimal.

Figure 8.5: A Graph and its Multiport Decomposition

Next find the vector $\mathbf{x}_{P_1} \oplus \mathbf{x}_{P_2}$ in $\mathcal{V}_{P_1P_2}$.
Finally find the collection of vectors $\mathbf{x}_{E_2} \oplus \mathbf{x}_{P_2}$ in $\mathcal{V}_{E_2P_2}$.The restriction of this collection to E_2 gives the range of poissible values that $\mathbf{x}/E_2$ can take.

E 8.3: The decomposition shown in Figure 8.5 provides such an example.

E 8.4:
i. In $\mathcal{G}_{E_1Q_1}$ of Figure 8.2, the currents entering e_1, e_4 are always negatives of each other. However, in $\mathcal{G}_{EP}$, because of the circulating current in $\{e_4, e_5, e_6, e_7\}$, this condition is not **necessarily satisfied**.

ii. The port conditions of $\mathcal{G}_{E_1Q_1}, \mathcal{G}_{E_2Q_2}$ would be satisfied in $\mathcal{G}_{EP}$ iff the current vectors that can exist in $\{e_1, \cdots, e_{10}\}$ in $\mathcal{G}_{EP}$ can also exist in $\mathcal{G}_{E_1Q_1}, \mathcal{G}_{E_2Q_2}$. Equivalently, iff

$$\mathcal{V}_i(\mathcal{G}_{EP} \times E) \subseteq \oplus_i \mathcal{V}_i(\mathcal{G}_{E_iQ_i} \times E_i).$$

The easiest way to check this is to see if each row of the incidence matrix of the graph on the right can be obtained by adding appropriate rows of the incidence matrix on the left.
More generally, one can check if the rows of a circuit matrix of $\mathcal{G}_{EP} \times E$ are orthogonal to an incidence matrix of $\oplus \mathcal{G}_{E_iQ_i} \times E_i$.

Remark: In practice we need to know whether the port conditions of electrical multiports $\mathcal{N}_{E_1P_1}, \mathcal{N}_{E_2P_2}$ **with devices present** are satisfied in the multiport $\mathcal{N}_{EQ}$ **with devices present**. However, the above procedure is still useful. This is because the characteristics of electrical devices, except those of short circuits and open circuits, cannot be constructed exactly. So one performs the above test after fusing the end points of short circuit branches and deleting open circuit branches. If the test succeeds then of course the connection is permissible. If it fails, the probability of the possible current solution vectors of $\mathcal{N}_{E_1P_1} \oplus \mathcal{N}_{E_2P_2}$ (restricted to E) lying in the set of possible current solution vectors of $\mathcal{N}_{EQ}$ (restricted to E) can be seen to be zero. This can be shown by using the fact that if one picks a vector randomly out of a vector space, the probability of its lying in a given proper subspace is zero.

E 8.5: Let $\mathcal{V}_{EP} \equiv \bigoplus_j \mathcal{V}_{E_jP_j}$. Then $\mathcal{V}_E = \mathcal{V}_{EP} \leftrightarrow \mathcal{V}_P$.
Hence, $(\mathcal{V}_E \leftrightarrow \mathcal{V}_Q) = (\mathcal{V}_{EP} \leftrightarrow \mathcal{V}_P) \leftrightarrow \mathcal{V}_Q$.
Since $P \cap Q = \emptyset$, the RHS can be written as $(\mathcal{V}_{EP} \leftrightarrow \mathcal{V}_Q) \leftrightarrow \mathcal{V}_P$.
Now $\mathcal{V}_Q = \mathcal{V}_{Q_1} \oplus \cdots \oplus \mathcal{V}_{Q_k}$, where $Q_j \subseteq E_j, j = 1, \cdots, k$.
It is then easily seen that

$$(\bigoplus_j \mathcal{V}_{E_jP_j}) \leftrightarrow (\bigoplus_j V_{Q_j}) = \bigoplus_j (\mathcal{V}_{E_jP_j} \leftrightarrow \mathcal{V}_{Q_j}).$$

The result follows.
Next we have

$$\mathcal{V}_E \times S \cdot T = \mathcal{V}_E \leftrightarrow \mathcal{V}_Q,$$

where $Q = E - T$ and $\mathcal{V}_Q$ has the representative matrix $\begin{array}{c} \scriptstyle E-S \quad S-T \\ [\ \mathbf{0} \quad \mathbf{I}\ ] \end{array}$. It is clear that $\mathcal{V}_Q$ is the direct sum of spaces $\mathcal{V}_{Q_j}$ whose representative matrices are given by $\begin{array}{c} \scriptstyle E_j-S_j \quad S_j-T_j \\ [\ \mathbf{0} \quad \mathbf{I}\ ] \end{array}$. Hence

$$\mathcal{V}_{E_jP_j} \leftrightarrow \mathcal{V}_{Q_j} = \mathcal{V}_{E_jP_j} \times (S_j \cup P_j) \cdot (T_j \cup P_j).$$

Since we have shown above that

$$\mathcal{V}_E \leftrightarrow \mathcal{V}_Q = (\bigoplus_j (\mathcal{V}_{E_jP_j} \leftrightarrow \mathcal{V}_{Q_j})) \leftrightarrow \mathcal{V}_P,$$

the required result about decomposition of a minor follows.

E 8.6: Let $\bigoplus_j \mathbf{f}_{E_j}, \bigoplus_j \mathbf{f}_{E_j}{}' \in \mathcal{V}_E$. We will show that for any i, $(\bigoplus_{j \neq i} \mathbf{f}_{E_j}) \oplus \mathbf{f}_{E_j}{}' \in \mathcal{V}_E$. This is clearly equivalent to showing that $\mathcal{V}_E = \bigoplus(\mathcal{V}_E \cdot E_j)$. We have, $(\bigoplus_j \mathcal{V}_{E_jP_j}) \leftrightarrow \mathcal{V}_P = \mathcal{V}_E$. So there exist $\bigoplus_j \mathbf{f}_{P_j}, \bigoplus_j \mathbf{f}_{P_j}{}' \in \mathcal{V}_P$ s.t. $\mathbf{f}_{E_j} \oplus \mathbf{f}_{P_j} \in \mathcal{V}_{E_jP_j}$ and $\mathbf{f}_{E_j}{}' \oplus \mathbf{f}_{P_j}{}' \in \mathcal{V}_{E_jP_j}, j = 1, \cdots, k$.
Now $\bigoplus_{j \neq i} \mathbf{f}_{P_j} \oplus \mathbf{f}_{P_i}{}' \in \mathcal{V}_P$, since $\mathcal{V}_P = \bigoplus_j (\mathcal{V}_P \cdot P_j)$.
Further, $\mathbf{f}_{E_j} \oplus \mathbf{f}_{P_j} \in \mathcal{V}_{E_jP_j}, j \neq i$
and $\mathbf{f}'_{E_i} \oplus \mathbf{f}_{P_i}{}' \in \mathcal{V}_{E_iP_i}$.
Hence, $(\bigoplus_{j \neq i} \mathbf{f}_{E_j}) \oplus \mathbf{f}^{'}_{E_i} \in \mathcal{V}_E$.

E 8.7: Let $I_k \equiv \{1, \cdots, k\}$.

i. Let $\bigoplus_{j\in I_1} \mathbf{f}_{E_j}$ be a vector in the LHS. Then there exists a vector

$$(\bigoplus_{j\in I_1} \mathbf{f}_{E_j}) \oplus (\bigoplus_{j\in I_1} \mathbf{f}_{P_j})$$

in $\bigoplus_{j\in I_1} \mathcal{V}_{E_jP_j}$ and a vector $\bigoplus_{j\in I_1} \mathbf{f}_{P_j}$ in $\mathcal{V}_P \cdot P_{I_1}$. Now

$$\bigoplus_{j\in I_1} \mathbf{f}_{P_j} \in (\mathcal{V}_P \cdot P_{I_1}) \subseteq ((\bigoplus_{j\in I_k} \mathcal{V}_{E_jP_j}) \cdot P_{I_1}).$$

Hence there exists a vector

$$\bigoplus_{j\in I_k} \mathbf{f}_{P_j} \in \mathcal{V}_P \subseteq ((\bigoplus_{j\in I_k} \mathcal{V}_{E_jP_j}) \cdot P)$$

We would then have

$$\bigoplus_{j\in I_2} \mathbf{f}_{P_j} \in (\mathcal{V}_P \cdot P_{I_2}) \subseteq ((\bigoplus_{j\in I_2} \mathcal{V}_{E_jP_j}) \cdot P_{I_2})$$

where $I_2 = \{1, \cdots, k\} - I_1$
Hence there exists a vector

$$((\bigoplus_{j\in I_2} \mathbf{f}_{E_j}) \oplus (\bigoplus_{j\in I_2} \mathbf{f}_{P_j})) \in \bigoplus_{j\in I_2} \mathcal{V}_{E_jP_j}.$$

Thus there exists a vector

$$((\bigoplus_{j\in I_k} \mathbf{f}_{E_j}) \oplus (\bigoplus_{j\in I_k} \mathbf{f}_{P_j})) \in \bigoplus_{j\in I_k} \mathcal{V}_{E_jP_j}.$$

It is therefore clear that $(\bigoplus_{j\in I_k} \mathbf{f}_{E_j}) \in \mathcal{V}_E$ and hence, $(\bigoplus_{j\in I_1} \mathbf{f}_{E_j}) \in \mathcal{V}_E \cdot E_{I_1}$. Thus LHS $\subseteq$ RHS. The reverse containment is easier to see.

ii. Using compatibility we have, $((\bigoplus_{j\in I_1} \mathcal{V}^{\perp}_{E_jP_j}) \leftrightarrow \mathcal{V}^{\perp}_P \cdot P_{I_1}) = \mathcal{V}^{\perp}_E \cdot E_{I_1}$.
Hence, $((\bigoplus_{j\in I_1} \mathcal{V}^{\perp}_{E_jP_j})^{\perp} \leftrightarrow (\mathcal{V}^{\perp}_P \cdot P_{I_1})^{\perp} = (\mathcal{V}^{\perp}_E \cdot E_{I_1})^{\perp}$
This is clearly equivalent to the required result.

iii. This is a direct consequence of the previous two sections of this problem.

iv. We verify the compatibility condition only for the first section of this problem. We have

$$\mathcal{V}_{E_jP_j} \cdot P_j \supseteq \mathcal{V}_P \cdot P_j = (\mathcal{V}_P \cdot P_{I_1}) \cdot P_j, j \in I_1$$

$$\mathcal{V}_{E_jP_j} \times P_j \subseteq \mathcal{V}_P \times P_j \subseteq (\mathcal{V}_P \cdot P_{I_1}) \times P_j, j \in I_1$$

E 8.8: Figure 8.2 shows component multiports $\mathcal{G}_{E_1Q_1}, \mathcal{G}_{E_2Q_2}$ and a port connection diagram $\mathcal{G}_{QP}$ such that when the multiports are connected according to the latter the graph $\mathcal{G}_{EP}$ shown in the figure results. But the voltage space of this graph $\mathcal{G}_{EP}$ cannot be decomposed into the voltage spaces of the given component multiports and the port connection diagram. The reason is that KVE corresponding to $\{e_4, e_5, e_6, e_7\}$ in $\mathcal{G}_{EP}$ is not indirectly imposed through the component multiports and the port connection diagram. Equivalently, if $\mathcal{V}_{EP} = (\mathcal{V}_v(\mathcal{G}_{E_1}Q_1) \oplus \mathcal{V}_v(\mathcal{G}_{E_2}Q_2)) \leftrightarrow \mathcal{V}_v(\mathcal{G}_{QP})$, it is clear that $\mathcal{V}_{EP} \neq \mathcal{V}_v(\mathcal{G}_{EP})$.

E 8.9: We have

$$\mathcal{V}_{EQ} = (\bigoplus_j(\bigoplus_i \mathcal{V}_{E_{ji}T_{ji}} \leftrightarrow \mathcal{V}_{T_jP_j})) \leftrightarrow \mathcal{V}_{PQ} \qquad (*)$$

We first prove the following

Lemma 8.7.1

$$(\mathcal{V}_{S_1T_1} \leftrightarrow \mathcal{V}_{T_1P_1}) \oplus (\mathcal{V}_{S_2T_2} \leftrightarrow \mathcal{V}_{T_2P_2}) = (\mathcal{V}_{S_1T_1} \oplus \mathcal{V}_{S_2T_2}) \leftrightarrow (\mathcal{V}_{T_1P_1} \oplus \mathcal{V}_{T_2P_2})$$

where $S_1, S_2, T_1, T_2, P_1, P_2$ are all pairwise disjoint.

Proof : : A vector $\mathbf{g}_{SP} \in$ L.H.S. iff there exist

$$\mathbf{g}_{S_1} \oplus \mathbf{g}_{T_1}, \mathbf{g}_{T_1} \oplus \mathbf{g}_{P_1}, \mathbf{g}_{S_2} \oplus \mathbf{g}_{T_2}, \mathbf{g}_{T_2} \oplus \mathbf{g}_{P_2}$$

belonging respectively to $\mathcal{V}_{S_1T_1}, \mathcal{V}_{T_1P_1}, \mathcal{V}_{S_2T_2}, \mathcal{V}_{T_2P_2}$ s.t.

$$\mathbf{g}_{SP} = \mathbf{g}_{S_1} \oplus \mathbf{g}_{S_2} \oplus \mathbf{g}_{P_1} \oplus \mathbf{g}_{P_2}$$

i.e., iff there exist vectors

$$\mathbf{g}_{S_1} \oplus \mathbf{g}_{T_1} \oplus \mathbf{g}_{S_2} \oplus \mathbf{g}_{T_2} \in \mathcal{V}_{S_1T_1} \oplus \mathcal{V}_{S_2T_2}, \quad \mathbf{g}_{T_1} \oplus \mathbf{g}_{P_1} \oplus \mathbf{g}_{T_2} \oplus \mathbf{g}_{P_2} \in \mathcal{V}_{T_1P_1} \oplus \mathcal{V}_{T_2P_2}$$

s.t. $\mathbf{g}_{SP} = \mathbf{g}_{S_1} \oplus \mathbf{g}_{S_2} \oplus \mathbf{g}_{P_1} \oplus \mathbf{g}_{P_2}$. The result follows.

□

Applying this lemma to the RHS of $(*)$ we get

$$\mathcal{V}_{EQ} = ((\bigoplus_{j,i} \mathcal{V}_{E_{ji}T_{ji}}) \leftrightarrow (\bigoplus_j \mathcal{V}_{T_jP_j})) \leftrightarrow \mathcal{V}_{PQ}$$

It is easy to see in general that

$$(\mathcal{V}_{ET} \leftrightarrow \mathcal{V}_{TP}) \leftrightarrow \mathcal{V}_{PQ} = \mathcal{V}_{ET} \leftrightarrow (\mathcal{V}_{TP} \leftrightarrow \mathcal{V}_{PQ})$$

where E, T, P, Q are pairwise disjoint.

Thus,

$$\mathcal{V}_{EQ} = (\bigoplus_{j,i} \mathcal{V}_{E_{ji}T_{ji}}) \leftrightarrow ((\bigoplus_j \mathcal{V}_{T_jP_j}) \leftrightarrow \mathcal{V}_{PQ}).$$

E 8.10:

i. As discussed in Section 8.3, when we solve a multiport in terms of $\mathbf{i}_{P_{j1}}$, $\mathbf{v}_{P_{j2}}$ the essential structure of the equations corresponds to $\mathcal{V}_{E_jP_j} \times (E_j \cup P_{j1}) \cdot E_j$. If $\mathcal{V}_{E_jP_j}$ is the voltage space of a graph $\mathcal{G}_{E_jP_j}$ then this structure corresponds to

$\mathcal{G}_{E_jP_j} \times (E_j \cup P_{j1}) \cdot E_j$. If we have freedom in choosing P_{j1}, P_{j2} we should choose that partition which gives us a large number of separators, preferably of uniform size, for the space $\mathcal{V}_{E_jP_j} \times (E_j \cup P_{j1}) \cdot E_j$. If the separators $E_{j1}, \cdots, E_{jt}$ of this space are decoupled in the device characteristic we would have an advantage during analysis.

ii. It can be observed, in Figure 8.4, that if P_{11} were shorted and P_{12} opened we get two separators. The other three options: opening both, shorting both, opening P_{11} and shorting P_{12} do not give this advantage. Therefore, while solving for $\mathcal{N}_{E_1P_1}$ we should solve in terms of $\mathbf{v}_{P_{11}}, \mathbf{i}_{P_{12}}$.

iii. Shorting all the port edges gives us the structure $\mathcal{V}_{E_jP_j} \times E_j$. Deleting all port edges gives us the structure $\mathcal{V}_{E_jP_j} \cdot E_j$.
Now if we perform nonsingular transformations on the columns P_j of representative matrices of $\mathcal{V}_{E_jP_j}$ and the same transformations also on columns P_j of representative matrices of $\mathcal{V}_P$, it can be seen that we would get a new multiport decomposition of $\mathcal{V}_E$. For any vector space $\mathcal{V}'_E$, s.t.

$$\bigoplus_j (\mathcal{V}_{E_jP_j} \times E_j) \subseteq \mathcal{V}'_E \subseteq \bigoplus_j (\mathcal{V}_{E_jP_j} \cdot E_j)$$

we can find a $\mathcal{V}_{P'}$ s.t.

$$\bigoplus_j \mathcal{V}_{EjP'_j} \leftrightarrow \mathcal{V}_{P'} = \mathcal{V}'_E$$

(using Exercise 7.5). By column transformation if required we can convert the representative matrix of $\mathcal{V}_{P'}$ to the form $[\mathbf{0}\ \mathbf{I}]$. We may, therefore, without loss of generality assume that the representative matrix has this form. But this implies that the generalized minor operation is equivalent to an ordinary minor operation. Further, since $\mathcal{V}_{EP'}$ has the form $\bigoplus_j \mathcal{V}_{E_jP'_j}$ this would correspond to performing a minor operation on each $\mathcal{V}_{E_jP'_j}$.
Thus, for any $\mathcal{V}'_E = \oplus_j \mathcal{V}'_{E_j}$ s.t.

$$\bigoplus_j (\mathcal{V}'_{E_jP_j} \times E_j) \subseteq \mathcal{V}'_E \subseteq \bigoplus_j (\mathcal{V}_{E_jP_j} \cdot E_j)$$

We can find a set of transformed ports P' as well as a partition of each $P_j{}'$ s.t.

$$\mathcal{V}'_E = \bigoplus_j \mathcal{V}'_{E_j} = \bigoplus_j (\mathcal{V}_{E_jP'_j} \times (E_j \cup P_j{}') \cdot E_j)$$

If we use minimal multiport decomposition we must have

$$\begin{array}{lrcl} & \mathcal{V}_{E_jP_j} \times E_j & = & \mathcal{V}_E \times E_j \\ \textit{and} & \mathcal{V}_{E_jP_j} \cdot E_j & = & \mathcal{V}_E \cdot E_j \end{array}$$

and the range of possible structures lies between $\mathcal{V}_E \times E_j$ and $\mathcal{V}_E \cdot E_j$.

E 8.11:

i. We need to show that $((\mathcal{V}_{E_jP_j})_k;\mathcal{V}_P)$ is a compatible decomposition iff

$$\bigoplus_j \mathcal{V}_{E_jP_j}\cdot P \supseteq \mathcal{V}_P \qquad (*)$$

$$\bigoplus_j \mathcal{V}_{E_jP_j}\times P \subseteq \mathcal{V}_P. \qquad (**)$$

Only if part is obvious. To prove the **if** part we observe that

$$(\bigoplus_j \mathcal{V}_{E_jP_j})\cdot P = \bigoplus_j (\mathcal{V}_{E_jP_j}\cdot P_j).$$

So the condition $(*)$ implies that $\mathcal{V}_{E_jP_j}\cdot P_j \supseteq \mathcal{V}_P\cdot P_j \quad \forall j$. The condition $(**)$ implies

$$\bigoplus_j (\mathcal{V}^{\perp}_{E_jP_j})\cdot P \supseteq \mathcal{V}^{\perp}_P$$

from which we conclude

$$\mathcal{V}^{\perp}_{E_jP_j}\cdot P_j \supseteq \mathcal{V}^{\perp}_P\cdot P_j \quad \forall j$$

and therefore, $\mathcal{V}_{E_jP_j}\times P_j \subseteq \mathcal{V}_P\times P_j \quad \forall j.$

ii. (a) By the previous section of this problem we have that compatibility is equivalent to the conditions $(*)$ and $(**)$. But by Theorem 7.6.1 in Problem 7.5 these conditions are equivalent to

$$\mathcal{V}_P = (\bigoplus_j \mathcal{V}_{E_jP_j}) \leftrightarrow \mathcal{V}_E$$

along with

$$\mathcal{V}_{E_jP_j}\cdot E_j \supseteq \mathcal{V}_E\cdot E_j \quad \forall j \qquad (\surd)$$

and

$$\mathcal{V}_{E_jP_j}\times E_j \subseteq \mathcal{V}_E\times E_j \quad \forall j \qquad (\surd\surd)$$

Now conditions $(\surd)$ and $(\surd\surd)$ are equivalent to the compatibility of the decomposition $((\mathcal{V}_{E_jP_j})_k,\mathcal{V}_E)$ of $\mathcal{V}_P$.
The 'strongly compatible' case is proved similarly.

ii. (b) By the Implicit Duality Theorem $((\mathcal{V}_{E_jP_j})_k;\mathcal{V}_P)$ is a decomposition of $\mathcal{V}_E$ iff $((\mathcal{V}^{\perp}_{E_jP_j})_k;\mathcal{V}^{\perp}_P)$ is a decomposition of $\mathcal{V}^{\perp}_E$. Compatibility of the decomposition of $\mathcal{V}^{\perp}_E$ follows since $\mathcal{V}_{E_jP_j}\cdot P_j \supseteq \mathcal{V}_P\cdot P_j$ is equivalent to $\mathcal{V}^{\perp}_{E_jP_j}\times P_j \subseteq \mathcal{V}^{\perp}_P\times P_j$ and $\mathcal{V}_{E_jP_j}\times P_j \subseteq \mathcal{V}_P\times P_j$ is equivalent to $\mathcal{V}^{\perp}_{E_jP_j}\cdot P_j \supseteq \mathcal{V}^{\perp}_P\cdot P_j$.

iii. The equivalence of the strong compatibility of the decompositions $((\mathcal{V}_{E_jP_j})_k;\mathcal{V}_P)$ and $((\mathcal{V}^{\perp}_{E_jP_j})_k;\mathcal{V}^{\perp}_P)$ is proved similar to (ii) (b) above.

iv. We have $\mathcal{V}_E \equiv (\bigoplus_j \mathcal{V}_{E_jP_j}) \leftrightarrow \mathcal{V}_P$. Let $\mathcal{V}'_E \equiv (\bigoplus_j \mathcal{V}_{E_jP'_j}) \leftrightarrow \mathcal{V}_{P'}$, $\hat{\mathbf{f}}_{P_i} \in \mathcal{V}_{E_iP_i}\times P_i$.

We will first show that $\mathcal{V}_E = \mathcal{V}'_E$.
Let $\mathbf{f}_E \in \mathcal{V}_E$. Then $\mathbf{f}_E = \bigoplus \mathbf{f}_{E_j}$ and there exists $\mathbf{f}_{P_j}$ s.t. $\mathbf{f}_P \equiv \bigoplus_j \mathbf{f}_{P_j} \in \mathcal{V}_P$ and $\mathbf{f}_{E_j} \oplus \mathbf{f}_{P_j} \in \mathcal{V}_{E_jP_j}$ for each j. By strong compatibility, $\mathcal{V}_{E_iP_i} \times P_i = \mathcal{V}_P \times P_i$. Hence $\mathbf{O}_{P-P_i} + \hat{\mathbf{f}}_{P_i}$ belongs to $\mathcal{V}_P$. Let $\hat{\mathbf{f}}_P$ denote this vector. Suppose $\mathbf{f}_{P_i}(e) = \lambda\hat{\mathbf{f}}_{P_i}(e)$. Then, $\mathbf{f}_P - \lambda\hat{\mathbf{f}}_P \in \mathcal{V}_P$. Let $\mathbf{f}_{P'} \equiv \mathbf{f}_P - \lambda\hat{\mathbf{f}}_P/P'$. Since $(\mathbf{f}_P - \lambda\hat{\mathbf{f}}_P)(e) = 0$, it is clear that $\mathbf{f}_{P'} \in \mathcal{V}_{P'}$. We know that $\hat{\mathbf{f}}_{P_i} \in \mathcal{V}_{E_iP_i} \times P_i$. Thus, $\mathbf{f}_{E_i} \oplus (\mathbf{f}_{P_i} - \lambda\hat{\mathbf{f}}_{P_i}) \in \mathcal{V}_{E_iP_i}$. But $(\mathbf{f}_{P_i} - \lambda\hat{\mathbf{f}}_{P_i})(e) = 0$. So $\mathbf{f}_{E_i} \oplus (\mathbf{f}_{P'}/P'_i) \in \mathcal{V}_{E_iP'_i}$. Noting that $P_j = P'_j \ \ \forall j \neq i$, it is clear that $\bigoplus_j \mathbf{f}_{E_j} \in \mathcal{V}'_E$. So $\mathcal{V}_E \subseteq \mathcal{V}'_E$. The reverse containment is easier to see.
Next we need to show that the decomposition $((\mathcal{V}_{E_jP'_j})_k; \mathcal{V}_P)$,where $P'_j = P_j \ \ \forall j \neq i$, is strongly compatible.
A vector $\mathbf{f}_{P'_i} \in \mathcal{V}_{E_iP'_i} \times P'_i$ iff $\mathbf{f}_{P'_i} \in \mathcal{V}_{E_iP_i} \times (E_i \uplus P'_i) \times P'_i (= \mathcal{V}_{E_iP_i} \times P_i \times P'_i)$, i.e., iff $\mathbf{0}_e + \mathbf{f}_{P'_i} \in \mathcal{V}_{E_iP_i} \times P_i$.
Now by strong compatibility of $((\mathcal{V}_{E_jP_j})_k; \mathcal{V}_P)$ we have $\mathcal{V}_{E_iP_i} \times P_i = \mathcal{V}_P \times P_i$.
Hence, $\mathbf{f}_{P'_i} \in \mathcal{V}_{E_iP'_i} \times P'_i$ iff $\mathbf{0}_e + \mathbf{f}_{P'_i} \in \mathcal{V}_P \times P_i$, i.e., iff $\mathbf{f}_{P'_i} \in \mathcal{V}_P \times P_i \times P'_i$, i.e., iff $\mathbf{f}_{P'_i} \in \mathcal{V}_P \times P' \times P'_i$, i.e., iff $\mathbf{f}_{P'_i} \in \mathcal{V}_{P'} \times P'_i$.
Next, a vector $\mathbf{f}_{P'_i} \in \mathcal{V}_{E_iP'_i} \cdot P'_i$ iff $\mathbf{f}_{P'_i} \in \mathcal{V}_{E_iP_i} \times (E_i \uplus P'_i) \cdot P'_i$, i.e., iff $\mathbf{f}_{P'_i} \in \mathcal{V}_{E_iP_i} \cdot P_i \times P'_i$, i.e., iff $\mathbf{f}_{P'_i} \in \mathcal{V}_P \cdot P_i \times P'_i$ (using strong compatibility), i.e., iff $\mathbf{f}_{P'_i} \in \mathcal{V}_P \times P' \cdot P'_i (= \mathcal{V}_{P'} \cdot P'_i)$. This proves the strong compatibility of $((\mathcal{V}_{E_jP'_j})_k; \mathcal{V}_P)$.

v. It is clear by the previous section of this problem that $((\mathcal{V}^{\perp}_{E_jQ_j})_k; \mathcal{V}^{\perp}_Q)$ is a strongly compatible decomposition of $\mathcal{V}^{\perp}_E$. (Observe that $\mathcal{V}^{\perp}_{E_iQ_i} = \mathcal{V}^{\perp}_{E_iP_i} \times (E_i \cup Q_i)$). But by an earlier section of the present problem this is equivalent to $((\mathcal{V}_{E_jQ_j})_k, \mathcal{V}_Q)$ being a strongly compatible decomposition of $\mathcal{V}_E$.
vi. The algorithm terminates in a strongly compatible decomposition $((\mathcal{V}_{E_jT_j})_k; \mathcal{V}_T)$, for which $\mathcal{V}_{E_jT_j} \times T_j = \mathcal{V}_T \times T_j = \mathbf{0}$ and $\mathcal{V}^{\perp}_{E_jT_j} \times T_j = \mathcal{V}^{\perp}_T \times T_j = \mathbf{0}$. By Theorem 8.4.1 it follows that the decomposition is minimal.
By the same theorem it also follows that every minimal decomposition is strongly compatible.

E 8.12:

i. Suppose there exist distinct vectors $\bigoplus_j \mathbf{f}_{P_j}$ and $\bigoplus_j \mathbf{f}_{P_j}{}'$ in $\mathcal{V}_P$ s.t. $\mathbf{f}_{E_j} \oplus \mathbf{f}_{P_j}$ as well as $\mathbf{f}_{E_j} \oplus \mathbf{f}_{P_j}{}'$ belong to $\mathcal{V}_{E_jP_j}, j = 1, \cdots, k$. But then $\mathbf{f}_{P_j} - \mathbf{f}_{P_j}{}' \in \mathcal{V}_{E_jP_j} \times P_j, j = 1, \cdots, k$. For minimal decompositions $\mathcal{V}_{E_jP_j} \times P_j$ has zero dimension (Theorem 8.4.1). Hence,

$$\bigoplus_j \mathbf{f}_{P_j}{}' = \bigoplus_j \mathbf{f}_{P_j}.$$

The linearity of this correspondence is clear since, if for each j,

$$\mathbf{f}_{E_j}{}^1 \oplus \mathbf{f}_{P_j}{}^1 \in \mathcal{V}_{E_jP_j}, \mathbf{f}_{E_j}{}^2 \oplus \mathbf{f}_{P_j}{}^2 \in \mathcal{V}_{E_jP_j}, \bigoplus_j \mathbf{f}_{P_j}{}^1 \in \mathcal{V}_P, \bigoplus_j \mathbf{f}_{P_j}{}^2 \in \mathcal{V}_P, \ \mathit{then}$$

$$(\lambda_1 \mathbf{f}_{E_j}{}^1 + \lambda_2 \mathbf{f}_{E_j}{}^2) \oplus (\lambda_1 \mathbf{f}_{P_j}{}^1 + \lambda_2 \mathbf{f}_{P_j}{}^2) \in \mathcal{V}_{E_jP_j}.$$

and

$$\lambda_1 \bigoplus_j \mathbf{f}_{P_j}{}^1 + \lambda_2 \bigoplus_j \mathbf{f}_{P_j}{}^2$$

belongs to $\mathcal{V}_P$. Thus, if $\bigoplus_j \mathbf{f}_{E_j}{}^1$ corresponds to $\mathbf{f}_{P_j}{}^1$ and $\bigoplus_j \mathbf{f}_{E_j}{}^2$ correspond to $\bigoplus_j \mathbf{f}_{P_j}{}^2$ it follows that $\bigoplus_j(\lambda_1\mathbf{f}_{E_j}{}^1 + \lambda_2\mathbf{f}_{E_j}{}^2)$ corresponds to $\bigoplus_j(\lambda_1\mathbf{f}_{P_j}{}^1 + \lambda_2\mathbf{f}_{P_j}{}^2)$.

ii. We use the following facts:

$$\begin{aligned} \mathcal{V}_{E_jP_j} \cdot P_j &= \mathcal{V}_P \cdot P_j, \qquad \textit{and} \\ \mathcal{V}_{E_jP_j} \times E_j &= \mathcal{V}_E \times E_j. \end{aligned}$$

(Theorem 8.2.2 and Theorem 8.4.1).
The existence of a vector $\bigoplus_j \mathbf{f}_{E_j} \in \mathcal{V}_E$ corresponding to $\bigoplus_j \mathbf{f}_{P_j}$ is clear from the fact that $\mathcal{V}_{E_jP_j} \cdot P_j \supseteq \mathcal{V}_P \cdot P_j, j = 1, \cdots, k$. If both $\bigoplus_j \mathbf{f}_{E_j}{}^1$ and $\bigoplus_j \mathbf{f}_{E_j}{}^2$ of $\mathcal{V}_E$ correspond to $\bigoplus_j \mathbf{f}_{P_j}$, it is clear that

$$(\mathbf{f}_{E_j}{}^1 - \mathbf{f}_{E_j}{}^2) \oplus \mathbf{0}_{P_j} \in \mathcal{V}_{E_jP_j}, j = 1, \cdots, k$$

$$\mathbf{f}_{E_j}{}^1 - \mathbf{f}_{E_j}{}^2 \in \mathcal{V}_{E_jP_j} \times E_j = \mathcal{V}_E \times E_j, j = 1, \cdots, k.$$

E 8.13: We use the results in Exercise 8.11. We see that both the decompositions being minimal are also compatible decompositions of $\mathcal{V}_E$. But then $((\mathcal{V}_{E_jP_j})_k; \mathcal{V}_E)$ is a decomposition of both $\mathcal{V}_P$ and $\hat{\mathcal{V}}_P$. We conclude that $\mathcal{V}_P = \hat{\mathcal{V}}_P$.

E 8.14:

i. It is easily seen that $\mathbf{f}_{E_j} \oplus \mathbf{f}_{P_j} \in \mathcal{V}_{E_jP_j}, j = 1, \cdots, k$
and $\bigoplus \mathbf{f}_{P_j} \in \mathcal{V}$ iff $\mathbf{f}_{E_j} \oplus \mathbf{f}_{P'_j} \in \mathcal{V}_{E_jP'_j}$ and $\bigoplus \mathbf{f}_{P'j} \in \mathcal{V}_{P'}$, where $\mathbf{f}_{P'_j} \equiv \mathbf{T}_j(\mathbf{f}_{P_j})$.
This proves that $((\mathcal{V}_{E_jP'j})_k, \mathcal{V}_{P'})$ is a decomposition of $\mathcal{V}_E$. Minimality follows from the fact that $\mid P' \mid = \mid P \mid$.

ii. We have

$$\mathcal{V}_{E_jP_j} \cdot E_j = \mathcal{V}_{E_jP'j} \cdot E_j$$

$$\mathcal{V}_{E_jP_j} \times E_j = \mathcal{V}_{E_jP'j} \times E_j$$

Hence, we have a representative matrix for $\mathcal{V}_{E_jP_j}$ of the form
$\begin{bmatrix} \mathbf{R}_{jj} & \mathbf{0} \\ \mathbf{R}_{2j} & \mathbf{Q}_{2j} \end{bmatrix}$ and for $\mathcal{V}_{E_jP'j}$ of the form $\begin{bmatrix} \mathbf{R}_{jj} & \mathbf{0} \\ \mathbf{R}_{2j} & \mathbf{Q}'_{2j} \end{bmatrix}$,
where $\begin{pmatrix} \mathbf{R}_{jj} \\ \mathbf{R}_{2j} \end{pmatrix}$ is a representative matrix for $\mathcal{V}_{E_jP_j} \cdot E_j$ and $\mathbf{R}_{jj}$ is a representative matrix for $\mathcal{V}_{E_jP_j} \times E_j$. Now by the minimality of the decomposition (using Theorem 8.4.1) we have

$$\begin{aligned} \mid P_j \mid = \mid P_j{}' \mid &= r(\mathcal{V}_{E_jP_j} \cdot E_j) - r(\mathcal{V}_{E_jP_j} \times E_j) \\ &= r(\mathcal{V}_{E_jP_j} \cdot P_j) = r(\mathcal{V}_{EjP'j} \cdot P_j{}') \end{aligned}$$

Thus, $\mathbf{Q}_{2j}, \mathbf{Q}'_{2j}$ are representative matrices of $\mathcal{V}_{E_jP_j} \cdot P_j, \mathcal{V}_{E_jP'j} \cdot P_j{}'$ and are square and nonsingular.
Clearly, there is a matrix $\mathbf{T}_j$ s.t. $\mathbf{Q}_{2j}\mathbf{T}_j = \mathbf{Q}'_{2j}$. Now let $((\hat{\mathcal{V}}_{E_jP'j})_k, \hat{\mathcal{V}}_{P'})$ be the decomposition derived from $((\mathcal{V}_{E_jP_j})_k; \mathcal{V}_P)$ by using the nonsingular transformation $\mathbf{T}_j$ on $\mathbf{f}_{P_j}$ as in the previous section of the present problem. Thus, $((\hat{\mathcal{V}}_{E_jP'j})_k; \hat{\mathcal{V}}_{P'})$ is a minimal k-decomposition of $\mathcal{V}_E$. We claim that the decomposition is identical

to $((\mathcal{V}_{E_jP'j})_k;\mathcal{V}_{P'})$. For it is clear that $\hat{\mathcal{V}}_{E_jP'j} = \mathcal{V}_{E_jP'j}$ by construction. By Exercise 8.13, since the decompositions are minimal and therefore compatible we can conclude that $\hat{\mathcal{V}}_{P'} = \mathcal{V}_{P'}$.

E 8.15:

i. This is condition (iv) of Theorem 8.4.1.

ii. Since the set of columns P_j are linearly independent in a representative matrix of $\mathcal{V}_P$ it is clear that $P_j \cap K_1$ would be linearly independent in a representative matrix of $\mathcal{V}_P \cdot K_1$. Similarly, since P_j are linearly independent in a representative matrix of $\mathcal{V}_P^{\perp}, P_j \cap (P - K_1)$ would be linearly independent in a representative matrix of $\mathcal{V}_P^{\perp} \cdot (P - K_1)$. Thus,

$$\mid P_j \mid \leq r(\mathcal{V}_P \cdot K_1) + r(\mathcal{V}_P^{\perp} \cdot (P - K_1))$$

for any $K_1 \subseteq P$.
The result follows.

iii. Let $\mathcal{V}_P, \mathcal{V}_P^{\perp}$ have the representative matrices $(\mathbf{R}_1 \vdots, \cdots \vdots \mathbf{R}_k)$, $(\mathbf{B}_1 \vdots \cdots \vdots \mathbf{B}_k)$ and let $\mathcal{V}_{P'}, \mathcal{V}_{P'}^{\perp}$ have the representative matrices $(\mathbf{R}'_1 \vdots, \cdots \vdots \mathbf{R}'_k)$, $(\mathbf{B}'_1 \vdots \cdots \vdots \mathbf{B}'_k)$, where the columns of $\mathbf{R}_j, \mathbf{B}_j$ correspond to P_j and those of $\mathbf{R}'_j, \mathbf{B}'_j$ correspond to P'_j.
Then it is clear that

$$\mathbf{R}'_j = \mathbf{R}_j \mathbf{T}_j$$

for an appropriate nonsingular matrix $\mathbf{T}_j, j = 1, \cdots, k$.
Since $(\mathbf{R}')(\mathbf{B}')^T = \mathbf{0} = (\mathbf{R})(\mathbf{B})^T$ we must have

$$\mathbf{B}'_j = \mathbf{B}_j({\mathbf{T}_j}^T)^{-1}.$$

It is clear therefore, that the columns ${P_j}'$ are linearly independent in the representative matrices of $\mathcal{V}'_P$, as well as $\mathcal{V}_{P'}^{\perp}$. Hence, $\mid {P_j}' \mid \leq$ hybrid rank of $\mathcal{V}_{P'}$ by the previous section of the present problem. But $\mid {P_j}' \mid = \mid P_j \mid$. So the result follows.

8.8 Solutions of Problems

P 8.1: Let

$$\hat{\mathcal{V}}_P \equiv (\bigoplus_j (\mathcal{V}_{E_jP_j} \leftrightarrow \mathcal{V}_{E_jQ_j})) \leftrightarrow \mathcal{V}_Q.$$

Now $\hat{\mathbf{f}}_{P_j} \oplus \mathbf{f}_{Q_j}$ belongs to $\mathcal{V}_{E_jP_j} \leftrightarrow \mathcal{V}_{E_jQ_j}$ iff there exists $\mathbf{f}_{E_j}$ s.t.

$$\mathbf{f}_{E_j} \oplus \hat{\mathbf{f}}_{P_j} \in \mathcal{V}_{E_jP_j} \quad and \quad \mathbf{f}_{E_j} \oplus \mathbf{f}_{Q_j} \in \mathcal{V}_{E_jQ_j}.$$

Next $\bigoplus_j \hat{\mathbf{f}}_{P_j} \in \hat{\mathcal{V}}_P$ iff there exist $\mathbf{f}_{E_j}, j = 1, \cdots, k$, s.t.

$$\mathbf{f}_{E_j} \oplus \hat{\mathbf{f}}_{P_j} \in \mathcal{V}_{E_jP_j}, \mathbf{f}_{E_j} \oplus \mathbf{f}_{Q_j} \in \mathcal{V}_{E_jQ_j} \quad and \quad \bigoplus_j \mathbf{f}_{Q_j} \in \mathcal{V}_Q.$$

Now $\mathbf{f}_{E_j}$, $j = 1, \cdots, k$, satisfy the above condition only if $\bigoplus_j \mathbf{f}_{E_j} \in \mathcal{V}_E$ since $((\mathcal{V}_{E_jQ_j})_k; \mathcal{V}_Q)$ is a decomposition of $\mathcal{V}_E$. Hence, this happens only if there exists $\mathbf{f}_{P_j}, j = 1, \cdots, k$, s.t. $\mathbf{f}_{E_j} \oplus \mathbf{f}_{P_j} \in \mathcal{V}_{E_jP_j}, j = 1, \cdots, k$, and $\bigoplus_j \mathbf{f}_{P_j} \in \mathcal{V}_P$, since $((\mathcal{V}_{E_jP_j})_k; \mathcal{V}_P)$ is a decomposition of $\mathcal{V}_E$. But such $\mathbf{f}_{P_j}$ must satisfy,

$$\mathbf{f}_{P_j} - \hat{\mathbf{f}}_{P_j} \in \mathcal{V}_{E_jP_j} \times P_j, \quad j = 1, \cdots, k.$$

Since the decomposition $((\mathcal{V}_{E_jP_j})_k; \mathcal{V}_P)$ is compatible this means that

$$\mathbf{f}_{P_j} - \hat{\mathbf{f}}_{P_j} \in \mathcal{V}_P \times P_j.$$

We conclude that $\bigoplus_j \hat{\mathbf{f}}_{P_j} \in \mathcal{V}_P$. Hence, $\hat{\mathcal{V}}_P \subseteq \mathcal{V}_P$.
On the other hand let $\bigoplus_j \mathbf{f}_{P_j} \in \mathcal{V}_P$. Since the decomposition is compatible,

$$\mathcal{V}_P \cdot P_j \subseteq \mathcal{V}_{E_jP_j} \cdot P_j, \quad j = 1, \cdots, k.$$

Hence, there exist $\mathbf{f}_{E_j}$, $j = 1, \cdots, k$, s.t.

$$\mathbf{f}_{E_j} \oplus \mathbf{f}_{P_j} \in \mathcal{V}_{E_jP_j}, \quad j = 1, \cdots, k.$$

But this means $\bigoplus_j \mathbf{f}_{E_j} \in \mathcal{V}_E$. Hence there exist $\mathbf{f}_{Q_j}, j = 1, \cdots, k$, s.t. $\mathbf{f}_{E_j} \oplus \mathbf{f}_{Q_j} \in \mathcal{V}_{E_jQ_j}$, $j = 1, \cdots, k$ and $\bigoplus_j \mathbf{f}_{Q_j} \in \mathcal{V}_Q$.
Thus,

$$\mathbf{f}_{P_j} \oplus \mathbf{f}_{Q_j} \in \mathcal{V}_{E_jP_j} \leftrightarrow \mathcal{V}_{E_jQ_j}, \quad j = 1, \cdots, k$$

and

$$\bigoplus_j \mathbf{f}_{Q_j} \in \mathcal{V}_Q.$$

Hence, $\bigoplus_j \mathbf{f}_{P_j} \in \hat{\mathcal{V}}_P$, as required.

P 8.2:

i. Suppose $\hat{\mathcal{V}}_E \cap (\bigoplus_j (\mathcal{V}_E \times E_j)) \subset \bigoplus_j (\mathcal{V}_E \times E_j)$.
Consider the space $\mathcal{V}'_E \equiv \hat{\mathcal{V}}_E + (\bigoplus_j (\mathcal{V}_E \times E_j))$. Since both $\hat{\mathcal{V}}_E$ and $\bigoplus_j (\mathcal{V}_E \times E_j)$ have the E_j as separators, $\mathcal{V}'_E$ also will have the E_j as separators. But

$$\bigoplus_j (\mathcal{V}_E \times E_j) \subseteq \mathcal{V}_E.$$

Hence, $\mathcal{V}_E + \mathcal{V}'_E = \mathcal{V}_E + \hat{\mathcal{V}}_E$. But $\mathcal{V}_E \cap \mathcal{V}'_E \supset \mathcal{V}_E \cap \hat{\mathcal{V}}_E$.
Hence, $d(\mathcal{V}_E, \mathcal{V}'_E) < d(\mathcal{V}_E, \hat{\mathcal{V}}_E)$, a contradiction.
Hence, $\hat{\mathcal{V}}_E \supseteq \bigoplus_j (\mathcal{V}_E \times E_j)$. We can similarly prove that

$$\hat{\mathcal{V}}_E \subseteq \bigoplus_j (\mathcal{V}_E \cdot E_j).$$

(The result also follows by using duality, i.e., working with $\mathcal{V}^\perp$ in place of $\mathcal{V}$ and using the facts that

$$(\mathcal{V} \cdot E_j)^\perp = \mathcal{V}^\perp \times E_j, (\mathcal{V} + \mathcal{V}')^\perp = \mathcal{V}^\perp \cap (\mathcal{V}')^\perp$$

and $$d(\mathcal{V},\mathcal{V}') = d(\mathcal{V}^{\perp},(\mathcal{V}')^{\perp}).$$

ii. Let $\mathcal{V}'_E$ be a vector space on E which has the E_j as separators. Let $\mathcal{V}'_P \equiv (\bigoplus_j \mathcal{V}_{E_jP_j}) \leftrightarrow \mathcal{V}'_E$. We know by Exercise 8.6 that $\mathcal{V}'_P$ has $P_j,\ j=1,\cdots,k$ as separators. The decomposition $((\mathcal{V}_{E_jP_j})_k;\mathcal{V}_P)$ is compatible and therefore by Exercise 8.11

$$\mathcal{V}_P = (\bigoplus_j(\mathcal{V}_{E_jP_j})_k) \leftrightarrow \mathcal{V}_E.$$

Hence by Problem 7.8

$$d(\mathcal{V}_E,\mathcal{V}'_E) = d(\mathcal{V}_P,\mathcal{V}'_P).$$

Thus, the hybrid rank of $\mathcal{V}_E$ relative to $\{E_1,\cdots,E_k\}$ is not less than the hybrid rank of $\mathcal{V}_P$ relative to $\{P_1,\cdots,P_k\}$.
Next let $\hat{\mathcal{V}}_P$ have the P_j as separators. Then the vector space

$$\hat{\mathcal{V}}_E \equiv ((\bigoplus \mathcal{V}_{E_jP_j}) \leftrightarrow \hat{\mathcal{V}}_P$$

has the E_j as separators. Again $d(\mathcal{V}_P,\hat{\mathcal{V}}_P) = d(\mathcal{V}_E,\hat{\mathcal{V}}_E)$. So the hybrid rank of $\mathcal{V}_P$ relative to $\{P_1,\cdots,P_k\}$ is not less than the hybrid rank of $\mathcal{V}_E$ relative to $\{E_1,\cdots,E_k\}$. This proves the required result.

P 8.3: We need to use the ideas of Exercise 8.11.
Let $a_1,\cdots,a_p$ be the edges of t_j and let $e_1,\cdots,e_q$ be the edges of L_j. Let $T_j(\supseteq t_j), j=1,\cdots,k$ be forests of $\mathcal{G}_{E_jP_j}$ respectively. We observe that whenever $1 \leq r \leq p$, the f-cutsets of T_j with respect to $a_{r+1},\cdots,a_p$ in the graph $\mathcal{G}_{E_jP_j}$ are contained in P_j and remain as f-cutsets even after $a_1,\cdots,a_r$ are contracted. The corresponding vectors (appropriately padded with zeros whenever required) belong to the voltage spaces of $\mathcal{G}_{E_jP_j} \times P_j, \mathcal{G}_{E_jP_j}$ as well as that of the graph obtained from $\mathcal{G}_{E_jP_j}$ by contracting $a_1,\cdots,a_r$.
Since $\mathcal{V}_{E_jP_j} \times P_j = \mathcal{V}_P \times P_j$, (strong compatibility) these vectors (again appropriately padded with zeros) belong to $\mathcal{V}_v(\mathcal{G}_P)$ as well as the voltage spaces of the graph obtained from $\mathcal{G}_P$ by contracting $a_1,\cdots,a_r$. Thus, the branches $a_1,\cdots,a_r$ may be successively contracted (i.e.,t_j may be contracted) in $\mathcal{G}_{E_jP_j}$ as well as $\mathcal{G}_P$ leaving the remaining components of the decomposition as they were earlier.
The voltage spaces of the resulting graphs would continue to be a strongly compatible decomposition of $\mathcal{V}_v(\mathcal{G})$. This process may be repeated for each of the components of the decomposition since when $t_j,\ j \neq i$, is contracted the f-cutsets of t_i remain as they were before the contraction.
At the end of this process involving all the j we would have the strongly compatible decomposition, $((\mathcal{G}_{E_jT_j})_k, \mathcal{G}_T)$, where

$$\mathcal{G}_{E_jT_j} = \mathcal{G}_{E_jP_j} \times (P_j - t_j)$$

$$\mathcal{G}_T = \mathcal{G}_P \times (P - \bigcup_j t_j).$$

The edges L_j would be contained in a coforest of $\mathcal{G}_{E_jT_j}$ as well as $\mathcal{G}_T$. (Because contraction of some tree edges would not disturb the corresponding coforest).

We have

$$(\mathcal{V}_v(\mathcal{G}_{E_jT_j}))\cdot T_j = (\mathcal{V}_v(\mathcal{G}_T)\cdot T_j)$$
$$i.e.,\quad (\mathcal{V}_i(\mathcal{G}_{E_jT_j}))\times T_j = (\mathcal{V}_i(\mathcal{G}_T))\times T_j$$
$$i.e.,\quad \mathcal{V}_i(\mathcal{G}_{E_jT_j}\cdot T_j) = (\mathcal{V}_i(\mathcal{G}_T\cdot T_j),\ j=1,\cdots,k.$$

We repeat the argument now in terms of circuits. It would then follow that L_j can be deleted from all the $\mathcal{G}_{E_jT_j}$ as well as $\mathcal{G}_T$ and the result would be the strongly compatible decomposition $((\mathcal{G}_{E_jQ_j})_k;\mathcal{G}_Q)$ of $\mathcal{G}$.
This decomposition would be minimal by Theorem 8.4.1 for the following reasons:
i. $r((\mathcal{G}_{E_jQ_j})\cdot Q_j) = \mid Q_j \mid = r(\mathcal{G}_Q\cdot Q_j)$
(since the coforest edges L_j have been deleted from $\mathcal{G}_{E_jT_j}\cdot T_j$).

ii. $r((\mathcal{G}_{E_jQ_j})\times Q_j) = 0 = r(\mathcal{G}_Q\times Q_j)$
(since the forest edges of $\mathcal{G}_{E_jP_j}\times P_j$ have been contracted).

P 8.4:
i. is trivial since the components and the port connection diagram are copies of the same graphs and the relevant sets for application of strong compatibility conditions are also trivial.

ii. This is an immediate specialization of the algorithm of Problem 8.3.

P 8.5: In the $\mathcal{N}_{AL}-\mathcal{N}_{BK}$ method, we have a graph whose edge set $E(\mathcal{G})$ is partitioned into $\{A,B\}$. Sets $K\subseteq A, L\subseteq B$ are such that $\mathcal{G}\times(A\cup L))\cdot A\cong\mathcal{G}\ .\ A$ and $\mathcal{G}\ .\ (B\cup K)\times B\cong\mathcal{G}\times B$. We need to show, using multiport decomposition,
i. $\mathcal{V}_v(\mathcal{G})$ equals the collection of all vectors $\mathbf{f}_{A-K}\oplus\mathbf{f}_K\oplus\mathbf{f}_B$ s.t. there exist vectors

$$\mathbf{f}_{A-K}\oplus\mathbf{f}_K\oplus\hat{\mathbf{f}}_L\in\mathcal{V}_v(\mathcal{G}\times(A\cup L))$$
$$and\qquad \mathbf{f}_B\oplus\mathbf{f}_K\in\mathcal{V}_v(\mathcal{G}\ .\ (B\cup K))$$

ii. $\mathcal{V}_i(\mathcal{G})$ equal the collection of all vectors

$$\mathbf{g}_{B-L}\oplus\mathbf{g}_L\oplus\mathbf{g}_A\qquad s.t.$$

there exist vectors

$$\mathbf{g}_{B-L}\oplus\mathbf{g}_L\oplus\hat{\mathbf{g}}_K\in\mathcal{V}_i(\mathcal{G}\ .\ (B\cup K))$$
$$and\qquad \mathbf{g}_A\oplus\mathbf{g}_L\in\mathcal{V}_i(\mathcal{G}\times(A\cup L)).$$

We will only prove the statement about voltage vectors. The statement about current vectors can be proved similarly (dually).
For the discussion to follow we need to build copies of graphs derived from $\mathcal{G}$. We use the following notation:
The sets P_A, P_B are copies of A, B. If $\mathcal{G}$ is alternatively denoted by $\mathcal{G}_{AB}$, $\mathcal{G}_{P_AP_B}$ would denote its copy on $P_A\cup P_B$. In general if $\mathcal{G}_{ST}, S\subseteq A, T\subseteq B$ is a graph derived from $\mathcal{G}_{AB}$ by a sequence of operations, then $\mathcal{G}_{P_SP_T}$ would denote the graph derived from $\mathcal{G}_{P_AP_B}$ by the same sequence of operations on corresponding elements of the copy. Denote the graphs $\mathcal{G}\times(A\cup L), \mathcal{G}\ .\ (B\cup K)$ respectively by $\mathcal{G}_{AL}, \mathcal{G}_{BK}$

and the graph $\mathcal{G} \,.\, (B \cup K) \times (K \cup L) \cong \mathcal{G} \times (A \cup L) \cdot (K \cup L)$ by $\mathcal{G}_{KL}$.

We will first show that $(\mathcal{G}_{AP_L}, \mathcal{G}_{BP_K}; \mathcal{G}_{P_K P_L})$ is a 2-multiport decomposition of $\mathcal{G}$.
We start with the strongly compatible decomposition (Exercise 8.11) $(\mathcal{V}_{AP_B}, \mathcal{V}_{BP_A}; \mathcal{V}_{P_B P_A})$ of $\mathcal{V}_v(\mathcal{G})$, where $\mathcal{V}_{ST}$ denotes the voltage space of $\mathcal{V}_v(\mathcal{G}_{ST})$. From Exercise 8.11 we will use the idea that if $(\mathcal{V}_{AP_1}, \mathcal{V}_{BP_2}; \mathcal{V}_{P_1 P_2})$ is a strongly compatible decomposition of $\mathcal{V}_{AB}$ and if $\mathbf{f}_{P_1} \in \mathcal{V}_{AP_1} \times P_1, e$ an element in the support of $\mathbf{f}_{P_1}$, then e can be contracted in $\mathcal{V}_{AP_1}$ as well as $\mathcal{V}_{P_1 P_2}$ and we would be left with a new strongly compatible decomposition of $\mathcal{V}_{AB}$. Dually if $\mathbf{g}_{P_1} \in \mathcal{V}^{\perp}_{AP} \times P_1, e$ an element in the support of $\mathbf{g}_{P_1}$, then e can be deleted in $\mathcal{V}_{AP_1}$ as well as $\mathcal{V}_{P_1 P_2}$ yielding a new strongly compatible decomposition of $\mathcal{V}_{AB}$. If $\mathcal{V}_{AP_1}$ is the voltage space of $\mathcal{G}_{AP_1}$ then a cutset (circuit) vector with support contained in P_1 would belong to $\mathcal{V}_{AP_1} \times P_1$ $(\mathcal{V}^{\perp}_{AP} \times P_1)$. So we can work directly with cutsets and circuits of $\mathcal{G}_{AP_1} \times P_1, \mathcal{G}_{AP_1} \cdot P_1$, respectively.
Build a forest t_k of $\mathcal{G} \,.\, K$, extend it to a forest t_A of $\mathcal{G} \,.\, A$. Let $t_{(B-L)}$ be a forest of $\mathcal{G} \,.\, (B-L)$. Since $\mathcal{G} \times (A \cup L) \cdot A \cong \mathcal{G} \,.\, A$, t_A contains no circuits in $\mathcal{G} \times (A \cup L)$. Hence, observing that $\mathcal{G} \times (A \cup L)$ is obtained by contracting the branches of $B-L$, $t_{(B-L)} \cup t_A$ contains no circuits of $\mathcal{G}$. Extend this set to a forest t of $\mathcal{G}$. Let $\bar{t}$ be the corresponding coforest.
We will denote by $t_y, \bar{t_y}$ the sets $t \cap Y, \bar{t} \cap Y$ respectively. For simplicity the copies of these sets in $P_A \cup P_B$ would also be denoted by the same symbols.
Observe that the f-cutsets of t with respect to edges in $t_{(B-L)}$ do not intersect A. Let $e_1 \cdots, e_p$ be the edges in $t_{(B-L)}$. When $e_1, \cdots, e_r$ $(1 \leq r < p)$, are contracted, it is clear that the f-cutsets of $t - \{e_1, \cdots, e_r\}$ with respect to $e_{r+1}, \cdots, e_p$ remain as cutsets in the contracted graph. We can, therefore, contract $t_{(B-L)}$ in $\mathcal{G}_{AP_B}$ and $\mathcal{G}_{P_A P_B}$ while leaving $\mathcal{G}_{BP_A}$ unaltered and the resulting voltage spaces would continue to be a strongly compatible decomposition of $\mathcal{V}_{AB}$. The edges of $\bar{t}_{(B-L)}$ would now have become self loops in $\mathcal{G}_{AP_B} \times (A \cup P_B - t_{(B-L)})$ and $\mathcal{G}_{P_A P_B} \times (P_A \cup P_B - t_{B-L})$. They can therefore, be deleted in $\mathcal{G}_{AP_B} \times (A \cup P_B - t_{B-L})$ as well as $\mathcal{G}_{P_A P_B} \times (P_A \cup P_B - t_{B-L})$. But deleting or contracting selfloops has the same effect. So they may be contracted in both the graphs. Thus, at this stage we are left with the graphs $\mathcal{G}_{AP_L} \equiv \mathcal{G}_{AP_B} \times (A \cup P_L), \mathcal{G}_{BP_A}$ and $\mathcal{G}_{P_L P_A} \equiv \mathcal{G}_{P_A P_B} \times (P_A \cup P_L)$, whose voltage spaces constitute a strongly compatible decomposition of $\mathcal{V}_{AB}$.

Next consider the f-circuits of t with respect to edges in $\bar{t}_{(A-K)}$ in the graph $\mathcal{G}$. These would remain as such in $\mathcal{G} \,.\, A \cong \mathcal{G} \times (A \cup L) \cdot A$ and therefore also in $\mathcal{G}_{BP_A}$ and $\mathcal{G}_{P_A P_L}$. So $\bar{t}_{(A-K)}$ may be deleted in $\mathcal{G}_{BP_A}$ and $\mathcal{G}_{P_A P_L}$ while leaving $\mathcal{G}_{AP_L}$ unaltered. The resulting voltage spaces would continue to be a strongly compatible decomposition of $\mathcal{V}_{AB}$. The edges of t_{A-K} would now have become coloops. They can therefore be contracted in $\mathcal{G}_{BP_A} \cdot (B \cup P_A - \bar{t}_{A-K})$ and $\mathcal{G}_{P_A P_L} \cdot (P_A \cup P_L - \bar{t}_{A-K})$. But deleting or contacting coloops has the same effect as far as voltage spaces are concerned. So they may be deleted in both the graphs. Thus at this stage we are left with $\mathcal{G}_{AP_L}, \mathcal{G}_{BP_K} \equiv \mathcal{G}_{BP_A} \cdot (B \cup P_K)$, $\mathcal{G}_{P_K P_L} \equiv \mathcal{G}_{P_A P_L} \cdot (P_K \cup P_L)$. Their voltage

spaces constitute a 2-multiport decomposition of $\mathcal{V}_{AB}$.
We will next show that the port voltage matching conditions are equivalent to the condition that voltage vector on K be the same in $\mathcal{G} \times (A \cup L)$ and $\mathcal{G} \cdot (B \cup K)$. The equivalence of port current matching conditions to the condition that current vector on L be the same in $\mathcal{G} \cdot (B \cup K)$ and $\mathcal{G} \times (A \cup L)$, can be proved similarly (dually).
In what follows if $\mathbf{f}_y$ is a vector on Y, $\mathbf{f}_{P_y}$ would denote that vector on P_y whose value on e' is the value of $\mathbf{f}_y$ on e, where e' in P_y is the copy of e in y.

A vector $\mathbf{f}_A \oplus \mathbf{f}_B \in \mathcal{V}_v(\mathcal{G})$ iff there exist vectors $\mathbf{f}_A \oplus \hat{\mathbf{f}}_{P_L}$, $\mathbf{f}_B \oplus \hat{\mathbf{f}}_{P_K}$, $\hat{\mathbf{f}}_{P_L} \oplus \hat{\mathbf{f}}_{P_K}$ of $\mathcal{V}_v(\mathcal{G}_{AP_L}), \mathcal{V}_v(\mathcal{G}_{BP_K})$, $\mathcal{V}_v(\mathcal{G}_{P_K P_L})$ respectively. Let us write $\mathbf{f}_A$ as $\mathbf{f}_{A-K} \oplus \mathbf{f}_K$. The vector $\mathbf{f}_K \oplus \hat{\mathbf{f}}_{P_L} \in \mathcal{V}_v(\mathcal{G}_{AP_L} \cdot (K \cup P_L))$. Hence, noting that $\mathcal{G}_{P_K P_L} \equiv \mathcal{G}_{P_A P_L} \cdot (P_K \cup P_L)$, the vector $\mathbf{f}_{P_K} \oplus \hat{\mathbf{f}}_{P_L} \in \mathcal{V}_v(\mathcal{G}_{P_K P_L})$. The vector $\hat{\mathbf{f}}_{P_K} \oplus \hat{\mathbf{f}}_{P_L}$ belongs to $\mathcal{V}_v(\mathcal{G}_{P_A P_L} \cdot (P_K \cup P_L))(= \mathcal{V}_v(\mathcal{G}_{P_K P_L}))$. Hence, the vector $\hat{\mathbf{f}}_K \oplus \hat{\mathbf{f}}_{P_L} \in \mathcal{V}_v(\mathcal{G}_{AP_L} \cdot (K \cup P_L))$. Thus, $\mathbf{f}_{P_K} - \hat{\mathbf{f}}_{P_K} \in \mathcal{V}_v(\mathcal{G}_{P_A P_L} \cdot (P_K \cup P_L) \times P_K)$ $(= \mathcal{V}_v(\mathcal{G}_{P_K P_L} \times P_K))$ and $\mathbf{f}_K - \hat{\mathbf{f}}_K \in \mathcal{V}_v(\mathcal{G}_{AP_L} \cdot (K \cup P_L) \times K)$.
Now $\mathcal{G} \cdot (B \cup K) \times K = \mathcal{G} \cdot (B \cup K) \times (K \cup L) \times K \cong \mathcal{G} \times (A \cup L) \cdot (K \cup L) \times K$. So $\mathcal{G}_{BP_A} \cdot (B \cup P_K) \times P_K$ is a copy of $\mathcal{G}_{AP_L} \cdot (K \cup P_L) \times K$. Hence,

$$\mathbf{f}_{P_K} - \hat{\mathbf{f}}_{P_K} \in \mathcal{V}_v(\mathcal{G}_{BP_A} \cdot (B \cup P_K) \times P_K)(= \mathcal{V}_v(\mathcal{G}_{BP_K} \times P_K).$$

It follows that $(\mathbf{f}_B \oplus \hat{\mathbf{f}}_{P_K}) + (\mathbf{f}_{P_K} - \hat{\mathbf{f}}_{P_K}) \in \mathcal{V}_v(\mathcal{G}_{BP_K})$ iff $\mathbf{f}_B \oplus \hat{\mathbf{f}}_{P_K}$ does and $(\hat{\mathbf{f}}_{P_L} \oplus \hat{\mathbf{f}}_{P_K}) + \mathbf{f}_{P_K} - \hat{\mathbf{f}}_{P_K} \in \mathcal{V}_v(\mathcal{G}_{P_K P_L})$ iff $\hat{\mathbf{f}}_{P_L} \oplus \hat{\mathbf{f}}_{P_K}$ does.
We conclude that $\mathbf{f}_A \oplus \mathbf{f}_B \in \mathcal{V}_v(\mathcal{G})$ iff there exist vectors $\mathbf{f}_{A-K} \oplus \mathbf{f}_K \oplus \hat{\mathbf{f}}_{P_L}$, $\mathbf{f}_B \oplus \mathbf{f}_{P_K}$, $\hat{\mathbf{f}}_{P_L} \oplus \mathbf{f}_{P_K}$, respectively in $\mathcal{V}_v(\mathcal{G}_{AP_L})$, $\mathcal{V}_v(\mathcal{G}_{BP_K})$, $\mathcal{V}_v(\mathcal{G}_{P_K P_L})$. This is equivalent to saying that $\mathbf{f}_A \oplus \mathbf{f}_B \in \mathcal{V}_v(\mathcal{G})$ iff there exist vectors

$$\mathbf{f}_{A-K} \oplus \mathbf{f}_K \oplus \hat{\mathbf{f}}_L \in \mathcal{G} \times (A \cup L), \quad \textit{and}$$

$$\mathbf{f}_B \oplus \mathbf{f}_K \in \mathcal{G} \cdot (B \cup K).$$

This proves the required result.

P 8.6:

i. Let $(\mathbf{v}, \mathbf{i})$ be a solution of $\mathcal{N}$. Then for each j we have

$$\mathbf{v}_{E_j} \in \mathcal{V}_E \cdot E_j,$$

$$\mathbf{i}_{E_j} \in \mathcal{V}_E^{\perp} \cdot E_j$$

$$(\mathbf{v}_{E_j}, \mathbf{i}_{E_j}) \in \mathcal{D}_{E_j},$$

where $\mathbf{v}_{E_j}, \mathbf{i}_{E_j}$ denote the restrictions of $\mathbf{v}, \mathbf{i}$ to E_j. Since $\mathcal{V}_E = (\bigoplus_j \mathcal{V}_{E_j P_j}) \leftrightarrow \mathcal{V}_P$, there must exist $\mathbf{v}_P \in \mathcal{V}_P$ s.t.

$$\mathbf{v}_{E_j} \oplus \mathbf{v}_{P_j} \in \mathcal{V}_{E_j P_j},$$

where $\mathbf{v}_{P_j} = \mathbf{v}_P / P_j, \ j = 1, \cdots, k$.
Dually, there must exist $\mathbf{i}_P \in \mathcal{V}_P^{\perp}$ s.t.

$$\mathbf{i}_{E_j} \oplus \mathbf{i}_{P_j} \in \mathcal{V}_{E_j P_j}^{\perp},$$

where $\mathbf{i}_{P_j} = \mathbf{i}_P/P_j,\ j = 1, \cdots, k$.
Thus, if $\mathbf{v}'_P = -\mathbf{v}_P, \mathbf{i}'_P = \mathbf{i}_P$, since $(\mathbf{v}_{E_j} \oplus (-\mathbf{v}'_{P_j}), \mathbf{i}_{E_j} \oplus \mathbf{i}'_{P_j})$, in addition to the above conditions, also belongs to $\mathcal{D}_{E_j} \times \delta_{P_j}$, it must be a solution of $\mathcal{N}_{E_j P_j}$. So $(\mathbf{v}'_{P_j}, \mathbf{i}'_{P_j}) \in \mathcal{D}'_{P_j}$. Further $\bigoplus_j \mathbf{v}'_{P_j} \in \mathcal{V}_P$ and $\bigoplus_j \mathbf{i}'_{P_j} \in \mathcal{V}_P^{\perp}$. Thus, $(\mathbf{v}'_P, \mathbf{i}'_P)$ is a solution of $\mathcal{N}'_P$. If $(\mathbf{v}"_P, \mathbf{i}"_P)$ is another such solution, $\mathbf{v}'_{P_j} - \mathbf{v}"_{P_j} \in \mathcal{V}_{E_j P_j} \times P_j$ for each j.
Since the decomposition is minimal (using Theorem 8.4.1), each of these spaces has zero dimension. Hence, $\mathbf{v}'_P = \mathbf{v}"_P$. Similarly one can show that $\mathbf{i}'_P = \mathbf{i}"_P$.

ii. $(\mathbf{v}'_E, \mathbf{i}'_E), (\mathbf{v}"_E, \mathbf{i}"_E)$ are both solutions of $\mathcal{N}$ corresponding to $(\mathbf{v}'_P, \mathbf{i}'_P)$ iff
$\mathbf{v}'_{E_j} - \mathbf{v}"_{E_j} \in \mathcal{V}_{E_j P_j} \times E_j \subseteq \mathcal{V}_E \times E_j$ for each j, and
$\mathbf{i}'_{E_j} - \mathbf{i}"_{E_j} \in \mathcal{V}_{E_j P_j}^{\perp} \times E_j \subseteq \mathcal{V}_E^{\perp} \times E_j$ for each j.

iii $\mathcal{N}'_Q$ would be defined in an identical manner to $\mathcal{N}'_P$. A solution $(\mathbf{v}'_Q, \mathbf{i}'_Q)$ of $\mathcal{N}'_Q$ corresponds to a collection $\mathcal{S}_{\mathcal{N}}$ of solutions of $\mathcal{N}$ such that whenever $(\mathbf{v}_1, \mathbf{i}_1), (\mathbf{v}_2, \mathbf{i}_2)$ belong to $\mathcal{S}_{\mathcal{N}}$ we must have $(\mathbf{v}_1 - \mathbf{v}_2) \in \bigoplus_j \mathcal{V}_E \times E_j$ and $(\mathbf{i}_1 - \mathbf{i}_2) \in \bigoplus_j \mathcal{V}_E^{\perp} \times E_j$. But whenever two solutions of $\mathcal{N}$ differ in this manner (by the previous sections of the present problem) they correspond to the same solution $(\mathbf{v}'_P, \mathbf{i}'_P)$ of $\mathcal{N}'_P$. Hence, each solution of $\mathcal{N}'_Q$ corresponds to a unique solution of $\mathcal{N}'_P$.

P 8.7:
i. Using equations 8.33 and 8.34 we can show that $\mathbf{i}_{P_R} = -\mathbf{G}_P \mathbf{v}_{P_R}$ where $\mathbf{G}_P$ is a positive definite matrix.
The edges P_R in $\mathcal{N}'_P$ would therefore have the device characteristic

$$\mathbf{i}'_{P_R} = \mathbf{G}_P \mathbf{v}'_{P_R}.$$

P_C, P_L have the device characteristic

$$\mathcal{C}_P \dot{\mathbf{v}}'_{P_C} = \mathbf{i}'_{P_C}$$

$$\mathcal{L}_P \dot{\mathbf{i}}'_{P_L} = \mathbf{v}'_{P_L}$$

where $\mathcal{C}_P, \mathcal{L}_P$ are positive definite.
By the discussion on trapped solutions in RLMC networks in the above section we know that in $\mathcal{N}'_P$, if $(\mathbf{v}, \mathbf{i})$ is trapped relative to P_R then $\mathbf{v}/P_R = \mathbf{0}$ and $\mathbf{i}/P_R = \mathbf{0}$; if trapped relative to P_L then $\mathbf{v}/P_L = \mathbf{0}$ and $\mathbf{i}/P_L \in \mathcal{V}_P^{\perp} \times P_L$; if trapped relative to P_C then $\mathbf{i}/P_C = 0$ and $\mathbf{v}/P_C \in \mathcal{V}_P \times P_C$. But the decomposition is given to be minimal. So by Theorem 8.4.1 we have $r(\mathcal{V}_P \times P_C) = 0$ and $r(\mathcal{V}_P^{\perp} \times P_L) = 0$. So $\mathbf{i}/P_L = 0$ and $\mathbf{v}/P_C = \mathbf{0}$. This proves the required result.

ii. We will show that in $\mathcal{N}'_P$, a zero eigen value solution has to be trapped solution relative to P_C, P_R, P_L. A zero eigen value solution implies that $\dot{\mathbf{v}}'_{P_C}, \dot{\mathbf{i}}'_{P_L}$ are zero. (This means $\mathbf{v}'_{P_C}, \mathbf{i}'_{P_L}$ are constant vectors).
Let $\mathbf{v}'_{P_C} \notin \mathcal{V}_P \times P_C$. This means that either $\mathbf{v}'_{P_L} \neq 0$ or $\mathbf{v}'_{P_R} \neq 0$. Since $\dot{\mathbf{i}}'_{P_L} = \mathbf{0}$ we must have $\mathbf{v}'_{P_L} = \mathbf{0}$. So $\mathbf{v}'_{P_R} \neq \mathbf{0}$. Since $\mathbf{G}_P$ is positive definite we conclude that $< \mathbf{v}'_{P_R}, \mathbf{i}'_{P_R} > \neq 0$. Now we have

$$< \mathbf{v}'_{P_C}, \mathbf{i}'_{P_C} > + < \mathbf{v}'_{P_L}, \mathbf{i}'_{P_L} > + < \mathbf{v}'_{P_R}, \mathbf{i}'_{P_R} > = 0.$$

So either $< \mathbf{v}'_{P_C}, \mathbf{i}'_{P_C} > \neq \mathbf{0}$ or $< \mathbf{v}'_{P_L}, \mathbf{i}'_{P_L} > \neq 0$. As we have seen, $\mathbf{v}'_{P_L} = \mathbf{0}$, and since $\dot{\mathbf{v}}'_{P_C} = \mathbf{0}$, we must have $\mathbf{i}'_{P_C} = \mathbf{0}$. This is a contradiction. A similar contradiction can be derived from the assumption that

$$\mathbf{i}'_{P_L} \notin \mathcal{V}_P^{\perp} \times P_L.$$

Thus, a zero eigen value solution corresponds to a trapped solution. But from the previous section of this problem the only trapped solution to this problem is the zero solution. This proves the required result.

P 8.8:

i. is routine.

ii. We have

$$r(\mathcal{V}_{EP} \leftrightarrow \mathcal{V}_P) = r(\mathcal{V}_{EP} \times E) + r((\mathcal{V}_{EP} \cdot P) \cap \mathcal{V}_P) - r((\mathcal{V}_{EP} \times P) \cap \mathcal{V}_P).$$

(see Problem 7.1).
Let $\mathcal{V}_{EP} \equiv \bigoplus_j \mathcal{V}_{E_j P_j}$.
Let $((\mathcal{V}_{E_j P_j})_k ; \mathcal{V}_P)$ be a compatible decomposition of $\mathcal{V}_E$. We know that

$$\mathcal{V}_{E_j P_j} \cdot P_j \supseteq \mathcal{V}_P \cdot P_j, j = 1, \cdots, k$$

$$\mathcal{V}_{E_j P_j} \times P_j \subseteq \mathcal{V}_P \times P_j, j = 1, \cdots, k.$$

By Exercise 8.7 we have, $((\mathcal{V}_{E_j P_j})_{j \in I} ; \mathcal{V}_P \cdot P_I)$ is a compatible decomposition of $\mathcal{V}_E \cdot E_I$. Thus,

$$\begin{aligned} r(\mathcal{V}_E \cdot E_I) &= r((\bigoplus_{j \in I} \mathcal{V}_{E_j P_j}) \times E_I) + r(\mathcal{V}_P \cdot P_I) - r((\bigoplus_{j \in I} \mathcal{V}_{E_j P_j}) \times P_I) \\ &= \sum_{j \in I} r(\mathcal{V}_{E_j P_j} \times E_j) + r(\mathcal{V}_P \cdot P_I) - \sum_{j \in I} r(\mathcal{V}_{E_j P_j} \times P_j) \end{aligned}$$

as required.

P 8.9:

i. If the vector spaces are over GF2, $\mathcal{M}(\mathcal{V}_S)$ fully determines $\mathcal{V}_S$. Hence if we know $\mathcal{M}(\mathcal{V}_{EP}), \mathcal{M}(\mathcal{V}_P)$, we know $\mathcal{V}_{EP}, \mathcal{V}_P$ and therefore, $\mathcal{V}_E$ and $\mathcal{M}(\mathcal{V}_E)$.

ii. See Example 7.1 p.95 [Narayanan86a].

Chapter 9

Submodular Functions

9.1 Introduction

In combinatorial mathematics submodular functions are a relatively recent phenomenon. Systematic interest in this area perhaps began with the work of Edmonds in the late sixties [Edmonds70]. By then matroids were well studied with numerous applications to engineering systems already known. Submodular functions could be regarded as a generalization of matroid rank functions and it is natural to wonder whether they are really required. The answer is that, even if we ignore considerations of theory, we come across them far more often in practical problems than we come across matroids. The method of attack for these problems using submodular function theory is usually quite simple and the algorithms generated, very efficient. Study of basic 'submodular' operations such as convolution and Dilworth truncation is likely to prove fruitful for practical algorithm designers since, in addition to completely capturing the essence of many practical situations, they also allow us to give acceptable approximate solutions to several intractable problems.

In this chapter we begin with simple equivalent restatements of the definition of submodularity along with a number of instances where submodular functions are found in 'nature'. Next we discuss some standard operations by which we get new submodular / supermodular (more compactly, semimodular) functions starting from such functions. We then pay special attention to the important special cases of matroid and polymatroid rank functions. Next we give a sketch of the polyhedral approach to the study of semimodular functions. Finally we give a brief outline of some recent work on minimization of symmetric submodular functions. The important notions of convolution and Dilworth truncation of submodular functions are relegated to subsequent chapters.

9.2 Submodularity

We begin with a few definitions of submodularity which are easily proved to be equivalent. The most useful form appears to be one which essentially states that the 'rate of increase' of submodular functions is less on 'larger' sets (analogous to 'cap' functons over the real line). Thereafter we present a number of simple examples of submodular functions.

Definition 9.2.1 *Let S be a finite set.*
Let $f : 2^S \longrightarrow \Re$, f is said to be a **submodular** *(***supermodular***) function iff*

$$f(X) + f(Y) \geq f(X \cup Y) + f(X \cap Y) \quad \forall X, Y \subseteq S \tag{9.1}$$

$$(f(X) + f(Y) \leq f(X \cup Y) + f(X \cap Y) \quad \forall X, Y \subseteq S)$$

The function f is **modular** *if the inequality is replaced by equality. A function is* **semimodular** *if it is submodular or supermodular.*[1]

The following theorem gives equivalent conditions for submodularity / supermodularity. These conditions are often easier to apply in practice than the original definitions. The conditions for supermodularity are obtained by reversing the submodular inequalities and the proof of the equivalence is obtained by reversing the inequalities line by line in the submodular case proof.

Theorem 9.2.1 *(k)*

i. *A function $f : 2^S \longrightarrow \Re$ is submodular iff it satisfies any one of the following properties.*

$$f(X \cup a) - f(X) \geq f(X \cup b \cup a) - f(X \cup b) \quad \forall X \subseteq S, \forall a, b \in S - X \tag{9.2}$$

$$f(X \cup a) - f(X) \geq f(Y \cup a) - f(Y) \quad \forall X \subseteq Y \subseteq S, \forall a \in S - Y \tag{9.3}$$

$$f(X \cup Z) - f(X) \geq f(Y \cup Z) - f(Y) \quad \forall X \subseteq Y \subseteq S, \forall Z \subseteq S - Y \tag{9.4}$$

ii. *The function f is supermodular iff it satisfies any one of the above three properties with the inequalities reversed.*

Proof: i.

$$(10.2) \Rightarrow (10.3) \Rightarrow (10.4) \Rightarrow (10.1) \Rightarrow (10.2)$$

$$(10.2) \Rightarrow (10.3)$$

Let $Y = X \uplus b_1 \uplus b_2 \uplus \cdots \uplus b_k$. If a is not in Y, we have, by (10.2)

$$f(X \cup a) - f(X) \geq f(X \cup b_1 \cup a) - f(X \cup b_1)$$

$$\geq f(X \cup b_1 \cup b_2 \cup a) - f(X \cup b_1 \cup b_2)$$

[1] Warning: before the early 70's submodular functions used to be referred to as semimodular functions in the literature.

$$\cdots$$

$$\geq f(X \cup b_1 \cup \cdots \cup b_k \cup a) - f(X \cup b_1 \cup \cdots \cup b_k).$$

$$(10.3) \Rightarrow (10.4)$$

Let $Z = \{a_1, a_2, \cdots a_t\}$ and let $X \subseteq Y \subseteq S, Z \subseteq S - Y$. Then

$$f(X \cup a_1) - f(X) \geq f(Y \cup a_1) - f(Y)$$

$$f(X \cup a_1 \cup a_2) - f(X \cup a_1) \geq f(Y \cup a_1 \cup a_2) - f(Y \cup a_1)$$

$$\cdots$$

$$f(X \cup a_1 \cup \cdots \cup a_t) - f(X \cup a_1 \cup \cdots \cup a_{t-1}) \geq f(Y \cup a_1 \cup \cdots \cup a_t) - f(Y \cup a_1 \cup \cdots \cup a_{t-1})$$

Adding all the inequalities we get (10.4).

$$(10.4) \Rightarrow (10.1)$$

This is immediate by setting (in (10.4))

$$X \textit{ to } X \cap Y, Z \textit{ to } (X - X \cap Y)(= (X \cup Y - Y)) \textit{ and } Y \textit{ to } Y$$

$$(10.1) \Rightarrow (10.2)$$

This is immediate by setting (in (10.1))

$$X \textit{ to } X \cup a \textit{ and } Y \textit{ to } X \cup b$$

ii. The supermodular case is similar.

□

We now give some common examples of submodular functions. The first set involves graphs.
Let $\mathcal{G}$ be a graph on vertices V and edges E.

Example 9.2.1 *(k) Let $V(X) \equiv$ set of endpoints of edges of X, $X \subseteq E(\mathcal{G})$. Then $|V|(\cdot)$* [2] *(called the* **vertex function** *of $\mathcal{G}$) is submodular.*

Example 9.2.2 *(k) Let $E(V_1) \equiv$ set of edges with both endpoints in V_1, $V_1 \subseteq V(\mathcal{G})$. Then $|E|(\cdot)$ (called the* **interior edge function** *of $\mathcal{G}$) is supermodular.*

Example 9.2.3 *(k) Let $I(V_1) \equiv$ set of edges with atleast one end point in V_1, $V_1 \subseteq V(\mathcal{G})$. Then $|I|(\cdot)$ (called the* **incidence function** *of $\mathcal{G}$)is submodular.*

Example 9.2.4 *(k) Let $\Gamma(V_1) \equiv$set of vertices adjacent to some vertex in V_1, $V_1 \subseteq V(\mathcal{G})$. Then $|\Gamma|(\cdot)$ (called the* **adjacency function***of $\mathcal{G}$) is submodular.*

Example 9.2.5 *(k) Let $cut(V_1) \equiv$ set of all branches with only one endpoint in V_1, $V_1 \subseteq V(\mathcal{G})$. Then $|cut|(\cdot)$, called the* **cut function** *of $\mathcal{G}$, is submodular.*

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[2] Throughout $|X(\cdot)|$ and $|X|(\cdot)$ are used interchangeably to specify the cardinality of $X(\cdot)$, where $X(\cdot)$ is any set function.

Example 9.2.6 *(k) Let $\mathcal{G}$ be a directed graph. We can now define, analogous to the definition in Example 10.2.1, $V_{tail}(X), V_{head}(X)$ over the subsets of the edge set of the directed graph. These lead to submodular functions. Analogous to the definitions in Examples 10.2.3, 10.2.4, 10.2.5, we could define $I_{out}(\cdot), I_{in}(\cdot)$, $\Gamma_{out}(\cdot), \Gamma_{in}(\cdot), cut_{out}(\cdot)$,*
$cut_{in}(\cdot)$ etc. In each case the functions are submodular. (The functions $I_{out}(\cdot)$, $I_{in}(\cdot)$ are actually modular).

The above examples can be generalized to the context of hypergraphs as represented by bipartite graphs. The reader might like to think of the right vertex set of the bipartite graph as the edge set of the hypergraph.

Example 9.2.7 *Let $B \equiv (V_L, V_R, E)$. Let $E_L(X) \equiv$ set of all vertices in V_R adjacent only to vertices in X, $X \subseteq V_L$. $E_R(\cdot)$ is defined similarly on subsets of V_R. Then $|E_L|(\cdot), |E_R|(\cdot)$ (called the* **left exclusivity function** *and the* **the right exclusivity function** *respectively of B) are supermodular.*

Example 9.2.8 *Let $B \equiv (V_L, V_R, E)$ be a bipartite graph. For $X \subseteq V_L$ define $c(X)$ to be the set of all vertices in V_R whose images under $\Gamma(\cdot)$ intersect both X and $V_L - X$. Then $|c|(\cdot)$ is submodular.*

The next couple of examples are of the matroid kind.

Example 9.2.9 *(k) Let E be the set of columns of a matrix over any field $\mathcal{F}$. Then, the rank function $r(\cdot)$ on the subsets of E is submodular.*

Example 9.2.10 *(k) Let $\mathcal{G}$ be a graph on the set of edges $E(\mathcal{G})$. Let $r(X), X \subseteq E$ be the number of edges in a forest of $\mathcal{G}.X$, the subgraph on X. Let $r'(X), X \subseteq E$ be the number of edges in a forest of $\mathcal{G} \times X$, the graph obtained by shorting and removing all edges in $E - X$. Let $\nu(X), X \subseteq E$ be the number of edges in a coforest of $\mathcal{G} \times X$. Let $\nu'(X), X \subseteq E$ be the number of edges in a coforest of $\mathcal{G}.X$. Then, $r(\cdot), \nu(\cdot)$ (called the rank and nullity functions of the graph respectively) are submodular, while $r'(\cdot), \nu'(\cdot)$ are supermodular.*

Exercise 9.1 *Show that the functions listed in the above examples are submodular or supermodular as the case may be.*

Remark: In Examples 10.2.9, 10.2.10 above, the size of the 'maximal independent set' contained in a subset turns out to be submodular. Weaker notions of independence do not always yield submodular functions. Exercise 10.2 presents such an instance.

Exercise 9.2 *Let $\mathcal{G}$ be a graph. Let a set of vertices $V_1 \subseteq V(\mathcal{G})$ be called e-independent if no two vertices of V_1 are joined by an edge. Let $k(V_i), V_i \subseteq V(\mathcal{G})$ be the maximum size of an e-independent set contained in V_i. Show that $k(\cdot)$ is not in general submodular or supermodular.*

Exercise 9.3 *Let (V_L, V_R, E) be a bipartite graph. Let $w(\cdot)$ be a nonnegative weight function assigned to V_R. Let $q : 2^{V_L} \longrightarrow \Re$ be defined by $q(X) \equiv w(Y)$ where Y is the set of all vertices in V_R which are adjacent to every vertex in X. Let $q' : 2^{V_L} \longrightarrow \Re$ be defined by $q'(X) \equiv w(Z)$ where Z is the set of all vertices in V_R*

which are adjacent to none of the vertices in X.

i. *Show that $q'(X) = w(E_L(V_L - X))$ and hence supermodular.*

ii. *Define the complementary bipartite graph $\overline{B}(V_L, V_R, \overline{E})$ of B as follows: $e \in \overline{E}$ iff the endpoints of e are not connected by an edge of E. Show that $q_B(X) \equiv q'_{\overline{B}}(X)$ where q_B and $q'_{\overline{B}}$ denote the appropriate functions defined for B and $\overline{B}$ respectively. Hence show that q_B is supermodular.*

The terms 'submodular' and 'supermodular' have arisen from the well known notion of modular set functions. However in our framework the latter functions are essentially trivial. For completeness we define modularity and give equivalent definitions below.

Definition 9.2.2 *A function $w : 2^S \longrightarrow \Re$ is* **modular** *iff it satisfies the relation*

$$w(X) + w(Y) = w(X \cup Y) + w(X \cap Y) \quad \forall X, Y \subseteq S.$$

If $w(X) \equiv \sum_{e \in X} w(e), X \subseteq S$, then we call $w(\cdot)$ a **weight function**.

Theorem 9.2.2 *(k)*

i. *Let $w(\cdot)$ be a modular function on subsets of S. Then, $w(X) = \sum_{e \in X}(w(e) - w(\emptyset)) + w(\emptyset)$. Hence, if $w(\emptyset) = 0, w(X) = \sum_{e \in X} w(e)$.*

ii. *if $w^E(X) \equiv |X \cup E|$, where E is a fixed subset of S, then $w^E(\cdot)$ is modular. (Note that $w^E(\emptyset) = |E|$ which could be nonzero.)*

iii. *the function*

$$\alpha(X) \equiv \sum_{e \in X} \alpha(e)$$

is modular. In particular $|\cdot|$ is modular. So is the function $w_E(X) \equiv |X \cap E|$, where E is a fixed subset of S. (In this case observe that the weight of elements in $S - E$ is zero.)

Proof: We prove only the part (i). Let $X \subseteq S$ and let $a \in S - X$. We then have

$$w(X) + w(a) = w(X \cup a) + w(\emptyset),$$

The result then follows by induction.

□

9.3 Basic Operations on Semimodular Functions

We now present a number of operations which act on submodular / supermodular / modular functions and convert them to submodular or supermodular functions.

We begin with addition and scalar multiplication. Here the underlying set does not change. In the case of 'direct sum' it becomes the disjoint union of the original sets while in the case of 'fusion' it is a partition of the original set. We next consider the fundamental operations of 'restriction', 'contraction' and two types of dualization. In the case of contraction and restriction the new functions are over the power sets of appropriate subsets of the old set while the dualization operations do not change the underlying set.

Definition 9.3.1 *Let $\mu_1(\cdot), \mu_2(\cdot)$ be real valued set functions on the subsets of S_1, S_2, where $S_1 \cap S_2 = \emptyset$. The* **direct sum** *of $\mu_1(\cdot), \mu_2(\cdot)$, denoted by $(\mu_1 \oplus \mu_2)(\cdot)$, is defined over subsets of $S_1 \uplus S_2$ by*

$$(\mu_1 \oplus \mu_2)(X_1 \uplus X_2) \equiv \mu_1(X_1) + \mu_2(X_2) \quad \forall X_1 \subseteq S_1, X_2 \subseteq S_2.$$

Exercise 9.4 *(k) Let $\mu_1(\cdot), \mu_2(\cdot)$ be submodular functions and let $w(\cdot)$ be a modular function on the subsets of S. Then $(\mu_1 + \mu_2)(\cdot)$, $(\mu_1 + w)(\cdot), (\mu_1 - w)(\cdot)$, $\lambda\mu_1(\cdot), (\lambda \geq 0)$ are submodular while $-\mu_1(\cdot)$ is supermodular. If μ_1, μ_2 are submodular (supermodular) on subsets of disjoint sets S_1, S_2 then $(\mu_1 \oplus \mu_2(\cdot))$ is submodular (supermodular).*

A common technique in optimization problems which involve finding the 'best subset' is to somehow show that the optimum subset can be thought to be a union of some of the blocks of an appropriate partition of the underlying set. In this manner the size of the problem is reduced since each block can be treated as a single element. The fusion operation defined below formalizes this notion.

Definition 9.3.2 *Let $\mu(\cdot)$ be a set function on subsets of S and let Π be a partition $(\equiv S_1, \cdots S_k)$ of S. Then the* **fusion of μ relative to Π** *, denoted by $\mu_{fus.\Pi}(\cdot)$, is defined on subsets of Π by*

$$\mu_{fus.\Pi}(X_f) \equiv \mu(\bigcup_{T \in X_f} T), X_f \subseteq \Pi.$$

It is immediate that $\mu_{fus.\Pi}(\cdot)$ is submodular (supermodular) [modular] if $\mu(\cdot)$ is such a function.
Contraction, restriction, dualization are fundamental matroidal operations. For graphs these ideas correspond to short circuiting (contracting), open circuiting (deleting) and taking planar duals. These ideas generalize naturally to submodular functions. We prefer to define them for real valued set functions and then specialize them.

Definition 9.3.3 *Let $\mu(\cdot)$ be a real valued set function on subsets of S. The* **restriction** *of $\mu(\cdot)$ to $X \subseteq S$ denoted by $\mu/\mathbf{X}(\cdot)$ is defined by*

$$\mu/\mathbf{X}(Y) \equiv \mu(Y) \quad \forall Y \subseteq X \subseteq S.$$

(Note that there is an abuse of notation here. The original function is on 2^S, while the restriction according to the definition of page 16 is on 2^X).
The **contraction** *of $\mu(\cdot)$ to $X \subseteq S$, denoted by $\mu\diamond\mathbf{X}(\cdot)$, is defined by*

$$\mu\diamond\mathbf{X}(Y) \equiv \mu(Y \cup (S - X)) - \mu(S - X) \quad \forall Y \subseteq X \subseteq S.$$

Definition 9.3.4 *Let $\mu(\cdot)$ be a real valued set function on subsets of S. Let $\alpha(\cdot)$ be a modular function with $\alpha(\emptyset) = 0$ (i.e., $\alpha(\cdot)$ is 'essentially' a real vector) and let $\alpha(e) \geq \mu(e) \quad \forall e \in S$. The* **comodular dual of** $\mu(\cdot)$ **relative to** $\alpha(\cdot)$, *denoted by $\mu^*(\cdot)$, is defined by*

$$\mu^*(X) \equiv \sum_{e \in X} \alpha(e) - [\mu(S) - \mu(S - X)].$$

(If $\alpha(\cdot)$ is unspecified we take $\alpha(e) \equiv \mu(e) \quad \forall e \in S$). The **contramodular dual of** $\mu(\cdot)$, *denoted by $\mu^d(\cdot)$, is defined by*

$$\mu^d(X) \equiv \mu(S) - \mu(S - X).$$

Let $\mu(\cdot), \Pi, \alpha(\cdot)$ be as in the above definitions. We collect properties of contraction, restriction, fusion, comodular and contramodular dualization in the following theorems. The first of these speaks of how to reverse the order of contraction and restriction without affecting the outcome. The reader might like to compare it with Theorems 3.4.1 and 3.4.2. The routine proof is omitted.

Theorem 9.3.1 *(k)*

i. *Let $P \subseteq Q \subseteq S$. Then,*

$$(\mu \diamond (\mathbf{S} - \mathbf{P}) / (\mathbf{Q} - \mathbf{P}))(\cdot) = (\mu / \mathbf{Q} \diamond (\mathbf{Q} - \mathbf{P}))(\cdot)$$

ii. *Let $P \subseteq Q \subseteq S$ and let P, Q be unions of blocks of Π. Let P_f, Q_f be the sets of blocks of Π contained in P, Q respectively. Then, $(\mu \diamond (\mathbf{S} - \mathbf{P}) / (\mathbf{Q} - \mathbf{P}))(Y) = (\mu_{fus.\Pi} \diamond (\mathbf{\Pi} - \mathbf{P_f}) / (\mathbf{Q_f} - \mathbf{P_f}))(Y_f) = (\mu_{fus.\Pi}) / \mathbf{Q_f} \diamond (\mathbf{Q_f} - \mathbf{P_f})(Y_f)$, where Y is the union of the blocks in $Y_f \subseteq \Pi$*

The next theorem speaks of the dual of a dual of a function (itself), the duality of contraction and restriction, and of the self dual nature of fusion. We note that if the comodular dual of $\mu(\cdot)$ is taken with respect to $\alpha(\cdot)$ then we would take that of $\mu_{fus.\Pi}(\cdot)$ with respect to $\alpha_{fus.\Pi}(\cdot)$.

Theorem 9.3.2 *(k) If $\mu(\emptyset) = 0$, then*

i. (a) $(\mu^*)^*(\cdot) = \mu(\cdot)$
   (b) $(\mu / \mathbf{X})^*(\cdot) = \mu^* \diamond \mathbf{X}(\cdot)$
   (c) $(\mu \diamond \mathbf{X})^*(\cdot) = \mu^* / \mathbf{X}(\cdot)$
   (d) $(\mu_{fus.\Pi})^*(\cdot) = \mu^*_{fus.\Pi}(\cdot)$

ii. (a) $(\mu^d)^d(\cdot) = \mu(\cdot)$
   (b) $(\mu / \mathbf{X})^d(\cdot) = \mu^d \diamond \mathbf{X}(\cdot)$
   (c) $(\mu \diamond \mathbf{X})^d(\cdot) = \mu^d / \mathbf{X}(\cdot)$

(d) $(\mu_{fus.\Pi})^d(\cdot) = \mu^d_{fus.\Pi}(\cdot)$

Proof:

i. (a) We have,

$$(\mu^*)^*(X) \equiv \sum_{e\in X}\alpha(e) - [\mu^*(S) - \mu^*(S-X)]$$

$$= \sum_{e\in X}\alpha(e) - [\sum_{e\in S}\alpha(e) - \sum_{e\in S-X}\alpha(e) - (\mu(S)-\mu(\emptyset)) + (\mu(S)-\mu(X))]$$

$$= \mu(X).$$

(b) Let $Z \subseteq X$. Then,

$$(\mu^*\diamond\mathbf{X})(Z) \equiv \mu^*(Z\cup(S-X)) - \mu^*(S-X)$$

$$= \sum_{e\in Z\cup(S-X)}\alpha(e) - \sum_{e\in S-X}\alpha(e) - (\mu(S)-\mu(X-Z)) + (\mu(S)-\mu(X))]$$

$$= \sum_{e\in Z}\alpha(e) - (\mu(X)-\mu(X-Z))$$

$$= (\mu/\mathbf{X})^*(Z).$$

(c) This follows by using the above two results.

(d) This follows from the definitions of fusion and comodular duals of $\mu(\cdot)$ and $\mu_{fus.\Pi}(\cdot)$.

ii. The proof is similar to the '*' case and is omitted.

□

The next theorem is a generalization of Corollaries 3.4.3 and 3.4.2. Its routine proof is omitted.

Theorem 9.3.3 *(k) Let $A \subseteq S$ and let $X \subseteq (S-A)$. Then*

$$\mu(X\cup A) = \mu/\mathbf{A}(A) + \mu\diamond(\mathbf{S}-\mathbf{A})(X)$$

We now show that contraction and restriction preserve submodularity and supermodularity while comodular and contramodular dualizations behave as the names indicate.

Theorem 9.3.4 *(k)*

i. Let $X \subseteq S$. Then, $\mu/\mathbf{X}(\cdot), \mu\diamond\mathbf{X}(\cdot)$ are submodular (supermodular) if $\mu(\cdot)$ is submodular (supermodular).

ii. If $\mu(\cdot)$ is submodular (supermodular), then $\mu^*(\cdot)$ is submodular (supermodular) while $\mu^d(\cdot)$ is supermodular (submodular).

Proof: We consider only the submodular case.
i. The submodularity of the restriction of a submodular function is obvious.
We now consider contraction. We have

$$\mu(Y_1 \cup (S - X)) + \mu(Y_2 \cup (S - X))$$
$$\geq \mu(Y_1 \cup Y_2 \cup (S - X)) + \mu(Y_1 \cap Y_2 \cup (S - X)) \quad \forall Y_1, Y_2 \subseteq X.$$

Further, $\mu(S - X)$ is a constant for subsets of X. The submodularity of $\mu \diamond \mathbf{X}(\cdot)$ follows.

ii. We have

$$\mu(S - X) + \mu(S - Y) \geq \mu(S - (X \cup Y)) + \mu(S - (X \cap Y)) \quad \forall X, Y \subseteq S.$$

Further, $w(X) \equiv \sum_{e \in X} \alpha(e)$ is a modular function, $\mu(S)$ is a constant. The submodularity of $\mu^*(\cdot)$ follows. The supermodularity of $\mu^d(\cdot)$ follows noting that the negative of a submodular function is supermodular.

□

Submodular and supermodular functions associated with graphs, hypergraphs (as represented by bipartite graphs) etc usually behave in an interesting way. A basic operation such as the ones described above on a semimodular function associated say with a graph $\mathcal{G}$ takes it to another such function associated with a second graph which can be derived from $\mathcal{G}$ in a simple way. The exercises given below illustrate this idea.

Exercise 9.5 *(k)* **Rank and nullity functions of a graph**
Let $\mathcal{G}$ be a graph. Let $X \subseteq E(\mathcal{G})$. We remind the reader that $\mathcal{G}.X$ is the subgraph of $\mathcal{G}$ on X and $\mathcal{G} \times X$ is the graph obtained from $\mathcal{G}$ by fusing the end points of edges in $E(\mathcal{G}) - X$ and removing them. Let $r'(X), X \subseteq E$, be the number of edges in a forest of $\mathcal{G} \times X$. We will call $r'(\cdot)$ the prime rank function of $\mathcal{G}$. Let $\nu'(X), X \subseteq E$, be the number of edges in a coforest of $\mathcal{G}.X$. We will call $\nu'(\cdot)$ the prime nullity function of $\mathcal{G}$. Prove

i. *The rank function of $\mathcal{G}.X = r/\mathbf{X}(\cdot)$.*

ii. *The rank function of $\mathcal{G} \times X = r \diamond \mathbf{X}(\cdot)$.*

iii. $r^d(\cdot) = r'(\cdot)$.

iv. $r^*(\cdot) = \nu(\cdot)$.

v. *The nullity function of $\mathcal{G} \times X = \nu/\mathbf{X}(\cdot)$.*

vi. *The nullity function of $\mathcal{G}.X = \nu \diamond \mathbf{X}(\cdot)$.*

vii. *The prime rank function of $\mathcal{G} \times X = r'/\mathbf{X}(\cdot)$.*

viii. *The prime rank function of $\mathcal{G}.X = r' \diamond \mathbf{X}(\cdot)$.*

ix. *The prime nullity function of* $\mathcal{G}.X = \nu'/\mathbf{X}(\cdot)$.

x. *The prime nullity function of* $\mathcal{G} \times X = \nu'\diamond\mathbf{X}(\cdot)$.

Exercise 9.6 The incidence function $|I|(\cdot)$ and the interior edge function $|E|(\cdot)$ of a graph $\mathcal{G}$
(see Examples 10.2.2 and 10.2.3) Let $\mathcal{G}'$ denote the graph obtained by fusing $V(\mathcal{G})-X$ into a single node and deleting all edges with both end points in $V(\mathcal{G})-X$ and let $|I'|(\cdot)$, $|E'|(\cdot)$ be its incidence and interior edge functions respectively. Let $|I"|(\cdot)$, $|E"|(\cdot)$ be the incidence and interior edge functions respectively of the subgraph of $\mathcal{G}$ on $X \cup \Gamma(X)$. $|\hat{I}|(\cdot)$ be the incidence function of the subgraph of $\mathcal{G}$ on X. Let Π be a partition of $V(\mathcal{G})$ and let $|I_\Pi|(\cdot)$ be the incidence function of the graph obtained from $\mathcal{G}$ by fusing the blocks of Π into single vertices but not deleting any edges.
Prove

i. $|I|/\mathbf{X}(\cdot) = |I'|/\mathbf{X}(\cdot) = |I"|/\mathbf{X}(\cdot)$ *and* $|I|\diamond\mathbf{X}(\cdot) = |I'|\diamond\mathbf{X}(\cdot) = |I"|\diamond\mathbf{X}(\cdot)$.

ii. $|I|\diamond\mathbf{X}(\cdot) = |\hat{I}|(\cdot)$.

iii. $|I|_{fus.\Pi}(\cdot) = |I_\Pi|(\cdot)$.

iv. $|I|^d(\cdot) = |E|(\cdot)$.

v. *Let $\mathcal{G}$ have no self loops. Then, $|I|^*(\cdot) = |I|(\cdot)$, where the dual is defined with respect to the weight vector $\alpha(\cdot)$ with $\alpha(v) \equiv |I|(v) \quad \forall v \in V(\mathcal{G})$.*

vi. $|E|\diamond\mathbf{X}(\cdot) = |E'|\diamond\mathbf{X}(\cdot) = |E"|\diamond\mathbf{X}(\cdot)$ *and* $|E|/\mathbf{X}(\cdot) = |E'|/\mathbf{X}(\cdot) = |E"|/\mathbf{X}(\cdot)$.

vii. $|E|/\mathbf{X}(\cdot) = |\hat{E}|(\cdot)$.

viii. $|E|_{fus.\Pi}(\cdot) = |E_\Pi|(\cdot)$.

Exercise 9.7 The $|\Gamma_L|(\cdot), |E_L|(\cdot)$ functions of a bipartite graph
Let $B \equiv (V_L, V_R, E)$ be a bipartite graph. Let $\Gamma_L(\cdot) \equiv \Gamma/\mathbf{V_L}(\cdot)$ and let $\Gamma_R(\cdot) \equiv \Gamma/\mathbf{V_R}(\cdot)$. We will call $|\Gamma_L|(\cdot)$, $(|\Gamma_R|(\cdot))$ $|E_L|(\cdot)$, $(|E_R|(\cdot))$, the left(right) adjacency function, left (right) exclusivity function respectively of B (see Example 10.2.7). Let $X \subseteq V_L$. Let $B._{LX}$ be the subgraph of B on $X \uplus \Gamma(X)$ and let $B_{\diamond LX}$ be the graph obtained by first deleting $V_L - X \uplus \Gamma(V_L - X)$ and all edges with atleast one end point in this set ($B._{RX}, B_{\diamond RX}, X \subseteq V_R$ are similarly defined interchanging left and right). Let Π be a partition of V_L. Let B_Π be the graph obtained from B by fusing the blocks of Π into single vertices. Let $|\Gamma_{\Pi L}|(\cdot), |E_{\Pi L}|(\cdot)$ be the corresponding left adjacency and left exclusivity functions.
Show that

i. $|\Gamma_L|/\mathbf{X}(\cdot)$ *is the left adjacency function of* $B._{LX}$.

ii. $|\Gamma_L|\diamond\mathbf{X}(\cdot)$ *is the left adjacency function of* $B_{\diamond LX}$.

iii. $|\Gamma_L|_{fus.\Pi}(\cdot) = |\Gamma|_{\Pi L}(\cdot)$.

iv. $|\Gamma_L|^d(\cdot) = |E_L|(\cdot)$

v. $|E_L|/\mathbf{X}(\cdot)$ *is the left exclusivity function of* $B_{\diamond LX}$.

vi. $|E_L|\diamond\mathbf{X}(\cdot)$ *is the left exclusivity function of* $B_{\cdot LX}$.

vii. $|E_L|_{fus.\Pi}(\cdot) = |E_{\Pi L}|(\cdot)$.

9.4 *Other Operations on Semimodular Functions

We now consider a number of other operations on semimodular functions which yield other such functions. These operations, while being useful, are by no means standard. We therefore study them through a sequence of problems.

Problem 9.1 *(k)*

i. *Let* $f(\cdot), g(\cdot)$ *be submodular (supermodular) on subsets of* S *and let* $(f-g)(\cdot)$ *be monotone increasing or monotone decreasing. Show that* $h(\cdot) \equiv min(f(\cdot), g(\cdot))$ $(max(f(\cdot), g(\cdot)))$ *is a submodular (supermodular) function.*

ii. *Let* $f(\cdot)$ *be an increasing or decreasing submodular (supermodular) function and let k be a constant. Show that min(k, f(X)) (max(k, f(X)) is a submodular (supermodular) function.*

Solution: We consider only the monotone increasing case, since $(f-g)(\cdot)$ is monotone decreasing iff $(g-f)(\cdot)$ is monotone increasing. Further we confine ourselves to submodular functions.

i. Let $X, Y \subseteq S$. We will verify that

$$h(X) + h(Y) \geq h(X \cup Y) + h(X \cap Y).$$

This is clear if $h(\cdot)$ agrees with $f(\cdot)$ or with $g(\cdot)$ on both X and Y. Let us therefore assume that $h(X) = f(X)$ and $h(Y) = g(Y)$. We then have,

$$h(X) + h(Y) \geq f(X \cup Y) + f(X \cap Y) + g(Y) - f(Y).$$

But $f(X \cup Y) + g(Y) - f(Y) \geq g(X \cup Y)$. Hence

$$h(X) + h(Y) \geq g(X \cup Y) + f(X \cap Y) \geq h(X \cup Y) + h(X \cap Y).$$

ii. This is a direct consequence of the previous result.

The next problem involves an instance of the convolution operation to be discussed in the next chapter. As we shall see later, even more important than the $\mu_{min}(\cdot)$ and $\mu_{max}(\cdot)$ functions are the collection of subsets over which these functions become equal to $\mu(\cdot)$.

Problem 9.2 *(k)*

Definition 9.4.1 *Let $\mu(\cdot)$ be a real valued set function defined over subsets of S. Let*

$$\mu_{min}(X) \equiv min_{Y \subseteq X} \mu(Y),$$
$$\mu_{max}(X) \equiv max_{Y \subseteq X} \mu(Y),$$
$$\underline{\mu_{min}}(X) \equiv min_{Y \supseteq X} \mu(Y),$$
$$\underline{\mu_{max}}(X) \equiv max_{Y \supseteq X} \mu(Y).$$

Show that

i. $\mu_{min}(\cdot), \underline{\mu_{min}}(\cdot)$ are submodular if $\mu(\cdot)$ is submodular and $\mu_{max}(\cdot), \underline{\mu_{max}}(\cdot)$ are supermodular if $\mu(\cdot)$ is supermodular.

ii. if $\mu(\cdot)$ be a submodular (supermodular) function on subsets of S then the subsets over which it reaches a minimum (maximum) form a distributive lattice(i.e., the collection is closed under union and intersection).

Definition 9.4.2 *The collection of subsets of S over which a submodular (supermodular) function reaches a minimum (maximum) is called its* **principal structure**.

iii. Prove

Theorem 9.4.1 *(k) Let $\mu(\cdot)$ be a submodular function on subsets of S. Let $X \subseteq S$ have the property that*

$$\mu(X) \leq \mu(Y) \quad \forall Y \subseteq X$$
$$(\mu(X) < \mu(Y) \quad \forall Y \subset X).$$

Then X is contained in some (every) set that minimizes $\mu(\cdot)$.

Solution:
i. Let m_Z denote a subset of $Z \subseteq S$ at which $\mu(\cdot)$ reaches the minimum among all the subsets of Z and let m_{Xa}, m_{Ya} denote $m_{X \cup a}, m_{Y \cup a}$. Let $X \subseteq Y \subseteq S$ and let $a \in S - Y$. We now have,

$$\mu(m_{Xa}) + \mu(m_Y) \geq \mu(m_Y \cup m_{Xa}) + \mu(m_Y \cap m_{Xa}).$$

i.e.,

$$\mu(m_Y \cup m_{Xa}) - \mu(m_Y) \leq \mu(m_{Xa}) - \mu(m_Y \cap m_{Xa}).$$

Now $m_Y \cup m_{Xa} \subseteq Y \cup a$ and $m_Y \cap m_{Xa} \subseteq X$. But then

$$\mu(m_{Ya}) \leq \mu(m_Y \cup m_{Xa})$$

and

$$\mu(m_X) \leq \mu(m_Y \cap m_{Xa}).$$

We therefore have,

$$\mu(m_{Ya}) - \mu(m_Y) \leq \mu(m_{Xa}) - \mu(m_X).$$

Since $\mu_{min}(T) = \mu(m_T) \quad \forall T \subseteq S$, it is clear that $\mu_{min}(\cdot)$ is submodular. The proof of the supermodular case $\mu_{max}(\cdot)$ is similar.
Next let

$$\sigma(X) \equiv \mu(S) - \mu(S - X).$$

Observe that

$$\underline{\mu_{min}}(X) = \mu(S) - \sigma_{max}(S - X)$$

and

$$\underline{\mu_{max}}(X) = \mu(S) - \sigma_{min}(S - X).$$

The required results follow by noting that $\sigma(\cdot)$ is supermodular (submodular) iff μ is submodular (supermodular).

ii. Let X, Y minimize the submodular function $\mu(\cdot)$. By the basic inequality

$$\mu(X) + \mu(Y) \geq \mu(X \cup Y) + \mu(X \cap Y).$$

The only way this inequality can be satisfied is to have the values of $\mu(\cdot)$ on all four sets to be the same. The result follows. The proof for the supermodular case is similar.

iii. Proof of Theorem 10.4.1: Let $\mu(\cdot)$ reach the minimum, among all subsets of S, at Z. If X is a subset of Z, there is nothing to prove. Suppose X is not. Then $X \cap Z \subset X$. By the submodularity of $\mu(\cdot)$, we then have,

$$\mu(X) + \mu(Z) \geq \mu(X \cup Z) + \mu(X \cap Z).$$

Case1 $\mu(X) \leq \mu(X \cap Z)$. In this case $\mu(Z) \geq \mu(X \cup Z)$. Thus X is contained in a subset that minimizes $\mu(\cdot)$ viz $X \cup Z$.
Case2 $\mu(X) < \mu(X \cap Z)$. In this case $\mu(Z) > \mu(X \cup Z)$, which is a contradiction. We conclude that X must be a subset of Z.

□

Problem 9.3 *Let $s(\cdot)$ be an increasing set function taking subsets of a finite set S_1 to subsets of another finite set S_2 i.e., $s(Y) \supseteq s(X) \quad \forall X \subseteq Y \subseteq S_1$. Suppose*

$$s(X) \cup s(Y) = s(X \cup Y)$$

$$(s(X) \cap s(Y) = s(X \cap Y)).$$

i. Let $w(\cdot)$ assign each element of S_2 a nonnegative weight. Define $\hat{s}(X) \equiv \sum_{e \in s(X)} w(e)$. Show that

(a) the function $\hat{s}(\cdot)$ is submodular (supermodular).

(b) (k) *the set functions (defined in Exercise10.1)* $V(\cdot), \Gamma(\cdot), I(\cdot), V_{tail}(\cdot)$, $V_{head}(\cdot), \Gamma_{in}(\cdot), \Gamma_{out}(\cdot), I_{in}(\cdot), I_{out}(\cdot)$ *yield submodular functions when 'weighted nonnegatively' while* $E(\cdot)$ *yields a supermodular function. Hence show that the nonnegatively weighted versions of* $cut(\cdot), cut_{in}(\cdot), cut_{out}$ *are also submodular.*

ii. (k) *Let* $\mu(\cdot)$ *be an increasing submodular(supermodular) function on subsets of* S_2. *Define* $\sigma(\cdot)$ *on subsets of* S_1 *by* $\sigma(X) \equiv \mu(s(X)), X \subseteq S_1$. *Show that* $\sigma(\cdot)$ *is submodular (supermodular).*
Example: *(k) Let* B *be a bipartite graph on* V_L, V_R. *Let the function* $s(\cdot)$ *be taken as the* $\Gamma(\cdot)$ $(E_L(\cdot))$ *function of the bipartite graph. Let* $\mu(\cdot)$ *be any increasing submodular (supermodular) function on subsets of* V_R. *Then* $\mu(\Gamma(\cdot))(\mu(E_L(\cdot)))$ *is a submodular (supermodular) function on subsets of* V_L.

Solution:
i(a) For any increasing set function, we must have,

$$s(X) \cup s(Y) \subseteq s(X \cup Y),$$

$$s(X) \cap s(Y) \supseteq s(X \cap Y).$$

If

$$s(X) \cup s(Y) = s(X \cup Y),$$

it is clear that

$$\begin{aligned}\hat{s}(X) + \hat{s}(Y) &= w(s(X)) + w(s(Y)) \\ &= w(s(X) \cup s(Y)) + w(s(X) \cap s(Y)) \\ &\geq \hat{s}(X \cup Y) + \hat{s}(X \cap Y).\end{aligned}$$

The supermodular case (where $s(X) \cap s(Y) = s(X \cap Y)$) is handled similarly.

i(b) It is easily verified that the functions $V(\cdot), \Gamma(\cdot), I(\cdot), V_{tail}(\cdot), V_{head}(\cdot)$, $\Gamma_{in}(\cdot), \Gamma_{out}(\cdot), I_{in}(\cdot), I_{out}(\cdot)$ all satisfy the property $s(X) \cup s(Y) = s(X \cup Y)$, while $E(\cdot)$ satisfies $s(X) \cap s(Y) = s(X \cap Y)$. Further they are all increasing set functions. Thus it is clear that they must yield sub- or supermodular functions (as the case may be) when weighted. To study the weighted version of $cut(\cdot)$, we observe that $cut(X) = \Gamma(X) - E(X)$ and $\Gamma(X) \supseteq E(X)$. So

$$w(cut(X)) = w(\Gamma(X)) - w(E(X))$$

and the submodularity of this function follows from the sub- and supermodularity of the functions $w(\Gamma(\cdot))$ and $w(E(\cdot)$. One can similarly prove that $cut_{in}(\cdot), cut_{out}(\cdot)$ yield submodular functions when weighted.

ii. We consider only the case where $s(X \cup Y) = s(X) \cup s(Y)$ and $\mu(\cdot)$ submodular. The supermodular case can be handled similarly. We have $\sigma(\cdot) \equiv \mu(s(\cdot))$. Now

$$\begin{aligned}\mu(s(X)) + \mu(s(Y)) &\geq \mu(s(X) \cup s(Y)) + \mu(s(X) \cap s(Y)) \\ &\geq \mu(s(X \cup Y)) + \mu(s(X \cap Y))\end{aligned}$$

(since

$$s(X) \cap s(Y) \supseteq s(X \cap Y), s(X) \cup s(Y) = s(X \cup Y)$$

and $\mu(\cdot)$ is increasing). Thus $\sigma(\cdot)$ is submodular on subsets of S_1.
To verify the correctness of the example we need only verify that $\Gamma(\cdot), E_L(\cdot)$ are increasing and respectively satisfy the property $s(X) \cup s(Y) = s(X \cup Y)$ and the property $s(X) \cap s(Y) = s(X \cap Y)$. This, as mentioned before, is routine.

Problem 9.4 *(k) Let f be any increasing cap (increasing cup) function from $\Re$ to $\Re$. Let μ be an increasing submodular (increasing supermodular) function. Show that $f(\mu(\cdot))$ is submodular (supermodular). Examples:*

i. Let $r(\cdot)$ be the rank function of a graph $\mathcal{G}$. Let $f(t) \equiv (r(\mathcal{G}))^2 - (t - r(\mathcal{G}))^2$. Note that $f(\cdot)$ is increasing in the interval $[0, r(\mathcal{G})]$. Then $f(r(\cdot))$ is submodular.

ii. Let $w(\cdot)$ be a weight function on S. Let $f(t) \equiv (w(S))^2 - (t - w(S))^2$ and let $g(t) \equiv t^2$. Then $f(w(\cdot))$ is submodular while $g(w(\cdot)), g(r'(\cdot)), g(\nu'(\cdot))$ (the latter of Example 10.2.10) are supermodular.

Solution: If the function $f(\cdot)$ is an increasing cap function from $\Re$ to $\Re$ then

$$f(x+h) - f(x) \geq f(y+h) - f(y) \quad \forall y \geq x, h \geq 0.$$

Let $\mu(\cdot)$ be an increasing submodular function on the subsets of S and let $X \subseteq Y \subseteq S, a \in S - Y$. Then

$$\mu(X \cup a) - \mu(X) \geq \mu(Y \cup a) - \mu(Y).$$

Let δ, ϵ represent the left and right sides of the above inequality. Then, since $f(\cdot)$ is increasing,

$$\begin{aligned} f(\mu(X) + \delta) - f(\mu(X)) &\geq f(\mu(X) + \epsilon) - f(\mu(X)) \\ &\geq f(\mu(Y) + \epsilon) - f(\mu(Y)). \end{aligned}$$

This is equivalent to saying that

$$f(\mu(X \cup a)) - f(\mu(X)) \geq f(\mu(Y \cup a)) - f(\mu(Y)).$$

This proves the submodularity of $f(\mu(\cdot))$. The supermodular case is similar and the examples are direct applications of the result.

9.5 Polymatroid and Matroid Rank Functions

Matroid rank functions are the most important class of submodular functions. Polymatroid rank functions are their immediate generalization. As we shall show, any submodular function is a translate of a polymatroid rank function by a modular function. In this section we define these functions and study the results of applying on them some of the basic operations introduced in Section 10.3.

Definition 9.5.1 *A submodular function is a* **polymatroid rank function** *iff it takes zero value on* $\emptyset$ *and is nonnegative and increasing.*

Definition 9.5.2 *A polymatroid rank function is a* **matroid rank function** *iff it takes integral values and does not exceed* 1 *on any of the singletons.*

Exercise 9.8 *(k) Show that the rank and nullity functions of a graph are matroid rank functions.*

Exercise 9.9 *(k) Show that the vertex function, incidence function and adjacency function of a graph are polymatroid rank functions while the cut function* $|cut|(\cdot)$ *is not (see Examples 10.2.1, 10.2.3, 10.2.4, 10.2.5).*

Both matroid and polymatroid rank functions behave nicely with respect to comodular dualization.

Exercise 9.10 ***Theorem 9.5.1*** *(k)* $\mu^*(\cdot)$ *is a polymatroid (matroid) rank function if* $\mu(\cdot)$ *is one.*

Remark: In the definition of $\mu^*(\cdot)$, if $\alpha(e)$ is less than $\mu(e)$ for some e, the dual of a polymatroid rank function would not be a polymatroid rank function. But we would still have $\mu^*(\cdot)$ as submodular.

The next theorem states among other things that every submodular function is a translate of a polymatroid rank function. This idea would be useful when we relate minimization of submodular functions to the operation of convolution in the next chapter.

Theorem 9.5.2 *(k) Let* $\mu(\cdot)$ *be a submodular function on subsets of a finite set* S

i. $\mu(\cdot)$ *is an increasing function iff*

$$\mu(S) - \mu(S-e) \geq 0 \quad \forall e \in S.$$

It is nonnegative increasing iff it satisfies, in addition, $\mu(\emptyset) \geq 0$.

ii. If

$$\mu(S) - \mu(S-e) = \mu(e) - \mu(\emptyset) \quad \forall e \in S,$$

then $\mu(\cdot)$ *is modular.*

iii. Let a weight function $w(\cdot)$ *on* S *be defined by* $w(e) \equiv \mu(S) - \mu(S-e)$. *Then* $\mu(\cdot) - w(\cdot) - \mu(\emptyset)$ *is a polymatroid rank function.*

Proof: 'only if' is clear.
if:

i. Since $\mu(\cdot)$ is submodular, we have

$$\mu(S) - \mu(S-e) \leq \mu(X \cup e) - \mu(X) \quad \forall X \subseteq S, e \in S - X.$$

The result follows. The nonnegative increasing case is trivial.

ii. Since

$$\mu(S) - \mu(S-e) \leq \mu(X \cup e) - \mu(X) \leq \mu(e) - \mu(\emptyset) \quad \forall X \subseteq S, e \in S - X,$$

it follows that the given condition implies that

$$\mu(X \cup e) - \mu(X) = \mu(e) - \mu(\emptyset) \quad \forall X \subseteq S, e \in S - X.$$

Thus

$$\mu(X) = \sum_{e \in X} [\mu(e) - \mu(\emptyset)] + \mu(\emptyset) \quad \forall X \subseteq S.$$

Clearly this means $\mu(\cdot)$ is modular.

iii. It is easily verified that the function $\mu(\cdot) - w(\cdot) - \mu(\emptyset)$ satisfies the above condition for being nonnegative increasing and takes zero value on $\emptyset$. Since it is clearly submodular, the function is a polymatroid rank function.

□

The next exercise speaks of contraction, restriction and dualization on natural functions associated with a matroid. The reader might like to compare it with Exercise 10.5.

Exercise 9.11 *(k)* **The rank and nullity functions of a matroid**
As has already been pointed out in Chapter 4, we can give a number of alternative descriptions of a matroid in terms of

- *independent sets*
- *circuits*
- *bases*
- *matroid rank function.*

Let us assume the last description (of a matroid rank function on subsets of S) is available. Then, a set $X \subseteq S$ is said to be **independent** *iff $r(X) = |X|$, it is a* **base** *iff $r(X) = |X| = r(S)$, it is a* **circuit** *iff $r(X) = |X|-1$ and all proper subsets of X are independent. We remind the reader that every independent set can be extended to a base and that circuits are minimal dependent (non independent) sets. It is verified elsewhere that these classes satisfy the conditions of the appropriate axiom sets. Let us denote the matroid corresponding to these classes by $\mathcal{M}$. The comodular dual of the rank function with respect to the $|.|$ function is called the* **nullity** *function of the matroid and denoted by $\nu(\cdot)$. Let Π be a partition of S. Show that*

i. $r/\mathbf{X}(\cdot), r\diamond\mathbf{X}(\cdot), \nu(\cdot)$ *are matroid rank functions.*

ii. $r_{fus.\Pi}(\cdot)$ *is an integral polymatroid rank function. (We show later, in the next chapter, that all integral polymatroid rank functions can be obtained by fusion of matroid rank functions).*

iii. *The independent sets of the matroid (denoted by $\mathcal{M}.X$) defined by $r/\mathbf{X}(\cdot)$ are independent sets of $\mathcal{M}$ contained in X. The bases of $\mathcal{M}.X$ are maximal intersections of bases of $\mathcal{M}$ with X. The circuits of $\mathcal{M}.X$ are circuits of $\mathcal{M}$ contained in X.*

iv. *The independent sets of the matroid (denoted by $\mathcal{M} \times X$) defined by $r\diamond\mathbf{X}(\cdot)$ are sets whose union with every independent set of $\mathcal{M}.(S-X)$ is independent in $\mathcal{M}$. The bases of $\mathcal{M} \times X$ are minimal intersections of bases of $\mathcal{M}$ with X. The circuits of $\mathcal{M} \times X$ are minimal intersections of circuits of $\mathcal{M}$ with X.*

v. *The bases of the matroid (denoted by $\mathcal{M}^*$) defined by $r^*(\cdot)$ are the complements of bases of $\mathcal{M}$.*

vi. *$r^d(X) =$ rank of $\mathcal{M} \times X$.*

vii. *The nullity function of $\mathcal{M} \times X = \nu/\mathbf{X}(\cdot)$. Thus $\mathcal{M}^*.X = (\mathcal{M} \times X)^*$.*

viii. *The nullity function of $\mathcal{M}.X = \nu\diamond\mathbf{X}(\cdot)$. Thus $\mathcal{M}^* \times X = (\mathcal{M}.X)^*$.*

ix. *The contramodular dual of the rank function of $\mathcal{M} \times X = r^d/\mathbf{X}(\cdot)$.*

x. *The contramodular dual of the rank function of $\mathcal{M}.X = r^d\diamond\mathbf{X}(\cdot)$.*

xi. *The contramodular dual of the nullity function of $\mathcal{M} \times X = \nu^d\diamond\mathbf{X}(\cdot)$.*

xii. *The contramodular dual of the nullity function of $\mathcal{M}.X = \nu^d/\mathbf{X}(\cdot)$.*

An elementary but useful notion in graphs is that of putting additional edges in parallel to existing ones. This notion immediately generalizes to matroids. For submodular functions however more is possible provided some minor conditions on monotonicity are satisfied.

Definition 9.5.3 *Let $\mu(\cdot)$ be a submodular function on subsets of S. Elements $e_1, e_2 \in S$ are* **parallel** *with respect to $\mu(\cdot)$ iff*

$$\mu(X \cup e_1) = \mu(X \cup e_2) = \mu(X \cup e_1 \cup e_2) \quad \forall X \subseteq S.$$

Observe that other elements cannot distinguish between e_1, e_2.

Definition 9.5.4 *[Lovász83] Let $\mu(\cdot)$ be a submodular function on subsets of S. Let $T \subseteq S$ and element $a_T \notin S$. The* **parallel extension of** $\mu(\cdot)$ **by** a_T **parallel to T**, *denoted by $\hat{\mu}(\cdot)$ on subsets of $S \cup a_T$ is defined by*

$$\hat{\mu}(X) \equiv \mu(X)$$

$$\hat{\mu}(X \cup a_T) \equiv \mu(X \cup T) \quad \forall X \subseteq S.$$

Theorem 9.5.3 *(k)*

i. *$\hat{\mu}(\cdot)$ is a submodular function, if, for each $e \in T$ we have $\mu(S) - \mu(S-e) \geq 0$.*

ii. *if $\mu(\cdot)$ is a polymatroid rank function, then, so is $\hat{\mu}(\cdot)$.*

iii. *if $\mu(\cdot)$ is a matroid rank function on subsets of S then e_1, e_2 are in parallel iff*

$$\mu(e_1) = \mu(e_2) = \mu(\{e_1, e_2\}).$$

Proof:

i. We need to verify if

$$\hat{\mu}(X \cup e) - \hat{\mu}(X) \geq \hat{\mu}(Y \cup e) - \hat{\mu}(Y) \quad \forall X \subseteq Y \subseteq S \cup a_T, \ \ \forall e \in (S \cup a_T) - Y.$$

We have the following cases

- $a_T \notin Y$
- $a_T \in X$
- $a_T \in Y - X$ and $e \notin T$
- $a_T \in Y - X$ and $e \in T$

In the first three cases the inequality holds by the submodularity of $\mu(\cdot)$ and the definition of $\hat{\mu}(\cdot)$. In the last case the RHS is zero while the LHS is nonnegative since

$$\mu(X \cup e) - \mu(X) \geq \mu(S) - \mu(S - e) \geq 0.$$

ii. By the above reasoning, since in this case $\mu(\cdot)$ is increasing, we must have the parallel extension as submodular. Further $\hat{\mu}(\emptyset) = 0$ and $\hat{\mu}(\cdot)$ is clearly increasing.

iii. We need only prove the 'if' part.
Let $\mu(\cdot)$ be a matroid rank function. We must have, for $i = 1, 2$,

$$\mu(X \cup e_i) - \mu(X) \leq \mu(X \cup e_1 \cup e_2) - \mu(X) \leq \mu(\{e_1, e_2\}) = \mu(e_i) \cdots (*)$$

If $\mu(e_1) = 0$ then it is easy to see that

$$\mu(X \cup e_1) = \mu(X \cup e_2) = \mu(X \cup e_1 \cup e_2) \quad \forall X \subseteq S \cdots (**)$$

Let $\mu(e_1) = 1$. Suppose $\mu(X \cup e_1) - \mu(X) = 1$. We claim $e_2 \notin X$, as otherwise

$$\mu(X \cup e_1) - \mu(X) \leq \mu(\{e_1, e_2\}) - \mu(e_2) = 0,$$

a contradiction. But then

$$\mu(X \cup e_2) + \mu(\{e_1, e_2\}) \geq \mu(X \cup e_1 \cup e_2) + \mu(e_2)$$

i.e., $\mu(X \cup e_2) = \mu(X \cup e_1 \cup e_2)$ (since $\mu(\cdot)$ is increasing and $\mu(\{e_1, e_2\}) = \mu(e_2)$). Similarly $\mu(X \cup e_1) = \mu(X \cup e_1 \cup e_2)$. Thus the desired equality $(**)$ holds. So we need only consider the case where

$$\mu(X \cup e_1) - \mu(X) = 0.$$

By the above argument $\mu(X \cup e_2) - \mu(X)$ cannot be 1 and is therefore 0. Now, by the submodularity of $\mu(\cdot)$, we must have

$$\mu(X \cup e_1 \cup e_2) - \mu(X \cup e_1) \leq \mu(X \cup e_2) - \mu(X) = 0,$$

from which the desired equality $(**)$ follows.

□

9.6 Connectedness for Semimodular Functions

When a submodular function can be expressed as the direct sum of other such functions, problems involving it drastically simplify. We essentially have to look at much smaller underlying sets which are disconnected under the function. We sketch elementary ideas regarding connectedness in this section.

We introduce the notion of an elementary separator of a submodular function below. This notion is a generalization of 2-connectedness for graphs and connectedness for matroids.

Definition 9.6.1 *Let $\mu(\cdot)$ be a submodular (supermodular) function on subsets of S with $\mu(\emptyset) = 0$. A set $E \subseteq S$ is a* **separator** *of $\mu(\cdot)$ iff*

$$\mu(E) + \mu(S - E) = \mu(S).$$

A minimal nonvoid separator is called an **elementary separator**.

Theorem 9.6.1 *(k) Let $\mu(\cdot)$ be a submodular (supermodular) function on subsets of S with $\mu(\emptyset) = 0$. Then,*

i.

$$\mu(X_1) + \cdots + \mu(X_n) \geq \mu(X_1 \cup \cdots \cup X_n) \quad \forall X_1, \cdots X_n \subseteq S, X_i \cap X_j = \emptyset, i \neq j$$

$$(\mu(X_1) + \cdots + \mu(X_n) \leq \mu(X_1 \cup \cdots \cup X_n) \quad \forall X_1, \cdots X_n \subseteq S, X_i \cap X_j = \emptyset, i \neq j).$$

ii. if E_1, E_2 are separators of $\mu(\cdot)$, then so are $E_1 \cup E_2, E_1 \cap E_2$.

iii. E is a separator of $\mu(\cdot)$ iff

$$\mu(X_1) + \mu(X_2) = \mu(X_1 \cup X_2) \quad \forall X_1 \subseteq E, X_2 \subseteq S - E$$

(Thus when E is a separator studying $\mu(\cdot)$ reduces to studying $\mu/\mathbf{E}(\cdot)$, $\mu/\mathbf{S} - \mathbf{E}(\cdot)$. In other words $\mu(\cdot) = (\mu/\mathbf{E} \oplus \mu/\mathbf{S} - \mathbf{E})(\cdot)$).

iv. E is a separator of $\mu(\cdot)$ iff

$$\mu/\mathbf{E}(\cdot) = \mu \diamond \mathbf{E}(\cdot).$$

v. if E is a separator of $\mu(\cdot)$, it is also a separator of $\mu^d(\cdot)$ and $\mu^*(\cdot)$.

Proof: We will handle only the submodular case. The supermodular situation is similar.

i. If X_1, X_2 do not intersect we have

$$\mu(X_1) + \mu(X_2) \geq \mu(X_1 \cup X_2) + \mu(\emptyset).$$

Since $\mu(\emptyset) = 0$, the result follows by induction on the number of sets.

ii. We have,

$$\mu(E_1) + \mu(E_2) \geq \mu(E_1 \cup E_2) + \mu(E_1 \cap E_2),$$
$$\mu(S - E_1) + \mu(S - E_2) \geq \mu(S - (E_1 \cup E_2)) + \mu(S - (E_1 \cap E_2)).$$

Adding the two inequalities we get,

$$\mu(E_1) + \mu(E_2) + \mu(S - E_1) + \mu(S - E_2)$$
$$\geq \mu(E_1 \cup E_2) + \mu(E_1 \cap E_2) + \mu(S - (E_1 \cup E_2)) + \mu(S - (E_1 \cap E_2))$$
$$\geq 2\mu(S).$$

But E_1, E_2 are separators and hence the $LHS = 2\mu(S)$. Thus the inequalities are throughout equalities and therefore

$$\mu(E_1 \cup E_2) + \mu(S - (E_1 \cup E_2)) = \mu(S)$$
$$\mu(E_1 \cap E_2) + \mu(S - (E_1 \cap E_2)) = \mu(S)$$

as required.

iii. The 'if' part is trivial. To show the 'only if' part, let E be a separator. Let E_2 denote $S - E$ and let $X_1 \subseteq E, X_2 \subseteq E_2$. We need to show that

$$\mu(X_1) + \mu(X_2) = \mu(X_1 \cup X_2)$$

We already have,

$$\mu(X_1) + \mu(X_2) \geq \mu(X_1 \cup X_2)$$
$$\mu(X_1 \cup X_2) + \mu(E_2) \geq \mu(X_1 \cup E_2) + \mu(X_2)$$
$$\mu(X_1 \cup X_2) + \mu(E) \geq \mu(X_2 \cup E) + \mu(X_1).$$

Hence (looking at the extreme LHS and RHS of the inequalities)

$$\mu(X_1 \cup X_2) + \mu(E_2) + \mu(X_1 \cup X_2) + \mu(E)$$
$$\geq \mu(X_1 \cup E_2) + \mu(X_2) + \mu(X_2 \cup E) + \mu(X_1)$$
$$\geq \mu(X_1 \cup X_2 \cup E \cup E_2) + \mu(X_1 \cup X_2) + \mu(X_1) + \mu(X_2)$$
$$\geq \mu(S) + \mu(X_1 \cup X_2) + \mu(X_1) + \mu(X_2).$$

Since E is a separator, we have,

$$\mu(S) = \mu(E) + \mu(E_2).$$

Hence (looking at the extreme LHS and the extreme RHS of the inequalities)

$$\mu(X_1 \cup X_2) \geq \mu(X_1) + \mu(X_2)$$

and the result follows, the reverse inequality already being shown.

iv. Observe that

$$\mu/\mathbf{E}(X) = \mu\diamond\mathbf{E}(X) \quad \forall X \subseteq E$$

iff

$$\mu(X) = \mu((S - E) \cup X) - \mu(S - E) \quad \forall X \subseteq E.$$

We see that this last condition, using the previous sections of the present problem, is equivalent to E being a separator.

v. If E is a separator of $\mu(\cdot)$, it is easily verified that

$$\mu^d(E) + \mu^d(S - E) = \mu^d(S),$$

$$\mu^*(E) + \mu^*(S - E) = \mu^*(S).$$

Since this condition is sufficient for E to be a separator both when the function is submodular as well as when it is supermodular, the result follows.

□

9.7 *Semimodular Polyhedra

A powerful technique to study a class of real(rational) valued set functions is to associate a polyhedron with it and study the geometry of the polyhedron. In this section we begin with the simple notions of a set polyhedron and its dual and their specialization to semimodular polyhedra. We show that any 'polyhedrally tight' set function can be naturally extended to a convex function over $\Re^n$. In particular submodular functions are polyhedrally tight. Their extension due to Lovász [Lovász83] (called Lovász extension [Fujishige91]), has a very simple alternative description.

Definition 9.7.1 *Let $f(\cdot)$ be a real valued set function on 2^S, $S \equiv \{e_1, \cdots, e_n\}$. Let χ_X denote the characteristic vector of $X \subseteq S$. When* $\mathbf{x}$ *is a real vector let*

$$x(X) \equiv (\chi_X)^T \mathbf{x} \quad \forall X \subseteq S.$$

Then the **polyhedron associated with** *$f(\cdot)$, denoted by P_f is defined as follows:A vector $\mathbf{x} \in \Re^S$ belongs to P_f iff*

$$x(X) \leq f(X) \quad \forall X \subseteq S.$$

We say $f(\cdot)$ is **polyhedrally tight** *iff for each $X \subseteq S$ there exists a vector $\mathbf{x} \in P_f$ such that $x(X) = f(X)$. The* **dual polyhedron associated with** *$f(\cdot)$ denoted by P_f^d is defined as follows:A vector $\mathbf{x} \in \Re^S$ belongs to P_f^d iff*

$$x(X) \geq f(X) \quad \forall X \subseteq S.$$

We say $f(\cdot)$ is **dually polyhedrally tight** *iff for each $X \subseteq S$ there exists a vector $\mathbf{x} \in P_f^d$ such that $\mathbf{x}(X) = f(X)$.*

We list some simple properties of polyhedrally tight functions in the next theorem.

Theorem 9.7.1 *(k)*

i. $\mathbf{x} \in P_f$ *and* $x(S) = f(S)$ *iff* $\mathbf{x} \in P^d_{f^d}$ *and* $x(S) = f(S)$.

ii. *if* $f(\cdot), g(\cdot)$ *are polyhedrally tight and* $\lambda \geq 0$ *then* $(\lambda f + g)(\cdot)$ *is polyhedrally tight.*

iii. *if* $A \subseteq S$ *and* $f(\cdot)$ *is polyhedrally tight, then* $f/\mathbf{A}(\cdot)$ *is polyhedrally tight.*

iv. *if* $f(\cdot)$ *is modular with* $f(\emptyset) = 0$ *then* P_f *is the set of all vectors beneath a single point and* $f(\cdot)$ *is polyhedrally tight.*

Proof:
i. We have, $x(X) \leq f(X) \quad \forall X \subseteq S$ and $x(S) = f(S)$ iff
$x(S-X) \geq f(S) - f(X) \quad \forall X \subseteq S$ and $\mathbf{x}(S) = f(S)$.

ii. If $\mathbf{x} \in P_f$ and $\mathbf{y} \in P_g$ such that $x(X) = f(X)$ and $\mathbf{y}(X) = g(X)$ then clearly $(\lambda\mathbf{x} + \mathbf{y}) \in P_{\lambda f + g}$ and further $(\lambda\mathbf{x} + \mathbf{y})(X) = (\lambda f + g)(X)$.

iii. This is immediate from the relevant definitions.

iv. Clearly $f(\cdot)$ is induced by the vector $\mathbf{x} \equiv (f(e_1), \cdots, f(e_n))$ (see Definition 10.2.2 and Theorem 10.2.2) and the vectors in P_f are precisely the set of all vectors less or equal to this vector. Since $f(X) = x(X) \quad \forall X \subseteq S$, $f(\cdot)$ is polyhedrally tight.

□

Exercise 9.12 *Let* $f(\cdot)$ *be a set function on subsets of* S *and let* $\mathbf{x}$ *denote also the modular function induced by the vector* $\mathbf{x}$. *Then* $P_{f+\mathbf{x}} = P_f + \mathbf{x}$, *where the latter addition denotes translation by the vector* $\mathbf{x}$.

When $f(\cdot)$ is submodular the problem of maximising a linear objective function over P_f is particularly easy. One need only use a greedy strategy. The next theorem speaks of this strategy. As a consequence it follows that $f(\cdot)$ must be polyhedrally tight. Further when $f(\cdot)$ is integral it turns out that it must have integral vertices.

Theorem 9.7.2 *(k) Let* $S = \{e_1, \cdots, e_n\}$ *and let* $f(\cdot)$ *be a submodular function on subsets of* S *such that* $f(\emptyset) = 0$. *Let* $\mathbf{c} \in \Re^S$ *be a nonnegative vector and let* $c(e_1) \geq \cdots \geq c(e_n)$. *Let* $\mathbf{x} \in \Re^S$ *be a vector such that*

$$x(e_1) = f(e_1), x(e_2) = f(\{e_1, e_2\}) - f(e_1)$$

$$\cdots x(e_n) = f(\{e_1 \cdots, e_n\}) - f(\{e_1 \cdots, e_{n-1}\}).$$

Then

i. $\mathbf{x}$ *is integral if* $f(\cdot)$ *is integral.*

ii. $\mathbf{x}$ *optimizes the linear program*

$$max\ \mathbf{c}^T\mathbf{z}, \mathbf{z} \in P_f.$$

Further if $\mathbf{c}$ *has a negative entry then the above linear program has no optimal solution.*

Proof:
i. This is immediate.

ii. Let T_j denote $\{e_1, \cdots, e_j\}$. We first show that $\mathbf{x} \in P_f$. If not then there exists a subset $T \subseteq S$ such that $x(T) > f(T)$. Let T have the smallest size consistent with this condition and let e_i be the element of T with the largest index. Observe that T cannot be null since $x(\emptyset) = f(\emptyset) = 0$. We have $x(T - e_i) \leq f(T - e_i)$. Next, $T_i \supseteq T$. Hence, by the submodularity of $f(\cdot)$,

$$f(T_i) - f(T_i - e_i) \leq f(T) - f(T - e_i).$$

But

$$x(e_i) = x(T_i) - x(T_i - e_i) = f(T_i) - f(T_i - e_i).$$

So $x(e_i) \leq f(T) - f(T - e_i)$. But

$$x(e_i) = x(T) - x(T - e_i).$$

Hence $x(T - e_i) > f(T - e_i)$, which contradicts the definition of T. Thus $\mathbf{x} \in P_f$. Next we show that $\mathbf{x}$ optimizes the linear program. We use LP duality. The dual linear program is

$$min \sum_{T \subseteq S} f(T) y_T$$

$$\sum_{T \ni e_i} y_T = c(e_i) \quad \forall e_i \in S, \mathbf{y} \geq \mathbf{0}.$$

(observe that $\mathbf{y}$ has one component for each subset of S). We select $y_T = c(e_i) - c(e_{i+1})$ if $T = T_i$, taking $c(e_{n+1})$ to be 0. Otherwise y_T is taken to be zero. It is easily verified that for such a selection $\mathbf{y} \geq \mathbf{0}$ and that

$$\sum_{T \ni e_i} y_T = c(e_i) \quad \forall e_i \in S.$$

Further,

$$\begin{aligned} &\sum f(T) y_T = \\ &(c(e_1) - c(e_2)) f(e_1) + (c(e_2) - c(e_3)) f(\{e_1, e_2\}) + \cdots c(e_n) f(\{e_1, \cdots e_n\}). \\ &= c(e_1) f(e_1) + c(e_2)(f(\{e_1, e_2\}) - f(e_1)) \\ &+ \cdots c(e_n)(f(\{e_1, \cdots e_n\}) - f(\{e_1, \cdots e_{n-1}\})). \\ &= c(e_1) x(e_1) + \cdots + c(e_n) x(e_n) = \mathbf{c}^T \mathbf{x}. \end{aligned}$$

This implies that $\mathbf{x}$ and $\mathbf{y}$ are optimal solutions to the primal and dual programs respectively.

Finally let us consider the situation where $\mathbf{c}$ has a negative entry. We note that decreasing the component of any vector in P_f will not take it out of the polyhedron. Therefore the component corresponding to the negative entry in $\mathbf{c}$ can be indefinitely decreased remaining in the polyhedron but arbitrarily increasing the objective function.

□

Corollary 9.7.1 *(k) If $f(\cdot)$ is submodular (supermodular) with $f(\emptyset) = 0$ then $f(\cdot)$ is polyhedrally tight (dually polyhedrally tight).*

Proof: We first consider the situation where $f(\cdot)$ is submodular. Let $X \subseteq S$. In the statement of Theorem 10.7.2 we select $\mathbf{c} = \chi_X$. The selection procedure for $\mathbf{x}$ used in the statement of the theorem ensures that $x(X) = f(X)$. Further $\mathbf{x} \in P_f$. Next let $f(\cdot)$ be supermodular. Let $X \subseteq S$. In the polyhedron P_{f^d} we select a vector $\mathbf{x}$ such that $x(S-X) = f^d(S-X)$ and $x(S) = f(S)$ (the procedure given in the statement of the theorem permits this). This vector belongs to P^d_f and satisfies $x(X) = f(X)$ (see Theorem 10.7.1 (i)).

□

Corollary 9.7.2 *(k) If $f(\cdot)$ is submodular (supermodular) and integral then all the vertices of P_f (P^d_f) are integral.*

Proof: We will consider only the submodular case . If $\mathbf{x}$ is a vertex of the polyhedron P_f then there exists a vector $\mathbf{c}$ such that $\mathbf{c}^T\mathbf{z}$ reaches its maximum value (among all vectors of the polyhedron) only at $\mathbf{x}$. Clearly this vector (by Theorem 10.7.2) must be nonnegative . But then the procedure outlined in the same theorem yields an integral optimum if $f(\cdot)$ is integral. We conclude that this integral optimum must be the given vertex.
The supermodular case follows by noting that

- $g(\cdot)$ is supermodular iff $-g(\cdot)$ is submodular,
- $$max\ \mathbf{c}^T\mathbf{z}, \mathbf{z} \in P_{-g}$$

  is equivalent to
  $$min\ \mathbf{c}^T(-\mathbf{z}), -\mathbf{z} \in P^d_g.$$

□

We now show that there is a natural convex extension to every polyhedrally tight set function. We need the following definitions.

Definition 9.7.2 *Let $S \equiv \{e_1, \cdots .e_n\}$. Let $f(\cdot) : \Re^S \longrightarrow \Re$ and let $g(\cdot)$ be a set function on subsets of S defined by $g(X) \equiv f(\chi_X) \quad \forall X \subseteq S$. Then we say that $g(\cdot)$ is the* **set function induced by** *$f(\cdot)$.*

Definition 9.7.3 *Let $P(A, b)$ denote the polyhedron defined by the system of inequalities $\mathbf{Ax} \leq \mathbf{b}$, where $\mathbf{x}$ is a vector in $\Re^S$. Let $f_{Ab}(\cdot)$ be the function on $\Re^S$ defined by*
$$f_{Ab}(\mathbf{c}) \equiv (max\ \mathbf{c}^T\mathbf{x}, \mathbf{x} \in P(A, b)).$$

Then we say the function $f_{Ab}(\cdot)$ *is* **induced by the polyhedron** $P(A,b)$. *If* $P(A,b)$ *is empty we take* $f_{Ab}(\mathbf{c}) = -\infty \quad \forall \mathbf{c} \in \Re^S$. *Let*

$$f^{Ab}(\mathbf{c}) \equiv (min\ \mathbf{c}^T\mathbf{x}, \mathbf{x} \in P(A,b)).$$

Then we say the function $f^{Ab}(\cdot)$ *is* **dually induced by the polyhedron** $P(A,b)$. *If* $P(A,b)$ *is empty we take* $f^{Ab}(\mathbf{c}) = \infty \quad \forall \mathbf{c} \in \Re^S$.

Theorem 9.7.3 *(k)*

i. $f_{Ab}(\lambda \mathbf{c}_1 + \mu \mathbf{c}_2) \leq \lambda f_{Ab}(\mathbf{c}_1) + \mu f_{Ab}(\mathbf{c}_2), \lambda, \mu \geq 0.$

ii. The collection of vectors on which $f_{Ab}(\cdot)$ *takes finite values is closed under addition and nonnegative scalar multiplication (i.e., forms a cone).*

iii. $f^{Ab}(\lambda \mathbf{c}_1 + \mu \mathbf{c}_2) \geq \lambda f^{Ab}(\mathbf{c}_1) + \mu f^{Ab}(\mathbf{c}_2), \lambda, \mu \geq 0$ *and the collection of vectors on which* $f^{Ab}(\cdot)$ *takes finite values is closed under addition and nonnegative scalar multiplication.*

iv. Let $f(\cdot)$ *be a submodular function on subsets of* S. *Let* $f'(\cdot)$ *be the function induced by* P_f *and let* $f"(\cdot)$ *be the set function induced by* $f'(\cdot)$. *Let* $\mathbf{c}$ *be a vector such that* $c(e_1) \geq \cdots \geq c(e_n)$. *Let* $T_i, i = 1, 2, \cdots n$ *denote* $\{e_1, \cdots, e_i\}$. *Then*

(a) $f'(\cdot)$ *takes finite values on all nonnegative vectors.*

(b) $f"(X) = f(X) \quad \forall X \subseteq S$.

(c) [Lovász83] $f'(\mathbf{c}) = (c(e_1) - c(e_2))f(\chi_{T_1}) + \cdots (c(e_{n-1}) - c(e_n))f(\chi_{T_{n-1}}) + (c(e_n))f(\chi_{T_n})$.

Proof:
i. Let $f_{Ab}(\lambda \mathbf{c}_1 + \mu \mathbf{c}_2) = (\lambda \mathbf{c}_1 + \mu \mathbf{c}_2)^T\mathbf{x}$ for some $\mathbf{x} \in P(A,b)$. By the definition of $f_{Ab}(\cdot)$ it follows that $(\mathbf{c}_1)^T\mathbf{x} \leq f_{Ab}(\mathbf{c}_1)$ and $(\mathbf{c}_2)^T\mathbf{x} \leq f_{Ab}(\mathbf{c}_2)$. The result follows by multiplying the first inequality by λ and the second by μ and adding.

ii. This follows immediately from the preceding result.

iii. Similar to the proof of the first two parts. Note that when P_{Ab} is not void, $f^{Ab}(\cdot)$ would take value $-\infty$ when it does not take a finite value).

iv(a) This follows from the definition of $f'(\cdot)$ and Theorem 10.7.2.

iv(b) This follows from the definition of $f"(\cdot)$, the fact that $f(\cdot)$ is polyhedrally tight (Corollary 10.7.1) and by use of the greedy strategy.

iv(c) By using the procedure given in the statement of Theorem 10.7.2 we can construct a vector $\mathbf{x}$ which optimizes

$$max\ \chi_{T_i}^T\mathbf{z}, \quad \mathbf{z} \in P_f$$

for $i = 1, 2 \cdots n$. We then have, by the definition of $f'(\cdot)$,

$$f'(\mathbf{c}) \geq \mathbf{c}^T\mathbf{x}.$$

But

$$\mathbf{c} = (c(e_1) - c(e_2))\chi_{T_1} + \cdots (c(e_{n-1}) - c(e_n))\chi_{T_{n-1}} + (c(e_n))\chi_{T_n}$$

and

$$\mathbf{c}^T\mathbf{x} = (c(e_1) - c(e_2))x(T_1) + \cdots (c(e_{n-1}) - c(e_n))x(T_{n-1}) + (c(e_n))x(T_n).$$

Noting that

$$x(T_i) = f(T_i) = f'(\chi_{T_i}), i = 1, 2, \ldots n,$$

we have,

$$f'(\mathbf{c}) \geq (c(e_1) - c(e_2))f'(\chi_{T_1}) + \cdots (c(e_{n-1}) - c(e_n))f'(\chi_{T_{n-1}}) + (c(e_n))f'(\chi_{T_n}).$$

The reverse inequality follows from the statement of the first part of the present theorem.

□

Remark: Observe that if $c(e_1) \geq \cdots \geq c(e_n)$, then

$$\mathbf{c} = (c(e_1) - c(e_2))(\chi_{T_1}) + \cdots (c(e_{n-1}) - c(e_n))(\chi_{T_{n-1}}) + (c(e_n))(\chi_{T_n}).$$

Thus $f'(\mathbf{c})$ is obtained by performing the above linear combination of the $f(\chi_{T_i})$.

The next result is one of the deepest in submodular function theory. It is a very good starting point for proving many of the important results in this area. Our proof is, however, not polyhedral even though the result is naturally polyhedral. A generalization of the result using polyhedral methods (in fact the Hahn-Banach Separation Theorem) is given in Problem 10.14

Theorem 9.7.4 (The 'Sandwich Theorem)' *[Frank82] Let $f(\cdot), g(\cdot)$ be submodular and supermodular functions defined on subsets of S such that $f(\cdot) \geq g(\cdot)$. Then there exists a modular function $h(\cdot)$ on subsets of S such that $f(\cdot) \geq h(\cdot) \geq g(\cdot)$ (equivalently, such that $h(\cdot)$* **separates** *or* **lies between** *$f(\cdot)$ and $g(\cdot)$). Further if $f(\cdot)$, $g(\cdot)$ are integral then $h(\cdot)$ can be chosen to be integral.*

Our proof of the theorem is an algorithmically more efficient version of that due to Lovász and Plummer [Lovász+Plummer86] and is based on the following lemma.

Lemma 9.7.1 *Let $\emptyset \subset A \subset S$ be such that $f(A) = g(A)$ and let there exist modular functions $h_A(\cdot), h_{S-A}(\cdot)$ on subsets of $A, S - A$ respectively such that $f/\mathbf{A}(\cdot) \geq h_A(\cdot) \geq g/\mathbf{A}(\cdot)$ and $f\diamond(\mathbf{S}-\mathbf{A})(\cdot) \geq h_{s-A}(\cdot) \geq g\diamond(\mathbf{S}-\mathbf{A})(\cdot)$. Then $f(\cdot) \geq (h_A \oplus h_{S-A})(\cdot) \geq g(\cdot)$.*

Proof of the Lemma: Let $X \subseteq S$. Clearly,

$$f(X \cap A) \geq h_A(X \cap A) \geq g(X \cap A)$$

$$\begin{aligned} f(X \cup A) &= f(A) + f\diamond(\mathbf{S}-\mathbf{A})(X \cap (S - A)) \\ &\geq h_A(A) + h_{S-A}(X \cap (S - A)) \\ &\geq g(A) + g\diamond(\mathbf{S}-\mathbf{A})(X \cap (S - A)) = g(X \cup A). \end{aligned}$$

Since $f(\cdot), g(\cdot)$ are submodular and supermodular respectively, we have

$$f(X) - f(X \cap A) \geq f(X \cup A) - f(A) = f\diamond(\mathbf{S}-\mathbf{A})(X \cap (S-A)),$$

$$g\diamond(\mathbf{S}-\mathbf{A})(X \cap (S-A)) = g(X \cup A) - g(A) \geq g(X) - g(X \cap A).$$

Further we are given that

$$f\diamond(\mathbf{S}-\mathbf{A})(X \cap (S-A)) \geq h_{S-A}(X \cap (S-A)) \geq g\diamond(\mathbf{S}-\mathbf{A})(X \cap (S-A)).$$

Thus

$$f(X) \geq h_{S-A}(X \cap (S-A)) + f(X \cap A) \geq h_S(X)$$

and

$$g(X) \leq h_{S-A}(X \cap (S-A)) + g(X \cap A) \leq h_S(X),$$

where $h_S \equiv h_A \oplus h_{S-A}$. Hence, $f(X) \geq h(X) \geq g(X)$.

□

Proof of Theorem 10.7.4: Assume without loss of generality that $f(\emptyset) = 0$. Suppose $f(\cdot)$ has every element e as a separator. Then $f(\cdot)$ is already modular and the theorem is trivially true. We therefore assume that the submodular function has atleast one element say e that is not a separator. Let Y be the subset of all elements of S which are singleton separators. We will assume that the theorem is true for ($|S| < n$) and for ($|S| = n, |S-Y| < m$). We note that the theorem is trivially true for $|S| = 1$ and also for $|S-Y| = 0$. We will now prove the result when ($|S| = n, |S-Y| = m$). We have $f(S) < f(S-e) + f(e)$. Let $f'(\cdot)$ be the function

$$f'(X) \equiv f(X-e) + (f(S) - f(S-e)), e \in X$$
$$\equiv f(X), e \notin X.$$

The function $f'(\cdot)$ is obviously a submodular function with e as a separator (see Theorem 10.6.1) and further $f(\cdot) \geq f'(\cdot)$ (since, by the submodularity of $f(\cdot)$, $f(X) - f(X-e) \geq f(S) - f(S-e)\quad \forall X$ such that $e \in X$). Let the set A minimize $(f'-g)(\cdot)$.

i. Case 1. If $(f'-g)(A)$ is nonnegative we have found a submodular function (namely $f'(\cdot)$), which lies between $f(\cdot)$ and $g(\cdot)$ and further has one more singleton separator (namely $\{e\}$), than $f(\cdot)$ has. Also $f'(\cdot)$ is integral if $f(\cdot)$ is. By induction on $|S-Y|$, the theorem is true for $f'(\cdot)$ and therefore for $f(\cdot)$.

ii. Case 2. Let $(f'-g)(A)$ be negative. Clearly A is not null or equal to S . Let $f"(\cdot)$ be defined by

$$f"(X) \equiv min(f(X), f'(X) + (g-f')(A)) \quad \forall X \subseteq S.$$

It can be verified that $f"(\cdot)$ is submodular (directly or by using the idea of convolution, to be introduced in the next chapter). Further $f(\cdot) \geq f"(\cdot) \geq$

$g(\cdot)$ and $f"(A) = g(A)$. It follows that $f"/\mathbf{A}(\cdot) \geq g/\mathbf{A}(\cdot)$, $f"\diamond(\mathbf{S}-\mathbf{A})(\cdot) \geq g\diamond(\mathbf{S}-\mathbf{A})(\cdot)$. By induction on $|S|$, we may assume that there are modular functions $h_A(\cdot)$ between $f"/\mathbf{A}(\cdot)$ and $g/\mathbf{A}(\cdot)$ and $h_{S-A}(\cdot)$ between $f"\diamond(\mathbf{S}-\mathbf{A})(\cdot)$ and $g\diamond(\mathbf{S}-\mathbf{A})(\cdot)$. Now by Lemma 10.7.1, $(h_A \oplus h_{S-A})(\cdot)$ lies between $f"(\cdot)$ and $g(\cdot)$ and therefore also lies between $f(\cdot)$ and $g(\cdot)$. Further, in case $f(\cdot), g(\cdot)$ are integral it is clear that $f"(\cdot)$ is integral and we may assume by induction that $h_A(\cdot), h_{S-A}(\cdot)$ are integral. It follows that $(h_A \oplus h_{S-A})(\cdot)$ is also integral. Thus the theorem is true when $|S| = n, |S-Y| = m$.

□

Remark: The above proof of the Sandwich Theorem contains an efficient algorithm for finding the separating modular function provided we have an efficient algorithm for minimising submodular functions (in this case the function $(f'-g)(\cdot)$). It may be verified that the algorithm requires no more than $|S|$ submodular function minimizations.

Exercise 9.13 *Let $\mathcal{M}$ be a matroid on S. Let $r(X) \equiv r(\mathcal{M}.X)$, $r'(X) \equiv r(\mathcal{M} \times X)$, $\nu(X) \equiv \nu(\mathcal{M} \times X), \nu'(X) \equiv \nu(\mathcal{M}.X)$ as in Exercise 10.11. We have already seen that $r(\cdot), \nu(\cdot)$ are submodular and $r'(\cdot), \nu'(\cdot)$ are supermodular. Show that*

i. $r(\cdot) \geq r'(\cdot)$ and $\nu(\cdot) \geq \nu'(\cdot)$.

ii. Find vectors $\mathbf{w}_r, \mathbf{w}_\nu$ such that $r(\cdot) \geq \mathbf{w}_r(\cdot) \geq r'(\cdot)$ and $\nu(\cdot) \geq \mathbf{w}_\nu(\cdot) \geq \nu'(\cdot)$.

Exercise 9.14 *Let $\rho(\cdot)$ be a polymatroid rank function on subsets of S. Let*

$$\rho'(X) = \rho(S) - \rho(S-X).$$

i. Show that $\rho(\cdot) \geq \rho'(\cdot)$.

ii. Choose a vector $w(\cdot)$ so that $\rho(\cdot) \geq w(\cdot) \geq \rho'(\cdot)$.

9.8 Symmetric Submodular Functions

A key problem in submodular function theory is that of minimization. To be precise, the search is for a practically efficient polynomial time algorithm for a general submodular function which is available through a rank oracle. (The oracle will give the value of the function on any given subset). The input size for such an algorithm is determined by the size of the underlying set and the maximum number of bits needed to represent a value of the submodular function. As we shall see later minimization is equivalent to convolution of an appropriate polymatroid rank function with a weight vector. The solution is known to the minimization problem in many practical situations: e.g. minimum directed cut in a graph, convolution of a matroid rank function with a weight vector [Cunningham84], [Narayanan95b].

For the general problem the ellipsoid method [Grötschel+Lovász+Schrijver81] does provide a polynomial algorithm which however is practically useless. There are a few algorithms for minimization [Cunningham85], [Sohoni92] which are practical but pseudo polynomial (for integral functions the algorithm is polynomial in the size of the underlying set and the maximum value of the function). Recently the case of symmetric submodular functions was solved in a surprisingly simple way. We describe this solution in this section.

Definition 9.8.1 *A set function $g : 2^S \longrightarrow \Re$ is symmetric iff $g(X) = g(S - X) \quad \forall X \subseteq S$.*

Example 9.8.1 *(k) The following are symmetric submodular functions.*

i. *Cut function on the vertex subsets of a graph.*

ii. *The function $|c|(X)$, $X \subseteq E(\mathcal{G})$ where $c(X) \equiv$ the set of vertices common to edges in X and $E(\mathcal{G}) - X$.*

iii. *$|c|(X)$ acting on the left vertex set of a bipartite graph. Here $|c|(X) \equiv$number of right side vertices adjacent to vertices in X as well as vertices in $V_L - X$ (see Example 10.2.8).*

iv. *The function $\xi(X) \equiv r(M.X) - r(M \times X), X \subseteq S$, where $\mathcal{M}$ is a matroid on S and $r(\cdot)$ denotes the rank function of the matroid.*

v. *$\theta(X) \equiv f(X) + f(S - X), X \subseteq S$, where $f(\cdot)$ is a submodular function on subsets of S.*

Exercise 9.15 *Show that the functions in Example 10.8.1 are symmetric submodular functions.*

Remark: Prior to the work of Nagamochi and Ibaraki [Nagamochi+Ibaraki92a] (see also [Nagamochi+Ibaraki92b] [Nagamochi+Ono+Ibaraki94]) the standard way of finding min cut was through flow techniques . The above authors used a special (linear time) decomposition of a graph into forests to identify a pair of vertices for which a minimum separating cut was immediately available. This was stored. This pair was fused and the process repeated until the graph had only two vertices. The minimum value cut among all the stored cuts gives the minimum cut. A simpler algorithm of the same complexity was found by M. Stoer and F. Wagner [Stoer+Wagner94] and, independently, by A. Frank [Frank94]). It is this algorithm that we generalize below. The version we present is essentially the same as the one due to M. Queyranne [Queyranne95]. However we have tried to bring out the relationship to the Stoer-Wagner algorithm more strongly.

We follow the notation of Stoer-Wagner and essentially do a line by line translation of their algorithm for finding a min cut to that for minimising a symmetric submodular function over all sets not equal to the full set or the void set.

Let $f : 2^S \longrightarrow \Re$ be a submodular function. Let $c(A, B) \equiv \frac{1}{2}(f(A)+f(B)-f(A\cup B)) \quad \forall A, B \subseteq S, A\cap B = \emptyset$. Observe that $c(A, B) = c(B, A)$.

We then have $c(X, S-X) = \frac{1}{2}(f(X)+f(S-X)-f(S))$. If $g(X) \equiv c(X, S-X)$ it is clear that $g(\cdot)$ is symmetric and submodular. We minimize $g(\cdot)$ over $X \subseteq S, X \neq \emptyset, X \neq S$.

Subroutine Minimum Phase (g, a, S_j)

BEGIN

$A \leftarrow \{a\}$

while $(A \neq S_j)$ {

add to A the element $e \notin A$ such that $c(A, e)$ is the largest i.e., $f(A \cup e) - f(e)$ is least.

}

Store the 'pair of the phase' $(e_n, S_j - e_n)$ and the 'value of the pair of the phase' $c(e_n, S_j - e_n)$.

Fuse e_n, e_{n-1}.

Replace the set S_j by

$S_{j+1} \equiv \{e_1, e_2, \ldots, e_{n-2}, e_{n-1} \cup e_n\}$.

(Note that e_{n-1}, e_n are treated as subsets of S.)

Replace the function g by $g_{fus.S_{j+1}}$.

END *Subroutine Minimum Phase.*

Algorithm Symmetric:

Initialize $S_0 = S$.

BEGIN

The algorithm has $|S|-1$ phases of *Subroutine Minimum Phase.*

Value of current minimum pair $= \infty$

while $(|S_j| > 1)$ {

Subroutine Minimum Phase (g, a, S_j).

If the value of the pair of the phase $(e_n, S_j - e_n)$ has a lower value than the current minimum pair then store $(e_n, S_j - e_n)$ as the current minimum pair.

}

Let the minimum pair be $(e_{n_i}, S_j - e_{n_i})$ at the end of the algorithm (i.e., when $|S_j| = 2$). Output sets $e_{n_i}, S - e_{n_i}$ as minimising sets (non void proper subsets of S) for $g(\cdot)$.

END **Algorithm Symmetric.**

We justify algorithm Symmetric through the following theorem. We first introduce some convenient notation and a couple of lemmas.

In any phase the elements of the current set S_i are ordered as say $(e_1, e_2, \ldots, e_k)$. Let $A_j \equiv \{e_1, e_2, \ldots, e_{j-1}\}, j = 2, \ldots, k+1$. (In particular $A_{k+1} = S_i$). Let $A \subseteq \{e_1, e_2, \ldots, e_k\}$ with $e_k \in A, e_{r-1} \in A$ and $\{e_r, \ldots, e_{k-1}\} \cap A = \emptyset$.

Lemma 9.8.1

$$c(A_k, e_k) - c(A_r, e_k) \leq c(A, A_{k+1} - A) - c(A_{r+1} \cap A, A_{r+1} - A_{r+1} \cap A)$$

Proof: We note that

$$c(X, Y) \equiv \frac{1}{2}[f(X) + f(Y) - f(X \cup Y)]$$

Hence the LHS of the inequality in the statement simplifies to

$$\frac{1}{2}[(f(A_k) - f(A_r) - f(A_{k+1}) + f(A_r \cup e_k)]$$

while the RHS simplifies to

$$\frac{1}{2}[(f(A) + f(A_{k+1} - A) - f(A_{k+1}) - f(A_{r+1} \cap A)$$

$$-f(A_{r+1} - A_{r+1} \cap A) + f(A_{r+1})].$$

We therefore have to show (after simplification) that the following inequality is valid. $f(A_k) - f(A_{r+1}) + f(A_r \cup e_k) - f(A_r)$

$$\leq \quad f(A_{k+1} - A) - f(A_{r+1} - A_{r+1} \cap A) + f(A) - f(A_{r+1} \cap A). \tag{9.5}$$

Now we note that

$$\begin{aligned} A_k - A_{r+1} &= (A_{k+1} - A) - (A_{r+1} - A_{r+1} \cap A) \\ &= \{e_{r+1}, \ldots, e_{k-1}\} \\ \textit{and } A_k &\supseteq A_{k+1} - A. \end{aligned}$$

Hence by submodularity

$$f(A_k) - f(A_{r+1}) \leq f(A_{k+1} - A) - f(A_{r+1} - A_{r+1} \cap A).$$

By a similar argument we see that

$$f(A_r \cup e_k) - f(A_r) \leq f(A) - f(A_{r+1} \cap A)$$

(since $A_r \cup e_k - A_r = e_k = A - A_{r+1} \cap A$). This proves the inequality(10.5) and hence the lemma.

□

Lemma 9.8.2 *Let $(e_k, S_i - e_k)$ be the pair of the phase. Then*

$$c(e_k, S_i - e_k) \leq c(A, S_i - A)$$

where $e_k \in A$ and $e_{k-1} \in S_i - A$.

Proof: The proof is by induction on the size of S_i. The lemma is true for $|S_i| = 2$. Suppose it to be true for $|S_i| \leq k$.
Let $A \subseteq S_i$ with $e_k \in A, e_{r-1} \in A, \{e_r, \ldots, e_{k-1}\} \cap A = \emptyset$.
By induction it follows that

$$c(A_{r+1} \cap A, A_{r+1} - A_{r+1} \cap A) \geq c(A_r, e_r) \tag{9.6}$$

(since $e_{r-1} \in A, e_r \in A_{r+1}$)
We have

$$\begin{aligned} c(A_k, e_k) &= c(A_r, e_k) + (c(A_k, e_k) - c(A_r, e_k)) \\ &\leq c(A_r, e_r) + (c(A_k, e_k) - c(A_r, e_k)) \\ &\quad (\textit{since in the algorithm } e_r \textit{ has been chosen so that} \\ c(A_r, e_r) &\geq c(A_r, e) \quad \forall e \in (S_i - A_r)). \end{aligned}$$

Hence,

$$\begin{aligned} c(A_k, e_k) &\leq c(A_{r+1} \cap A, A_{r+1} - A_{r+1} \cap A) + (c(A_k, e_k) - c(A_r, e_k)) \ (\textit{by } (10.6)) \\ &\leq c(A_{r+1} \cap A, A_{r+1} - A_{r+1} \cap A) + (c(A, A_{k+1} - A) \\ &\quad - c(A_{r+1} \cap A, A_{r+1} - A_{r+1} \cap A)) \ \textit{by } (\textit{Lemma } 10.8.1). \end{aligned}$$

Thus

$$c(A_k, e_k) \leq c(A, A_{k+1} - A),$$

which is the required inequality.

□

Theorem 9.8.1 *The current minimum pair $(e_{n_j}, S_j - e_{n_j})$ at the end of the Algorithm Symmetric yields the minimising set e_{n_j} for $g(\cdot)$.*

Proof: The proof is by induction on the size of the set S. The theorem is clearly valid when the $|S| = 2$. Suppose it to be valid when the size of the set is $n - 1$. Let $|S| = n$. Now let the last two elements of the first phase be e_{n-1}, e_n. If the minimum value pair $(X, S - X), X \subseteq S, X \neq \emptyset, X \neq S$ had $\{e_{n-1}, e_n\} \subseteq X$ then we can fuse $\{e_{n-1}, e_n\}$ and work with the set $S_2 = \{e_1, e_2, \ldots, e_{n-2}, \{e_{n-1}, e_n\}\}$ and the function $g_{fus.S_2}$. By induction, in the subsequent phases of the algorithm the minimum pair will be revealed as a pair of some phase. On the other hand if the minimum pair $(X, S - X)$ had $e_n \in X, e_{n-1} \in S - X$, by Lemma 10.8.2 the pair of the first phase has this minimum value. This completes the proof of the theorem.

□

Complexity of Algorithm Symmetric

Suppose there is an oracle that gives the value of $c(X, Y)$ for given $X, Y \subseteq S,\ X \cap Y = \emptyset$. If the current set is S_i the number of calls to this oracle is $O(|S_i|^2)$ during the phase. At the end of the phase the set reduces in size by one. We continue until the set reaches size two. Hence the total number of calls to the oracle is $O(|S_i|^3)$.

Special cases can be handled much faster: for instance in the case of $cut(\cdot)$ function of a graph Stoer & Wagner show that the complexity is $O(|E| + |V| log\ |V|)$ elementary operations per phase, giving an overall complexity of $O(|V||E| + |V|^2 log\ |V|)$.

The case of the symmetric function $(\Gamma_L - E_L)(\cdot)$ of a bipartite graph $B \equiv (V_L, V_R, E)$ is almost identical. The complexity here per phase is $O(|E| + |V_L| log\ |V_L|)$ and overall is $O(|V_L||E| + |V_L|^2 log\ |V_L|)$.

Exercise 9.16 *(k) For any function $f(\cdot)$ show that $g(X) = \frac{1}{2}(f(X)+f(S-X))$ is symmetric. If $f(\cdot)$ is a symmetric function show that $c(X, S-X) = f(X)$ provided $f(S) = 0$.*

Exercise 9.17 *If $f(\cdot)$ is a symmetric function which of the following operations preserve symmetry ?*

i. Contraction.

ii. Restriction.

iii. Comodular dual.

iv. Contramodular dual.

Exercise 9.18 *For any submodular function $f(\cdot)$ show (using the notation of Section 10.8) that $c(A, B) \geq c(X, Y)$ whenever $X \subseteq A, Y \subseteq B, A \cap B = \emptyset$. Interpret this statement for the cut function of a graph.*

Exercise 9.19 *Let $g(\cdot) : 2^S \longrightarrow \Re$ be a symmetric submodular function. Give an algorithm to minimize $g(\cdot)$ over*

i. all subsets $X,\ A \subseteq X \subset S$.

ii. all subsets $X,\ \emptyset \subset X \subseteq A$.

Exercise 9.20 *Specialize* **Algorithm Symmetric** *to the case of*

i. the cut function $|cut|(\cdot)$ of a graph $\mathcal{G} \equiv (V, E)$.

ii. the function $(|\Gamma_L| - |E_L|)(\cdot)$ of the bipartite graph $B \equiv (V_L, V_R, E)$.

iii. the symmetric function arising from $|\Gamma_L|(\cdot)$ in B above.

Exercise 9.21 *In Theorem 10.5.2 we have shown how to subtract a modular function from a given submodular function to yield a polymatroid rank function. Work this case out for $|cut|(\cdot)$.*

9.9 Problems

Problem 9.5 *(k) Prove the principle of inclusion and exclusion for weight functions. (The intersection form: $w(X_1 \cup \cdots \cup X_n) = \sum w(X_i) - \sum w(X_i \cap X_j) + w(X_i \cap X_j \cap X_k) + \cdots$). State the union form of the principle. Suppose the value of such a function is known for arbitrary unions of sets from a collection $\{X_1, X_2 \cdots X_n\}$, show how to obtain its value on sets obtained by arbitrary intersection of such sets and their complements.*

Problem 9.6 *(k) Let $\rho : 2^S \longrightarrow \Re$ be a submodular (supermodular) function. Then show that*

$$\rho(A_1 \cup \ldots \cup A_n) \leq \sum \rho(A_i) - \sum_{i \neq j} \rho(A_i \cap A_j) + \sum_{i,j,k \ distinct} \rho(A_i \cap A_j \cap A_k) + \ldots + (-1)^{r+1} \sum_{i_1,\ldots,i_r \ distinct} \rho(A_{i_1} \cap \ldots \cap A_{i_r}) + \cdots$$

$$(\rho(A_1 \cup \ldots \cup A_n) \geq \sum \rho(A_i) - \sum_{i \neq j} \rho(A_i \cap A_j) + \sum_{i,j,k \ distinct} \rho(A_i \cap A_j \cap A_k) + \ldots + (-1)^{r+1} \sum_{i_1,\ldots,i_r \ distinct} \rho(A_{i_1} \cap \ldots \cap A_{i_r}) \cdots).$$

Problem 9.7 *Let $\mathcal{G}$ be a graph on vertices V and edges E. Let w be a nonnegative weight function on the edges of $\mathcal{G}$. As usual $w(X) \equiv \sum_{e \in X} w(e)$. Let $\rho : 2^E \longrightarrow \Re_+$ be defined by $\rho(X) \equiv$ maximum weight of a forest of the subgraph of $\mathcal{G}$ on X. Show that $\rho(\cdot)$ is submodular. Examine the case where $\rho(\cdot)$ is defined by minimum weight instead of maximum weight.*

Problem 9.8 *How would you generalize Problem 10.6 above for the adjacency and exclusivity functions of a bipartite graph, given a nonnegative weight function of the left vertex set?*

Problem 9.9 *(Generalization of Problem 10.6 to the matroid base case) Let $\mathcal{M}$ be a matroid on S. Let w be a nonnegative weight function on the elements of $\mathcal{M}$. Let $\rho : 2^S \longrightarrow \Re_+$ be defined by*

$$\rho(X) \equiv maximum\ weight\ base\ of\ \mathcal{M}.X$$

Show that $\rho(\cdot)$ is a polymatroid rank function.

Problem 9.10 *(k) Let $\mathbf{A}$ be a matrix with real entries. Let $R \equiv \{\mathbf{r_1}, \ldots, \mathbf{r_k}\}, C \equiv \{\mathbf{c_1}, \ldots, \mathbf{c_n}\}$ be the sets of rows and columns of $\mathbf{A}$ respectively. Let $\mathbf{A}_X$ denote the submatrix of $\mathbf{A}$ using only the rows in $X \subseteq R$ but all columns. Let $f : 2^R \longrightarrow \Re$ be defined as follows*

$$f(X) = log(det\ (\mathbf{A}_X \mathbf{A}_X^T)) \quad \forall X \subseteq R$$

Prove that $f(\cdot)$ is submodular.

Problem 9.11 *Let $\mathbf{A}$ be a matrix with linearly independent rows. Let T be a set of maximally independent columns of $\mathbf{A}$ and let the submatrix corresponding to the columns of T be the identity matrix. Let $\mathbf{A}_X, X \subseteq T$, denote the submatrix of $\mathbf{A}$ composed of those rows of $\mathbf{A}$ which have nonzero entries in one of the columns corresponding to X. Let $f(X) \equiv log(\mathbf{A}_X \mathbf{A}_X^T)$, $X \subseteq T$. Prove that $f(\cdot)$ is submodular.*

Problem 9.12 *Let* $\mathbf{A}$ *be a totally unimodular matrix (every subdeterminant of* $\mathbf{A}$ *is* $0, \pm 1$*) with columns* S*. Let* $\mathcal{M}$ *be the matroid on* S *associated with* $\mathbf{A}$*. Let* T *be a base of* $\mathcal{M}$*. Let*

$$f(X) \equiv log(number\ of\ bases\ of\ \mathcal{M} \times (S - (T - X))), X \subseteq T.$$

Prove that $f(\cdot)$ *is submodular.*

Problem 9.13 *[Fujishige78a], [Fujishige91] We consider some analogies between polymatroids and matroids in this problem. We note that if we perform the fusion operation on a matroid rank function we obtain a polymatroid rank function. Later, in the next chapter, we show that every polymatroid rank function can be so obtained. In other words every polymatroid rank function can be 'expanded' into a matroid rank function. If* $\rho : 2^S \longrightarrow \Re$ *is a polymatroid rank function we say that* $P_\rho \equiv \{\mathbf{x} \in \Re^S : \ \mathbf{x}(X) \leq \rho(X) \quad \forall X \subseteq S\}$ *is a polymatroid.*

An **independent vector** *of a polymatroid is a nonnegative vector in* P_ρ*, i.e., it is a vector* $\mathbf{x}$ *s.t.* $x(X) \leq \rho(X) \quad \forall X \subseteq S$*. We will show later that if* $\mathbf{x}$ *is an integral independent vector, in the expanded matroid there is an independent set* $T \subseteq \hat{S}$ *whose intersection with the set* $e \in S$ *has size* $x(e)$ *(Here* $\hat{S}$ *is the expanded version of the set* S*, the latter being a partition of* $\hat{S}$*).*

i. Let $\mathbf{x}$ *be a real vector on* S*. Define*

$$D(\mathbf{x}) \equiv \{X | X \subseteq S, x(X) = \rho(X)\}$$

Show that $D(\mathbf{x})$ *is closed with respect to union and intersection and hence has a unique maximal and a unique minimal set.*

ii. If $\mathbf{x}$ *is independent define the* **saturation function**

$$sat(\mathbf{x}) \equiv \bigcup \{X | X \subseteq S, \mathbf{x}(X) = \rho(X)\}$$

Show that

$$sat(\mathbf{x}) \equiv \{e | e \in S \quad \forall \alpha > 0 : \mathbf{x} + \alpha \chi_e \notin P_\rho \ \}$$

Observe that the saturation function generalizes the closure function of a matroid.

iii. Let $D(\mathbf{x}, e)$ *denote the collection of sets in* $D(X)$ *which have* e *as a member. Show that* $D(\mathbf{x}, e)$ *is closed under union and intersection and hence has a unique maximal and a unique minimal element.*

iv. For an independent vector $\mathbf{x} \in P_\rho$ *and* $e \in sat(\mathbf{x})$*, define the* **dependence function**

$$dep(\mathbf{x}, e) \equiv \bigcap \{X | e \in X \subseteq S, \mathbf{x}(X) = \rho(X)\}$$

Let $x(e) < \rho(e)$ *and let* $0 < \alpha < \rho(e) - \mathbf{x}(e)$*. Let* $\mathbf{x}' \equiv \mathbf{x} + \alpha \chi_e$ *and let* $\theta(X) \equiv \mathbf{x}'(X) - \rho(X) \quad \forall X \subseteq S$*. Let* $D'(\mathbf{x}')$ *denote the collection of sets where* $\theta(\cdot)$ *reaches a maximum. Show that*

(a) $D'(\mathbf{x}')$ *is closed under union and intersection and hence has a unique maximal and a unique minimal set.*

(b) *Let* K *be the minimal member of* $D'(\mathbf{x}')$*. Then* $K = dep(\mathbf{x}, e)$*.*

(c)

$$dep(\mathbf{x}, e) = \{e' | e' \in S, \exists \alpha > 0 : \mathbf{x} + \alpha(\chi_e - \chi_{e'}) \in P_\rho\}$$

Observe that the dependence function generalizes the fundamental circuit of a matroid.

Problem 9.14 *Let* $f : 2^S \longrightarrow \Re$ *be a submodular function with* $f(\emptyset) = 0$*. Let* $\mathbf{c}$ *be a nonnegative weight function on* S*. Let* $\mathbf{c}_X$ *denote the row vector on* S *with*

$$\mathbf{c}_X(e) = 0, \; e \notin X$$

$$\mathbf{c}_X(e) = c(e), \; e \in X$$

Let $\rho : 2^S \longrightarrow \Re$ *be defined by*

$$\rho(X) \equiv max \; \mathbf{c}_X^T \mathbf{x}, \; \mathbf{x} \in P_f.$$

Show that $\rho(\cdot)$ *is a submodular function.*

Problem 9.15 *We state the* **Hahn-Banach Separation Theorem** *below.*
Let $\mathcal{V}$ *be a normed space over* $\Re$ *and let* E_1, E_2 *be nonempty disjoint convex subsets of* $\mathcal{V}$*, where* E_1 *is open in* $\mathcal{V}$*. Then there is a real hyperplane in* $\mathcal{V}$ *which separates* E_1 *and* E_2 *in the following sense: For some linear functional* $g(\cdot)$ *over* $\mathcal{V}$ *and* $t \in \Re$*, we have*

$$g(y_1) > t \geq g(y_2) \quad \forall y_1 \in E_1, y_2 \in E_2.$$

Use this result to prove the following:

Let $f_1, f_2 : 2^S \longrightarrow \Re$ *be polyhedrally tight and dually polyhedrally tight set functions respectively with* $f_1(\cdot) \geq f_2(\cdot)$*. Then there is a modular function* $w(\cdot)$ *such that* $f_1(\cdot) \geq w(\cdot) \geq f_2(\cdot)$*.*

Problem 9.16 *Use Sandwich Theorem to prove the following:*
Let $\mathcal{M}_1, \mathcal{M}_2$ *be two matroids on* S*. Then the maximum size of a common independent set of the two matroids* $= min_{X \subseteq S} \; r_1(X) + r_2(S - X)$*, where* $r_1(\cdot), r_2(\cdot)$ *are the rank functions of the matroids* $\mathcal{M}_1, \mathcal{M}_2$ *respectively.*

Problem 9.17 *Let* $f : 2^S \longrightarrow \Re$ *be a submodular function. For each of the following cases construct an algorithm for minimising this function.*

i. $f(X - e) - f(X) \geq f((S - X) \cup e) - f(S - X) \quad \forall X \subseteq S, \; e \in X.$

ii. $f(X - e) - f(X) \leq f((S - X) \cup e) - f(S - X) \quad \forall X \subseteq S, \; e \in X.$

9.10 Notes

Submodular functions became an active area of research in optimization after Edmond's pioneering work. Earlier prominent workers who used such ideas were H. Whitney [Whitney35] and W.T. Tutte [Tutte58], [Tutte59], [Tutte61], [Tutte65], [Tutte71] (in their work on graphs and matroids), G. Choquet [Choquet55] (in his work on capacity theory) and O. Ore [Ore56] (in his work on graph theory). The present chapter follows Lovász's excellent review paper [Lovász83]. We have also made use of Fujishige's comprehensive monograph [Fujishige91] on the subject. In the present book, the polyhedral approach to submodular functions has not been emphasized. A thorough treatment of this subject through the polyhedral approach may be found in [Frank+Tardos88] and in the above mentioned monograph of Fujishige. The notion of comodular duality is due to McDiarmid [McDiarmid75]. The Sandwich theorem first appeared as a technical result in [Frank82]. Its importance as a central result in submodular function theory was recognized by Lovász and Plummer [Lovász+Plummer86].

9.11 Solutions of Exercises

E 10.1: For most of the examples the easiest route for proving sub or super modularity is to show that Inequality 10.3 holds.

Example 10.2.1: Let $X \subseteq Y \subseteq E(\mathcal{G})$ and let $a \in E(\mathcal{G}) - Y$. Then

$$V(X \cup a) = V(X) \biguplus (V(a) - V(X)),$$

$$V(Y \cup a) = V(Y) \biguplus (V(a) - V(Y)).$$

Clearly since $V(Y) \supseteq V(X)$ it follows that

$$V(a) - V(Y) \subseteq V(a) - V(X).$$

Hence

$$|V|(X \cup a) - |V|(X) \geq |V|(Y \cup a) - |V|(Y).$$

Example 10.2.2: Let $X \subseteq Y \subseteq V(\mathcal{G})$ and let $a \in V(\mathcal{G}) - Y$. Now

$$E(Z \cup a) = E(Z) \biguplus E_{Za}, Z \subseteq V(\mathcal{G}), a \in V(\mathcal{G}) - Z,$$

where $E_{Za}$is the set of all edges, with a as one end point, the other endpoint lying in $Z \cup a$. Clearly

$$E_{Xa} \subseteq E_{Ya}.$$

Thus

$$|E|(X \cup a) - |E|(X) \leq |E|(Y \cup a) - E(Y).$$

Example 10.2.3: Let $X \subseteq Y \subseteq V(\mathcal{G})$ and let $a \in V(\mathcal{G}) - Y$. Then

$$I(X \cup a) = I(X) \biguplus [I(a) - I(X)],$$

$$I(Y \cup a) = I(Y) \biguplus [I(a) - I(Y)],$$

Clearly

$$I(a) - I(X) \supseteq I(a) - I(Y).$$

The result follows.
Example 10.2.4: Let $X \subseteq Y \subseteq V(\mathcal{G}), a \in V(\mathcal{G}) - Y$. We have

$$\Gamma(X \cup a) = \Gamma(X) \biguplus (\Gamma(a) - \Gamma(X)),$$

$$\Gamma(Y \cup a) = \Gamma(Y) \biguplus (\Gamma(a) - \Gamma(Y)).$$

Clearly

$$(\Gamma(a) - \Gamma(X)) \supseteq (\Gamma(a) - \Gamma(Y)).$$

The result follows.
Example 10.2.5: Let $X \subseteq Y \subseteq V(\mathcal{G})$ and let $a \in V(\mathcal{G}) - Y$. Then

$$cut(X \cup a) = (cut(X) - cut(X) \cap cut(a)) \biguplus (cut(a) - cut(X)),$$

$$cut(Y \cup a) = (cut(Y) - cut(Y) \cap cut(a)) \biguplus (cut(a) - cut(Y)),$$

Clearly

$$cut(a) - cut(X) \supseteq cut(a) - cut(Y).$$

and $cut(X) \cap cut(a) \subseteq cut(Y) \cap cut(a)$ The result follows.
Example 10.2.6: In every one of these cases the method of the above solutions works.
Example 10.2.7: The proof is similar to that of Example 10.2.2. We only have to define E_{Za} to be the set of all vertices in V_R adjacent only to vertices in $Z \cup a$ and adjacent to a.
Example 10.2.8: We have seen that the function $|E_L|(\cdot)$ is supermodular in Example10.2.7. Next let $\Gamma_L(\cdot)$ denote $\Gamma/\mathbf{V_L}(\cdot)$. Now

$$c(X) = \Gamma_L(X) - E_L(X), X \subseteq V_L,$$

and

$$\Gamma_L(X) \supseteq E_L(X).$$

Hence

$$|c|(X) = |\Gamma_L|(X) - |E_L|(X).$$

Now $|\Gamma_L|(\cdot)$ is submodular while $|E|(\cdot)$ is supermodular. The result follows from the fact that subtraction of a supermodular function from a submodular function yields another submodular function. Such facts are presented in Exercise 10.4 .
Example 10.2.9: Let $X \subseteq Y \subseteq S$ and let $e \in E - Y$. Clearly if e is independent of Y it must be independent of X. Hence

$$r(X \cup e) - r(X) \geq r(Y \cup e) - r(Y).$$

The submodularity of $r(\cdot)$ follows.

Example 10.2.10: We parallel the argument of the previous example. Let $X \subseteq Y \subseteq E(\mathcal{G})$ and let $e \in E(\mathcal{G}) - Y$. If e does not form a circuit with edges in Y it is clear that e will not do so with edges in X either. Hence

$$r(X \cup e) - r(X) \geq r(Y \cup e) - r(Y).$$

So $r(\cdot)$ is submodular. We know that

$$\nu'(X) = |X| - r(X).$$

Since $|\cdot|$ is modular and $r(\cdot)$ is submodular it is easily seen that the function $\nu'(\cdot)$ is supermodular. Next observe that $r'(X \cup e) > r'(X)$ iff $E - X$ contains no circuit with e as a member. Clearly, whenever $Y \supseteq X$, if $E - X$ contains no circuit with e as a member neither will $E - Y$ contain a circuit with e as a member. Hence

$$r'(X \cup e) - r'(X) \leq r'(Y \cup e) - r'(Y).$$

This is equivalent to supermodularity of $r'(\cdot)$. The submodularity of $\nu(\cdot)$ follows since $\nu(X) =\mid X \mid - r'(X)$.

E 10.2: Let $\mathcal{G}$ be the graph on the vertex set $\{a, b, c, d\}$ with edges between $(a, b), (b,$
$c), (c, d), (d, a)$. Let $X = \{b\}, Y = \{c, b\}, Z = \{a, b, c\}$. Observe that

$$k(Y \cup a) - k(Y) = 1,$$

$$k(X \cup a) - k(X) = 0.$$

So $k(\cdot)$ cannot be submodular. On the other hand

$$k(Z \cup d) - k(Z) = 0,$$

$$k(X \cup d) - k(X) = 1.$$

So $k(\cdot)$ cannot be supermodular.

E 10.5:
i. This is immediate.

ii. Let the rank function of $\mathcal{G} \times X$ be $r'(\cdot)$. It is easily verified that $r'(\cdot)$ and $r \diamond \mathbf{X}(\cdot)$ are matroid rank functions (increasing, integral, submodular with zero value on $\emptyset$ and value not exceeding one on singletons). A matroid rank function $r(\cdot)$ is fully determined by its independent sets (sets on which $r(Y) = |Y|$). So we need to show that $r'(\cdot)$ and $r \diamond \mathbf{X}(\cdot)$ have the same independent sets. Now independent sets of $r'(\cdot)$ are the circuit free sets of $\mathcal{G} \times X$. Let $Y \subseteq X$. We know that this set contains no circuit of $\mathcal{G} \times X$ iff for each $Z \subseteq E(\mathcal{G}) - X$ that contains no circuit of $\mathcal{G}$, $Y \cup Z$ contains no circuit of $\mathcal{G}$, i.e., iff no circuit of $\mathcal{G}$ intersects X in a subset of Y i.e., iff $r(Y \cup (E(\mathcal{G}) - X)) = |Y| + r(E(\mathcal{G}) - X)$ i.e., iff $r \diamond \mathbf{X}(Y) = |Y|$.

iii. This follows immediately from the above proof and the definition of contramodular dual.

iv. Immediate from the definition of nullity function.
The remaining parts follow from the above through the use of Theorem 10.3.2 (i), (ii).

E 10.6:

i. This is immediate from the definitions of restriction, contraction and incidence functions of the relevant graphs.

ii. Let $Y \subseteq X \subseteq V(\mathcal{G})$. Then, $|I|\diamond\mathbf{X}(Y) \equiv |I|((V(\mathcal{G}) - X) \cup Y) - |I|(V(\mathcal{G}) - X)=$ the number of edges which are incident on Y but not on $V(\mathcal{G}) - X$. The result follows.

iii. This is immediate.

iv. We have,

$$|I|^d(X) \equiv |I|(V(\mathcal{G})) - |I|(V(\mathcal{G}) - X).$$
$$= |E|(X).$$

v. Let $X \subseteq V(\mathcal{G})$. We have

$$|I|^*(X) \equiv \alpha(X) - |I|(V(\mathcal{G})) + |I|(V(\mathcal{G}) - X).$$
$$= \alpha(X) - |E|(X)$$
$$= \sum_{v \in X} |I|(v) - |E|(X).$$
$$= |I|(X).$$

The remaining parts follow through the use of Theorem 10.3.2 (ii). on the corresponding results for the incidence function.

E 10.7:

i. This is immediate.

ii. Let $Y \subseteq X$. Then $|\Gamma_L \diamond \mathbf{X}|(Y) \equiv \Gamma_L((V_L - X) \cup Y) - \Gamma_L(V_L - X)$, i.e., the size of the set of vertices adjacent to Y but not in $\Gamma_L(V_L - X)$. This is clearly the size of the set adjacent to Y in $B \diamond_L X$.

iii. This is immediate from the definitions of $|(\Gamma_L)_{fus.\Pi}(\cdot)$ and $|\Gamma_{\Pi L}|(\cdot)$.

iv. Let $X \subseteq V_L$. Then

$$|\Gamma_L^d|(X) = |\Gamma_L|(V_L) - |\Gamma_L|(V_L - X).$$

This is the size of the set of vertices which are adjacent to X but not to $V_L - X$ i.e., the size of $E_L(X)$.
The remaining parts follow by the use of Theorem 10.3.2 (ii) on the above results.

E 10.8: The submodularity of these functions has already been shown (Example10.2.10). That they are increasing and integral functions and their values on singletons do not exceed 1 is clear.

E 10.9: The submodularity of all these functions has already been shown (see the Solution of Exercise 10.1). It is clear that they all take zero value on the null set. Except for the cut function all the functions are monotone increasing. The cut function is symmetric i.e., its value is the same on a set and its complement. So it cannot be monotone increasing if the graph is not trivial.

E 10.10: Proof of Theorem10.5.1: We will consider only the polymatroid case, since the matroid case is an easy consequence. We have

$$\mu^*(X) \equiv \sum_{e \in X} \alpha(e) - [\mu(S) - \mu(S - X)].$$

(Note that $\alpha(\cdot)$ satisfies $\alpha(e) \geq \mu(e) \quad \forall e \in S$). We have already seen that $\mu^*(\cdot)$ is submodular (Theorem 10.3.4).

- It is immediate from the definition that $\mu^*(\emptyset) = 0$
- Let $Y \supseteq X$. Since $\mu(\cdot)$ is submodular and $\mu(\emptyset) = 0$, we have

  $$\mu(S - X) - \mu(S - Y) \leq \sum_{e \in (Y - X)} \mu(e).$$

  Hence

  $$\mu^*(Y) - \mu^*(X) \equiv \sum_{e \in Y - X} \alpha(e) - [\mu(S - X) - \mu(S - Y)] \geq 0.$$

  Thus $\mu^*(\cdot)$ is increasing.
- $\mu^*(e) \geq 0 \quad \forall e \in S$, because $\mu^*(\emptyset) = 0$ and μ^* is increasing.

Thus $\mu^*(\cdot)$ is a polymatroid rank function.

□

E 10.11:
i. It is easily verified that all these functions are submodular, increasing, integral, take the value zero on the null set and zero or one on singleton sets.

ii. Clearly this function is submodular, increasing , integral and takes value zero on the null set.

iii. The statements about independent sets, bases and circuits are all immediate from the definitions of these sets in terms of the rank function and the definition of the notion of restriction.

iv. A set Y is independent in $\mathcal{M} \times X$ iff

$$|Y| = r \diamond \mathbf{X}(Y) = r((S - X) \cup Y) - r(S - X).$$

Now it can be verified that, whenever Z is a base of $\mathcal{M}.(S - X)$,

$$r((S - X) \cup Y) - r(S - X) = r(Z \cup Y) - r(Z)$$

and $r(Z) = |Z|$. Hence Y is independent in $\mathcal{M} \times X$ iff, whenever Z is a base of $\mathcal{M}.(S - X)$, we have

$$|Y| + |Z| = r(Z \cup Y),$$

equivalently, we have $Y \cup Z$ independent in $\mathcal{M}$. Noting that a base is a maximal independent set, we find that, Y is a base of $\mathcal{M} \times X$ iff whenever Z is a base of $\mathcal{M}.(S - X)$, $Y \cup Z$ is a base of $\mathcal{M}$. But a base of $\mathcal{M}.(S - X)$ is a maximal intersection of a base of $\mathcal{M}$ with $S - X$. The desired result follows.

A circuit of a matroid can be seen to be a minimal set not contained in any base of the matroid. A circuit of $\mathcal{M} \times X$ is a minimal set not contained in any minimal intersection of bases of $\mathcal{M}$ with X. Suppose C is a circuit of $\mathcal{M}$ with $C \cap X \neq \emptyset$. Now $C - X$ is independent in $\mathcal{M}$ and hence is contained in a base b of $\mathcal{M}.(S - X)$. If b_X is any base of $\mathcal{M} \times X$ we know that $b \cup b_X$ is a base of $\mathcal{M}$ which intersects X minimally. Clearly $b \cup b_X$ cannot contain C. Hence no base of $\mathcal{M} \times X$ can contain $C \cap X$. Hence $C \cap X$ contains a circuit of $\mathcal{M} \times X$. On the other hand if C' is a circuit of $\mathcal{M} \times X$, it is seen from the definition (in terms of rank) of such a circuit that $C' \cup Z$, when Z is a base of $\mathcal{M}.(S - X)$, contains a circuit of $\mathcal{M}$. Since Z contains no circuit of $\mathcal{M}$ it follows that C' contains the intersection of a circuit of $\mathcal{M}$ with X.

v. We have, B is a base of $\mathcal{M}^*$ iff

$$|B| = r^*(B) = r^*(S) = |S| - r(S).$$

Now $r^*(B) = |B| - r(S) + r(S - B)$. Thus B is a base of $\mathcal{M}^*$ iff $|S - B| = r(S - B)$ i.e., iff $S - B$ is a base of $\mathcal{M}$.

vi. We have,

$$r^d(X) \equiv r(S) - r(S - X) = r \diamond \mathbf{X}(X).$$

But by (iv) above we have, rank of $\mathcal{M} \times X = r \diamond \mathbf{X}(X)$. The result follows.
The remaining parts are consequences of Theorem 10.3.2 ((i), (ii)), when applied to the above results.

E 10.12: A vector $\mathbf{y}$ belongs to $P_{f+\mathbf{x}}$ iff $\mathbf{y}(X) \leq f(X) + x(X) \quad \forall X \subseteq S$ i.e., iff $(\mathbf{y} - \mathbf{x})(X) \leq f(X) \quad \forall X \subseteq S$ i.e., iff $(\mathbf{y} - \mathbf{x}) \in P_f$.

□

E 10.13:
i. Since $r(\emptyset) = 0$,

$$r(X) + r(S - X) \geq r(S).$$

Hence,

$$r(S) - r(S - X) = r'(X) \leq r(X).$$

The proof for the $\nu(\cdot)$ case is similar.

ii. Let b be any base of $\mathcal{M}$. If $X \subseteq S$ we have $X \cap b$ independent in $\mathcal{M}$ and hence $X \cap b$ is contained in a base of $\mathcal{M}.X$. Thus $|X \cap b| \leq r(X)$. On the other hand $X \cap b$ contains a base of $\mathcal{M} \times X$. Hence $|X \cap b| \geq r'(X)$. Thus we can choose $\mathbf{w}_r$

as follows. $\mathbf{w}_r(e) = 1,\ e \in b$ and $\mathbf{w}_r(e) = 0,\ e \notin b$. Similarly it can be seen that choosing $\mathbf{w}_\nu(\cdot)$ corresponding to a cobase would satisfy $\nu(\cdot) \geq \mathbf{w}_\nu(\cdot) \geq \nu'(\cdot)$.

E 10.14:

i. Similar to the matroid case.

ii. Choose w to be a base of the polymatroid, i.e., a vector on S such that $w(X) \leq \rho(X),\ X \subseteq S$ *and* $w(S) = \rho(S)$. We then have

$$\rho(X) \geq w(X) \quad \forall X \subseteq S$$

$$\rho(S) - \rho(X) \leq w(S) - w(X) \quad \forall X \subseteq S.$$

E 10.15: The functions in (i) and (ii) are special cases of the function in (iii). The latter has already been shown to be submodular in the solution to Exercise 10.1, Example 10.2.8. The symmetry follows by definition. The function $\xi(\cdot)$ in (iv) is the sum of two submodular functions. Further

$$\begin{aligned}\xi(S-X) &= r(M.(S-X)) - r(M \times (S-X)) \\ &= r(M) - r(M \times X) - r(M) + r(M.X) \\ &= \xi(X).\end{aligned}$$

The function $\theta(\cdot)$ in (v) is the sum of two submodular functions. The symmetry follows by definition.

E 10.17: Only contramodular dualization $(\mu^d(X) \equiv \mu(S) - \mu(S-X))$ preserves symmetry.

E 10.18: We need to show that

$$f(A) + f(B) - f(A \cup B) \geq f(X) + f(Y) - f(X \cup Y)$$

$$X \subseteq A, Y \subseteq B, A \cap B = \emptyset,$$

i.e., to show that,

$$f(A) + f(B) + f(X \cup Y) \geq f(X) + f(Y) + f(A \cup B)$$

By using submodularity twice the LHS

$$\geq f(A \cup Y) + f(X) + f(B) \geq f(X) + f(Y) + f(A \cup B).$$

as desired.

In the case of the cut function, $c(A, B)$ is the sum of the weights of edges with one end in A and another in B. But the set of edges with one end in A and the other in B contains the set of edges with one end in X and the other in Y. so the inequality is obvious in this case.

E 10.19:

i. Treat A as a single element and use Algorithm Symmetric on the set $(S-A) \cup \{A\}$

and the corresponding version of $g(\cdot)$. If (X, Y) is the output of the algorithm either X or Y would contain A.

ii. Treat $S - A$ as a single element.

E 10.20: In the case of the cut function the algorithm goes through without change if we define,

$$\begin{aligned} c(A,B) &\equiv \frac{1}{2}(|cut|(A) + |cut|(B) - |cut|(A \cup B)) \\ &\equiv \textit{weight of the edges with one end point in } A \textit{ and the other in } B. \end{aligned}$$

and in the case of $(|\Gamma_L| - |E_L|)(\cdot)$

$$c(A,B) \equiv \frac{1}{2}((|\Gamma_L| - |E_L|)(A) + (|\Gamma_L| - |E_L|)(B) - (|\Gamma_L| - |E_L|)(A \cup B))$$

In the case of $|\Gamma_L|(\cdot)$

$$\begin{aligned} c(A,B) &\equiv \frac{1}{2}(|\Gamma_L|(A) + |\Gamma_L|(B) - |\Gamma_L|(A \cup B)) \\ &\equiv halfweight\ of\ vertices\ (of\ V_R)\ adjacent\ to\ vertices\ in\ A\ and\ B. \end{aligned}$$

Notice that both $(|\Gamma_L| - |E_L|)(\cdot)$ and $|\Gamma_L|(\cdot)$ when processed by Algorithm Symmetric will yield the same optimal set. But the $c(A, B)$ function is simpler in the case of $|\Gamma_L|(\cdot)$.

E 10.21: Let $\mu(\cdot)$ denote the function $|cut|(\cdot)$ on the vertex subset of graph $\mathcal{G}$. Then the desired weight function $w : V \longrightarrow \Re$ is defined by

$$\begin{aligned} w(v) &\equiv \mu(V) - \mu(V - v) \\ &\equiv 0 - \mu(V - v) \\ &\equiv -(weight\ of\ edges\ incident\ on\ v). \end{aligned}$$

Then $\mu(A) - w(A) =$ weight of edges with one endpoint in A + [2(weight of edges with both endpoints in A) + (weight of edges with only one endpoint in A)] = $2|\Gamma|(A)$.

Thus minimising $\mu(\cdot)$ over nonvoid proper subsets is the same as minimising $|\Gamma|(\cdot) + \frac{1}{2}w(\cdot)$ over such subsets. The latter is the same as minimising $|\Gamma|(X) - \frac{1}{2}w(V - X)$. Since $-w(X)$ takes positive value on each vertex, this latter can be posed as a sequence of flow problems. The complexity is however worse than that of the Stoer - Wagner algorithm (mainly because mincuts corresponding to $X = S, X = \emptyset$ must be avoided).

9.12 Solutions of Problems

P 9.6: Analogous to the proof of the principle of inclusion and exclusion. Clearly true when $n = 2$. Now use induction.

P 10.6: The maximum weight case. Let $X \subseteq Y \subseteq E - e$. We need to show that

$$\rho(X \cup e) - \rho(X) \geq \rho(Y \cup e) - \rho(Y)$$

Let t_X, t_Y be the maximum weight forests of $\mathcal{G}.X, \mathcal{G}.Y$ respectively. Suppose e is not spanned by t_Y. Then it would not also be spanned by t_X. So in this case the inequality would be satisfied as an equality. Next let e be spanned by t_Y. Then

$$\begin{aligned} \rho(Y \cup e) - \rho(Y) &= w(e) - w(e_Y) \; if \; w(e) > w(e_Y) \\ &= 0, \; otherwise \end{aligned}$$

where e_Y is the edge with least weight in the fundamental circuit of e with respect to t_Y. If e is not spanned by t_X then

$$\rho(X \cup e) - \rho(X) = w(e)$$

and the inequality is satisfied. Suppose e is spanned by t_X. Let e_X be the edge with least weight in the fundamental circuit of e with respect to t_X. Clearly if $w(e_X) \leq w(e_Y)$ the desired inequality is satisfied. Suppose otherwise. Let $e_1, e_2, \ldots, e_k$ be the edges in the fundamental circuit of e with respect to t_X. Of these edges let $e_1, \ldots, e_r$ not belong to t_Y and let $e_{r+1}, \ldots, e_k$ belong to t_Y. Consider the fundamental circuits $C_1, C_2, \ldots, C_r$ of $e_1, \ldots, e_r$ with respect to t_Y. Now $(C_1 \cup C_2 \cup \ldots \cup C_r - \{e_1, e_2, \ldots, e_r\}) \cup \{e_{r+1}, \ldots, e_k\}$ span the edges $\{e_1, \ldots, e_k\}$. Hence they span the edge e. But the minimal set of edges in t_Y which span e is the set of edges of t_Y which lie in the fundamental circuit C_e of e with respect to t_Y. Hence each of $C_e - e$ lies in one of the C_i or is equal to one of $e_{r+1}, \ldots, e_k$. Now $e_Y \notin \{e_{r+1}, \ldots, e_k\}$ since we have assumed that $w(e_X) > w(e_Y)$. So e_Y belongs to one of the C_i say C_j. But $w(e_j) \geq w(e_X) > w(e_Y)$. This contradicts the fact that t_Y is a maximum weight forest of the subgraph on Y.

The minimum weight case: Observe that it could happen that e is spanned by t_Y but not by t_X. Assume $w(e_Y) > w(e) > 0$. So in this case

$$\rho(X \cup e) - \rho(X) = w(e) > \rho(Y \cup e) - \rho(Y) = w(e) - w(e_Y).$$

Suppose however e spanned by t_Y as well as t_X. By the argument that we used for the maximum weight case we must have $w(e_Y) \leq w(e_X)$. Assume $w(e) < w(e_Y) < w(e_X)$. In this case

$$\rho(X \cup e) - \rho(X) = w(e) - w(e_X) < \rho(Y \cup e) - \rho(Y) = w(e) - w(e_Y).$$

Thus in general $\rho(\cdot)$ is neither supermodular nor submodular.

P 10.7: See the general solution of the submodular function case. (Problem 10.13).

P 10.8: Routine generalization of the solution of Problem 10.6.

P 10.9: We need to show that

$$f(X) - f(X - \{\mathbf{r}\}) \geq f(Y) - f(Y - \{\mathbf{r}\}), \; X \subseteq Y, \; \mathbf{r} \in X$$

The following lemma contains the required proof.

Lemma 9.12.1 *Let $\mathbf{A}_Z$ be a submatrix of $\mathbf{A}$ on $Z \subseteq R$ and all columns of $\mathbf{A}$. Let $\hat{\mathbf{A}}_Z$ be obtained from $\mathbf{A}_Z$ by applying the Gram - Schmidt orthogonalization procedure on the rows of $\mathbf{A}_Z$ in the same order as the order of the rows. Then*

i. $\hat{\mathbf{A}}_Z\hat{\mathbf{A}}_Z^T = \mathbf{D}_Z$ *where $\mathbf{D}_Z$ is a diagonal matrix with its jth diagonal entry d_{jj} = square of the euclidian norm of the jth row of $\hat{\mathbf{A}}_Z$.*

ii. $det(\mathbf{A}_Z\mathbf{A}_Z^T) = det(\hat{\mathbf{A}}_Z\hat{\mathbf{A}}_Z^T) = det(\mathbf{D}_Z) = \prod(d_{jj})$.

iii. Let $\mathbf{A}_Z^1$ be obtained from $\mathbf{A}_Z$ by permuting the first $(j-1)$ rows of the latter. Let $\mathbf{D}_Z^1$ denote the diagonal matrix obtained from $\mathbf{A}_Z^1$ the way $\mathbf{D}_Z$ is obtained from $\mathbf{A}_Z$. Then

$$(\mathbf{D}_Z^1)_{kk} = (\mathbf{D}_Z)_{kk},\ k \geq j.$$

iv. If $\mathbf{r}$ is the last (say tth) row of $\mathbf{A}_Z$, then

$$f(Z) - f(Z - \{\mathbf{r}\}) = log(d_{tt}).$$

v. Let $Y \supseteq Z$. Let $\mathbf{r}$ be the last, say sth, row of $\mathbf{A}_Y$. If $\hat{\mathbf{A}}_Y\hat{\mathbf{A}}_Y^T = \mathbf{D}'_Y$ and Y has s rows then $d'_{ss} \leq d_{tt}$.

Proof of the Lemma: We remind the reader that the Gram - Schmidt orthogonalization procedure yields, while acting on the sequence of vectors $\mathbf{x}_1, \ldots, \mathbf{x}_t$,

$$\begin{aligned}
\hat{\mathbf{x}}_1 &= \mathbf{x}_1 \\
\hat{\mathbf{x}}_2 &= \mathbf{x}_2 - \frac{< \mathbf{x}_2, \hat{\mathbf{x}}_1 >}{< \hat{\mathbf{x}}_1, \hat{\mathbf{x}}_1 >}\hat{\mathbf{x}}_1 \\
&\vdots \\
\hat{\mathbf{x}}_t &= \mathbf{x}_t - \frac{< \mathbf{x}_t, \hat{\mathbf{x}}_{t-1} >}{< \hat{\mathbf{x}}_{t-1}, \hat{\mathbf{x}}_{t-1} >}\hat{\mathbf{x}}_{t-1} - \ldots - \frac{< \mathbf{x}_t, \hat{\mathbf{x}}_1 >}{< \hat{\mathbf{x}}_1, \hat{\mathbf{x}}_1 >}\hat{\mathbf{x}}_1
\end{aligned}$$

i. Since $\hat{\mathbf{x}}_1, \ldots, \hat{\mathbf{x}}_t$ are mutually orthogonal any matrix say $\hat{\mathbf{A}}_Z$ which has these as rows will satisfy $\hat{\mathbf{A}}_Z\hat{\mathbf{A}}_Z^T = \mathbf{D}_Z$ where $\mathbf{D}_Z$ satisfies the conditions specified in the statement (i) of the lemma.
ii. It is clear from the description of the procedure that $\hat{\mathbf{A}}_Z = (\mathbf{K})\mathbf{A}_Z$ where $\mathbf{K}$ is a lower triangular matrix with 1s along the diagonal. Hence

$$\begin{aligned}
det(\hat{\mathbf{A}}_Z\hat{\mathbf{A}}_Z^T) &= det(\mathbf{K}).det(\mathbf{A}_Z\mathbf{A}_Z^T).det(\mathbf{K}^T) \\
&= det(\mathbf{A}_Z\mathbf{A}_Z^T)
\end{aligned}$$

But $det(\hat{\mathbf{A}}_Z\hat{\mathbf{A}}_Z^T) = det(\mathbf{D}_Z) = \prod(d_{jj})$. Thus the result follows.

iii. This is a consequence of the previous parts of this lemma and the Gram-Schmidt orthogonalization procedure.

iv. If $\mathbf{r}$ is the last say tth row of $\mathbf{A}_Z$, it is clear that

$$\frac{det(\mathbf{A}_Z\mathbf{A}_Z^T)}{det(\mathbf{A}_{Z-\mathbf{r}}\mathbf{A}_{Z-\mathbf{r}}^T)} = \frac{\prod_{j=1}^{t} d_{jj}}{\prod_{j=1}^{t-1} d_{jj}} = d_{tt}.$$

Hence

$$f(Z) - f(Z - \{\mathbf{r}\}) = log(d_{tt}).$$

v. We see that in the procedure

$$\mathbf{x}_t = \hat{\mathbf{x}}_t + \frac{<\mathbf{x}_t, \hat{\mathbf{x}}_{t-1}>}{<\hat{\mathbf{x}}_{t-1}, \hat{\mathbf{x}}_{t-1}>}\hat{\mathbf{x}}_{t-1} + \ldots + \frac{<\mathbf{x}_t, \hat{\mathbf{x}}_1>}{<\hat{\mathbf{x}}_1, \hat{\mathbf{x}}_1>}\hat{\mathbf{x}}_1$$

$$\mathbf{y}_s = \hat{\mathbf{y}}_s + \frac{<\mathbf{y}_s, \hat{\mathbf{y}}_{s-1}>}{<\hat{\mathbf{y}}_{s-1}, \hat{\mathbf{y}}_{s-1}>}\hat{\mathbf{y}}_{s-1} + \ldots + \frac{<\mathbf{y}_s, \hat{\mathbf{y}}_1>}{<\hat{\mathbf{y}}_1, \hat{\mathbf{y}}_1>}\hat{\mathbf{y}}_1$$

Suppose the vector $\mathbf{r}$ occurs as the tth vector in the first case and the sth vector in the second case and the first $(t-1)$ vectors are the same in both expressions. Then the terms

$$\frac{<\mathbf{r}, \hat{\mathbf{x}}_{t-1}>}{<\hat{\mathbf{x}}_{t-1}, \hat{\mathbf{x}}_{t-1}>}\hat{\mathbf{x}}_{t-1} + \ldots + \frac{<\mathbf{r}, \hat{\mathbf{x}}_1>}{<\hat{\mathbf{x}}_1, \hat{\mathbf{x}}_1>}\hat{\mathbf{x}}_1$$

denoted as α, appear in both expressions. Also $\hat{\mathbf{x}}_1, \ldots, \hat{\mathbf{x}}_s$ are orthogonal. Hence

$$<\mathbf{r}, \mathbf{r}> = <\hat{\mathbf{x}}_t, \hat{\mathbf{x}}_t> + <\alpha, \alpha> \geq <\hat{\mathbf{y}}_s, \hat{\mathbf{y}}_s> + <\alpha, \alpha>$$

It follows that $<\hat{\mathbf{x}}_t, \hat{\mathbf{x}}_t> \geq <\hat{\mathbf{y}}_s, \hat{\mathbf{y}}_s>$.
Equivalently, $\mathbf{r}$ is being decomposed into perpendicular vectors $\hat{\mathbf{x}}_t, \alpha$ in the first case, $\hat{\mathbf{y}}_s, \beta$ in the second, where α, β are the projections of $\mathbf{r}$ onto subspaces say $\mathcal{V}_\alpha, \mathcal{V}_\beta$ with $\mathcal{V}_\alpha \subseteq \mathcal{V}_\beta$. Hence

$$<\hat{\mathbf{x}}_t, \hat{\mathbf{x}}_t> \geq <\hat{\mathbf{y}}_s, \hat{\mathbf{y}}_s>.$$

□

P 10.10: We observe that this is essentially the same as the previous problem except that the underlying set has been changed from R to T. However a natural way of identifying elements of R with those of T is given in the problem.

P 10.11: By Binet - Cauchy Theorem $det(\mathbf{A}_X\mathbf{A}_X^T)$ = sum of the squares of nonzero major subdeterminants of $\mathbf{A}_X$, which is equal to number of bases of the matroid associated with $\mathbf{A}_X$ (since every subdeterminant of $\mathbf{A}$ is $0, \pm 1$). The result follows from the solution to the previous problem .

P 10.12:
i. Let $X, Y \in D(\mathbf{x})$. We have

$$x(X) + x(Y) = x(X \cup Y) + x(X \cap Y)$$

Now $\rho(\cdot)$ is submodular. Suppose

$$\rho(X) + \rho(Y) > \rho(X \cup Y) + \rho(X \cap Y)$$

We then must have $x(X \cup Y) > \rho(X \cup Y)$ or $\mathbf{x}(X \cap Y) > \rho(X \cap Y)$. But this is impossible since $\mathbf{x}$ is an independent vector. Hence

$$\rho(X) + \rho(Y) = \rho(X \cup Y) + \rho(X \cap Y)$$

and $x(X \cup Y) = \rho(X \cup Y)$, $x(X \cap Y) = \rho(X \cap Y)$.

ii. Let $e \in sat(\mathbf{x})(\equiv Z)$. If $\alpha > 0$,

$$(\mathbf{x} + \alpha\chi_e)(Z) > \rho(Z).$$

On the other hand if $\forall \alpha > 0 : \mathbf{x} + \alpha\chi_e \notin P_\rho$ then there exists a set $Y \subseteq S$ s.t. $e \in Y$ and $x(Y) \geq \rho(Y)$. Since $\mathbf{x}$ is independent we conclude that $x(Y) = \rho(Y)$ and $e \in sat(\mathbf{x})$.

iii. As in (i).

iv(a) Let X, Y maximize $\theta(\cdot)$. We have

$$\theta(X) + \theta(Y) \leq \theta(X \cup Y) + \theta(X \cap Y).$$

Hence the inequality can only be satisfied as an equality and $\theta(X \cup Y) = \theta(X \cap Y) = \theta(X)$. An immediate consequence is that $D'(\mathbf{x}')$ has a unique maximal and a unique minimal member set.

iv(b) We have $x(K) \leq \rho(K)$ and $\mathbf{x}'(K) > \rho(K)$. Suppose $x(K) < \rho(K)$ then

$$\mathbf{x}'(K) - \rho(K) < \mathbf{x}'(Z) - \rho(Z)$$

where $Z = sat(\mathbf{x})$. This is a contradiction. We conclude that $x(K) = \rho(K)$. Thus $K \in D(\mathbf{x}, e)$. Suppose $K \supset L \in D(\mathbf{x}, e)$. We must have

$$\mathbf{x}'(L) - \rho(L) = \mathbf{x}'(K) - \rho(K) = \alpha.$$

Hence K would not be the minimal member of $D'(\mathbf{x}')$. We conclude that K is the minimal member of $D(\mathbf{x}, e)$ ie $K = dep(\mathbf{x}, e)$.

iv(c) Let $K \equiv dep(\mathbf{x}, e)$ and let K' denote the set in the RHS of the equation in the statement. If $x(e) = \rho(e)$, the statement is obvious with $dep(\mathbf{x}, e) = \{e\}$. So let $x(e) < \rho(e)$ and let $e' \in K - e$. Select α s.t.

$$0 < \alpha < \rho(L) - x(L) \quad \forall L : e \in L, L \notin D(\mathbf{x}, e).$$

Now if $\mathbf{x}'' \equiv \mathbf{x} + \alpha(\chi_e - \chi_{e'})$, it is clear that $\mathbf{x}'' \in P_\rho$. Hence $K \subseteq K'$. On the other hand if for some $\alpha > 0, \mathbf{x} + \alpha(\chi_e - \chi_{e'}) \in P_\rho$, we must have $e' \in K$ as otherwise $\mathbf{x}''(K) > \rho(K)$. Hence $K \supseteq K'$.

P 10.13: We need to show that

$$\rho(X \cup e) - \rho(X) \geq \rho(Y \cup e) - \rho(Y), \; X \subseteq Y \subseteq S - e$$

We note that we can use the greedy algorithm to compute the value of $\rho(\cdot)$ on any set. Thus to compute $\rho(X)$ we first order the elements in X as $e_{i1}, \ldots, e_{ik}$ such that

$$c(e_{i1}) \geq c(e_{i2}) \geq \ldots \geq c(e_{ik})$$

We set

$$\begin{aligned} x(e_{i1}) &= f(e_{i1}) \\ x(e_{i2}) &= f(e_{i1}, e_{i2}) - f(e_{i1}) \\ \vdots &= \\ x(e_{ik}) &= f(X) - f(X - e_{ik}) \end{aligned}$$

Then,

$$\rho(X) \equiv \sum_{j=1}^{k} c(e_{ij})x(e_{ij}) \equiv \sum_{j=1}^{k}(c(e_{ij}) - c(e_{ij+1})(f(e_{i1}, \ldots, e_{ij}))$$

where $c(e_{ik+1})$ is taken to be zero. Let us now order the elements of the set $X, X \cup e, Y, Y \cup e$ according to their decreasing $\mathbf{c}(\cdot)$ values. Let the ordering be

$$(e_{i1}, \ldots, e_{ik}),\ (e_{i1}, \ldots, e_{ir}, e, e_{ir+1}, \ldots, e_{ik})$$

$$(e_{j1}, \ldots, e_{jq}),\ (e_{j1}, \ldots, e_{jp}, e, e_{jp+1}, \ldots, e_{jq})$$

We will adopt the convention that the elements of $X, X \cup e$ etc will appear in the larger sets in the same order but possibly with additional intermediate elements. We then have,
$\rho(X \cup e) - \rho(X)$

$$\begin{aligned} &= (c(e) - c(e_{ir+1}))(f(e_{i1}, \ldots, e_{ir}, e) - f(e_{i1}, \ldots, e_{ir})) \\ &+(c(e_{ir+1}) - c(e_{ir+2}))(f(e_{i1}, \ldots, e_{ir}, e, e_{ir+1}) - f(e_{i1}, \ldots, e_{ir}, e_{ir+1})) \\ \vdots\ &\ \vdots \\ &+c(e_{ik})(f(X \cup e) - f(X)) \end{aligned}$$

and
$\rho(Y \cup e) - \rho(Y)$

$$\begin{aligned} = \quad & (c(e) - c(e_{jp+1}))(f(e_{j1}, \ldots, e_{jp}, e) - f(e_{j1}, \ldots, e_{jp})) \\ & +(c(e_{jp+1}) - c(e_{jp+2}))(f(e_{j1}, \ldots, e_{jp}, e, e_{jp+1}) - f(e_{j1}, \ldots, e_{jp}, e_{jp+1})) \\ \vdots \quad & \vdots \\ & +c(e_{jk})(f(Y \cup e) - f(Y)) \end{aligned}$$

We observe that the elements $e_{ir+1}, \ldots, e_{ik}$ also occur among the above elements $e_{jp+1}, \ldots, e_{jq}$ in the same order. Let

$$
\begin{array}{rcl}
e_{ir+1} & = & e_{jp+s1} \\
\vdots & \vdots & \vdots \\
e_{ir+t} & = & e_{jp+st} \\
\vdots & \vdots & \vdots \\
e_{ik} & = & e_{jp+sk}
\end{array}
$$

We can therefore rewrite $\rho(X \cup e) - \rho(X)$ as

$$[(c(e) - c(e_{jp+1}))(f(e_{i1}, \ldots, e_{ir}, e) - f(e_{i1}, \ldots, e_{ir})) + \ldots +$$

$$(c(e_{jp+s1-1}) - c(e_{jp+s1}))(f(e_{i1}, \ldots, e_{ir}, e) - f(e_{i1}, \ldots, e_{ir}))]$$

$$+ \ldots + (c(e_{jp+sk}) - c(e_{jp+sk+1}))(f(X \cup e) - f(X)) + \ldots + c(e_{jq})(f(X \cup e) - f(X)).$$

Now we could compare the expansion for $\rho(Y \cup e) - \rho(Y)$ and that for $\rho(X \cup e) - \rho(X)$ term by term (identifying a product term by the expression $(c(e_{je}) - c(e_{je+1}))$ in the first bracket). But by submodularity the expression in the second bracket of a product term of $\rho(X \cup e) - \rho(X)$ will never be smaller than the second bracket expression of the corresponding term of $\rho(Y \cup e) - \rho(Y)$. Example

$$(f(e_{i1}, \ldots, e_{ir}, e) - f(e_{i1}, \ldots, e_{ir})) \geq (f(e_{j1}, \ldots, e_{jp}, e) - f(e_{j1}, \ldots, e_{jp}))$$

since $\{e_{j1}, \ldots, e_{jp}\} \supseteq \{e_{i1}, \ldots, e_{ir}\}$, while the first bracket expression is always nonnegative because of the way the elements have been ordered. Thus

$$\rho(X \cup e) - \rho(X) \geq \rho(Y \cup e) - \rho(Y)$$

as required.

P 10.14: Let $f_1'(\cdot), f_2'(\cdot)$ be the extensions of $f_1(\cdot), f_2(\cdot)$ induced by $P_{f_1}, P_{f_2}^d$ respectively. By Theorem 10.7.3 $f_1'(\cdot), f_2'(\cdot)$ are convex (cup like) and concave (cap like) functions respectively. We can associate two convex regions E_1, E_2 in $\Re^{|S|+1}$ with $f_1'(\cdot)$ and $f_2'(\cdot)$ respectively in the usual way: $(\mathbf{c}, x) \in E_1$ (E_2), where $\mathbf{c} \in \Re^S, x \in \Re$ iff $f_1'(\mathbf{c}) < x$ $(f_2'(\mathbf{c}) \geq x)$. It is easily verified that these regions are convex. For instance let $(\mathbf{c}_1, x_1), (\mathbf{c}_2, x_2) \in E_1$. Then $f_1'(\mathbf{c}_1) < x_1$ *and* $f_1'(\mathbf{c}_2) < x_2$. Hence

$$\lambda f_1'(\mathbf{c}_1) + (1-\lambda) f_1'(\mathbf{c}_2) < \lambda x_1 + (1-\lambda)x_2, \ 0 \leq \lambda \leq 1$$

$$ie \ f_1'(\lambda \mathbf{c}_1 + (1-\lambda)\mathbf{c}_2) < \lambda x_1 + (1-\lambda)x_2$$

$$ie \ (\lambda \mathbf{c}_1 + (1-\lambda)\mathbf{c}_2, \lambda x_1 + (1-\lambda)x_2) \in E_1.$$

We note that E_1 is open because of the strict inequality in its definition and also that $E_1 \cap E_2 = \emptyset$. Further since $f_1'(\mathbf{0}) = f_2'(\mathbf{0}) = 0$ (by definition) we must have $(\mathbf{0}, x) \in E_1$ iff $x > 0$ and $(\mathbf{0}, x) \in E_2$ iff $x \leq 0$.

By the Separation Theorem we have a hyperplane H (which is the set of all points in $\Re^{|S|+1}$ on which the functional $g(\cdot)$ in the separation theorem takes the value t) separating E_1 and E_2. Clearly $(\mathbf{0}, 0) \in H$ and H is a linear subspace. Now if $(\mathbf{c}, x_1), (\mathbf{c}, x_2) \in H$ we must have $x_1 = x_2$. For the hypothesis implies that $(\mathbf{0}, x_1 - x_2) \in H$, and $(\mathbf{0}, x_2 - x_1) \in H$. So neither $(\mathbf{0}, x_1 - x_2)$ nor $(\mathbf{0}, x_2 - x_1)$ belongs to E_1. Hence $x_1 - x_2 \leq 0$ and $x_2 - x_1 \leq 0$. We conclude $x_1 = x_2$.

Thus H induces a function $w(\cdot)$ on $\Re^{|S|}$ through $w(\mathbf{c}) \equiv x$ iff $(\mathbf{c}, x) \in H$. Since H is a linear subspace it is easy to verify that $w(\cdot)$ is a linear functional. Further $w(\cdot)$ induces a modular function on 2^S. For

$$\begin{aligned} w(\chi_X) + w(\chi_Y) &= w(\chi_X + \chi_Y) \\ &= w(\chi_{X\cup Y} + \chi_{X\cap Y}) \\ &= w(\chi_{X\cup Y}) + w(\chi_{X\cap Y}). \end{aligned}$$

Next let $(\mathbf{c}, x_1), (\mathbf{c}, x_0), (\mathbf{c}, x_2)$ belong respectively to E_1, H, E_2. We must have $x_1 > x_0$. Otherwise $(\mathbf{c}, x_0) \in E_1$ (by definition of E_1), which is a contradiction. Suppose $x_0 < x_2$. Then $w(\mathbf{c}, x_2) \neq t$ as otherwise $(\mathbf{c}, x_2) \in H$ and $x_2 = x_0$. Hence $w(\mathbf{c}, x_2) < t$. Now $w(\mathbf{c}, x_1) > t$. Hence $w(\mathbf{c}, \lambda x_1 + (1-\lambda)x_2) = t$ for some $0 \leq \lambda \leq 1$. But then $x_0 = \lambda x_1 + (1-\lambda)x_2$, which is impossible since $x_0 < x_1, x_2$. Hence we conclude that $x_1 > x_0 \geq x_2$. Thus $f_1'(\mathbf{c}) \geq w(\mathbf{c}) \geq f_2'(\mathbf{c})$. Restricting these functions to characteristic vectors of subsets of S we get the desired result.

P 10.15: We first show that

$$\forall X \subseteq S, r_1(X) + r_2(S - X) \geq max\ size\ of\ common\ independent\ set\ . \quad (9.7)$$

Let T be a common independent set. Then

$$|T| = r_1(X \cap T) + r_2((S - X) \cap T) \leq r_1(X) + r_2(S - X).$$

So the desired result (10.7) follows.

We will next construct a common independent set T whose size equals $r_1(X) + r_2(S - X)$ for an appropriate subset X. We will use the Sandwich Theorem for this purpose. Consider the submodular and supermodular functions $r_1(\cdot), r_2^d(\cdot)$ (where $r_2^d(X) = r_2(S) - r_2(S - X) \quad \forall X \subseteq S$). Let k be the least integer for which $r_1(\cdot) \geq r_2^d(\cdot) - k$. (Note that this means there exists some $X \subseteq S$ for which $r_1(\cdot) = r_2^d(\cdot) - k$). By Sandwich Theorem there is an integral weight function $h(\cdot)$ such that

$$r_1(\cdot) \geq h(\cdot) \geq r_2^d(\cdot) - k$$

It is clear that $h(e) = 0\ or 1$ on each $e \in S$. Let A be the support of $h(\cdot)$. We have

$$r_1(A) \geq h(A) = |A|.$$

Therefore A is independent in the matroid $\mathcal{M}_1$. Next,

$$h(S - A) = 0 \geq r_2^d(S - A) - k$$

$$i.e., \ r_2^d(S-A) \leq k.$$

It follows that $r_2(A) \geq r_2(S) - k$. But k is the least integer such that $r_1(\cdot) \geq r_2^d(\cdot) - k$. Hence there exists $X \subseteq S$ such that $r_1(X) = r_2^d(X) - k$. Therefore $r_2(A) \geq r_1(X) + r_2(S-X)$ for some $X \subseteq S$. Let $T \subseteq A$ be independent in $\mathcal{M}_2$ with $r_2(A) = T$. Now T is a common independent set for $\mathcal{M}_1$ and $\mathcal{M}_2$ and we have

$$|T| = r_2(A) \geq min_{X \subseteq S} \ r_1(X) + r_2(S-X).$$

Thus T is the maximum size common independent set.
Remark: The generalization of the above result is given in the next chapter. The above technique has to be modified slightly to prove that result.

P 10.16:
i. Let $a \notin S$. Let $g : 2^{S \cup a} \longrightarrow \Re$ be defined as follows.

$$g(X \cup a) \equiv f(X), \ X \subseteq S.$$

$$g(X) \equiv f(S-X), \ X \subseteq S.$$

Observe that $g(\cdot)$ is symmetric, since if $a \notin Y$, $g(Y) = f(S-Y) = g((S-Y) \cup a)$ and if $a \in Y, g(Y) = f(Y-a) = g(S-(Y-a))$. For proving that $g(\cdot)$ is submodular we consider a number of cases.

i(a) If $X \subseteq Y \subseteq S$ and $e \notin Y, \ e \neq a$.

$$g(X \cup e) - g(X) = f(S-X-e) - f(S-X)$$

$$g(Y \cup e) - g(Y) = f(S-Y-e) - f(S-Y)$$

and $f(S-Y) - f(S-Y-e) \geq f(S-X) - f(S-X-e)$.

i(b) If $X \subseteq Y \subseteq S$ and $e = a$.

$$g(X \cup e) - g(X) = f(X) - f(S-X)$$

$$g(Y \cup e) - g(Y) = f(Y) - f(S-Y)$$

so we need to show

$$f(X) - f(Y) \geq f(S-X) - f.(S-Y)$$

Let $Y - X = \{e_1, \ldots, e_n\}$. We then have,

$$(f(X) - f(X \cup e_1)) + f(X \cup e_1) - f(X \cup e_1 \cup e_2)) + \ldots + (f(Y-e_n) - f(Y))$$
$$\geq (f(S-X) - f(S-X-e_1)) + (f(S-X-e_1) - f(S-X-e_1-e_2)) + \ldots$$
$$+(f(S-(Y-e_n)) - f(S-Y)).$$

But this follows from the conditions of the problem.

i(c) If $X \subseteq Y \subseteq S \cup a$ and $a \in X, \ e \notin Y$.

$$g(X \cup e) - g(X) = f(X \cup e - a) - f(X-a)$$

$$g(Y \cup e) - g(Y) = f(Y \cup e - a) - f(Y - a)$$

So the desired inequality is immediate.

i(d) If $X \subseteq Y \subseteq S \cup a$ and $a \in Y - X,\ e \notin Y$.

$$g(X \cup e) - g(X) = f(S - X - e) - f(S - X)$$

$$g(Y \cup e) - g(Y) = f(Y \cup e - a) - f(Y - a)$$

Now $f(S - X - e) - f(S - X) \geq f(X \cup e) - f(X)$ (given). But,

$$f(X \cup e) - f(X) \geq f(Y \cup e - a) - f(Y - a)$$

since $a \in Y - X$. Thus the desired inequality holds.

Now the function $g(\cdot)$ is symmetric and submodular so that we can apply Algorithm Symmetric to it. If $g(\cdot)$ reaches its minimum among all nonvoid proper subsets of $S \cup a$ at X, either X or $S \cup a - X$ contains a. So $f(\cdot)$ reaches its minimum at $X - a$ or $(S \cup a - X)$. However $g(\cdot)$ might reach its minimum at $\{a\}$. To avoid this eventuality minimize $g(\cdot)$ over all subsets that contain $\{a, e_i\}, e_i \in S$. Repeat the minimization for each e_i.

ii. In this case we define $g : 2^{S \cup a} \longrightarrow \Re$ as follows.

$$g(X \cup a) \equiv f(S - X),\ X \subseteq S.$$

$$g(X) \equiv f(X),\ X \subseteq S.$$

It can be seen that $g(\cdot)$ is symmetric and submodular by arguments similar to those of the previous section of the present problem. We can minimize $g(\cdot)$ over nonvoid proper subsets of $S \cup a$, using Algorithm Symmetric. If the minimum occurs at X, either X or $S \cup a - X$, does not contain a. Then $f(\cdot)$ reaches its minimum on that set. If the minimum falls on S we use the strategy described in the previous section of the present problem.

Chapter 10

Submodular Functions

10.1 Introduction

In combinatorial mathematics submodular functions are a relatively recent phenomenon. Systematic interest in this area perhaps began with the work of Edmonds in the late sixties [Edmonds70]. By then matroids were well studied with numerous applications to engineering systems already known. Submodular functions could be regarded as a generalization of matroid rank functions and it is natural to wonder whether they are really required. The answer is that, even if we ignore considerations of theory, we come across them far more often in practical problems than we come across matroids. The method of attack for these problems using submodular function theory is usually quite simple and the algorithms generated, very efficient. Study of basic 'submodular' operations such as convolution and Dilworth truncation is likely to prove fruitful for practical algorithm designers since, in addition to completely capturing the essence of many practical situations, they also allow us to give acceptable approximate solutions to several intractable problems.

In this chapter we begin with simple equivalent restatements of the definition of submodularity along with a number of instances where submodular functions are found in 'nature'. Next we discuss some standard operations by which we get new submodular / supermodular (more compactly, semimodular) functions starting from such functions. We then pay special attention to the important special cases of matroid and polymatroid rank functions. Next we give a sketch of the polyhedral approach to the study of semimodular functions. Finally we give a brief outline of some recent work on minimization of symmetric submodular functions. The important notions of convolution and Dilworth truncation of submodular functions are relegated to subsequent chapters.

10.2 Submodularity

We begin with a few definitions of submodularity which are easily proved to be equivalent. The most useful form appears to be one which essentially states that the 'rate of increase' of submodular functions is less on 'larger' sets (analogous to 'cap' functons over the real line). Thereafter we present a number of simple examples of submodular functions.

Definition 10.2.1 *Let S be a finite set.*
Let $f : 2^S \longrightarrow \Re$, f is said to be a **submodular** *(***supermodular***) function iff*

$$f(X) + f(Y) \geq f(X \cup Y) + f(X \cap Y) \quad \forall X, Y \subseteq S \tag{10.1}$$

$$(f(X) + f(Y) \leq f(X \cup Y) + f(X \cap Y) \quad \forall X, Y \subseteq S)$$

The function f is **modular** *if the inequality is replaced by equality. A function is* **semimodular** *if it is submodular or supermodular.*[1]

The following theorem gives equivalent conditions for submodularity / supermodularity. These conditions are often easier to apply in practice than the original definitions. The conditions for supermodularity are obtained by reversing the submodular inequalities and the proof of the equivalence is obtained by reversing the inequalities line by line in the submodular case proof.

Theorem 10.2.1 *(k)*

i. *A function $f : 2^S \longrightarrow \Re$ is submodular iff it satisfies any one of the following properties.*

$$f(X \cup a) - f(X) \geq f(X \cup b \cup a) - f(X \cup b) \quad \forall X \subseteq S, \forall a, b \in S - X \tag{10.2}$$

$$f(X \cup a) - f(X) \geq f(Y \cup a) - f(Y) \quad \forall X \subseteq Y \subseteq S, \forall a \in S - Y \tag{10.3}$$

$$f(X \cup Z) - f(X) \geq f(Y \cup Z) - f(Y) \quad \forall X \subseteq Y \subseteq S, \forall Z \subseteq S - Y \tag{10.4}$$

ii. *The function f is supermodular iff it satisfies any one of the above three properties with the inequalities reversed.*

Proof: i.

$$(10.2) \Rightarrow (10.3) \Rightarrow (10.4) \Rightarrow (10.1) \Rightarrow (10.2)$$

$$(10.2) \Rightarrow (10.3)$$

Let $Y = X \uplus b_1 \uplus b_2 \uplus \cdots \uplus b_k$. If a is not in Y, we have, by (10.2)

$$f(X \cup a) - f(X) \geq f(X \cup b_1 \cup a) - f(X \cup b_1)$$

$$\geq f(X \cup b_1 \cup b_2 \cup a) - f(X \cup b_1 \cup b_2)$$

[1] Warning: before the early 70's submodular functions used to be referred to as semimodular functions in the literature.

$$\cdots$$

$$\geq f(X \cup b_1 \cup \cdots \cup b_k \cup a) - f(X \cup b_1 \cup \cdots \cup b_k).$$

$$(10.3) \Rightarrow (10.4)$$

Let $Z = \{a_1, a_2, \cdots a_t\}$ and let $X \subseteq Y \subseteq S, Z \subseteq S - Y$. Then

$$f(X \cup a_1) - f(X) \geq f(Y \cup a_1) - f(Y)$$

$$f(X \cup a_1 \cup a_2) - f(X \cup a_1) \geq f(Y \cup a_1 \cup a_2) - f(Y \cup a_1)$$

$$\cdots$$

$$f(X \cup a_1 \cup \cdots \cup a_t) - f(X \cup a_1 \cup \cdots \cup a_{t-1}) \geq f(Y \cup a_1 \cup \cdots \cup a_t) - f(Y \cup a_1 \cup \cdots \cup a_{t-1})$$

Adding all the inequalities we get (10.4).

$$(10.4) \Rightarrow (10.1)$$

This is immediate by setting (in (10.4))

$$X \ to \ X \cap Y, Z \ to \ (X - X \cap Y)(= (X \cup Y - Y)) \ and \ Y \ to \ Y$$

$$(10.1) \Rightarrow (10.2)$$

This is immediate by setting (in (10.1))

$$X \ to \ X \cup a \ and \ Y \ to \ X \cup b$$

ii. The supermodular case is similar.

□

We now give some common examples of submodular functions. The first set involves graphs.
Let $\mathcal{G}$ be a graph on vertices V and edges E.

Example 10.2.1 *(k) Let $V(X) \equiv$ set of endpoints of edges of X, $X \subseteq E(\mathcal{G})$. Then $|V|(\cdot)$* [2] *(called the* **vertex function** *of $\mathcal{G}$) is submodular.*

Example 10.2.2 *(k) Let $E(V_1) \equiv$ set of edges with both endpoints in V_1, $V_1 \subseteq V(\mathcal{G})$. Then $|E|(\cdot)$ (called the* **interior edge function** *of $\mathcal{G}$) is supermodular.*

Example 10.2.3 *(k) Let $I(V_1) \equiv$ set of edges with atleast one end point in V_1, $V_1 \subseteq V(\mathcal{G})$. Then $|I|(\cdot)$ (called the* **incidence function** *of $\mathcal{G}$)is submodular.*

Example 10.2.4 *(k) Let $\Gamma(V_1) \equiv$set of vertices adjacent to some vertex in V_1, $V_1 \subseteq V(\mathcal{G})$. Then $|\Gamma|(\cdot)$ (called the* **adjacency function***of $\mathcal{G}$) is submodular.*

Example 10.2.5 *(k) Let $cut(V_1) \equiv$ set of all branches with only one endpoint in V_1, $V_1 \subseteq V(\mathcal{G})$. Then $|cut|(\cdot)$, called the* **cut function** *of $\mathcal{G}$, is submodular.*

---

[2] Throughout $|X(\cdot)|$ and $|X|(\cdot)$ are used interchangeably to specify the cardinality of $X(\cdot)$, where $X(\cdot)$ is any set function.

Example 10.2.6 *(k) Let $\mathcal{G}$ be a directed graph. We can now define, analogous to the definition in Example 10.2.1, $V_{tail}(X), V_{head}(X)$ over the subsets of the edge set of the directed graph. These lead to submodular functions. Analogous to the definitions in Examples 10.2.3, 10.2.4, 10.2.5, we could define $I_{out}(\cdot), I_{in}(\cdot)$, $\Gamma_{out}(\cdot), \Gamma_{in}(\cdot), cut_{out}(\cdot)$,*
$cut_{in}(\cdot)$ etc. In each case the functions are submodular. (The functions $I_{out}(\cdot)$, $I_{in}(\cdot)$ are actually modular).

The above examples can be generalized to the context of hypergraphs as represented by bipartite graphs. The reader might like to think of the right vertex set of the bipartite graph as the edge set of the hypergraph.

Example 10.2.7 *Let $B \equiv (V_L, V_R, E)$. Let $E_L(X) \equiv$ set of all vertices in V_R adjacent only to vertices in X, $X \subseteq V_L$. $E_R(\cdot)$ is defined similarly on subsets of V_R. Then $|E_L|(\cdot), |E_R|(\cdot)$ (called the* **left exclusivity function** *and the* **the right exclusivity function** *respectively of B) are supermodular.*

Example 10.2.8 *Let $B \equiv (V_L, V_R, E)$ be a bipartite graph. For $X \subseteq V_L$ define $c(X)$ to be the set of all vertices in V_R whose images under $\Gamma(\cdot)$ intersect both X and $V_L - X$. Then $|c|(\cdot)$ is submodular.*

The next couple of examples are of the matroid kind.

Example 10.2.9 *(k) Let E be the set of columns of a matrix over any field $\mathcal{F}$. Then, the rank function $r(\cdot)$ on the subsets of E is submodular.*

Example 10.2.10 *(k) Let $\mathcal{G}$ be a graph on the set of edges $E(\mathcal{G})$. Let $r(X), X \subseteq E$ be the number of edges in a forest of $\mathcal{G}.X$, the subgraph on X. Let $r'(X), X \subseteq E$ be the number of edges in a forest of $\mathcal{G} \times X$, the graph obtained by shorting and removing all edges in $E - X$. Let $\nu(X), X \subseteq E$ be the number of edges in a coforest of $\mathcal{G} \times X$. Let $\nu'(X), X \subseteq E$ be the number of edges in a coforest of $\mathcal{G}.X$. Then, $r(\cdot), \nu(\cdot)$ (called the rank and nullity functions of the graph respectively) are submodular, while $r'(\cdot), \nu'(\cdot)$ are supermodular.*

Exercise 10.1 *Show that the functions listed in the above examples are submodular or supermodular as the case may be.*

Remark: In Examples 10.2.9, 10.2.10 above, the size of the 'maximal independent set' contained in a subset turns out to be submodular. Weaker notions of independence do not always yield submodular functions. Exercise 10.2 presents such an instance.

Exercise 10.2 *Let $\mathcal{G}$ be a graph. Let a set of vertices $V_1 \subseteq V(\mathcal{G})$ be called e-independent if no two vertices of V_1 are joined by an edge. Let $k(V_i), V_i \subseteq V(\mathcal{G})$ be the maximum size of an e-independent set contained in V_i. Show that $k(\cdot)$ is not in general submodular or supermodular.*

Exercise 10.3 *Let (V_L, V_R, E) be a bipartite graph. Let $w(\cdot)$ be a nonnegative weight function assigned to V_R. Let $q : 2^{V_L} \longrightarrow \Re$ be defined by $q(X) \equiv w(Y)$ where Y is the set of all vertices in V_R which are adjacent to every vertex in X. Let $q' : 2^{V_L} \longrightarrow \Re$ be defined by $q'(X) \equiv w(Z)$ where Z is the set of all vertices in*

V_R which are adjacent to none of the vertices in X.

i. *Show that $q'(X) = w(E_L(V_L - X))$ and hence supermodular.*

ii. *Define the complementary bipartite graph $\overline{B}(V_L, V_R, \overline{E})$ of B as follows: $e \in \overline{E}$ iff the endpoints of e are not connected by an edge of E. Show that $q_B(X) \equiv q'_{\overline{B}}(X)$ where q_B and $q'_{\overline{B}}$ denote the appropriate functions defined for B and $\overline{B}$ respectively. Hence show that q_B is supermodular.*

The terms 'submodular' and 'supermodular' have arisen from the well known notion of modular set functions. However in our framework the latter functions are essentially trivial. For completeness we define modularity and give equivalent definitions below.

Definition 10.2.2 *A function $w : 2^S \longrightarrow \Re$ is* **modular** *iff it satisfies the relation*

$$w(X) + w(Y) = w(X \cup Y) + w(X \cap Y) \quad \forall X, Y \subseteq S.$$

If $w(X) \equiv \sum_{e \in X} w(e), X \subseteq S$, then we call $w(\cdot)$ a **weight function**.

Theorem 10.2.2 *(k)*

i. *Let $w(\cdot)$ be a modular function on subsets of S. Then, $w(X) = \sum_{e \in X}(w(e) - w(\emptyset)) + w(\emptyset)$. Hence, if $w(\emptyset) = 0, w(X) = \sum_{e \in X} w(e)$.*

ii. *if $w^E(X) \equiv |X \cup E|$, where E is a fixed subset of S, then $w^E(\cdot)$ is modular. (Note that $w^E(\emptyset) = |E|$ which could be nonzero.)*

iii. *the function*
$$\alpha(X) \equiv \sum_{e \in X} \alpha(e)$$
is modular. In particular $|\cdot|$ is modular. So is the function $w_E(X) \equiv |X \cap E|$, where E is a fixed subset of S. (In this case observe that the weight of elements in $S - E$ is zero.)

Proof: We prove only the part (i). Let $X \subseteq S$ and let $a \in S - X$. We then have

$$w(X) + w(a) = w(X \cup a) + w(\emptyset),$$

The result then follows by induction.

$\square$

10.3 Basic Operations on Semimodular Functions

We now present a number of operations which act on submodular / supermodular / modular functions and convert them to submodular or supermodular functions.

We begin with addition and scalar multiplication. Here the underlying set does not change. In the case of 'direct sum' it becomes the disjoint union of the original sets while in the case of 'fusion' it is a partition of the original set. We next consider the fundamental operations of 'restriction', 'contraction' and two types of dualization. In the case of contraction and restriction the new functions are over the power sets of appropriate subsets of the old set while the dualization operations do not change the underlying set.

Definition 10.3.1 *Let $\mu_1(\cdot), \mu_2(\cdot)$ be real valued set functions on the subsets of S_1, S_2, where $S_1 \cap S_2 = \emptyset$. The* **direct sum** *of $\mu_1(\cdot), \mu_2(\cdot)$, denoted by $(\mu_1 \oplus \mu_2)(\cdot)$, is defined over subsets of $S_1 \uplus S_2$ by*

$$(\mu_1 \oplus \mu_2)(X_1 \uplus X_2) \equiv \mu_1(X_1) + \mu_2(X_2) \quad \forall X_1 \subseteq S_1, X_2 \subseteq S_2.$$

Exercise 10.4 *(k) Let $\mu_1(\cdot), \mu_2(\cdot)$ be submodular functions and let $w(\cdot)$ be a modular function on the subsets of S. Then $(\mu_1 + \mu_2)(\cdot)$, $(\mu_1 + w)(\cdot), (\mu_1 - w)(\cdot)$, $\lambda\mu_1(\cdot), (\lambda \geq 0)$ are submodular while $-\mu_1(\cdot)$ is supermodular. If μ_1, μ_2 are submodular (supermodular) on subsets of disjoint sets S_1, S_2 then $(\mu_1 \oplus \mu_2(\cdot))$ is submodular (supermodular).*

A common technique in optimization problems which involve finding the 'best subset' is to somehow show that the optimum subset can be thought to be a union of some of the blocks of an appropriate partition of the underlying set. In this manner the size of the problem is reduced since each block can be treated as a single element. The fusion operation defined below formalizes this notion.

Definition 10.3.2 *Let $\mu(\cdot)$ be a set function on subsets of S and let Π be a partition $(\equiv S_1, \cdots S_k)$ of S. Then the* **fusion of μ relative to Π** *, denoted by $\mu_{fus.\Pi}(\cdot)$, is defined on subsets of Π by*

$$\mu_{fus.\Pi}(X_f) \equiv \mu(\bigcup_{T \in X_f} T), X_f \subseteq \Pi.$$

It is immediate that $\mu_{fus.\Pi}(\cdot)$ is submodular (supermodular) [modular] if $\mu(\cdot)$ is such a function.

Contraction, restriction, dualization are fundamental matroidal operations. For graphs these ideas correspond to short circuiting (contracting), open circuiting (deleting) and taking planar duals. These ideas generalize naturally to submodular functions. We prefer to define them for real valued set functions and then specialize them.

Definition 10.3.3 *Let $\mu(\cdot)$ be a real valued set function on subsets of S. The* **restriction** *of $\mu(\cdot)$ to $X \subseteq S$ denoted by $\mu/\mathbf{X}(\cdot)$ is defined by*

$$\mu/\mathbf{X}(Y) \equiv \mu(Y) \quad \forall Y \subseteq X \subseteq S.$$

(Note that there is an abuse of notation here. The original function is on 2^S, while the restriction according to the definition of page 16 is on 2^X).

The **contraction** *of $\mu(\cdot)$ to $X \subseteq S$, denoted by $\mu\diamond\mathbf{X}(\cdot)$, is defined by*

$$\mu\diamond\mathbf{X}(Y) \equiv \mu(Y \cup (S - X)) - \mu(S - X) \quad \forall Y \subseteq X \subseteq S.$$

Definition 10.3.4 *Let $\mu(\cdot)$ be a real valued set function on subsets of S. Let $\alpha(\cdot)$ be a modular function with $\alpha(\emptyset) = 0$ (i.e., $\alpha(\cdot)$ is 'essentially' a real vector) and let $\alpha(e) \geq \mu(e) \quad \forall e \in S$. The* **comodular dual of** $\mu(\cdot)$ **relative to** $\alpha(\cdot)$, *denoted by $\mu^*(\cdot)$, is defined by*

$$\mu^*(X) \equiv \sum_{e \in X} \alpha(e) - [\mu(S) - \mu(S - X)].$$

(If $\alpha(\cdot)$ is unspecified we take $\alpha(e) \equiv \mu(e) \quad \forall e \in S$). The **contramodular dual of** $\mu(\cdot)$, *denoted by $\mu^d(\cdot)$, is defined by*

$$\mu^d(X) \equiv \mu(S) - \mu(S - X).$$

Let $\mu(\cdot), \Pi, \alpha(\cdot)$ be as in the above definitions. We collect properties of contraction, restriction, fusion, comodular and contramodular dualization in the following theorems. The first of these speaks of how to reverse the order of contraction and restriction without affecting the outcome. The reader might like to compare it with Theorems 3.4.1 and 3.4.2. The routine proof is omitted.

Theorem 10.3.1 *(k)*

i. Let $P \subseteq Q \subseteq S$. Then,

$$(\mu \diamond (\mathbf{S} - \mathbf{P}) / (\mathbf{Q} - \mathbf{P}))(\cdot) = (\mu / \mathbf{Q} \diamond (\mathbf{Q} - \mathbf{P}))(\cdot)$$

ii. Let $P \subseteq Q \subseteq S$ and let P, Q be unions of blocks of Π. Let P_f, Q_f be the sets of blocks of Π contained in P, Q respectively. Then, $(\mu \diamond (\mathbf{S} - \mathbf{P}) / (\mathbf{Q} - \mathbf{P}))(Y) = (\mu_{fus.\Pi} \diamond (\mathbf{\Pi} - \mathbf{P_f}) / (\mathbf{Q_f} - \mathbf{P_f}))(Y_f) = (\mu_{fus.\Pi}) / \mathbf{Q_f} \diamond (\mathbf{Q_f} - \mathbf{P_f})(Y_f)$, where Y is the union of the blocks in $Y_f \subseteq \Pi$

The next theorem speaks of the dual of a dual of a function (itself), the duality of contraction and restriction, and of the self dual nature of fusion. We note that if the comodular dual of $\mu(\cdot)$ is taken with respect to $\alpha(\cdot)$ then we would take that of $\mu_{fus.\Pi}(\cdot)$ with respect to $\alpha_{fus.\Pi}(\cdot)$.

Theorem 10.3.2 *(k) If $\mu(\emptyset) = 0$, then*

i. (a) $(\mu^*)^*(\cdot) = \mu(\cdot)$
   (b) $(\mu / \mathbf{X})^*(\cdot) = \mu^* \diamond \mathbf{X}(\cdot)$
   (c) $(\mu \diamond \mathbf{X})^*(\cdot) = \mu^* / \mathbf{X}(\cdot)$
   (d) $(\mu_{fus.\Pi})^*(\cdot) = \mu^*_{fus.\Pi}(\cdot)$

ii. (a) $(\mu^d)^d(\cdot) = \mu(\cdot)$
   (b) $(\mu / \mathbf{X})^d(\cdot) = \mu^d \diamond \mathbf{X}(\cdot)$
   (c) $(\mu \diamond \mathbf{X})^d(\cdot) = \mu^d / \mathbf{X}(\cdot)$

(d) $(\mu_{fus.\Pi})^d(\cdot) = \mu^d_{fus.\Pi}(\cdot)$

Proof:

i. (a) We have,

$$(\mu^*)^*(X) \equiv \sum_{e \in X} \alpha(e) - [\mu^*(S) - \mu^*(S - X)]$$

$$= \sum_{e \in X} \alpha(e) - [\sum_{e \in S} \alpha(e) - \sum_{e \in S-X} \alpha(e) - (\mu(S) - \mu(\emptyset)) + (\mu(S) - \mu(X))]$$

$$= \mu(X).$$

(b) Let $Z \subseteq X$. Then,

$$(\mu^* \diamond \mathbf{X})(Z) \equiv \mu^*(Z \cup (S - X)) - \mu^*(S - X)$$

$$= \sum_{e \in Z \cup (S-X)} \alpha(e) - \sum_{e \in S-X} \alpha(e) - (\mu(S) - \mu(X - Z)) + (\mu(S) - \mu(X))]$$

$$= \sum_{e \in Z} \alpha(e) - (\mu(X) - \mu(X - Z))$$

$$= (\mu/\mathbf{X})^*(Z).$$

(c) This follows by using the above two results.

(d) This follows from the definitions of fusion and comodular duals of $\mu(\cdot)$ and $\mu_{fus.\Pi}(\cdot)$.

ii. The proof is similar to the '*' case and is omitted.

□

The next theorem is a generalization of Corollaries 3.4.3 and 3.4.2. Its routine proof is omitted.

Theorem 10.3.3 *(k) Let $A \subseteq S$ and let $X \subseteq (S - A)$. Then*

$$\mu(X \cup A) = \mu/\mathbf{A}(A) + \mu\diamond(\mathbf{S} - \mathbf{A})(X)$$

We now show that contraction and restriction preserve submodularity and supermodularity while comodular and contramodular dualizations behave as the names indicate.

Theorem 10.3.4 *(k)*

i. Let $X \subseteq S$. Then, $\mu/\mathbf{X}(\cdot), \mu\diamond\mathbf{X}(\cdot)$ are submodular (supermodular) if $\mu(\cdot)$ is submodular (supermodular).

ii. If $\mu(\cdot)$ is submodular (supermodular), then $\mu^*(\cdot)$ is submodular (supermodular) while $\mu^d(\cdot)$ is supermodular (submodular).

Proof: We consider only the submodular case.
i. The submodularity of the restriction of a submodular function is obvious.
We now consider contraction. We have

$$\mu(Y_1 \cup (S-X)) + \mu(Y_2 \cup (S-X))$$
$$\geq \mu(Y_1 \cup Y_2 \cup (S-X)) + \mu(Y_1 \cap Y_2 \cup (S-X)) \quad \forall Y_1, Y_2 \subseteq X.$$

Further, $\mu(S-X)$ is a constant for subsets of X. The submodularity of $\mu \diamond \mathbf{X}(\cdot)$ follows.

ii. We have

$$\mu(S-X) + \mu(S-Y) \geq \mu(S-(X \cup Y)) + \mu(S-(X \cap Y)) \quad \forall X, Y \subseteq S.$$

Further, $w(X) \equiv \sum_{e \in X} \alpha(e)$ is a modular function, $\mu(S)$ is a constant. The submodularity of $\mu^*(\cdot)$ follows. The supermodularity of $\mu^d(\cdot)$ follows noting that the negative of a submodular function is supermodular.

□

Submodular and supermodular functions associated with graphs, hypergraphs (as represented by bipartite graphs) etc usually behave in an interesting way. A basic operation such as the ones described above on a semimodular function associated say with a graph $\mathcal{G}$ takes it to another such function associated with a second graph which can be derived from $\mathcal{G}$ in a simple way. The exercises given below illustrate this idea.

Exercise 10.5 *(k)* **Rank and nullity functions of a graph**
Let $\mathcal{G}$ be a graph. Let $X \subseteq E(\mathcal{G})$. We remind the reader that $\mathcal{G}.X$ is the subgraph of $\mathcal{G}$ on X and $\mathcal{G} \times X$ is the graph obtained from $\mathcal{G}$ by fusing the end points of edges in $E(\mathcal{G}) - X$ and removing them. Let $r'(X), X \subseteq E$, be the number of edges in a forest of $\mathcal{G} \times X$. We will call $r'(\cdot)$ the prime rank function of $\mathcal{G}$. Let $\nu'(X), X \subseteq E$, be the number of edges in a coforest of $\mathcal{G}.X$. We will call $\nu'(\cdot)$ the prime nullity function of $\mathcal{G}$. Prove

i. *The rank function of* $\mathcal{G}.X = r/\mathbf{X}(\cdot)$.

ii. *The rank function of* $\mathcal{G} \times X = r \diamond \mathbf{X}(\cdot)$.

iii. $r^d(\cdot) = r'(\cdot)$.

iv. $r^*(\cdot) = \nu(\cdot)$.

v. *The nullity function of* $\mathcal{G} \times X = \nu/\mathbf{X}(\cdot)$.

vi. *The nullity function of* $\mathcal{G}.X = \nu \diamond \mathbf{X}(\cdot)$.

vii. *The prime rank function of* $\mathcal{G} \times X = r'/\mathbf{X}(\cdot)$.

viii. *The prime rank function of* $\mathcal{G}.X = r' \diamond \mathbf{X}(\cdot)$.

ix. *The prime nullity function of* $\mathcal{G}.X = \nu'/\mathbf{X}(\cdot)$.

x. *The prime nullity function of* $\mathcal{G} \times X = \nu' \diamond \mathbf{X}(\cdot)$.

Exercise 10.6 The incidence function $|I|(\cdot)$ and the interior edge function $|E|(\cdot)$ of a graph $\mathcal{G}$
(see Examples 10.2.2 and 10.2.3) Let $\mathcal{G}'$ denote the graph obtained by fusing $V(\mathcal{G})-X$ into a single node and deleting all edges with both end points in $V(\mathcal{G})-X$ and let $|I'|(\cdot)$, $|E'|(\cdot)$ be its incidence and interior edge functions respectively. Let $|I"|(\cdot)$, $|E"|(\cdot)$ be the incidence and interior edge functions respectively of the subgraph of $\mathcal{G}$ on $X \cup \Gamma(X)$. $|\hat{I}|(\cdot)$ be the incidence function of the subgraph of $\mathcal{G}$ on X. Let Π be a partition of $V(\mathcal{G})$ and let $|I_\Pi|(\cdot)$ be the incidence function of the graph obtained from $\mathcal{G}$ by fusing the blocks of Π into single vertices but not deleting any edges.
Prove

i. $|I|/\mathbf{X}(\cdot) = |I'|/\mathbf{X}(\cdot) = |I"|/\mathbf{X}(\cdot)$ *and* $|I|\diamond\mathbf{X}(\cdot) = |I'|\diamond\mathbf{X}(\cdot) = |I"|\diamond\mathbf{X}(\cdot)$.

ii. $|I|\diamond\mathbf{X}(\cdot) = |\hat{I}|(\cdot)$.

iii. $|I|_{fus.\Pi}(\cdot) = |I_\Pi|(\cdot)$.

iv. $|I|^d(\cdot) = |E|(\cdot)$.

v. *Let $\mathcal{G}$ have no self loops. Then, $|I|^*(\cdot) = |I|(\cdot)$, where the dual is defined with respect to the weight vector $\alpha(\cdot)$ with $\alpha(v) \equiv |I|(v) \quad \forall v \in V(\mathcal{G})$.*

vi. $|E|\diamond\mathbf{X}(\cdot) = |E'|\diamond\mathbf{X}(\cdot) = |E"|\diamond\mathbf{X}(\cdot)$ *and* $|E|/\mathbf{X}(\cdot) = |E'|/\mathbf{X}(\cdot) = |E"|/\mathbf{X}(\cdot)$.

vii. $|E|/\mathbf{X}(\cdot) = |\hat{E}|(\cdot)$.

viii. $|E|_{fus.\Pi}(\cdot) = |E_\Pi|(\cdot)$.

Exercise 10.7 The $|\Gamma_L|(\cdot), |E_L|(\cdot)$ functions of a bipartite graph
Let $B \equiv (V_L, V_R, E)$ be a bipartite graph. Let $\Gamma_L(\cdot) \equiv \Gamma/\mathbf{V_L}(\cdot)$ and let $\Gamma_R(\cdot) \equiv \Gamma/\mathbf{V_R}(\cdot)$. We will call $|\Gamma_L|(\cdot)$, ($|\Gamma_R|(\cdot)$) $|E_L|(\cdot)$, ($|E_R|(\cdot)$), the left(right) adjacency function, left (right) exclusivity function respectively of B (see Example 10.2.7). Let $X \subseteq V_L$. Let $B._{LX}$ be the subgraph of B on $X \uplus \Gamma(X)$ and let $B_{\diamond LX}$ be the graph obtained by first deleting $V_L - X \uplus \Gamma(V_L - X)$ and all edges with atleast one end point in this set ($B._{RX}, B_{\diamond RX}, X \subseteq V_R$ are similarly defined interchanging left and right). Let Π be a partition of V_L. Let B_Π be the graph obtained from B by fusing the blocks of Π into single vertices. Let $|\Gamma_{\Pi L}|(\cdot), |E_{\Pi L}|(\cdot)$ be the corresponding left adjacency and left exclusivity functions.
Show that

i. $|\Gamma_L|/\mathbf{X}(\cdot)$ *is the left adjacency function of* $B._{LX}$.

ii. $|\Gamma_L|\diamond\mathbf{X}(\cdot)$ *is the left adjacency function of* $B_{\diamond LX}$.

iii. $|\Gamma_L|_{fus.\Pi}(\cdot) = |\Gamma|_{\Pi L}(\cdot)$.

iv. $|\Gamma_L|^d(\cdot) = |E_L|(\cdot)$

v. $|E_L|/\mathbf{X}(\cdot)$ *is the left exclusivity function of* $B_{\diamond LX}$.

vi. $|E_L|\diamond\mathbf{X}(\cdot)$ *is the left exclusivity function of* $B_{\cdot LX}$.

vii. $|E_L|_{fus.\Pi}(\cdot) = |E_{\Pi L}|(\cdot)$.

10.4 *Other Operations on Semimodular Functions

We now consider a number of other operations on semimodular functions which yield other such functions. These operations, while being useful, are by no means standard. We therefore study them through a sequence of problems.

Problem 10.1 *(k)*

i. Let $f(\cdot), g(\cdot)$ *be submodular (supermodular) on subsets of* S *and let* $(f-g)(\cdot)$ *be monotone increasing or monotone decreasing. Show that* $h(\cdot) \equiv min(f(\cdot), g(\cdot))$ $(max(f(\cdot), g(\cdot)))$ *is a submodular (supermodular) function.*

ii. Let $f(\cdot)$ *be an increasing or decreasing submodular (supermodular) function and let k be a constant. Show that min(k, f(X)) (max(k, f(X)) is a submodular (supermodular) function.*

Solution: We consider only the monotone increasing case, since $(f-g)(\cdot)$ is monotone decreasing iff $(g-f)(\cdot)$ is monotone increasing. Further we confine ourselves to submodular functions.

i. Let $X, Y \subseteq S$. We will verify that

$$h(X) + h(Y) \geq h(X \cup Y) + h(X \cap Y).$$

This is clear if $h(\cdot)$ agrees with $f(\cdot)$ or with $g(\cdot)$ on both X and Y. Let us therefore assume that $h(X) = f(X)$ and $h(Y) = g(Y)$. We then have,

$$h(X) + h(Y) \geq f(X \cup Y) + f(X \cap Y) + g(Y) - f(Y).$$

But $f(X \cup Y) + g(Y) - f(Y) \geq g(X \cup Y)$. Hence

$$h(X) + h(Y) \geq g(X \cup Y) + f(X \cap Y) \geq h(X \cup Y) + h(X \cap Y).$$

ii. This is a direct consequence of the previous result.

The next problem involves an instance of the convolution operation to be discussed in the next chapter. As we shall see later, even more important than the $\mu_{min}(\cdot)$ and $\mu_{max}(\cdot)$ functions are the collection of subsets over which these functions become equal to $\mu(\cdot)$.

Problem 10.2 *(k)*

Definition 10.4.1 *Let $\mu(\cdot)$ be a real valued set function defined over subsets of S. Let*

$$\mu_{min}(X) \equiv min_{Y \subseteq X} \mu(Y),$$

$$\mu_{max}(X) \equiv max_{Y \subseteq X} \mu(Y),$$

$$\underline{\mu_{min}}(X) \equiv min_{Y \supseteq X} \mu(Y),$$

$$\underline{\mu_{max}}(X) \equiv max_{Y \supseteq X} \mu(Y).$$

Show that

i. *$\mu_{min}(\cdot), \underline{\mu_{min}}(\cdot)$ are submodular if $\mu(\cdot)$ is submodular and $\mu_{max}(\cdot), \underline{\mu_{max}}(\cdot)$ are supermodular if $\mu(\cdot)$ is supermodular.*

ii. *if $\mu(\cdot)$ be a submodular (supermodular) function on subsets of S then the subsets over which it reaches a minimum (maximum) form a distributive lattice(i.e., the collection is closed under union and intersection).*

Definition 10.4.2 *The collection of subsets of S over which a submodular (supermodular) function reaches a minimum (maximum) is called its* **principal structure**.

iii. *Prove*

Theorem 10.4.1 *(k) Let $\mu(\cdot)$ be a submodular function on subsets of S. Let $X \subseteq S$ have the property that*

$$\mu(X) \leq \mu(Y) \quad \forall Y \subseteq X$$

$$(\mu(X) < \mu(Y) \quad \forall Y \subset X).$$

Then X is contained in some (every) set that minimizes $\mu(\cdot)$.

Solution:
i. Let m_Z denote a subset of $Z \subseteq S$ at which $\mu(\cdot)$ reaches the minimum among all the subsets of Z and let m_{Xa}, m_{Ya} denote $m_{X \cup a}, m_{Y \cup a}$. Let $X \subseteq Y \subseteq S$ and let $a \in S - Y$. We now have,

$$\mu(m_{Xa}) + \mu(m_Y) \geq \mu(m_Y \cup m_{Xa}) + \mu(m_Y \cap m_{Xa}).$$

i.e.,

$$\mu(m_Y \cup m_{Xa}) - \mu(m_Y) \leq \mu(m_{Xa}) - \mu(m_Y \cap m_{Xa}).$$

Now $m_Y \cup m_{Xa} \subseteq Y \cup a$ and $m_Y \cap m_{Xa} \subseteq X$. But then

$$\mu(m_{Ya}) \leq \mu(m_Y \cup m_{Xa})$$

and

$$\mu(m_X) \leq \mu(m_Y \cap m_{Xa}).$$

We therefore have,

$$\mu(m_{Ya}) - \mu(m_Y) \leq \mu(m_{Xa}) - \mu(m_X).$$

Since $\mu_{min}(T) = \mu(m_T) \quad \forall T \subseteq S$, it is clear that $\mu_{min}(\cdot)$ is submodular. The proof of the supermodular case $\mu_{max}(\cdot)$ is similar.
Next let

$$\sigma(X) \equiv \mu(S) - \mu(S - X).$$

Observe that

$$\underline{\mu_{min}}(X) = \mu(S) - \sigma_{max}(S - X)$$

and

$$\underline{\mu_{max}}(X) = \mu(S) - \sigma_{min}(S - X).$$

The required results follow by noting that $\sigma(\cdot)$ is supermodular (submodular) iff μ is submodular (supermodular).

ii. Let X, Y minimize the submodular function $\mu(\cdot)$. By the basic inequality

$$\mu(X) + \mu(Y) \geq \mu(X \cup Y) + \mu(X \cap Y).$$

The only way this inequality can be satisfied is to have the values of $\mu(\cdot)$ on all four sets to be the same. The result follows. The proof for the supermodular case is similar.

iii. Proof of Theorem 10.4.1: Let $\mu(\cdot)$ reach the minimum, among all subsets of S, at Z. If X is a subset of Z, there is nothing to prove. Suppose X is not. Then $X \cap Z \subset X$. By the submodularity of $\mu(\cdot)$, we then have,

$$\mu(X) + \mu(Z) \geq \mu(X \cup Z) + \mu(X \cap Z).$$

Case1 $\mu(X) \leq \mu(X \cap Z)$. In this case $\mu(Z) \geq \mu(X \cup Z)$. Thus X is contained in a subset that minimizes $\mu(\cdot)$ viz $X \cup Z$.
Case2 $\mu(X) < \mu(X \cap Z)$. In this case $\mu(Z) > \mu(X \cup Z)$, which is a contradiction. We conclude that X must be a subset of Z.

□

Problem 10.3 *Let $s(\cdot)$ be an increasing set function taking subsets of a finite set S_1 to subsets of another finite set S_2 i.e., $s(Y) \supseteq s(X) \quad \forall X \subseteq Y \subseteq S_1$. Suppose*

$$s(X) \cup s(Y) = s(X \cup Y)$$

$$(s(X) \cap s(Y) = s(X \cap Y)).$$

i. Let $w(\cdot)$ assign each element of S_2 a nonnegative weight. Define $\hat{s}(X) \equiv \sum_{e \in s(X)} w(e)$. Show that

(a) the function $\hat{s}(\cdot)$ is submodular (supermodular).

(b) (k) *the set functions (defined in Exercise10.1)* $V(\cdot), \Gamma(\cdot), I(\cdot), V_{tail}(\cdot)$, $V_{head}(\cdot), \Gamma_{in}(\cdot), \Gamma_{out}(\cdot), I_{in}(\cdot), I_{out}(\cdot)$ *yield submodular functions when 'weighted nonnegatively' while* $E(\cdot)$ *yields a supermodular function. Hence show that the nonnegatively weighted versions of* $cut(\cdot), cut_{in}(\cdot), cut_{out}$ *are also submodular.*

ii. (k) *Let* $\mu(\cdot)$ *be an increasing submodular(supermodular) function on subsets of* S_2. *Define* $\sigma(\cdot)$ *on subsets of* S_1 *by* $\sigma(X) \equiv \mu(s(X)), X \subseteq S_1$. *Show that* $\sigma(\cdot)$ *is submodular (supermodular).*
Example: *(k) Let* B *be a bipartite graph on* V_L, V_R. *Let the function* $s(\cdot)$ *be taken as the* $\Gamma(\cdot)$ $(E_L(\cdot))$ *function of the bipartite graph. Let* $\mu(\cdot)$ *be any increasing submodular (supermodular) function on subsets of* V_R. *Then* $\mu(\Gamma(\cdot))(\mu(E_L(\cdot)))$ *is a submodular (supermodular) function on subsets of* V_L.

Solution:
i(a) For any increasing set function, we must have,

$$s(X) \cup s(Y) \subseteq s(X \cup Y),$$

$$s(X) \cap s(Y) \supseteq s(X \cap Y).$$

If

$$s(X) \cup s(Y) = s(X \cup Y),$$

it is clear that

$$\begin{aligned} \hat{s}(X) + \hat{s}(Y) &= w(s(X)) + w(s(Y)) \\ &= w(s(X) \cup s(Y)) + w(s(X) \cap s(Y)) \\ &\geq \hat{s}(X \cup Y) + \hat{s}(X \cap Y). \end{aligned}$$

The supermodular case (where $s(X) \cap s(Y) = s(X \cap Y)$) is handled similarly.

i(b) It is easily verified that the functions $V(\cdot), \Gamma(\cdot), I(\cdot), V_{tail}(\cdot), V_{head}(\cdot)$, $\Gamma_{in}(\cdot), \Gamma_{out}(\cdot), I_{in}(\cdot), I_{out}(\cdot)$ all satisfy the property $s(X) \cup s(Y) = s(X \cup Y)$, while $E(\cdot)$ satisfies $s(X) \cap s(Y) = s(X \cap Y)$. Further they are all increasing set functions. Thus it is clear that they must yield sub- or supermodular functions (as the case may be) when weighted. To study the weighted version of $cut(\cdot)$, we observe that $cut(X) = \Gamma(X) - E(X)$ and $\Gamma(X) \supseteq E(X)$. So

$$w(cut(X)) = w(\Gamma(X)) - w(E(X))$$

and the submodularity of this function follows from the sub- and supermodularity of the functions $w(\Gamma(\cdot))$ and $w(E(\cdot)$. One can similarly prove that $cut_{in}(\cdot), cut_{out}(\cdot)$ yield submodular functions when weighted.

ii. We consider only the case where $s(X \cup Y) = s(X) \cup s(Y)$ and $\mu(\cdot)$ submodular. The supermodular case can be handled similarly. We have $\sigma(\cdot) \equiv \mu(s(\cdot))$. Now

$$\begin{aligned} \mu(s(X)) + \mu(s(Y)) &\geq \mu(s(X) \cup s(Y)) + \mu(s(X) \cap s(Y)) \\ &\geq \mu(s(X \cup Y)) + \mu(s(X \cap Y)) \end{aligned}$$

(since

$$s(X) \cap s(Y) \supseteq s(X \cap Y), s(X) \cup s(Y) = s(X \cup Y)$$

and $\mu(\cdot)$ is increasing). Thus $\sigma(\cdot)$ is submodular on subsets of S_1.
To verify the correctness of the example we need only verify that $\Gamma(\cdot), E_L(\cdot)$ are increasing and respectively satisfy the property $s(X) \cup s(Y) = s(X \cup Y)$ and the property $s(X) \cap s(Y) = s(X \cap Y)$. This, as mentioned before, is routine.

Problem 10.4 *(k) Let f be any increasing cap (increasing cup) function from $\Re$ to $\Re$. Let μ be an increasing submodular (increasing supermodular) function. Show that $f(\mu(\cdot))$ is submodular (supermodular). Examples:*

i. Let $r(\cdot)$ be the rank function of a graph $\mathcal{G}$. Let $f(t) \equiv (r(\mathcal{G}))^2 - (t - r(\mathcal{G}))^2$. Note that $f(\cdot)$ is increasing in the interval $[0, r(\mathcal{G})]$. Then $f(r(\cdot))$ is submodular.

ii. Let $w(\cdot)$ be a weight function on S. Let $f(t) \equiv (w(S))^2 - (t - w(S))^2$ and let $g(t) \equiv t^2$. Then $f(w(\cdot))$ is submodular while $g(w(\cdot)), g(r'(\cdot)), g(\nu'(\cdot))$ (the latter of Example 10.2.10) are supermodular.

Solution: If the function $f(\cdot)$ is an increasing cap function from $\Re$ to $\Re$ then

$$f(x+h) - f(x) \geq f(y+h) - f(y) \quad \forall y \geq x, h \geq 0.$$

Let $\mu(\cdot)$ be an increasing submodular function on the subsets of S and let $X \subseteq Y \subseteq S, a \in S - Y$. Then

$$\mu(X \cup a) - \mu(X) \geq \mu(Y \cup a) - \mu(Y).$$

Let δ, ϵ represent the left and right sides of the above inequality. Then, since $f(\cdot)$ is increasing,

$$\begin{aligned} f(\mu(X) + \delta) - f(\mu(X)) &\geq f(\mu(X) + \epsilon) - f(\mu(X)) \\ &\geq f(\mu(Y) + \epsilon) - f(\mu(Y)). \end{aligned}$$

This is equivalent to saying that

$$f(\mu(X \cup a)) - f(\mu(X)) \geq f(\mu(Y \cup a)) - f(\mu(Y)).$$

This proves the submodularity of $f(\mu(\cdot))$. The supermodular case is similar and the examples are direct applications of the result.

10.5 Polymatroid and Matroid Rank Functions

Matroid rank functions are the most important class of submodular functions. Polymatroid rank functions are their immediate generalization. As we shall show, any submodular function is a translate of a polymatroid rank function by a modular function. In this section we define these functions and study the results of applying on them some of the basic operations introduced in Section 10.3.

Definition 10.5.1 *A submodular function is a* **polymatroid rank function** *iff it takes zero value on $\emptyset$ and is nonnegative and increasing.*

Definition 10.5.2 *A polymatroid rank function is a* **matroid rank function** *iff it takes integral values and does not exceed* 1 *on any of the singletons.*

Exercise 10.8 *(k) Show that the rank and nullity functions of a graph are matroid rank functions.*

Exercise 10.9 *(k) Show that the vertex function, incidence function and adjacency function of a graph are polymatroid rank functions while the cut function $|cut|(\cdot)$ is not (see Examples 10.2.1, 10.2.3, 10.2.4, 10.2.5).*

Both matroid and polymatroid rank functions behave nicely with respect to comodular dualization.

Exercise 10.10 ***Theorem 10.5.1*** *(k) $\mu^*(\cdot)$ is a polymatroid (matroid) rank function if $\mu(\cdot)$ is one.*

Remark: In the definition of $\mu^*(\cdot)$, if $\alpha(e)$ is less than $\mu(e)$ for some e, the dual of a polymatroid rank function would not be a polymatroid rank function. But we would still have $\mu^*(\cdot)$ as submodular.

The next theorem states among other things that every submodular function is a translate of a polymatroid rank function. This idea would be useful when we relate minimization of submodular functions to the operation of convolution in the next chapter.

Theorem 10.5.2 *(k) Let $\mu(\cdot)$ be a submodular function on subsets of a finite set S*

i. *$\mu(\cdot)$ is an increasing function iff*

$$\mu(S) - \mu(S-e) \geq 0 \quad \forall e \in S.$$

It is nonnegative increasing iff it satisfies, in addition, $\mu(\emptyset) \geq 0$.

ii. *If*

$$\mu(S) - \mu(S-e) = \mu(e) - \mu(\emptyset) \quad \forall e \in S,$$

then $\mu(\cdot)$ is modular.

iii. *Let a weight function $w(\cdot)$ on S be defined by $w(e) \equiv \mu(S) - \mu(S-e)$. Then $\mu(\cdot) - w(\cdot) - \mu(\emptyset)$ is a polymatroid rank function.*

Proof: 'only if' is clear.
if:

i. Since $\mu(\cdot)$ is submodular, we have

$$\mu(S) - \mu(S-e) \leq \mu(X \cup e) - \mu(X) \quad \forall X \subseteq S, e \in S - X.$$

The result follows. The nonnegative increasing case is trivial.

ii. Since

$$\mu(S) - \mu(S-e) \leq \mu(X \cup e) - \mu(X) \leq \mu(e) - \mu(\emptyset) \quad \forall X \subseteq S, e \in S - X,$$

it follows that the given condition implies that

$$\mu(X \cup e) - \mu(X) = \mu(e) - \mu(\emptyset) \quad \forall X \subseteq S, e \in S - X.$$

Thus

$$\mu(X) = \sum_{e \in X} [\mu(e) - \mu(\emptyset)] + \mu(\emptyset) \quad \forall X \subseteq S.$$

Clearly this means $\mu(\cdot)$ is modular.

iii. It is easily verified that the function $\mu(\cdot) - w(\cdot) - \mu(\emptyset)$ satisfies the above condition for being nonnegative increasing and takes zero value on $\emptyset$. Since it is clearly submodular, the function is a polymatroid rank function.

□

The next exercise speaks of contraction, restriction and dualization on natural functions associated with a matroid. The reader might like to compare it with Exercise 10.5.

Exercise 10.11 *(k)* **The rank and nullity functions of a matroid**
As has already been pointed out in Chapter 4, we can give a number of alternative descriptions of a matroid in terms of

- *independent sets*
- *circuits*
- *bases*
- *matroid rank function.*

Let us assume the last description (of a matroid rank function on subsets of S) is available. Then, a set $X \subseteq S$ is said to be **independent** *iff $r(X) = |X|$, it is a* **base** *iff $r(X) = |X| = r(S)$, it is a* **circuit** *iff $r(X) = |X| - 1$ and all proper subsets of X are independent. We remind the reader that every independent set can be extended to a base and that circuits are minimal dependent (non independent) sets. It is verified elsewhere that these classes satisfy the conditions of the appropriate axiom sets. Let us denote the matroid corresponding to these classes by $\mathcal{M}$. The comodular dual of the rank function with respect to the $|.|$ function is called the* **nullity** *function of the matroid and denoted by $\nu(\cdot)$. Let Π be a partition of S. Show that*

i. $r/\mathbf{X}(\cdot), r\diamond\mathbf{X}(\cdot), \nu(\cdot)$ *are matroid rank functions.*

ii. $r_{fus.\Pi}(\cdot)$ *is an integral polymatroid rank function. (We show later, in the next chapter, that all integral polymatroid rank functions can be obtained by fusion of matroid rank functions).*

iii. *The independent sets of the matroid (denoted by* $\mathcal{M}.X$*) defined by* $r/\mathbf{X}(\cdot)$ *are independent sets of* $\mathcal{M}$ *contained in* X*. The bases of* $\mathcal{M}.X$ *are maximal intersections of bases of* $\mathcal{M}$ *with* X*. The circuits of* $\mathcal{M}.X$ *are circuits of* $\mathcal{M}$ *contained in* X*.*

iv. *The independent sets of the matroid (denoted by* $\mathcal{M} \times X$*) defined by* $r\diamond\mathbf{X}(\cdot)$ *are sets whose union with every independent set of* $\mathcal{M}.(S-X)$ *is independent in* $\mathcal{M}$*. The bases of* $\mathcal{M} \times X$ *are minimal intersections of bases of* $\mathcal{M}$ *with* X*. The circuits of* $\mathcal{M} \times X$ *are minimal intersections of circuits of* $\mathcal{M}$ *with* X*.*

v. *The bases of the matroid (denoted by* $\mathcal{M}^*$*) defined by* $r^*(\cdot)$ *are the complements of bases of* $\mathcal{M}$*.*

vi. $r^d(X) =$ *rank of* $\mathcal{M} \times X$*.*

vii. *The nullity function of* $\mathcal{M} \times X = \nu/\mathbf{X}(\cdot)$*. Thus* $\mathcal{M}^*.X = (\mathcal{M} \times X)^*$*.*

viii. *The nullity function of* $\mathcal{M}.X = \nu\diamond\mathbf{X}(\cdot)$*. Thus* $\mathcal{M}^* \times X = (\mathcal{M}.X)^*$*.*

ix. *The contramodular dual of the rank function of* $\mathcal{M} \times X = r^d/\mathbf{X}(\cdot)$*.*

x. *The contramodular dual of the rank function of* $\mathcal{M}.X = r^d\diamond\mathbf{X}(\cdot)$*.*

xi. *The contramodular dual of the nullity function of* $\mathcal{M} \times X = \nu^d\diamond\mathbf{X}(\cdot)$*.*

xii. *The contramodular dual of the nullity function of* $\mathcal{M}.X = \nu^d/\mathbf{X}(\cdot)$*.*

An elementary but useful notion in graphs is that of putting additional edges in parallel to existing ones. This notion immediately generalizes to matroids. For submodular functions however more is possible provided some minor conditions on monotonicity are satisfied.

Definition 10.5.3 *Let* $\mu(\cdot)$ *be a submodular function on subsets of* S*. Elements* $e_1, e_2 \in S$ *are* **parallel** *with respect to* $\mu(\cdot)$ *iff*

$$\mu(X \cup e_1) = \mu(X \cup e_2) = \mu(X \cup e_1 \cup e_2) \quad \forall X \subseteq S.$$

Observe that other elements cannot distinguish between e_1, e_2.

Definition 10.5.4 *[Lovász83] Let* $\mu(\cdot)$ *be a submodular function on subsets of* S*. Let* $T \subseteq S$ *and element* $a_T \notin S$*. The* **parallel extension of** $\mu(\cdot)$ **by** a_T **parallel to T***, denoted by* $\hat{\mu}(\cdot)$ *on subsets of* $S \cup a_T$ *is defined by*

$$\hat{\mu}(X) \equiv \mu(X)$$

$$\hat{\mu}(X \cup a_T) \equiv \mu(X \cup T) \quad \forall X \subseteq S.$$

Theorem 10.5.3 *(k)*

i. $\hat{\mu}(\cdot)$ *is a submodular function, if, for each* $e \in T$ *we have* $\mu(S) - \mu(S-e) \geq 0$.

ii. if $\mu(\cdot)$ *is a polymatroid rank function, then, so is* $\hat{\mu}(\cdot)$.

iii. if $\mu(\cdot)$ *is a matroid rank function on subsets of* S *then* e_1, e_2 *are in parallel iff*

$$\mu(e_1) = \mu(e_2) = \mu(\{e_1, e_2\}).$$

Proof:
i. We need to verify if

$$\hat{\mu}(X \cup e) - \hat{\mu}(X) \geq \hat{\mu}(Y \cup e) - \hat{\mu}(Y) \quad \forall X \subseteq Y \subseteq S \cup a_T, \ \ \forall e \in (S \cup a_T) - Y.$$

We have the following cases

- $a_T \notin Y$
- $a_T \in X$
- $a_T \in Y - X$ and $e \notin T$
- $a_T \in Y - X$ and $e \in T$

In the first three cases the inequality holds by the submodularity of $\mu(\cdot)$ and the definition of $\hat{\mu}(\cdot)$. In the last case the RHS is zero while the LHS is nonnegative since

$$\mu(X \cup e) - \mu(X) \geq \mu(S) - \mu(S-e) \geq 0.$$

ii. By the above reasoning, since in this case $\mu(\cdot)$ is increasing, we must have the parallel extension as submodular. Further $\hat{\mu}(\emptyset) = 0$ and $\hat{\mu}(\cdot)$ is clearly increasing.

iii. We need only prove the 'if' part.
Let $\mu(\cdot)$ be a matroid rank function. We must have, for $i = 1, 2$,

$$\mu(X \cup e_i) - \mu(X) \leq \mu(X \cup e_1 \cup e_2) - \mu(X) \leq \mu(\{e_1, e_2\}) = \mu(e_i) \cdots (*)$$

If $\mu(e_1) = 0$ then it is easy to see that

$$\mu(X \cup e_1) = \mu(X \cup e_2) = \mu(X \cup e_1 \cup e_2) \quad \forall X \subseteq S \cdots (**)$$

Let $\mu(e_1) = 1$. Suppose $\mu(X \cup e_1) - \mu(X) = 1$. We claim $e_2 \notin X$, as otherwise

$$\mu(X \cup e_1) - \mu(X) \leq \mu(\{e_1, e_2\}) - \mu(e_2) = 0,$$

a contradiction. But then

$$\mu(X \cup e_2) + \mu(\{e_1, e_2\}) \geq \mu(X \cup e_1 \cup e_2) + \mu(e_2)$$

i.e., $\mu(X \cup e_2) = \mu(X \cup e_1 \cup e_2)$ (since $\mu(\cdot)$ is increasing and $\mu(\{e_1, e_2\}) = \mu(e_2)$). Similarly $\mu(X \cup e_1) = \mu(X \cup e_1 \cup e_2)$. Thus the desired equality $(**)$ holds. So we need only consider the case where

$$\mu(X \cup e_1) - \mu(X) = 0.$$

By the above argument $\mu(X \cup e_2) - \mu(X)$ cannot be 1 and is therefore 0. Now, by the submodularity of $\mu(\cdot)$, we must have

$$\mu(X \cup e_1 \cup e_2) - \mu(X \cup e_1) \leq \mu(X \cup e_2) - \mu(X) = 0,$$

from which the desired equality $(**)$ follows.

□

10.6 Connectedness for Semimodular Functions

When a submodular function can be expressed as the direct sum of other such functions, problems involving it drastically simplify. We essentially have to look at much smaller underlying sets which are disconnected under the function. We sketch elementary ideas regarding connectedness in this section.

We introduce the notion of an elementary separator of a submodular function below. This notion is a generalization of 2-connectedness for graphs and connectedness for matroids.

Definition 10.6.1 *Let $\mu(\cdot)$ be a submodular (supermodular) function on subsets of S with $\mu(\emptyset) = 0$. A set $E \subseteq S$ is a* **separator** *of $\mu(\cdot)$ iff*

$$\mu(E) + \mu(S - E) = \mu(S).$$

A minimal nonvoid separator is called an **elementary separator**.

Theorem 10.6.1 *(k) Let $\mu(\cdot)$ be a submodular (supermodular) function on subsets of S with $\mu(\emptyset) = 0$. Then,*

i.

$$\mu(X_1) + \cdots + \mu(X_n) \geq \mu(X_1 \cup \cdots \cup X_n) \quad \forall X_1, \cdots X_n \subseteq S, X_i \cap X_j = \emptyset, i \neq j$$

$$(\mu(X_1) + \cdots + \mu(X_n) \leq \mu(X_1 \cup \cdots \cup X_n) \quad \forall X_1, \cdots X_n \subseteq S, X_i \cap X_j = \emptyset, i \neq j).$$

ii. if E_1, E_2 are separators of $\mu(\cdot)$, then so are $E_1 \cup E_2, E_1 \cap E_2$.

iii. E is a separator of $\mu(\cdot)$ iff

$$\mu(X_1) + \mu(X_2) = \mu(X_1 \cup X_2) \quad \forall X_1 \subseteq E, X_2 \subseteq S - E$$

(Thus when E is a separator studying $\mu(\cdot)$ reduces to studying $\mu/\mathbf{E}(\cdot)$, $\mu/\mathbf{S} - \mathbf{E}(\cdot)$. In other words $\mu(\cdot) = (\mu/\mathbf{E} \oplus \mu/\mathbf{S} - \mathbf{E})(\cdot)$).

iv. E is a separator of $\mu(\cdot)$ iff

$$\mu/\mathbf{E}(\cdot) = \mu\diamond\mathbf{E}(\cdot).$$

v. if E is a separator of $\mu(\cdot)$, it is also a separator of $\mu^d(\cdot)$ and $\mu^*(\cdot)$.

Proof: We will handle only the submodular case. The supermodular situation is similar.

i. If X_1, X_2 do not intersect we have

$$\mu(X_1) + \mu(X_2) \geq \mu(X_1 \cup X_2) + \mu(\emptyset).$$

Since $\mu(\emptyset) = 0$, the result follows by induction on the number of sets.

ii. We have,

$$\mu(E_1) + \mu(E_2) \geq \mu(E_1 \cup E_2) + \mu(E_1 \cap E_2),$$
$$\mu(S - E_1) + \mu(S - E_2) \geq \mu(S - (E_1 \cup E_2)) + \mu(S - (E_1 \cap E_2)).$$

Adding the two inequalities we get,

$$\mu(E_1) + \mu(E_2) + \mu(S - E_1) + \mu(S - E_2)$$
$$\geq \mu(E_1 \cup E_2) + \mu(E_1 \cap E_2) + \mu(S - (E_1 \cup E_2)) + \mu(S - (E_1 \cap E_2))$$
$$\geq 2\mu(S).$$

But E_1, E_2 are separators and hence the $LHS = 2\mu(S)$. Thus the inequalities are throughout equalities and therefore

$$\mu(E_1 \cup E_2) + \mu(S - (E_1 \cup E_2)) = \mu(S)$$
$$\mu(E_1 \cap E_2) + \mu(S - (E_1 \cap E_2)) = \mu(S)$$

as required.

iii. The 'if' part is trivial. To show the 'only if' part, let E be a separator. Let E_2 denote $S - E$ and let $X_1 \subseteq E, X_2 \subseteq E_2$. We need to show that

$$\mu(X_1) + \mu(X_2) = \mu(X_1 \cup X_2)$$

We already have,

$$\mu(X_1) + \mu(X_2) \geq \mu(X_1 \cup X_2)$$
$$\mu(X_1 \cup X_2) + \mu(E_2) \geq \mu(X_1 \cup E_2) + \mu(X_2)$$
$$\mu(X_1 \cup X_2) + \mu(E) \geq \mu(X_2 \cup E) + \mu(X_1).$$

Hence (looking at the extreme LHS and RHS of the inequalities)

$$\mu(X_1 \cup X_2) + \mu(E_2) + \mu(X_1 \cup X_2) + \mu(E)$$
$$\geq \mu(X_1 \cup E_2) + \mu(X_2) + \mu(X_2 \cup E) + \mu(X_1)$$

$$\geq \mu(X_1 \cup X_2 \cup E \cup E_2) + \mu(X_1 \cup X_2) + \mu(X_1) + \mu(X_2)$$

$$\geq \mu(S) + \mu(X_1 \cup X_2) + \mu(X_1) + \mu(X_2).$$

Since E is a separator, we have,

$$\mu(S) = \mu(E) + \mu(E_2).$$

Hence (looking at the extreme LHS and the extreme RHS of the inequalities)

$$\mu(X_1 \cup X_2) \geq \mu(X_1) + \mu(X_2)$$

and the result follows, the reverse inequality already being shown.

iv. Observe that

$$\mu/\mathbf{E}(X) = \mu\diamond\mathbf{E}(X) \quad \forall X \subseteq E$$

iff

$$\mu(X) = \mu((S-E) \cup X) - \mu(S-E) \quad \forall X \subseteq E.$$

We see that this last condition, using the previous sections of the present problem, is equivalent to E being a separator.

v. If E is a separator of $\mu(\cdot)$, it is easily verified that

$$\mu^d(E) + \mu^d(S-E) = \mu^d(S),$$

$$\mu^*(E) + \mu^*(S-E) = \mu^*(S).$$

Since this condition is sufficient for E to be a separator both when the function is submodular as well as when it is supermodular, the result follows.

□

10.7 *Semimodular Polyhedra

A powerful technique to study a class of real(rational) valued set functions is to associate a polyhedron with it and study the geometry of the polyhedron. In this section we begin with the simple notions of a set polyhedron and its dual and their specialization to semimodular polyhedra. We show that any 'polyhedrally tight' set function can be naturally extended to a convex function over $\Re^n$. In particular submodular functions are polyhedrally tight. Their extension due to Lovász [Lovász83] (called Lovász extension [Fujishige91]), has a very simple alternative description.

Definition 10.7.1 *Let $f(\cdot)$ be a real valued set function on 2^S, $S \equiv \{e_1, \cdots, e_n\}$. Let χ_X denote the characteristic vector of $X \subseteq S$. When $\mathbf{x}$ is a real vector let*

$$x(X) \equiv (\chi_X)^T \mathbf{x} \quad \forall X \subseteq S.$$

Then the **polyhedron associated with** $f(\cdot)$*, denoted by* P_f *is defined as follows:A vector* $\mathbf{x} \in \Re^S$ *belongs to* P_f *iff*

$$x(X) \leq f(X) \quad \forall X \subseteq S.$$

We say $f(\cdot)$ *is* **polyhedrally tight** *iff for each* $X \subseteq S$ *there exists a vector* $\mathbf{x} \in P_f$ *such that* $x(X) = f(X)$*. The* **dual polyhedron associated with** $f(\cdot)$ *denoted by* P_f^d *is defined as follows:A vector* $\mathbf{x} \in \Re^S$ *belongs to* P_f^d *iff*

$$x(X) \geq f(X) \quad \forall X \subseteq S.$$

We say $f(\cdot)$ *is* **dually polyhedrally tight** *iff for each* $X \subseteq S$ *there exists a vector* $\mathbf{x} \in P_f^d$ *such that* $\mathbf{x}(X) = f(X)$*.*

We list some simple properties of polyhedrally tight functions in the next theorem.

Theorem 10.7.1 *(k)*

i. $\mathbf{x} \in P_f$ *and* $x(S) = f(S)$ *iff* $\mathbf{x} \in P_{f^d}^d$ *and* $x(S) = f(S)$*.*

ii. if $f(\cdot), g(\cdot)$ *are polyhedrally tight and* $\lambda \geq 0$ *then* $(\lambda f + g)(\cdot)$ *is polyhedrally tight.*

iii. if $A \subseteq S$ *and* $f(\cdot)$ *is polyhedrally tight, then* $f/\mathbf{A}(\cdot)$ *is polyhedrally tight.*

iv. if $f(\cdot)$ *is modular with* $f(\emptyset) = 0$ *then* P_f *is the set of all vectors beneath a single point and* $f(\cdot)$ *is polyhedrally tight.*

Proof:
i. We have, $x(X) \leq f(X) \quad \forall X \subseteq S$ and $x(S) = f(S)$ iff
$x(S - X) \geq f(S) - f(X) \quad \forall X \subseteq S$ and $\mathbf{x}(S) = f(S)$.

ii. If $\mathbf{x} \in P_f$ and $\mathbf{y} \in P_g$ such that $x(X) = f(X)$ and $\mathbf{y}(X) = g(X)$ then clearly $(\lambda\mathbf{x} + \mathbf{y}) \in P_{\lambda f + g}$ and further $(\lambda\mathbf{x} + \mathbf{y})(X) = (\lambda f + g)(X)$.

iii. This is immediate from the relevant definitions.

iv. Clearly $f(\cdot)$ is induced by the vector $\mathbf{x} \equiv (f(e_1), \cdots, f(e_n))$ (see Definition 10.2.2 and Theorem 10.2.2) and the vectors in P_f are precisely the set of all vectors less or equal to this vector. Since $f(X) = x(X) \quad \forall X \subseteq S$, $f(\cdot)$ is polyhedrally tight.

□

Exercise 10.12 *Let* $f(\cdot)$ *be a set function on subsets of* S *and let* $\mathbf{x}$ *denote also the modular function induced by the vector* $\mathbf{x}$*. Then* $P_{f+\mathbf{x}} = P_f + \mathbf{x}$*, where the latter addition denotes translation by the vector* $\mathbf{x}$*.*

When $f(\cdot)$ is submodular the problem of maximising a linear objective function over P_f is particularly easy. One need only use a greedy strategy. The next theorem speaks of this strategy. As a consequence it follows that $f(\cdot)$ must be polyhedrally tight. Further when $f(\cdot)$ is integral it turns out that it must have integral vertices.

Theorem 10.7.2 *(k) Let $S = \{e_1, \cdots, e_n\}$ and let $f(\cdot)$ be a submodular function on subsets of S such that $f(\emptyset) = 0$. Let $\mathbf{c} \in \Re^S$ be a nonnegative vector and let $c(e_1) \geq \cdots \geq c(e_n)$. Let $\mathbf{x} \in \Re^S$ be a vector such that*

$$x(e_1) = f(e_1), x(e_2) = f(\{e_1, e_2\}) - f(e_1)$$

$$\cdots x(e_n) = f(\{e_1 \cdots, e_n\}) - f(\{e_1 \cdots, e_{n-1}\}).$$

Then

i. $\mathbf{x}$ *is integral if $f(\cdot)$ is integral.*

ii. $\mathbf{x}$ *optimizes the linear program*

$$max\ \mathbf{c}^T \mathbf{z}, \mathbf{z} \in P_f.$$

Further if $\mathbf{c}$ *has a negative entry then the above linear program has no optimal solution.*

Proof:

i. This is immediate.

ii. Let T_j denote $\{e_1, \cdots, e_j\}$. We first show that $\mathbf{x} \in P_f$. If not then there exists a subset $T \subseteq S$ such that $x(T) > f(T)$. Let T have the smallest size consistent with this condition and let e_i be the element of T with the largest index. Observe that T cannot be null since $x(\emptyset) = f(\emptyset) = 0$. We have $x(T - e_i) \leq f(T - e_i)$. Next, $T_i \supseteq T$. Hence, by the submodularity of $f(\cdot)$,

$$f(T_i) - f(T_i - e_i) \leq f(T) - f(T - e_i).$$

But

$$x(e_i) = x(T_i) - x(T_i - e_i) = f(T_i) - f(T_i - e_i).$$

So $x(e_i) \leq f(T) - f(T - e_i)$. But

$$x(e_i) = x(T) - x(T - e_i).$$

Hence $x(T - e_i) > f(T - e_i)$, which contradicts the definition of T. Thus $\mathbf{x} \in P_f$. Next we show that $\mathbf{x}$ optimizes the linear program. We use LP duality. The dual linear program is

$$min \sum_{T \subseteq S} f(T) y_T$$

$$\sum_{T \ni e_i} y_T = c(e_i) \quad \forall e_i \in S, \mathbf{y} \geq \mathbf{0}.$$

(observe that $\mathbf{y}$ has one component for each subset of S). We select $y_T = c(e_i) - c(e_{i+1})$ if $T = T_i$, taking $c(e_{n+1})$ to be 0. Otherwise y_T is taken to be zero. It is easily verified that for such a selection $\mathbf{y} \geq \mathbf{0}$ and that

$$\sum_{T \ni e_i} y_T = c(e_i) \quad \forall e_i \in S.$$

Further,

$$\sum f(T)y_T =$$

$$(c(e_1) - c(e_2))f(e_1) + (c(e_2) - c(e_3))f(\{e_1, e_2\}) + \cdots c(e_n)f(\{e_1, \cdots e_n\}).$$

$$= c(e_1)f(e_1) + c(e_2)(f(\{e_1, e_2\}) - f(e_1))$$

$$+ \cdots c(e_n)(f(\{e_1, \cdots e_n\}) - f(\{e_1, \cdots e_{n-1}\})).$$

$$= c(e_1)x(e_1) + \cdots + c(e_n)x(e_n) = \mathbf{c}^T\mathbf{x}.$$

This implies that $\mathbf{x}$ and $\mathbf{y}$ are optimal solutions to the primal and dual programs respectively.

Finally let us consider the situation where $\mathbf{c}$ has a negative entry. We note that decreasing the component of any vector in P_f will not take it out of the polyhedron. Therefore the component corresponding to the negative entry in $\mathbf{c}$ can be indefinitely decreased remaining in the polyhedron but arbitrarily increasing the objective function.

□

Corollary 10.7.1 *(k) If $f(\cdot)$ is submodular (supermodular) with $f(\emptyset) = 0$ then $f(\cdot)$ is polyhedrally tight (dually polyhedrally tight).*

Proof: We first consider the situation where $f(\cdot)$ is submodular. Let $X \subseteq S$. In the statement of Theorem 10.7.2 we select $\mathbf{c} = \chi_X$. The selection procedure for $\mathbf{x}$ used in the statement of the theorem ensures that $x(X) = f(X)$. Further $\mathbf{x} \in P_f$. Next let $f(\cdot)$ be supermodular. Let $X \subseteq S$. In the polyhedron P_{f^d} we select a vector $\mathbf{x}$ such that $x(S - X) = f^d(S - X)$ and $x(S) = f(S)$ (the procedure given in the statement of the theorem permits this). This vector belongs to P^d_f and satisfies $x(X) = f(X)$ (see Theorem 10.7.1 (i)).

□

Corollary 10.7.2 *(k) If $f(\cdot)$ is submodular (supermodular) and integral then all the vertices of P_f (P^d_f) are integral.*

Proof: We will consider only the submodular case . If $\mathbf{x}$ is a vertex of the polyhedron P_f then there exists a vector $\mathbf{c}$ such that $\mathbf{c}^T\mathbf{z}$ reaches its maximum value (among all vectors of the polyhedron) only at $\mathbf{x}$. Clearly this vector (by Theorem 10.7.2) must be nonnegative . But then the procedure outlined in the same theorem yields an integral optimum if $f(\cdot)$ is integral. We conclude that this integral optimum must be the given vertex.
The supermodular case follows by noting that

- $g(\cdot)$ is supermodular iff $-g(\cdot)$ is submodular,
- $$max\ \mathbf{c}^T\mathbf{z}, \mathbf{z} \in P_{-g}$$

  is equivalent to
  $$min\ \mathbf{c}^T(-\mathbf{z}), -\mathbf{z} \in P^d_g.$$

□

We now show that there is a natural convex extension to every polyhedrally tight set function. We need the following definitions.

Definition 10.7.2 *Let $S \equiv \{e_1, \cdots .e_n\}$. Let $f(\cdot) : \Re^S \longrightarrow \Re$ and let $g(\cdot)$ be a set function on subsets of S defined by $g(X) \equiv f(\chi_X) \quad \forall X \subseteq S$. Then we say that $g(\cdot)$ is the* **set function induced by** *$f(\cdot)$.*

Definition 10.7.3 *Let $P(A, b)$ denote the polyhedron defined by the system of inequalities $\mathbf{Ax} \leq \mathbf{b}$, where $\mathbf{x}$ is a vector in $\Re^S$. Let $f_{Ab}(\cdot)$ be the function on $\Re^S$ defined by*
$$f_{Ab}(\mathbf{c}) \equiv (max\ \mathbf{c}^T\mathbf{x}, \mathbf{x} \in P(A, b)).$$

Then we say the function $f_{Ab}(\cdot)$ *is* **induced by the polyhedron** $P(A,b)$. *If* $P(A,b)$ *is empty we take* $f_{Ab}(\mathbf{c}) = -\infty \quad \forall \mathbf{c} \in \Re^S$. *Let*

$$f^{Ab}(\mathbf{c}) \equiv (min\ \mathbf{c}^T\mathbf{x}, \mathbf{x} \in P(A,b)).$$

Then we say the function $f^{Ab}(\cdot)$ *is* **dually induced by the polyhedron** $P(A,b)$. *If* $P(A,b)$ *is empty we take* $f^{Ab}(\mathbf{c}) = \infty \quad \forall \mathbf{c} \in \Re^S$.

Theorem 10.7.3 *(k)*

i. $f_{Ab}(\lambda\mathbf{c}_1 + \mu\mathbf{c}_2) \leq \lambda f_{Ab}(\mathbf{c}_1) + \mu f_{Ab}(\mathbf{c}_2), \lambda, \mu \geq 0$.

ii. *The collection of vectors on which* $f_{Ab}(\cdot)$ *takes finite values is closed under addition and nonnegative scalar multiplication (i.e., forms a cone).*

iii. $f^{Ab}(\lambda\mathbf{c}_1 + \mu\mathbf{c}_2) \geq \lambda f^{Ab}(\mathbf{c}_1) + \mu f^{Ab}(\mathbf{c}_2), \lambda, \mu \geq 0$ *and the collection of vectors on which* $f^{Ab}(\cdot)$ *takes finite values is closed under addition and nonnegative scalar multiplication.*

iv. *Let* $f(\cdot)$ *be a submodular function on subsets of* S. *Let* $f'(\cdot)$ *be the function induced by* P_f *and let* $f"(\cdot)$ *be the set function induced by* $f'(\cdot)$. *Let* $\mathbf{c}$ *be a vector such that* $c(e_1) \geq \cdots \geq c(e_n)$. *Let* $T_i, i = 1, 2, \cdots n$ *denote* $\{e_1, \cdots, e_i\}$. *Then*

  (a) $f'(\cdot)$ *takes finite values on all nonnegative vectors.*

  (b) $f"(X) = f(X) \quad \forall X \subseteq S$.

  (c) *[Lovász83]* $f'(\mathbf{c}) = (c(e_1) - c(e_2))f(\chi_{T_1}) + \cdots (c(e_{n-1}) - c(e_n))f(\chi_{T_{n-1}}) + (c(e_n))f(\chi_{T_n})$.

Proof:

i. Let $f_{Ab}(\lambda\mathbf{c}_1 + \mu\mathbf{c}_2) = (\lambda\mathbf{c}_1 + \mu\mathbf{c}_2)^T\mathbf{x}$ for some $\mathbf{x} \in P(A,b)$. By the definition of $f_{Ab}(\cdot)$ it follows that $(\mathbf{c}_1)^T\mathbf{x} \leq f_{Ab}(\mathbf{c}_1)$ and $(\mathbf{c}_2)^T\mathbf{x} \leq f_{Ab}(\mathbf{c}_2)$. The result follows by multiplying the first inequality by λ and the second by μ and adding.

ii. This follows immediately from the preceding result.

iii. Similar to the proof of the first two parts. Note that when P_{Ab} is not void, $f^{Ab}(\cdot)$ would take value $-\infty$ when it does not take a finite value).

iv(a) This follows from the definition of $f'(\cdot)$ and Theorem 10.7.2.

iv(b) This follows from the definition of $f"(\cdot)$, the fact that $f(\cdot)$ is polyhedrally tight (Corollary 10.7.1) and by use of the greedy strategy.

iv(c) By using the procedure given in the statement of Theorem 10.7.2 we can construct a vector $\mathbf{x}$ which optimizes

$$max\ \chi_{T_i}^T\mathbf{z}, \quad \mathbf{z} \in P_f$$

for $i = 1, 2 \cdots n$. We then have, by the definition of $f'(\cdot)$,

$$f'(\mathbf{c}) \geq \mathbf{c}^T\mathbf{x}.$$

But

$$\mathbf{c} = (c(e_1) - c(e_2))\chi_{T_1} + \cdots (c(e_{n-1}) - c(e_n))\chi_{T_{n-1}} + (c(e_n))\chi_{T_n}$$

and

$$\mathbf{c}^T\mathbf{x} = (c(e_1) - c(e_2))x(T_1) + \cdots (c(e_{n-1}) - c(e_n))x(T_{n-1}) + (c(e_n))x(T_n).$$

Noting that

$$x(T_i) = f(T_i) = f'(\chi_{T_i}), i = 1, 2, \ldots n,$$

we have,

$$f'(\mathbf{c}) \geq (c(e_1) - c(e_2))f'(\chi_{T_1}) + \cdots (c(e_{n-1}) - c(e_n))f'(\chi_{T_{n-1}}) + (c(e_n))f'(\chi_{T_n}).$$

The reverse inequality follows from the statement of the first part of the present theorem.

□

Remark: Observe that if $c(e_1) \geq \cdots \geq c(e_n)$, then

$$\mathbf{c} = (c(e_1) - c(e_2))(\chi_{T_1}) + \cdots (c(e_{n-1}) - c(e_n))(\chi_{T_{n-1}}) + (c(e_n))(\chi_{T_n}).$$

Thus $f'(\mathbf{c})$ is obtained by performing the above linear combination of the $f(\chi_{T_i})$.

The next result is one of the deepest in submodular function theory. It is a very good starting point for proving many of the important results in this area. Our proof is, however, not polyhedral even though the result is naturally polyhedral. A generalization of the result using polyhedral methods (in fact the Hahn-Banach Separation Theorem) is given in Problem 10.14

Theorem 10.7.4 (The 'Sandwich Theorem)' *[Frank82] Let $f(\cdot), g(\cdot)$ be submodular and supermodular functions defined on subsets of S such that $f(\cdot) \geq g(\cdot)$. Then there exists a modular function $h(\cdot)$ on subsets of S such that $f(\cdot) \geq h(\cdot) \geq g(\cdot)$ (equivalently, such that $h(\cdot)$* **separates** *or* **lies between** *$f(\cdot)$ and $g(\cdot)$). Further if $f(\cdot)$, $g(\cdot)$ are integral then $h(\cdot)$ can be chosen to be integral.*

Our proof of the theorem is an algorithmically more efficient version of that due to Lovász and Plummer [Lovász+Plummer86] and is based on the following lemma.

Lemma 10.7.1 *Let $\emptyset \subset A \subset S$ be such that $f(A) = g(A)$ and let there exist modular functions $h_A(\cdot), h_{S-A}(\cdot)$ on subsets of $A, S-A$ respectively such that $f/\mathbf{A}(\cdot) \geq h_A(\cdot) \geq g/\mathbf{A}(\cdot)$ and $f\diamond(\mathbf{S}-\mathbf{A})(\cdot) \geq h_{s-A}(\cdot) \geq g\diamond(\mathbf{S}-\mathbf{A})(\cdot)$. Then $f(\cdot) \geq (h_A \oplus h_{S-A})(\cdot) \geq g(\cdot)$.*

Proof of the Lemma: Let $X \subseteq S$. Clearly,

$$f(X \cap A) \geq h_A(X \cap A) \geq g(X \cap A)$$

$$\begin{aligned} f(X \cup A) &= f(A) + f\diamond(\mathbf{S}-\mathbf{A})(X \cap (S-A)) \\ &\geq h_A(A) + h_{S-A}(X \cap (S-A)) \\ &\geq g(A) + g\diamond(\mathbf{S}-\mathbf{A})(X \cap (S-A)) = g(X \cup A). \end{aligned}$$

Since $f(\cdot), g(\cdot)$ are submodular and supermodular respectively, we have

$$f(X) - f(X \cap A) \geq f(X \cup A) - f(A) = f \diamond (\mathbf{S} - \mathbf{A})(X \cap (S - A)),$$

$$g \diamond (\mathbf{S} - \mathbf{A})(X \cap (S - A)) = g(X \cup A) - g(A) \geq g(X) - g(X \cap A).$$

Further we are given that

$$f \diamond (\mathbf{S} - \mathbf{A})(X \cap (S - A)) \geq h_{S-A}(X \cap (S - A)) \geq g \diamond (\mathbf{S} - \mathbf{A})(X \cap (S - A)).$$

Thus

$$f(X) \geq h_{S-A}(X \cap (S - A)) + f(X \cap A) \geq h_S(X)$$

and

$$g(X) \leq h_{S-A}(X \cap (S - A)) + g(X \cap A) \leq h_S(X),$$

where $h_S \equiv h_A \oplus h_{S-A}$. Hence, $f(X) \geq h(X) \geq g(X)$.

□

Proof of Theorem 10.7.4: Assume without loss of generality that $f(\emptyset) = 0$. Suppose $f(\cdot)$ has every element e as a separator. Then $f(\cdot)$ is already modular and the theorem is trivially true. We therefore assume that the submodular function has atleast one element say e that is not a separator. Let Y be the subset of all elements of S which are singleton separators. We will assume that the theorem is true for $(|S| < n)$ and for $(|S| = n, |S - Y| < m)$. We note that the theorem is trivially true for $|S| = 1$ and also for $|S - Y| = 0$. We will now prove the result when $(|S| = n, |S - Y| = m)$. We have $f(S) < f(S - e) + f(e)$. Let $f'(\cdot)$ be the function

$$f'(X) \equiv f(X - e) + (f(S) - f(S - e)), e \in X$$
$$\equiv f(X), e \notin X.$$

The function $f'(\cdot)$ is obviously a submodular function with e as a separator (see Theorem 10.6.1) and further $f(\cdot) \geq f'(\cdot)$ (since, by the submodularity of $f(\cdot)$, $f(X) - f(X - e) \geq f(S) - f(S - e) \quad \forall X$ such that $e \in X$). Let the set A minimize $(f' - g)(\cdot)$.

i. Case 1. If $(f' - g)(A)$ is nonnegative we have found a submodular function (namely $f'(\cdot)$), which lies between $f(\cdot)$ and $g(\cdot)$ and further has one more singleton separator (namely $\{e\}$), than $f(\cdot)$ has. Also $f'(\cdot)$ is integral if $f(\cdot)$ is. By induction on $|S - Y|$, the theorem is true for $f'(\cdot)$ and therefore for $f(\cdot)$.

ii. Case 2. Let $(f' - g)(A)$ be negative. Clearly A is not null or equal to S . Let $f"(\cdot)$ be defined by

$$f"(X) \equiv min(f(X), f'(X) + (g - f')(A)) \quad \forall X \subseteq S.$$

It can be verified that $f"(\cdot)$ is submodular (directly or by using the idea of convolution, to be introduced in the next chapter). Further $f(\cdot) \geq f"(\cdot) \geq$

$g(\cdot)$ and $f"(A) = g(A)$. It follows that $f"/\mathbf{A}(\cdot) \geq g/\mathbf{A}(\cdot)$, $f"\diamond(\mathbf{S}-\mathbf{A})(\cdot) \geq g\diamond(\mathbf{S}-\mathbf{A})(\cdot)$. By induction on $|S|$, we may assume that there are modular functions $h_A(\cdot)$ between $f"/\mathbf{A}(\cdot)$ and $g/\mathbf{A}(\cdot)$ and $h_{S-A}(\cdot)$ between $f"\diamond(\mathbf{S}-\mathbf{A})(\cdot)$ and $g\diamond(\mathbf{S}-\mathbf{A})(\cdot)$. Now by Lemma 10.7.1, $(h_A \oplus h_{S-A})(\cdot)$ lies between $f"(\cdot)$ and $g(\cdot)$ and therefore also lies between $f(\cdot)$ and $g(\cdot)$. Further, in case $f(\cdot), g(\cdot)$ are integral it is clear that $f"(\cdot)$ is integral and we may assume by induction that $h_A(\cdot), h_{S-A}(\cdot)$ are integral. It follows that $(h_A \oplus h_{S-A})(\cdot)$ is also integral. Thus the theorem is true when $|S| = n, |S-Y| = m$.

□

Remark: The above proof of the Sandwich Theorem contains an efficient algorithm for finding the separating modular function provided we have an efficient algorithm for minimising submodular functions (in this case the function $(f'-g)(\cdot)$). It may be verified that the algorithm requires no more than $|S|$ submodular function minimizations.

Exercise 10.13 *Let $\mathcal{M}$ be a matroid on S. Let $r(X) \equiv r(\mathcal{M}.X)$, $r'(X) \equiv r(\mathcal{M} \times X)$, $\nu(X) \equiv \nu(\mathcal{M} \times X), \nu'(X) \equiv \nu(\mathcal{M}.X)$ as in Exercise 10.11. We have already seen that $r(\cdot), \nu(\cdot)$ are submodular and $r'(\cdot), \nu'(\cdot)$ are supermodular. Show that*

i. *$r(\cdot) \geq r'(\cdot)$ and $\nu(\cdot) \geq \nu'(\cdot)$.*

ii. *Find vectors $\mathbf{w}_r, \mathbf{w}_\nu$ such that $r(\cdot) \geq \mathbf{w}_r(\cdot) \geq r'(\cdot)$ and $\nu(\cdot) \geq \mathbf{w}_\nu(\cdot) \geq \nu'(\cdot)$.*

Exercise 10.14 *Let $\rho(\cdot)$ be a polymatroid rank function on subsets of S. Let*

$$\rho'(X) = \rho(S) - \rho(S-X).$$

i. *Show that $\rho(\cdot) \geq \rho'(\cdot)$.*

ii. *Choose a vector $w(\cdot)$ so that $\rho(\cdot) \geq w(\cdot) \geq \rho'(\cdot)$.*

10.8 Symmetric Submodular Functions

A key problem in submodular function theory is that of minimization. To be precise, the search is for a practically efficient polynomial time algorithm for a general submodular function which is available through a rank oracle. (The oracle will give the value of the function on any given subset). The input size for such an algorithm is determined by the size of the underlying set and the maximum number of bits needed to represent a value of the submodular function. As we shall see later minimization is equivalent to convolution of an appropriate polymatroid rank function with a weight vector. The solution is known to the minimization problem in many practical situations: e.g. minimum directed cut in a graph, convolution of a matroid rank function with a weight vector [Cunningham84], [Narayanan95b].

For the general problem the ellipsoid method [Grötschel+Lovász+Schrijver81] does provide a polynomial algorithm which however is practically useless. There are a few algorithms for minimization [Cunningham85], [Sohoni92] which are practical but pseudo polynomial (for integral functions the algorithm is polynomial in the size of the underlying set and the maximum value of the function). Recently the case of symmetric submodular functions was solved in a surprisingly simple way. We describe this solution in this section.

Definition 10.8.1 *A set function* $g : 2^S \longrightarrow \Re$ *is symmetric iff* $g(X) = g(S - X) \quad \forall X \subseteq S$.

Example 10.8.1 *(k) The following are symmetric submodular functions.*

i. Cut function on the vertex subsets of a graph.

ii. The function $|c|(X)$*,* $X \subseteq E(\mathcal{G})$ *where* $c(X) \equiv$ *the set of vertices common to edges in* X *and* $E(\mathcal{G}) - X$*.*

iii. $|c|(X)$ *acting on the left vertex set of a bipartite graph. Here* $|c|(X) \equiv$*number of right side vertices adjacent to vertices in* X *as well as vertices in* $V_L - X$ *(see Example 10.2.8).*

iv. The function $\xi(X) \equiv r(M.X) - r(M \times X), X \subseteq S$*, where* $\mathcal{M}$ *is a matroid on* S *and* $r(\cdot)$ *denotes the rank function of the matroid.*

v. $\theta(X) \equiv f(X) + f(S - X), X \subseteq S$*, where* $f(\cdot)$ *is a submodular function on subsets of* S*.*

Exercise 10.15 *Show that the functions in Example 10.8.1 are symmetric submodular functions.*

Remark: Prior to the work of Nagamochi and Ibaraki [Nagamochi+Ibaraki92a] (see also [Nagamochi+Ibaraki92b] [Nagamochi+Ono+Ibaraki94]) the standard way of finding min cut was through flow techniques . The above authors used a special (linear time) decomposition of a graph into forests to identify a pair of vertices for which a minimum separating cut was immediately available. This was stored. This pair was fused and the process repeated until the graph had only two vertices. The minimum value cut among all the stored cuts gives the minimum cut. A simpler algorithm of the same complexity was found by M. Stoer and F. Wagner [Stoer+Wagner94] and, independently, by A. Frank [Frank94]). It is this algorithm that we generalize below. The version we present is essentially the same as the one due to M. Queyranne [Queyranne95]. However we have tried to bring out the relationship to the Stoer-Wagner algorithm more strongly.

We follow the notation of Stoer-Wagner and essentially do a line by line translation of their algorithm for finding a min cut to that for minimising a symmetric submodular function over all sets not equal to the full set or the void set.

Let $f : 2^S \longrightarrow \Re$ be a submodular function. Let $c(A, B) \equiv \frac{1}{2}(f(A) + f(B) - f(A \cup B)) \quad \forall A, B \subseteq S, A \cap B = \emptyset$. Observe that $c(A, B) = c(B, A)$.

We then have $c(X,S-X)=\frac{1}{2}(f(X)+f(S-X)-f(S))$. If $g(X)\equiv c(X,S-X)$ it is clear that $g(\cdot)$ is symmetric and submodular. We minimize $g(\cdot)$ over $X\subseteq S, X\neq \emptyset, X\neq S$.

Subroutine Minimum Phase (g,a,S_j)

BEGIN

$A \leftarrow \{a\}$

while $(A\neq S_j)$ {

add to A the element $e\notin A$ such that $c(A,e)$ is the largest i.e., $f(A\cup e)-f(e)$ is least.

}

Store the 'pair of the phase' (e_n, S_j-e_n) and the 'value of the pair of the phase' $c(e_n,S_j-e_n)$.

Fuse e_n, e_{n-1}.

Replace the set S_j by

$S_{j+1}\equiv \{e_1,e_2,\ldots,e_{n-2},e_{n-1}\cup e_n\}$.

(Note that e_{n-1},e_n are treated as subsets of S.)

Replace the function g by $g_{fus.S_{j+1}}$.

END *Subroutine Minimum Phase.*

Algorithm Symmetric:

Initialize $S_0=S$.

BEGIN

The algorithm has $|S|-1$ phases of *Subroutine Minimum Phase.*

Value of current minimum pair $=\infty$

while $(|S_j|>1)$ {

Subroutine Minimum Phase (g,a,S_j).

If the value of the pair of the phase (e_n,S_j-e_n) has a lower value than the current minimum pair then store (e_n,S_j-e_n) as the current minimum pair.

}

Let the minimum pair be $(e_{n_i},S_j-e_{n_i})$ at the end of the algorithm (i.e., when $|S_j|=2$). Output sets $e_{n_i}, S-e_{n_i}$ as minimising sets (non void proper subsets of S) for $g(\cdot)$.

END **Algorithm Symmetric.**

We justify algorithm Symmetric through the following theorem. We first introduce some convenient notation and a couple of lemmas.

In any phase the elements of the current set S_i are ordered as say $(e_1,e_2,\ldots,e_k)$. Let $A_j\equiv\{e_1,e_2,\ldots,e_{j-1}\}, j=2,\ldots,k+1$. (In particular $A_{k+1}=S_i$). Let $A\subseteq\{e_1,e_2,\ldots,e_k\}$ with $e_k\in A, e_{r-1}\in A$ and $\{e_r,\ldots,e_{k-1}\}\cap A=\emptyset$.

Lemma 10.8.1

$$c(A_k,e_k)-c(A_r,e_k)\leq c(A,A_{k+1}-A)-c(A_{r+1}\cap A, A_{r+1}-A_{r+1}\cap A)$$

Proof: We note that

$$c(X,Y)\equiv\frac{1}{2}[f(X)+f(Y)-f(X\cup Y)]$$

Hence the LHS of the inequality in the statement simplifies to

$$\frac{1}{2}[(f(A_k) - f(A_r) - f(A_{k+1}) + f(A_r \cup e_k)]$$

while the RHS simplifies to

$$\frac{1}{2}[(f(A) + f(A_{k+1} - A) - f(A_{k+1}) - f(A_{r+1} \cap A)$$

$$-f(A_{r+1} - A_{r+1} \cap A) + f(A_{r+1})].$$

We therefore have to show (after simplification) that the following inequality is valid. $f(A_k) - f(A_{r+1}) + f(A_r \cup e_k) - f(A_r)$

$$\leq \quad f(A_{k+1} - A) - f(A_{r+1} - A_{r+1} \cap A) + f(A) - f(A_{r+1} \cap A). \quad (10.5)$$

Now we note that

$$\begin{aligned} A_k - A_{r+1} &= (A_{k+1} - A) - (A_{r+1} - A_{r+1} \cap A) \\ &= \{e_{r+1}, \ldots, e_{k-1}\} \\ \textit{and } A_k &\supseteq A_{k+1} - A. \end{aligned}$$

Hence by submodularity

$$f(A_k) - f(A_{r+1}) \leq f(A_{k+1} - A) - f(A_{r+1} - A_{r+1} \cap A).$$

By a similar argument we see that

$$f(A_r \cup e_k) - f(A_r) \leq f(A) - f(A_{r+1} \cap A)$$

(since $A_r \cup e_k - A_r = e_k = A - A_{r+1} \cap A$). This proves the inequality(10.5) and hence the lemma.

□

Lemma 10.8.2 *Let $(e_k, S_i - e_k)$ be the pair of the phase. Then*

$$c(e_k, S_i - e_k) \leq c(A, S_i - A)$$

where $e_k \in A$ and $e_{k-1} \in S_i - A$.

Proof: The proof is by induction on the size of S_i. The lemma is true for $|S_i| = 2$. Suppose it to be true for $|S_i| \leq k$.
Let $A \subseteq S_i$ with $e_k \in A, e_{r-1} \in A, \{e_r, \ldots, e_{k-1}\} \cap A = \emptyset$.
By induction it follows that

$$c(A_{r+1} \cap A, A_{r+1} - A_{r+1} \cap A) \geq c(A_r, e_r) \quad (10.6)$$

(since $e_{r-1} \in A, e_r \in A_{r+1}$)
We have

$$\begin{array}{rcl} c(A_k, e_k) & = & c(A_r, e_k) + (c(A_k, e_k) - c(A_r, e_k)) \\ & \leq & c(A_r, e_r) + (c(A_k, e_k) - c(A_r, e_k)) \\ & & (\textit{since in the algorithm } e_r \textit{ has been chosen so that} \\ c(A_r, e_r) & \geq & c(A_r, e) \quad \forall e \in (S_i - A_r)). \end{array}$$

Hence,

$$\begin{array}{rcl} c(A_k, e_k) & \leq & c(A_{r+1} \cap A, A_{r+1} - A_{r+1} \cap A) + (c(A_k, e_k) - c(A_r, e_k)) \ (\textit{by } (10.6)) \\ & \leq & c(A_{r+1} \cap A, A_{r+1} - A_{r+1} \cap A) + (c(A, A_{k+1} - A) \\ & & -c(A_{r+1} \cap A, A_{r+1} - A_{r+1} \cap A)) \ \textit{by } (\textit{Lemma } 10.8.1). \end{array}$$

Thus

$$c(A_k, e_k) \leq c(A, A_{k+1} - A),$$

which is the required inequality.

□

Theorem 10.8.1 *The current minimum pair* $(e_{n_j}, S_j - e_{n_j})$ *at the end of the Algorithm Symmetric yields the minimising set* e_{n_j} *for* $g(\cdot)$.

Proof: The proof is by induction on the size of the set S. The theorem is clearly valid when the $|S| = 2$. Suppose it to be valid when the size of the set is $n-1$. Let $|S| = n$. Now let the last two elements of the first phase be e_{n-1}, e_n. If the minimum value pair $(X, S-X), X \subseteq S, X \neq \emptyset, X \neq S$ had $\{e_{n-1}, e_n\} \subseteq X$ then we can fuse $\{e_{n-1}, e_n\}$ and work with the set $S_2 = \{e_1, e_2, \ldots, e_{n-2}, \{e_{n-1}, e_n\}\}$ and the function $g_{fus.S_2}$. By induction, in the subsequent phases of the algorithm the minimum pair will be revealed as a pair of some phase. On the other hand if the minimum pair $(X, S-X)$ had $e_n \in X, e_{n-1} \in S - X$, by Lemma 10.8.2 the pair of the first phase has this minimum value. This completes the proof of the theorem.

□

Complexity of Algorithm Symmetric

Suppose there is an oracle that gives the value of $c(X, Y)$ for given $X, Y \subseteq S$, $X \cap Y = \emptyset$. If the current set is S_i the number of calls to this oracle is $O(|S_i|^2)$ during the phase. At the end of the phase the set reduces in size by one. We continue until the set reaches size two. Hence the total number of calls to the oracle is $O(|S_i|^3)$.

Special cases can be handled much faster: for instance in the case of $cut(\cdot)$ function of a graph Stoer & Wagner show that the complexity is $O(|E|+|V|log\ |V|)$ elementary operations per phase, giving an overall complexity of $O(|V||E|+|V|^2 log\ |V|)$.

The case of the symmetric function $(\Gamma_L - E_L)(\cdot)$ of a bipartite graph $B \equiv (V_L, V_R, E)$ is almost identical. The complexity here per phase is $O(|E| + |V_L| log\ |V_L|)$ and overall is $O(|V_L||E| + |V_L|^2 log\ |V_L|)$.

Exercise 10.16 *(k) For any function $f(\cdot)$ show that $g(X) = \frac{1}{2}(f(X)+f(S-X))$ is symmetric. If $f(\cdot)$ is a symmetric function show that $c(X, S-X) = f(X)$ provided $f(S) = 0$.*

Exercise 10.17 *If $f(\cdot)$ is a symmetric function which of the following operations preserve symmetry ?*

i. Contraction.

ii. Restriction.

iii. Comodular dual.

iv. Contramodular dual.

Exercise 10.18 *For any submodular function $f(\cdot)$ show (using the notation of Section 10.8) that $c(A, B) \geq c(X, Y)$ whenever $X \subseteq A, Y \subseteq B, A \cap B = \emptyset$. Interpret this statement for the cut function of a graph.*

Exercise 10.19 *Let $g(\cdot) : 2^S \longrightarrow \Re$ be a symmetric submodular function. Give an algorithm to minimize $g(\cdot)$ over*

i. all subsets $X,\ A \subseteq X \subset S$.

ii. all subsets $X,\ \emptyset \subset X \subseteq A$.

Exercise 10.20 *Specialize* **Algorithm Symmetric** *to the case of*

i. the cut function $|cut|(\cdot)$ of a graph $\mathcal{G} \equiv (V, E)$.

ii. the function $(|\Gamma_L| - |E_L|)(\cdot)$ of the bipartite graph $B \equiv (V_L, V_R, E)$.

iii. the symmetric function arising from $|\Gamma_L|(\cdot)$ in B above.

Exercise 10.21 *In Theorem 10.5.2 we have shown how to subtract a modular function from a given submodular function to yield a polymatroid rank function. Work this case out for $|cut|(\cdot)$.*

10.9 Problems

Problem 10.5 *(k) Prove the principle of inclusion and exclusion for weight functions. (The intersection form: $w(X_1 \cup \cdots \cup X_n) = \sum w(X_i) - \sum w(X_i \cap X_j) + w(X_i \cap X_j \cap X_k) + \cdots$). State the union form of the principle. Suppose the value of such a function is known for arbitrary unions of sets from a collection $\{X_1, X_2 \cdots X_n\}$, show how to obtain its value on sets obtained by arbitrary intersection of such sets and their complements.*

Problem 10.6 *Let $\mathcal{G}$ be a graph on vertices V and edges E. Let w be a nonnegative weight function on the edges of $\mathcal{G}$. As usual $w(X) \equiv \sum_{e \in X} w(e)$. Let $\rho : 2^E \longrightarrow \Re_+$ be defined by $\rho(X) \equiv$ maximum weight of a forest of the subgraph of $\mathcal{G}$ on X. Show that $\rho(\cdot)$ is submodular. Examine the case where $\rho(\cdot)$ is defined by minimum weight instead of maximum weight.*

Problem 10.7 *How would you generalize Problem 10.6 above for the adjacency and exclusivity functions of a bipartite graph, given a nonnegative weight function of the left vertex set?*

Problem 10.8 *(Generalization of Problem 10.6 to the matroid base case) Let $\mathcal{M}$ be a matroid on S. Let w be a nonnegative weight function on the elements of $\mathcal{M}$. Let $\rho : 2^S \longrightarrow \Re_+$ be defined by*

$$\rho(X) \equiv maximum\ weight\ base\ of\ \mathcal{M}.X$$

Show that $\rho(\cdot)$ is a polymatroid rank function.

Problem 10.9 *(k) Let $\mathbf{A}$ be a matrix with real entries. Let $R \equiv \{\mathbf{r_1}, \ldots, \mathbf{r_k}\}, C \equiv \{\mathbf{c_1}, \ldots, \mathbf{c_n}\}$ be the sets of rows and columns of $\mathbf{A}$ respectively. Let $\mathbf{A}_X$ denote the submatrix of $\mathbf{A}$ using only the rows in $X \subseteq R$ but all columns. Let $f : 2^R \longrightarrow \Re$ be defined as follows*

$$f(X) = log(det\ (\mathbf{A}_X \mathbf{A}_X^T)) \quad \forall X \subseteq R$$

Prove that $f(\cdot)$ is submodular.

Problem 10.10 *Let $\mathbf{A}$ be a matrix with linearly independent rows. Let T be a set of maximally independent columns of $\mathbf{A}$ and let the submatrix corresponding to the columns of T be the identity matrix. Let $\mathbf{A}_X, X \subseteq T$, denote the submatrix of $\mathbf{A}$ composed of those rows of $\mathbf{A}$ which have nonzero entries in one of the columns corresponding to X. Let $f(X) \equiv log(\mathbf{A}_X \mathbf{A}_X^T),\ X \subseteq T$. Prove that $f(\cdot)$ is submodular.*

Problem 10.11 *Let $\mathbf{A}$ be a totally unimodular matrix (every subdeterminant of $\mathbf{A}$ is $0, \pm 1$) with columns S. Let $\mathcal{M}$ be the matroid on S associated with $\mathbf{A}$. Let T be a base of $\mathcal{M}$. Let*

$$f(X) \equiv log(number\ of\ bases\ of\ \mathcal{M} \times (S - (T - X))), X \subseteq T.$$

Prove that $f(\cdot)$ is submodular.

Problem 10.12 *[Fujishige78a], [Fujishige91] We consider some analogies between polymatroids and matroids in this problem. We note that if we perform the fusion operation on a matroid rank function we obtain a polymatroid rank function. Later, in the next chapter, we show that every polymatroid rank function can be so obtained. In other words every polymatroid rank function can be 'expanded' into a matroid rank function. If $\rho : 2^S \longrightarrow \Re$ is a polymatroid rank function we say that $P_\rho \equiv \{\mathbf{x} \in \Re^S : \mathbf{x}(X) \leq \rho(X) \quad \forall X \subseteq S\}$ is a polymatroid.*

An **independent vector** *of a polymatroid is a nonnegative vector in P_ρ, i.e., it is a vector $\mathbf{x}$ s.t. $x(X) \leq \rho(X) \quad \forall X \subseteq S$. We will show later that if $\mathbf{x}$ is an* *integral independent vector, in the expanded matroid there is an independent set* $T \subseteq \hat{S}$ *whose intersection with the set* $e \in S$ *has size* $x(e)$ *(Here* $\hat{S}$ *is the expanded version of the set* S*, the latter being a partition of* $\hat{S}$*).*

i. *Let* $\mathbf{x}$ *be a real vector on* S*. Define*

$$D(\mathbf{x}) \equiv \{X | X \subseteq S, x(X) = \rho(X)\}$$

Show that $D(\mathbf{x})$ *is closed with respect to union and intersection and hence has a unique maximal and a unique minimal set.*

ii. *If* $\mathbf{x}$ *is independent define the* **saturation function**

$$sat(\mathbf{x}) \equiv \bigcup \{X | X \subseteq S, \mathbf{x}(X) = \rho(X)\}$$

Show that

$$sat(\mathbf{x}) \equiv \{e | e \in S \quad \forall \alpha > 0 : \mathbf{x} + \alpha \chi_e \notin P_\rho \ \}$$

Observe that the saturation function generalizes the closure function of a matroid.

iii. *Let* $D(\mathbf{x}, e)$ *denote the collection of sets in* $D(X)$ *which have* e *as a member. Show that* $D(\mathbf{x}, e)$ *is closed under union and intersection and hence has a unique maximal and a unique minimal element.*

iv. *For an independent vector* $\mathbf{x} \in P_\rho$ *and* $e \in sat(\mathbf{x})$*, define the* **dependence function**

$$dep(\mathbf{x}, e) \equiv \bigcap \{X | e \in X \subseteq S, \mathbf{x}(X) = \rho(X)\}$$

Let $x(e) < \rho(e)$ *and let* $0 < \alpha < \rho(e) - \mathbf{x}(e)$*. Let* $\mathbf{x}' \equiv \mathbf{x} + \alpha \chi_e$ *and let* $\theta(X) \equiv \mathbf{x}'(X) - \rho(X) \quad \forall X \subseteq S$*. Let* $D'(\mathbf{x}')$ *denote the collection of sets where* $\theta(\cdot)$ *reaches a maximum. Show that*

(a) $D'(\mathbf{x}')$ *is closed under union and intersection and hence has a unique maximal and a unique minimal set.*

(b) *Let* K *be the minimal member of* $D'(\mathbf{x}')$*. Then* $K = dep(\mathbf{x}, e)$*.*

(c)

$$dep(\mathbf{x}, e) = \{e' | e' \in S, \exists \alpha > 0 : \mathbf{x} + \alpha(\chi_e - \chi_{e'}) \in P_\rho\}$$

Observe that the dependence function generalizes the fundamental circuit of a matroid.

Problem 10.13 *Let* $f : 2^S \longrightarrow \Re$ *be a submodular function with* $f(\emptyset) = 0$*. Let* $\mathbf{c}$ *be a nonnegative weight function on* S*. Let* $\mathbf{c}_X$ *denote the row vector on* S *with*

$$\mathbf{c}_X(e) = 0, \ \ e \notin X$$

$$\mathbf{c}_X(e) = c(e), \ \ e \in X$$

Let $\rho : 2^S \longrightarrow \Re$ *be defined by*

$$\rho(X) \equiv max \ \mathbf{c}_X^T \mathbf{x}, \ \ \mathbf{x} \in P_f.$$

Show that $\rho(\cdot)$ *is a submodular function.*

Problem 10.14 *We state the* **Hahn-Banach Separation Theorem** *below.*
Let $\mathcal{V}$ be a normed space over $\Re$ and let E_1, E_2 be nonempty disjoint convex subsets of $\mathcal{V}$, where E_1 is open in $\mathcal{V}$. Then there is a real hyperplane in $\mathcal{V}$ which separates E_1 and E_2 in the following sense: For some linear functional $g(\cdot)$ over $\mathcal{V}$ and $t \in \Re$, we have

$$g(y_1) > t \geq g(y_2) \quad \forall y_1 \in E_1, y_2 \in E_2.$$

Use this result to prove the following:

Let $f_1, f_2 : 2^S \longrightarrow \Re$ be polyhedrally tight and dually polyhedrally tight set functions respectively with $f_1(\cdot) \geq f_2(\cdot)$. Then there is a modular function $w(\cdot)$ such that $f_1(\cdot) \geq w(\cdot) \geq f_2(\cdot)$.

Problem 10.15 *Use Sandwich Theorem to prove the following:*
Let $\mathcal{M}_1, \mathcal{M}_2$ be two matroids on S. Then the maximum size of a common independent set of the two matroids $= min_{X \subseteq S}\ r_1(X) + r_2(S - X)$, where $r_1(\cdot), r_2(\cdot)$ are the rank functions of the matroids $\mathcal{M}_1, \mathcal{M}_2$ respectively.

Problem 10.16 *Let $f : 2^S \longrightarrow \Re$ be a submodular function. For each of the following cases construct an algorithm for minimising this function.*

i. $f(X - e) - f(X) \geq f((S - X) \cup e) - f(S - X) \quad \forall X \subseteq S,\ e \in X.$

ii. $f(X - e) - f(X) \leq f((S - X) \cup e) - f(S - X) \quad \forall X \subseteq S,\ e \in X.$

10.10 Notes

Submodular functions became an active area of research in optimization after Edmond's pioneering work. Earlier prominent workers who used such ideas were H. Whitney [Whitney35] and W.T. Tutte [Tutte58], [Tutte59], [Tutte61], [Tutte65], [Tutte71] (in their work on graphs and matroids), G. Choquet [Choquet55] (in his work on capacity theory) and O. Ore [Ore56] (in his work on graph theory). The present chapter follows Lovász's excellent review paper [Lovász83]. We have also made use of Fujishige's comprehensive monograph [Fujishige91] on the subject. In the present book, the polyhedral approach to submodular functions has not been emphasized. A thorough treatment of this subject through the polyhedral approach may be found in [Frank+Tardos88] and in the above mentioned monograph of Fujishige. The notion of comodular duality is due to McDiarmid [McDiarmid75]. The Sandwich theorem first appeared as a technical result in [Frank82]. Its importance as a central result in submodular function theory was recognized by Lovász and Plummer [Lovász+Plummer86].

10.11 Solutions of Exercises

E 10.1: For most of the examples the easiest route for proving sub or super modularity is to show that Inequality 10.3 holds.

Example 10.2.1: Let $X \subseteq Y \subseteq E(\mathcal{G})$ and let $a \in E(\mathcal{G}) - Y$. Then

$$V(X \cup a) = V(X) \biguplus (V(a) - V(X)),$$

$$V(Y \cup a) = V(Y) \biguplus (V(a) - V(Y)).$$

Clearly since $V(Y) \supseteq V(X)$ it follows that

$$V(a) - V(Y) \subseteq V(a) - V(X).$$

Hence

$$|V|(X \cup a) - |V|(X) \geq |V|(Y \cup a) - |V|(Y).$$

Example 10.2.2: Let $X \subseteq Y \subseteq V(\mathcal{G})$ and let $a \in V(\mathcal{G}) - Y$. Now

$$E(Z \cup a) = E(Z) \biguplus E_{Za}, Z \subseteq V(\mathcal{G}), a \in V(\mathcal{G}) - Z,$$

where $E_{Za}$is the set of all edges, with a as one end point, the other endpoint lying in $Z \cup a$. Clearly

$$E_{Xa} \subseteq E_{Ya}.$$

Thus

$$|E|(X \cup a) - |E|(X) \leq |E|(Y \cup a) - E(Y).$$

Example 10.2.3: Let $X \subseteq Y \subseteq V(\mathcal{G})$ and let $a \in V(\mathcal{G}) - Y$. Then

$$I(X \cup a) = I(X) \biguplus [I(a) - I(X)],$$

$$I(Y \cup a) = I(Y) \biguplus [I(a) - I(Y)],$$

Clearly

$$I(a) - I(X) \supseteq I(a) - I(Y).$$

The result follows.
Example 10.2.4: Let $X \subseteq Y \subseteq V(\mathcal{G}), a \in V(\mathcal{G}) - Y$. We have

$$\Gamma(X \cup a) = \Gamma(X) \biguplus (\Gamma(a) - \Gamma(X)),$$

$$\Gamma(Y \cup a) = \Gamma(Y) \biguplus (\Gamma(a) - \Gamma(Y)).$$

Clearly

$$(\Gamma(a) - \Gamma(X)) \supseteq (\Gamma(a) - \Gamma(Y)).$$

The result follows.
Example 10.2.5: Let $X \subseteq Y \subseteq V(\mathcal{G})$ and let $a \in V(\mathcal{G}) - Y$. Then

$$cut(X \cup a) = (cut(X) - cut(X) \cap cut(a)) \biguplus (cut(a) - cut(X)),$$

$$cut(Y \cup a) = (cut(Y) - cut(Y) \cap cut(a)) \biguplus (cut(a) - cut(Y)),$$

Clearly

$$cut(a) - cut(X) \supseteq cut(a) - cut(Y).$$

and $cut(X) \cap cut(a) \subseteq cut(Y) \cap cut(a)$ The result follows.

Example 10.2.6: In every one of these cases the method of the above solutions works.

Example 10.2.7: The proof is similar to that of Example 10.2.2. We only have to define E_{Za} to be the set of all vertices in V_R adjacent only to vertices in $Z \cup a$ and adjacent to a.

Example 10.2.8: We have seen that the function $|E_L|(\cdot)$ is supermodular in Example10.2.7. Next let $\Gamma_L(\cdot)$ denote $\Gamma/\mathbf{V_L}(\cdot)$. Now

$$c(X) = \Gamma_L(X) - E_L(X), X \subseteq V_L,$$

and

$$\Gamma_L(X) \supseteq E_L(X).$$

Hence

$$|c|(X) = |\Gamma_L|(X) - |E_L|(X).$$

Now $|\Gamma_L|(\cdot)$ is submodular while $|E|(\cdot)$ is supermodular. The result follows from the fact that subtraction of a supermodular function from a submodular function yields another submodular function. Such facts are presented in Exercise 10.4 .

Example 10.2.9: Let $X \subseteq Y \subseteq S$ and let $e \in E - Y$. Clearly if e is independent of Y it must be independent of X. Hence

$$r(X \cup e) - r(X) \geq r(Y \cup e) - r(Y).$$

The submodularity of $r(\cdot)$ follows.

Example 10.2.10: We parallel the argument of the previous example. Let $X \subseteq Y \subseteq E(\mathcal{G})$ and let $e \in E(\mathcal{G}) - Y$. If e does not form a circuit with edges in Y it is clear that e will not do so with edges in X either. Hence

$$r(X \cup e) - r(X) \geq r(Y \cup e) - r(Y).$$

So $r(\cdot)$ is submodular. We know that

$$\nu'(X) = |X| - r(X).$$

Since $|\cdot|$ is modular and $r(\cdot)$ is submodular it is easily seen that the function $\nu'(\cdot)$ is supermodular. Next observe that $r'(X \cup e) > r'(X)$ iff $E - X$ contains no circuit with e as a member. Clearly, whenever $Y \supseteq X$, if $E - X$ contains no circuit with e as a member neither will $E - Y$ contain a circuit with e as a member. Hence

$$r'(X \cup e) - r'(X) \leq r'(Y \cup e) - r'(Y).$$

This is equivalent to supermodularity of $r'(\cdot)$. The submodularity of $\nu(\cdot)$ follows since $\nu(X) =| X | - r'(X)$.

E 10.2: Let $\mathcal{G}$ be the graph on the vertex set $\{a, b, c, d\}$ with edges between $(a, b), (b,$
$c), (c, d), (d, a)$. Let $X = \{b\}, Y = \{c, b\}, Z = \{a, b, c\}$. Observe that

$$k(Y \cup a) - k(Y) = 1,$$

$$k(X \cup a) - k(X) = 0.$$

So $k(\cdot)$ cannot be submodular. On the other hand

$$k(Z \cup d) - k(Z) = 0,$$

$$k(X \cup d) - k(X) = 1.$$

So $k(\cdot)$ cannot be supermodular.

E 10.5:
i. This is immediate.

ii. Let the rank function of $\mathcal{G} \times X$ be $r'(\cdot)$. It is easily verified that $r'(\cdot)$ and $r \diamond \mathbf{X}(\cdot)$ are matroid rank functions (increasing, integral, submodular with zero value on $\emptyset$ and value not exceeding one on singletons). A matroid rank function $r(\cdot)$ is fully determined by its independent sets (sets on which $r(Y) = |Y|$). So we need to show that $r'(\cdot)$ and $r \diamond \mathbf{X}(\cdot)$ have the same independent sets. Now independent sets of $r'(\cdot)$ are the circuit free sets of $\mathcal{G} \times X$. Let $Y \subseteq X$. We know that this set contains no circuit of $\mathcal{G} \times X$ iff for each $Z \subseteq E(\mathcal{G}) - X$ that contains no circuit of $\mathcal{G}$, $Y \cup Z$ contains no circuit of $\mathcal{G}$, i.e., iff no circuit of $\mathcal{G}$ intersects X in a subset of Y i.e., iff $r(Y \cup (E(\mathcal{G}) - X)) = |Y| + r(E(\mathcal{G}) - X)$ i.e., iff $r \diamond \mathbf{X}(Y) = |Y|$.

iii. This follows immediately from the above proof and the definition of contramodular dual.

iv. Immediate from the definition of nullity function.
The remaining parts follow from the above through the use of Theorem 10.3.2 (i), (ii).

E 10.6:
i. This is immediate from the definitions of restriction, contraction and incidence functions of the relevant graphs.

ii. Let $Y \subseteq X \subseteq V(\mathcal{G})$. Then, $|I| \diamond \mathbf{X}(Y) \equiv |I|((V(\mathcal{G}) - X) \cup Y) - |I|(V(\mathcal{G}) - X)=$ the number of edges which are incident on Y but not on $V(\mathcal{G}) - X$. The result follows.

iii. This is immediate.

iv. We have,

$$|I|^d(X) \equiv |I|(V(\mathcal{G})) - |I|(V(\mathcal{G}) - X).$$
$$= |E|(X).$$

v. Let $X \subseteq V(\mathcal{G})$. We have

$$|I|^*(X) \equiv \alpha(X) - |I|(V(\mathcal{G})) + |I|(V(\mathcal{G}) - X).$$

$$= \alpha(X) - |E|(X)$$
$$= \sum_{v \in X} |I|(v) - |E|(X).$$
$$= |I|(X).$$

The remaining parts follow through the use of Theorem 10.3.2 (ii). on the corresponding results for the incidence function.

E 10.7:

i. This is immediate.

ii. Let $Y \subseteq X$. Then $|\Gamma_L \diamond \mathbf{X}|(Y) \equiv \Gamma_L((V_L - X) \cup Y) - \Gamma_L(V_L - X)$, i.e., the size of the set of vertices adjacent to Y but not in $\Gamma_L(V_L - X)$. This is clearly the size of the set adjacent to Y in $B \diamond_L X$.

iii. This is immediate from the definitions of $|(\Gamma_L)_{fus.\Pi}(\cdot)$ and $|\Gamma_{\Pi L}|(\cdot)$.

iv. Let $X \subseteq V_L$. Then

$$|\Gamma_L^d|(X) = |\Gamma_L|(V_L) - |\Gamma_L|(V_L - X).$$

This is the size of the set of vertices which are adjacent to X but not to $V_L - X$ i.e., the size of $E_L(X)$.
The remaining parts follow by the use of Theorem 10.3.2 (ii) on the above results.

E 10.8: The submodularity of these functions has already been shown (Example10.2.10). That they are increasing and integral functions and their values on singletons do not exceed 1 is clear.

E 10.9: The submodularity of all these functions has already been shown (see the Solution of Exercise 10.1). It is clear that they all take zero value on the null set. Except for the cut function all the functions are monotone increasing. The cut function is symmetric i.e., its value is the same on a set and its complement. So it cannot be monotone increasing if the graph is not trivial.

E 10.10: Proof of Theorem10.5.1: We will consider only the polymatroid case, since the matroid case is an easy consequence. We have

$$\mu^*(X) \equiv \sum_{e \in X} \alpha(e) - [\mu(S) - \mu(S - X)].$$

(Note that $\alpha(\cdot)$ satisfies $\alpha(e) \geq \mu(e) \quad \forall e \in S$). We have already seen that $\mu^*(\cdot)$ is submodular (Theorem 10.3.4).

- It is immediate from the definition that $\mu^*(\emptyset) = 0$
- Let $Y \supseteq X$. Since $\mu(\cdot)$ is submodular and $\mu(\emptyset) = 0$, we have

$$\mu(S - X) - \mu(S - Y) \leq \sum_{e \in (Y - X)} \mu(e).$$

Hence

$$\mu^*(Y) - \mu^*(X) \equiv \sum_{e \in Y-X} \alpha(e) - [\mu(S-X) - \mu(S-Y)] \geq 0.$$

Thus $\mu^*(\cdot)$ is increasing.

- $\mu^*(e) \geq 0 \quad \forall e \in S$, because $\mu^*(\emptyset) = 0$ and μ^* is increasing.

Thus $\mu^*(\cdot)$ is a polymatroid rank function.

□

E 10.11:
i. It is easily verified that all these functions are submodular, increasing, integral, take the value zero on the null set and zero or one on singleton sets.

ii. Clearly this function is submodular, increasing , integral and takes value zero on the null set.

iii. The statements about independent sets, bases and circuits are all immediate from the definitions of these sets in terms of the rank function and the definition of the notion of restriction.

iv. A set Y is independent in $\mathcal{M} \times X$ iff

$$|Y| = r \diamond \mathbf{X}(Y) = r((S-X) \cup Y) - r(S-X).$$

Now it can be verified that, whenever Z is a base of $\mathcal{M}.(S-X)$,

$$r((S-X) \cup Y) - r(S-X) = r(Z \cup Y) - r(Z)$$

and $r(Z) = |Z|$. Hence Y is independent in $\mathcal{M} \times X$ iff, whenever Z is a base of $\mathcal{M}.(S-X)$, we have

$$|Y| + |Z| = r(Z \cup Y),$$

equivalently, we have $Y \cup Z$ independent in $\mathcal{M}$. Noting that a base is a maximal independent set, we find that, Y is a base of $\mathcal{M} \times X$ iff whenever Z is a base of $\mathcal{M}.(S-X)$, $Y \cup Z$ is a base of $\mathcal{M}$. But a base of $\mathcal{M}.(S-X)$ is a maximal intersection of a base of $\mathcal{M}$ with $S-X$. The desired result follows.

A circuit of a matroid can be seen to be a minimal set not contained in any base of the matroid. A circuit of $\mathcal{M} \times X$ is a minimal set not contained in any minimal intersection of bases of $\mathcal{M}$ with X. Suppose C is a circuit of $\mathcal{M}$ with $C \cap X \neq \emptyset$. Now $C - X$ is independent in $\mathcal{M}$ and hence is contained in a base b of $\mathcal{M}.(S-X)$. If b_X is any base of $\mathcal{M} \times X$ we know that $b \cup b_X$ is a base of $\mathcal{M}$ which intersects X minimally. Clearly $b \cup b_X$ cannot contain C. Hence no base of $\mathcal{M} \times X$ can contain $C \cap X$. Hence $C \cap X$ contains a circuit of $\mathcal{M} \times X$. On the other hand if C' is a circuit of $\mathcal{M} \times X$, it is seen from the definition (in terms of rank) of such a circuit that $C' \cup Z$, when Z is a base of $\mathcal{M}.(S-X)$, contains a circuit of $\mathcal{M}$. Since Z contains no circuit of $\mathcal{M}$ it follows that C' contains the intersection of a circuit of $\mathcal{M}$ with X.

v. We have, B is a base of $\mathcal{M}^*$ iff

$$|B| = r^*(B) = r^*(S) = |S| - r(S).$$

Now $r^*(B) = |B| - r(S) + r(S - B)$. Thus B is a base of $\mathcal{M}^*$ iff $|S - B| = r(S - B)$ i.e., iff $S - B$ is a base of $\mathcal{M}$.

vi. We have,

$$r^d(X) \equiv r(S) - r(S - X) = r \diamond \mathbf{X}(X).$$

But by (iv) above we have, rank of $\mathcal{M} \times X = r \diamond \mathbf{X}(X)$. The result follows.
The remaining parts are consequences of Theorem 10.3.2 ((i), (ii)), when applied to the above results.

E 10.12: A vector $\mathbf{y}$ belongs to $P_{f+\mathbf{x}}$ iff $\mathbf{y}(X) \leq f(X) + x(X) \quad \forall X \subseteq S$ i.e., iff $(\mathbf{y} - \mathbf{x})(X) \leq f(X) \quad \forall X \subseteq S$ i.e., iff $(\mathbf{y} - \mathbf{x}) \in P_f$.

□

E 10.13:
i. Since $r(\emptyset) = 0$,

$$r(X) + r(S - X) \geq r(S).$$

Hence,

$$r(S) - r(S - X) = r'(X) \leq r(X).$$

The proof for the $\nu(\cdot)$ case is similar.

ii. Let b be any base of $\mathcal{M}$. If $X \subseteq S$ we have $X \cap b$ independent in $\mathcal{M}$ and hence $X \cap b$ is contained in a base of $\mathcal{M}.X$. Thus $|X \cap b| \leq r(X)$. On the other hand $X \cap b$ contains a base of $\mathcal{M} \times X$. Hence $|X \cap b| \geq r'(X)$. Thus we can choose $\mathbf{w}_r$ as follows. $\mathbf{w}_r(e) = 1,\ e \in b$ and $\mathbf{w}_r(e) = 0,\ e \notin b$. Similarly it can be seen that choosing $\mathbf{w}_\nu(\cdot)$ corresponding to a cobase would satisfy $\nu(\cdot) \geq \mathbf{w}_\nu(\cdot) \geq \nu'(\cdot)$.

E 10.14:
i. Similar to the matroid case.

ii. Choose w to be a base of the polymatroid, i.e., a vector on S such that $w(X) \leq \rho(X),\ X \subseteq S$ *and* $w(S) = \rho(S)$. We then have

$$\rho(X) \geq w(X) \quad \forall X \subseteq S$$

$$\rho(S) - \rho(X) \leq w(S) - w(X) \quad \forall X \subseteq S.$$

E 10.15: The functions in (i) and (ii) are special cases of the function in (iii). The latter has already been shown to be submodular in the solution to Exercise 10.1, Example 10.2.8. The symmetry follows by definition. The function $\xi(\cdot)$ in (iv) is the sum of two submodular functions. Further

$$\begin{aligned} \xi(S - X) &= r(M.(S - X)) - r(M \times (S - X)) \\ &= r(M) - r(M \times X) - r(M) + r(M.X) \\ &= \xi(X). \end{aligned}$$

The function $\theta(\cdot)$ in (v) is the sum of two submodular functions. The symmetry follows by definition.

E 10.17: Only contramodular dualization ($\mu^d(X) \equiv \mu(S) - \mu(S-X)$) preserves symmetry.

E 10.18: We need to show that

$$f(A) + f(B) - f(A \cup B) \geq f(X) + f(Y) - f(X \cup Y)$$
$$X \subseteq A, Y \subseteq B, A \cap B = \emptyset,$$

i.e., to show that,

$$f(A) + f(B) + f(X \cup Y) \geq f(X) + f(Y) + f(A \cup B)$$

By using submodularity twice the LHS

$$\geq f(A \cup Y) + f(X) + f(B) \geq f(X) + f(Y) + f(A \cup B).$$

as desired.

In the case of the cut function, $c(A, B)$ is the sum of the weights of edges with one end in A and another in B. But the set of edges with one end in A and the other in B contains the set of edges with one end in X and the other in Y. so the inequality is obvious in this case.

E 10.19:
i. Treat A as a single element and use Algorithm Symmetric on the set $(S-A)\cup\{A\}$ and the corresponding version of $g(\cdot)$. If (X, Y) is the output of the algorithm either X or Y would contain A.

ii. Treat $S - A$ as a single element.

E 10.20: In the case of the cut function the algorithm goes through without change if we define,

$$\begin{aligned} c(A,B) &\equiv \frac{1}{2}(|cut|(A) + |cut|(B) - |cut|(A \cup B)) \\ &\equiv \textit{weight of the edges with one end point in } A \textit{ and the other in } B. \end{aligned}$$

and in the case of $(|\Gamma_L| - |E_L|)(\cdot)$

$$c(A,B) \equiv \frac{1}{2}((|\Gamma_L| - |E_L|)(A) + (|\Gamma_L| - |E_L|)(B) - (|\Gamma_L| - |E_L|)(A \cup B))$$

In the case of $|\Gamma_L|(\cdot)$

$$\begin{aligned} c(A,B) &\equiv \frac{1}{2}(|\Gamma_L|(A) + |\Gamma_L|(B) - |\Gamma_L|(A \cup B)) \\ &\equiv \textit{halfweight of vertices (of } V_R\textit{) adjacent to vertices in } A \textit{ and } B. \end{aligned}$$

Notice that both $(|\Gamma_L| - |E_L|)(\cdot)$ and $|\Gamma_L|(\cdot)$ when processed by Algorithm Symmetric will yield the same optimal set. But the $c(A, B)$ function is simpler in the case of $|\Gamma_L|(\cdot)$.

E 10.21: Let $\mu(\cdot)$ denote the function $|cut|(\cdot)$ on the vertex subset of graph $\mathcal{G}$. Then the desired weight function $w : V \longrightarrow \Re$ is defined by

$$\begin{aligned} w(v) &\equiv \mu(V) - \mu(V - v) \\ &\equiv 0 - \mu(V - v) \\ &\equiv -(weight\ of\ edges\ incident\ on\ v). \end{aligned}$$

Then $\mu(A) - w(A) =$ weight of edges with one endpoint in A + [2(weight of edges with both endpoints in A) + (weight of edges with only one endpoint in A)] = $2|\Gamma|(A)$.

Thus minimising $\mu(\cdot)$ over nonvoid proper subsets is the same as minimising $|\Gamma|(\cdot) + \frac{1}{2}w(\cdot)$ over such subsets. The latter is the same as minimising $|\Gamma|(X) - \frac{1}{2}w(V - X)$. Since $-w(X)$ takes positive value on each vertex, this latter can be posed as a sequence of flow problems. The complexity is however worse than that of the Stoer - Wagner algorithm (mainly because mincuts corresponding to $X = S, X = \emptyset$ must be avoided).

10.12 Solutions of Problems

P 10.6: The maximum weight case. Let $X \subseteq Y \subseteq E - e$. We need to show that

$$\rho(X \cup e) - \rho(X) \geq \rho(Y \cup e) - \rho(Y)$$

Let t_X, t_Y be the maximum weight forests of $\mathcal{G}.X, \mathcal{G}.Y$ respectively. Suppose e is not spanned by t_Y. Then it would not also be spanned by t_X. So in this case the inequality would be satisfied as an equality. Next let e be spanned by t_Y. Then

$$\begin{aligned} \rho(Y \cup e) - \rho(Y) &= w(e) - w(e_Y)\ if\ w(e) > w(e_Y) \\ &= 0,\ otherwise \end{aligned}$$

where e_Y is the edge with least weight in the fundamental circuit of e with respect to t_Y. If e is not spanned by t_X then

$$\rho(X \cup e) - \rho(X) = w(e)$$

and the inequality is satisfied. Suppose e is spanned by t_X. Let e_X be the edge with least weight in the fundamental circuit of e with respect to t_X. Clearly if $w(e_X) \leq w(e_Y)$ the desired inequality is satisfied. Suppose otherwise. Let $e_1, e_2, \ldots, e_k$ be the edges in the fundamental circuit of e with respect to t_X. Of these edges let $e_1, \ldots, e_r$ not belong to t_Y and let $e_{r+1}, \ldots, e_k$ belong to t_Y. Consider the fundamental circuits $C_1, C_2, \ldots, C_r$ of $e_1, \ldots, e_r$ with respect to t_Y. Now $(C_1 \cup$

$C_2 \cup \ldots \cup C_r - \{e_1, e_2, \ldots, e_r\}) \cup \{e_{r+1}, \ldots, e_k\}$ span the edges $\{e_1, \ldots, e_k\}$. Hence they span the edge e. But the minimal set of edges in t_Y which span e is the set of edges of t_Y which lie in the fundamental circuit C_e of e with respect to t_Y. Hence each of $C_e - e$ lies in one of the C_i or is equal to one of $e_{r+1}, \ldots, e_k$. Now $e_Y \notin \{e_{r+1}, \ldots, e_k\}$ since we have assumed that $w(e_X) > w(e_Y)$. So e_Y belongs to one of the C_i say C_j. But $w(e_j) \geq w(e_X) > w(e_Y)$. This contradicts the fact that t_Y is a maximum weight forest of the subgraph on Y.

The minimum weight case: Observe that it could happen that e is spanned by t_Y but not by t_X. Assume $w(e_Y) > w(e) > 0$. So in this case

$$\rho(X \cup e) - \rho(X) = w(e) > \rho(Y \cup e) - \rho(Y) = w(e) - w(e_Y).$$

Suppose however e spanned by t_Y as well as t_X. By the argument that we used for the maximum weight case we must have $w(e_Y) \leq w(e_X)$. Assume $w(e) < w(e_Y) < w(e_X)$. In this case

$$\rho(X \cup e) - \rho(X) = w(e) - w(e_X) < \rho(Y \cup e) - \rho(Y) = w(e) - w(e_Y).$$

Thus in general $\rho(\cdot)$ is neither supermodular nor submodular.

P 10.7: See the general solution of the submodular function case. (Problem 10.13).

P 10.8: Routine generalization of the solution of Problem 10.6.

P 10.9: We need to show that

$$f(X) - f(X - \{\mathbf{r}\}) \geq f(Y) - f(Y - \{\mathbf{r}\}), \ X \subseteq Y, \ \mathbf{r} \in X$$

The following lemma contains the required proof.

Lemma 10.12.1 *Let $\mathbf{A}_Z$ be a submatrix of $\mathbf{A}$ on $Z \subseteq R$ and all columns of $\mathbf{A}$. Let $\hat{\mathbf{A}}_Z$ be obtained from $\mathbf{A}_Z$ by applying the Gram - Schmidt orthogonalization procedure on the rows of $\mathbf{A}_Z$ in the same order as the order of the rows. Then*

i. $\hat{\mathbf{A}}_Z \hat{\mathbf{A}}_Z^T = \mathbf{D}_Z$ *where $\mathbf{D}_Z$ is a diagonal matrix with its jth diagonal entry d_{jj} = square of the euclidian norm of the jth row of $\hat{\mathbf{A}}_Z$.*

ii. $det(\mathbf{A}_Z \mathbf{A}_Z^T) = det(\hat{\mathbf{A}}_Z \hat{\mathbf{A}}_Z^T) = det(\mathbf{D}_Z) = \prod(d_{jj})$.

iii. Let $\mathbf{A}_Z^1$ be obtained from $\mathbf{A}_Z$ by permuting the first $(j-1)$ rows of the latter. Let $\mathbf{D}_Z^1$ denote the diagonal matrix obtained from $\mathbf{A}_Z^1$ the way $\mathbf{D}_Z$ is obtained from $\mathbf{A}_Z$. Then

$$(\mathbf{D}_Z^1)_{kk} = (\mathbf{D}_Z)_{kk}, \ k \geq j.$$

iv. If $\mathbf{r}$ is the last (say tth) row of $\mathbf{A}_Z$, then

$$f(Z) - f(Z - \{\mathbf{r}\}) = log(d_{tt}).$$

v. Let $Y \supseteq Z$. Let $\mathbf{r}$ be the last, say sth, row of $\mathbf{A}_Y$. If $\hat{\mathbf{A}}_Y \hat{\mathbf{A}}_Y^T = \mathbf{D}'_Y$ and Y has s rows then $d'_{ss} \leq d_{tt}$.

Proof of the Lemma: We remind the reader that the Gram - Schmidt orthogonalization procedure yields, while acting on the sequence of vectors $\mathbf{x}_1, \ldots, \mathbf{x}_t$,

$$\begin{aligned}
\hat{\mathbf{x}}_1 &= \mathbf{x}_1 \\
\hat{\mathbf{x}}_2 &= \mathbf{x}_2 - \frac{<\mathbf{x}_2, \hat{\mathbf{x}}_1>}{<\hat{\mathbf{x}}_1, \hat{\mathbf{x}}_1>}\hat{\mathbf{x}}_1 \\
\vdots & \quad \vdots \\
\hat{\mathbf{x}}_t &= \mathbf{x}_t - \frac{<\mathbf{x}_t, \hat{\mathbf{x}}_{t-1}>}{<\hat{\mathbf{x}}_{t-1}, \hat{\mathbf{x}}_{t-1}>}\hat{\mathbf{x}}_{t-1} - \ldots - \frac{<\mathbf{x}_t, \hat{\mathbf{x}}_1>}{<\hat{\mathbf{x}}_1, \hat{\mathbf{x}}_1>}\hat{\mathbf{x}}_1
\end{aligned}$$

i. Since $\hat{\mathbf{x}}_1, \ldots, \hat{\mathbf{x}}_t$ are mutually orthogonal any matrix say $\hat{\mathbf{A}}_Z$ which has these as rows will satisfy $\hat{\mathbf{A}}_Z \hat{\mathbf{A}}_Z^T = \mathbf{D}_Z$ where $\mathbf{D}_Z$ satisfies the conditions specified in the statement (i) of the lemma.
ii. It is clear from the description of the procedure that $\hat{\mathbf{A}}_Z = (\mathbf{K})\mathbf{A}_Z$ where $\mathbf{K}$ is a lower triangular matrix with 1s along the diagonal. Hence

$$\begin{aligned}
det(\hat{\mathbf{A}}_Z \hat{\mathbf{A}}_Z^T) &= det(\mathbf{K}).det(\mathbf{A}_Z \mathbf{A}_Z^T).det(\mathbf{K}^T) \\
&= det(\mathbf{A}_Z \mathbf{A}_Z^T)
\end{aligned}$$

But $det(\hat{\mathbf{A}}_Z \hat{\mathbf{A}}_Z^T) = det(\mathbf{D}_Z) = \prod(d_{jj})$. Thus the result follows.

iii. This is a consequence of the previous parts of this lemma and the Gram-Schmidt orthogonalization procedure.

iv. If $\mathbf{r}$ is the last say tth row of $\mathbf{A}_Z$, it is clear that

$$\frac{det(\mathbf{A}_Z \mathbf{A}_Z^T)}{det(\mathbf{A}_{Z-\mathbf{r}} \mathbf{A}_{Z-\mathbf{r}}^T)} = \frac{\prod_{j=1}^{t} d_{jj}}{\prod_{j=1}^{t-1} d_{jj}} = d_{tt}.$$

Hence

$$f(Z) - f(Z - \{\mathbf{r}\}) = log(d_{tt}).$$

v. We see that in the procedure

$$\mathbf{x}_t = \hat{\mathbf{x}}_t + \frac{<\mathbf{x}_t, \hat{\mathbf{x}}_{t-1}>}{<\hat{\mathbf{x}}_{t-1}, \hat{\mathbf{x}}_{t-1}>}\hat{\mathbf{x}}_{t-1} + \ldots + \frac{<\mathbf{x}_t, \hat{\mathbf{x}}_1>}{<\hat{\mathbf{x}}_1, \hat{\mathbf{x}}_1>}\hat{\mathbf{x}}_1$$

$$\mathbf{y}_s = \hat{\mathbf{y}}_s + \frac{<\mathbf{y}_s, \hat{\mathbf{y}}_{s-1}>}{<\hat{\mathbf{y}}_{s-1}, \hat{\mathbf{y}}_{s-1}>}\hat{\mathbf{y}}_{s-1} + \ldots + \frac{<\mathbf{y}_s, \hat{\mathbf{y}}_1>}{<\hat{\mathbf{y}}_1, \hat{\mathbf{y}}_1>}\hat{\mathbf{y}}_1$$

Suppose the vector $\mathbf{r}$ occurs as the tth vector in the first case and the sth vector in the second case and the first $(t-1)$ vectors are the same in both expressions. Then the terms

$$\frac{<\mathbf{r}, \hat{\mathbf{x}}_{t-1}>}{<\hat{\mathbf{x}}_{t-1}, \hat{\mathbf{x}}_{t-1}>}\hat{\mathbf{x}}_{t-1} + \ldots + \frac{<\mathbf{r}, \hat{\mathbf{x}}_1>}{<\hat{\mathbf{x}}_1, \hat{\mathbf{x}}_1>}\hat{\mathbf{x}}_1$$

denoted as α, appear in both expressions. Also $\hat{\mathbf{x}}_1, \ldots, \hat{\mathbf{x}}_s$ are orthogonal. Hence

$$< \mathbf{r}, \mathbf{r} >=< \hat{\mathbf{x}}_t, \hat{\mathbf{x}}_t > + < \alpha, \alpha > \geq < \hat{\mathbf{y}}_s, \hat{\mathbf{y}}_s > + < \alpha, \alpha >$$

It follows that $< \hat{\mathbf{x}}_t, \hat{\mathbf{x}}_t > \geq < \hat{\mathbf{y}}_s, \hat{\mathbf{y}}_s >$.
Equivalently, $\mathbf{r}$ is being decomposed into perpendicular vectors $\hat{\mathbf{x}}_t, \alpha$ in the first case, $\hat{\mathbf{y}}_s, \beta$ in the second, where α, β are the projections of $\mathbf{r}$ onto subspaces say $\mathcal{V}_\alpha, \mathcal{V}_\beta$ with $\mathcal{V}_\alpha \subseteq \mathcal{V}_\beta$. Hence

$$< \hat{\mathbf{x}}_t, \hat{\mathbf{x}}_t > \geq < \hat{\mathbf{y}}_s, \hat{\mathbf{y}}_s > .$$

□

P 10.10: We observe that this is essentially the same as the previous problem except that the underlying set has been changed from R to T. However a natural way of identifying elements of R with those of T is given in the problem.

P 10.11: By Binet - Cauchy Theorem $det(\mathbf{A}_X \mathbf{A}_X^T)$ = sum of the squares of nonzero major subdeterminants of $\mathbf{A}_X$, which is equal to number of bases of the matroid associated with $\mathbf{A}_X$ (since every subdeterminant of $\mathbf{A}$ is $0, \pm 1$). The result follows from the solution to the previous problem .

P 10.12:
i. Let $X, Y \in D(\mathbf{x})$. We have

$$x(X) + x(Y) = x(X \cup Y) + x(X \cap Y)$$

Now $\rho(\cdot)$ is submodular. Suppose

$$\rho(X) + \rho(Y) > \rho(X \cup Y) + \rho(X \cap Y)$$

We then must have $x(X \cup Y) > \rho(X \cup Y)$ or $\mathbf{x}(X \cap Y) > \rho(X \cap Y)$. But this is impossible since $\mathbf{x}$ is an independent vector. Hence

$$\rho(X) + \rho(Y) = \rho(X \cup Y) + \rho(X \cap Y)$$

and $x(X \cup Y) = \rho(X \cup Y)$, $x(X \cap Y) = \rho(X \cap Y)$.

ii. Let $e \in sat(\mathbf{x})(\equiv Z)$. If $\alpha > 0$,

$$(\mathbf{x} + \alpha\chi_e)(Z) > \rho(Z).$$

On the other hand if $\forall \alpha > 0 : \mathbf{x} + \alpha\chi_e \notin P_\rho$ then there exists a set $Y \subseteq S$ s.t. $e \in Y$ and $x(Y) \geq \rho(Y)$. Since $\mathbf{x}$ is independent we conclude that $x(Y) = \rho(Y)$ and $e \in sat(\mathbf{x})$.

iii. As in (i).

iv(a) Let X, Y maximize $\theta(\cdot)$. We have

$$\theta(X) + \theta(Y) \leq \theta(X \cup Y) + \theta(X \cap Y).$$

Hence the inequality can only be satisfied as an equality and $\theta(X \cup Y) = \theta(X \cap Y) = \theta(X)$. An immediate consequence is that $D'(\mathbf{x}')$ has a unique maximal and a unique minimal member set.

iv(b) We have $x(K) \leq \rho(K)$ and $\mathbf{x}'(K) > \rho(K)$. Suppose $x(K) < \rho(K)$ then

$$\mathbf{x}'(K) - \rho(K) < \mathbf{x}'(Z) - \rho(Z)$$

where $Z = sat(\mathbf{x})$. This is a contradiction. We conclude that $x(K) = \rho(K)$. Thus $K \in D(\mathbf{x}, e)$. Suppose $K \supset L \in D(\mathbf{x}, e)$. We must have

$$\mathbf{x}'(L) - \rho(L) = \mathbf{x}'(K) - \rho(K) = \alpha.$$

Hence K would not be the minimal member of $D'(\mathbf{x}')$. We conclude that K is the minimal member of $D(\mathbf{x}, e)$ ie $K = dep(\mathbf{x}, e)$.

iv(c) Let $K \equiv dep(\mathbf{x}, e)$ and let K' denote the set in the RHS of the equation in the statement. If $x(e) = \rho(e)$, the statement is obvious with $dep(\mathbf{x}, e) = \{e\}$. So let $x(e) < \rho(e)$ and let $e' \in K - e$. Select α s.t.

$$0 < \alpha < \rho(L) - x(L) \quad \forall L : e \in L, L \notin D(\mathbf{x}, e).$$

Now if $\mathbf{x}'' \equiv \mathbf{x} + \alpha(\chi_e - \chi_{e'})$, it is clear that $\mathbf{x}'' \in P_\rho$. Hence $K \subseteq K'$. On the other hand if for some $\alpha > 0, \mathbf{x} + \alpha(\chi_e - \chi_{e'}) \in P_\rho$, we must have $e' \in K$ as otherwise $\mathbf{x}''(K) > \rho(K)$. Hence $K \supseteq K'$.

P 10.13: We need to show that

$$\rho(X \cup e) - \rho(X) \geq \rho(Y \cup e) - \rho(Y), \ \ X \subseteq Y \subseteq S - e$$

We note that we can use the greedy algorithm to compute the value of $\rho(\cdot)$ on any set. Thus to compute $\rho(X)$ we first order the elements in X as $e_{i1}, \ldots, e_{ik}$ such that

$$c(e_{i1}) \geq c(e_{i2}) \geq \ldots \geq c(e_{ik})$$

We set

$$\begin{aligned}
x(e_{i1}) &= f(e_{i1}) \\
x(e_{i2}) &= f(e_{i1}, e_{i2}) - f(e_{i1}) \\
\vdots &= \\
x(e_{ik}) &= f(X) - f(X - e_{ik})
\end{aligned}$$

Then,

$$\rho(X) \equiv \sum_{j=1}^{k} c(e_{ij})x(e_{ij}) \equiv \sum_{j=1}^{k}(c(e_{ij}) - c(e_{ij+1})(f(e_{i1}, \ldots, e_{ij}))$$

where $c(e_{ik+1})$ is taken to be zero. Let us now order the elements of the set $X, X \cup e, Y, Y \cup e$ according to their decreasing $\mathbf{c}(\cdot)$ values. Let the ordering be

$$(e_{i1}, \ldots, e_{ik}),\ (e_{i1}, \ldots, e_{ir}, e, e_{ir+1}, \ldots, e_{ik})$$

$$(e_{j1}, \ldots, e_{jq}),\ (e_{j1}, \ldots, e_{jp}, e, e_{jp+1}, \ldots, e_{jq})$$

We will adopt the convention that the elements of $X, X \cup e$ etc will appear in the larger sets in the same order but possibly with additional intermediate elements. We then have,
$\rho(X \cup e) - \rho(X)$

$$\begin{aligned} &= (c(e) - c(e_{ir+1}))(f(e_{i1}, \ldots, e_{ir}, e) - f(e_{i1}, \ldots, e_{ir})) \\ &+(c(e_{ir+1}) - c(e_{ir+2}))(f(e_{i1}, \ldots, e_{ir}, e, e_{ir+1}) - f(e_{i1}, \ldots, e_{ir}, e_{ir+1})) \\ \vdots\ &\vdots \\ &+c(e_{ik})(f(X \cup e) - f(X)) \end{aligned}$$

and
$\rho(Y \cup e) - \rho(Y)$

$$\begin{aligned} = \quad & (c(e) - c(e_{jp+1}))(f(e_{j1}, \ldots, e_{jp}, e) - f(e_{j1}, \ldots, e_{jp})) \\ & +(c(e_{jp+1}) - c(e_{jp+2}))(f(e_{j1}, \ldots, e_{jp}, e, e_{jp+1}) - f(e_{j1}, \ldots, e_{jp}, e_{jp+1})) \\ \vdots\ \ & \vdots \\ & +c(e_{jk})(f(Y \cup e) - f(Y)) \end{aligned}$$

We observe that the elements $e_{ir+1}, \ldots, e_{ik}$ also occur among the above elements $e_{jp+1}, \ldots, e_{jq}$ in the same order. Let

$$\begin{array}{rcl} e_{ir+1} & = & e_{jp+s1} \\ \vdots & \vdots & \vdots \\ e_{ir+t} & = & e_{jp+st} \\ \vdots & \vdots & \vdots \\ e_{ik} & = & e_{jp+sk} \end{array}$$

We can therefore rewrite $\rho(X \cup e) - \rho(X)$ as

$$[(c(e) - c(e_{jp+1}))(f(e_{i1}, \ldots, e_{ir}, e) - f(e_{i1}, \ldots, e_{ir})) + \ldots +$$

$$(c(e_{jp+s1-1}) - c(e_{jp+s1}))(f(e_{i1}, \ldots, e_{ir}, e) - f(e_{i1}, \ldots, e_{ir}))]$$

$$+ \ldots + (c(e_{jp+sk}) - c(e_{jp+sk+1}))(f(X \cup e) - f(X)) + \ldots + c(e_{jq})(f(X \cup e) - f(X)).$$

Now we could compare the expansion for $\rho(Y \cup e) - \rho(Y)$ and that for $\rho(X \cup e) - \rho(X)$ term by term (identifying a product term by the expression $(c(e_{je}) - c(e_{je+1}))$ in the first bracket). But by submodularity the expression in the second bracket of a product term of $\rho(X \cup e) - \rho(X)$ will never be smaller than the second bracket expression of the corresponding term of $\rho(Y \cup e) - \rho(Y)$. Example

$$(f(e_{i1}, \ldots, e_{ir}, e) - f(e_{i1}, \ldots, e_{ir})) \geq (f(e_{j1}, \ldots, e_{jp}, e) - f(e_{j1}, \ldots, e_{jp}))$$

since $\{e_{j1}, \ldots, e_{jp}\} \supseteq \{e_{i1}, \ldots, e_{ir}\}$, while the first bracket expression is always nonnegative because of the way the elements have been ordered. Thus

$$\rho(X \cup e) - \rho(X) \geq \rho(Y \cup e) - \rho(Y)$$

as required.

P 10.14: Let $f_1'(\cdot), f_2'(\cdot)$ be the extensions of $f_1(\cdot), f_2(\cdot)$ induced by $P_{f_1}, P_{f_2}^d$ respectively. By Theorem 10.7.3 $f_1'(\cdot), f_2'(\cdot)$ are convex (cup like) and concave (cap like) functions respectively. We can associate two convex regions E_1, E_2 in $\Re^{|S|+1}$ with $f_1'(\cdot)$ and $f_2'(\cdot)$ respectively in the usual way: $(\mathbf{c}, x) \in E_1$ (E_2), where $\mathbf{c} \in \Re^S, x \in \Re$ iff $f_1'(\mathbf{c}) < x$ $(f_2'(\mathbf{c}) \geq x)$. It is easily verified that these regions are convex. For instance let $(\mathbf{c}_1, x_1), (\mathbf{c}_2, x_2) \in E_1$. Then $f_1'(\mathbf{c}_1) < x_1$ *and* $f_1'(\mathbf{c}_2) < x_2$. Hence

$$\lambda f_1'(\mathbf{c}_1) + (1-\lambda) f_1'(\mathbf{c}_2) < \lambda x_1 + (1-\lambda) x_2, \ 0 \leq \lambda \leq 1$$

$$ie \ f_1'(\lambda \mathbf{c}_1 + (1-\lambda)\mathbf{c}_2) < \lambda x_1 + (1-\lambda) x_2$$

$$ie \ (\lambda \mathbf{c}_1 + (1-\lambda)\mathbf{c}_2, \lambda x_1 + (1-\lambda) x_2) \in E_1.$$

We note that E_1 is open because of the strict inequality in its definition and also that $E_1 \cap E_2 = \emptyset$. Further since $f_1'(\mathbf{0}) = f_2'(\mathbf{0}) = 0$ (by definition) we must have $(\mathbf{0}, x) \in E_1$ iff $x > 0$ and $(\mathbf{0}, x) \in E_2$ iff $x \leq 0$.

By the Separation Theorem we have a hyperplane H (which is the set of all points in $\Re^{|S|+1}$ on which the functional $g(\cdot)$ in the separation theorem takes the value t) separating E_1 and E_2. Clearly $(\mathbf{0}, 0) \in H$ and H is a linear subspace. Now if $(\mathbf{c}, x_1), (\mathbf{c}, x_2) \in H$ we must have $x_1 = x_2$. For the hypothesis implies that $(\mathbf{0}, x_1 - x_2) \in H$, and $(\mathbf{0}, x_2 - x_1) \in H$. So neither $(\mathbf{0}, x_1 - x_2)$ nor $(\mathbf{0}, x_2 - x_1)$ belongs to E_1. Hence $x_1 - x_2 \leq 0$ and $x_2 - x_1 \leq 0$. We conclude $x_1 = x_2$.

Thus H induces a function $w(\cdot)$ on $\Re^{|S|}$ through $w(\mathbf{c}) \equiv x$ iff $(\mathbf{c}, x) \in H$. Since H is a linear subspace it is easy to verify that $w(\cdot)$ is a linear functional. Further $w(\cdot)$ induces a modular function on 2^S. For

$$\begin{aligned} w(\chi_X) + w(\chi_Y) &= w(\chi_X + \chi_Y) \\ &= w(\chi_{X\cup Y} + \chi_{X\cap Y}) \\ &= w(\chi_{X\cup Y}) + w(\chi_{X\cap Y}). \end{aligned}$$

Next let $(\mathbf{c}, x_1), (\mathbf{c}, x_0), (\mathbf{c}, x_2)$ belong respectively to E_1, H, E_2. We must have $x_1 > x_0$. Otherwise $(\mathbf{c}, x_0) \in E_1$ (by definition of E_1), which is a contradiction. Suppose $x_0 < x_2$. Then $w(\mathbf{c}, x_2) \neq t$ as otherwise $(\mathbf{c}, x_2) \in H$ and $x_2 = x_0$. Hence $w(\mathbf{c}, x_2) < t$. Now $w(\mathbf{c}, x_1) > t$. Hence $w(\mathbf{c}, \lambda x_1 + (1-\lambda)x_2) = t$ for some $0 \leq \lambda \leq 1$. But then $x_0 = \lambda x_1 + (1-\lambda)x_2$, which is impossible since $x_0 < x_1, x_2$. Hence we conclude that $x_1 > x_0 \geq x_2$. Thus $f_1'(\mathbf{c}) \geq w(\mathbf{c}) \geq f_2'(\mathbf{c})$. Restricting these functions to characteristic vectors of subsets of S we get the desired result.

P 10.15: We first show that

$$\forall X \subseteq S, r_1(X) + r_2(S-X) \geq max\ size\ of\ common\ independent\ set\ . \qquad (10.7)$$

Let T be a common independent set. Then

$$|T| = r_1(X \cap T) + r_2((S-X)\cap T) \leq r_1(X) + r_2(S-X).$$

So the desired result (10.7) follows.

We will next construct a common independent set T whose size equals $r_1(X) + r_2(S-X)$ for an appropriate subset X. We will use the Sandwich Theorem for this purpose. Consider the submodular and supermodular functions $r_1(\cdot), r_2^d(\cdot)$ (where $r_2^d(X) = r_2(S) - r_2(S-X) \quad \forall X \subseteq S$). Let k be the least integer for which $r_1(\cdot) \geq r_2^d(\cdot) - k$. (Note that this means there exists some $X \subseteq S$ for which $r_1(\cdot) = r_2^d(\cdot) - k$). By Sandwich Theorem there is an integral weight function $h(\cdot)$ such that

$$r_1(\cdot) \geq h(\cdot) \geq r_2^d(\cdot) - k$$

It is clear that $h(e) = 0\ or 1$ on each $e \in S$. Let A be the support of $h(\cdot)$. We have

$$r_1(A) \geq h(A) = |A|.$$

Therefore A is independent in the matroid $\mathcal{M}_1$. Next,

$$h(S-A) = 0 \geq r_2^d(S-A) - k$$

$$i.e., \; r_2^d(S-A) \leq k.$$

It follows that $r_2(A) \geq r_2(S) - k$. But k is the least integer such that $r_1(\cdot) \geq r_2^d(\cdot) - k$. Hence there exists $X \subseteq S$ such that $r_1(X) = r_2^d(X) - k$. Therefore $r_2(A) \geq r_1(X) + r_2(S-X)$ for some $X \subseteq S$. Let $T \subseteq A$ be independent in $\mathcal{M}_2$ with $r_2(A) = T$. Now T is a common independent set for $\mathcal{M}_1$ and $\mathcal{M}_2$ and we have

$$|T| = r_2(A) \geq min_{X \subseteq S} \; r_1(X) + r_2(S-X).$$

Thus T is the maximum size common independent set.

Remark: The generalization of the above result is given in the next chapter. The above technique has to be modified slightly to prove that result.

P 10.16:

i. Let $a \notin S$. Let $g : 2^{S \cup a} \longrightarrow \Re$ be defined as follows.

$$g(X \cup a) \equiv f(X), \; X \subseteq S.$$

$$g(X) \equiv f(S-X), \; X \subseteq S.$$

Observe that $g(\cdot)$ is symmetric, since if $a \notin Y, \; g(Y) = f(S-Y) = g((S-Y) \cup a)$ and if $a \in Y, g(Y) = f(Y-a) = g(S-(Y-a))$. For proving that $g(\cdot)$ is submodular we consider a number of cases.

i(a) If $X \subseteq Y \subseteq S$ and $e \notin Y, \; e \neq a$.

$$g(X \cup e) - g(X) = f(S-X-e) - f(S-X)$$

$$g(Y \cup e) - g(Y) = f(S-Y-e) - f(S-Y)$$

and $f(S-Y) - f(S-Y-e) \geq f(S-X) - f(S-X-e)$.

i(b) If $X \subseteq Y \subseteq S$ and $e = a$.

$$g(X \cup e) - g(X) = f(X) - f(S-X)$$

$$g(Y \cup e) - g(Y) = f(Y) - f(S-Y)$$

so we need to show

$$f(X) - f(Y) \geq f(S-X) - f.(S-Y)$$

Let $Y - X = \{e_1, \ldots, e_n\}$. We then have,

$$(f(X) - f(X \cup e_1)) + f(X \cup e_1) - f(X \cup e_1 \cup e_2)) + \ldots + (f(Y-e_n) - f(Y))$$

$$\geq (f(S-X) - f(S-X-e_1)) + (f(S-X-e_1) - f(S-X-e_1-e_2)) + \ldots$$
$$+ (f(S-(Y-e_n)) - f(S-Y)).$$

But this follows from the conditions of the problem.

i(c) If $X \subseteq Y \subseteq S \cup a$ and $a \in X, \; e \notin Y$.

$$g(X \cup e) - g(X) = f(X \cup e - a) - f(X-a)$$

$$g(Y \cup e) - g(Y) = f(Y \cup e - a) - f(Y - a)$$

So the desired inequality is immediate.

i(d) If $X \subseteq Y \subseteq S \cup a$ and $a \in Y - X,\ e \notin Y$.

$$g(X \cup e) - g(X) = f(S - X - e) - f(S - X)$$

$$g(Y \cup e) - g(Y) = f(Y \cup e - a) - f(Y - a)$$

Now $f(S - X - e) - f(S - X) \geq f(X \cup e) - f(X)$ (given). But,

$$f(X \cup e) - f(X) \geq f(Y \cup e - a) - f(Y - a)$$

since $a \in Y - X$. Thus the desired inequality holds.

Now the function $g(\cdot)$ is symmetric and submodular so that we can apply Algorithm Symmetric to it. If $g(\cdot)$ reaches its minimum among all nonvoid proper subsets of $S \cup a$ at X, either X or $S \cup a - X$ contains a. So $f(\cdot)$ reaches its minimum at $X - a$ or $(S \cup a - X)$. However $g(\cdot)$ might reach its minimum at $\{a\}$. To avoid this eventuality minimize $g(\cdot)$ over all subsets that contain $\{a, e_i\}, e_i \in S$. Repeat the minimization for each e_i.

ii. In this case we define $g : 2^{S \cup a} \longrightarrow \Re$ as follows.

$$g(X \cup a) \equiv f(S - X),\ X \subseteq S.$$

$$g(X) \equiv f(X),\ X \subseteq S.$$

It can be seen that $g(\cdot)$ is symmetric and submodular by arguments similar to those of the previous section of the present problem. We can minimize $g(\cdot)$ over nonvoid proper subsets of $S \cup a$, using Algorithm Symmetric. If the minimum occurs at X, either X or $S \cup a - X$, does not contain a. Then $f(\cdot)$ reaches its minimum on that set. If the minimum falls on S we use the strategy described in the previous section of the present problem.

Chapter 11

Convolution of Submodular Functions

11.1 Introduction

The operations of **convolution** and **Dilworth truncation** are fundamental to the development of the theory of submodular functions. In this chapter we concentrate on convolution and the related notion of **principal partition.** We begin with a formal description of the convolution operation and study its properties. Convolution is important both in terms of the resulting function as well as in terms of the sets that arise in the course of the definition of the operation. The principal partition displays the relationships that exist between these sets when the convolved functions are scaled. In this chapter, among other things, we study the principal partitions of functions derived from the original functions in simple ways such as through restricted minor operations and dualization. We also present efficient algorithms for its construction and specialize these algorithms to an important instance based on the bipartite graph.

11.2 Convolution

11.2.1 Formal Properties

Definition 11.2.1 *Let* $f(\cdot), g(\cdot) : 2^S \longrightarrow \Re$. *The* **lower convolution of** $f(\cdot)$ **and** $g(\cdot)$, *denoted by* $f * g(\cdot)$, *is defined by*

$$f * g(X) \equiv min_{Y \subseteq X}[f(Y) + g(X - Y)].$$

The collection of subsets Y, *at which* $f(Y) + g(X - Y) = f * g(X)$, *is denoted by* $\mathcal{B}_{f,g}(X)$. *But if* $X = S$, *we will simply write* $\mathcal{B}_{f,g}$.

The **upper convolution of** $f(\cdot)$ **and** $g(\cdot)$, *denoted by* $f\overline{*}g(\cdot)$, *is defined by*

$$f\overline{*}g(X) \equiv max_{Y\subseteq X}[f(Y) + g(X - Y)].$$

It is clear that

i. $f * g(\cdot) = g * f(\cdot)$.

ii. if $\lambda, \sigma \in \Re$ then

$$(f(\cdot) + \lambda) * (g(\cdot) + \sigma) = f * g(\cdot) + (\lambda + \sigma).$$

iii. if $\mathbf{x}$ is a vector with all entries equal and the corresponding weight function is denoted by $x(\cdot)$, then

$$(f(\cdot) + x(\cdot)) * (g(\cdot) + x(\cdot)) = f * g(\cdot) + x(\cdot).$$

We now have the following elementary but important result.

Theorem 11.2.1 *(k) If* $f(\cdot)$ *is submodular (supermodular) and* $g(\cdot)$ *is modular then*
$f * g(\cdot)(f\overline{*}g(\cdot))$ *is submodular(supermodular).*

We need the following lemma for the proof of the theorem.

Lemma 11.2.1 *Let* $g(\cdot)$ *be a modular function on the subsets of* S*.Let* $A, B, C, D \subseteq S$ *such that* $A \cup B = C \cup D$, $A \cap B = C \cap D$*. Then*

$$g(A) + g(B) = g(C) + g(D).$$

Proof: Both LHS and RHS in the statement of the lemma are equal to

$$\sum_{e\in A\cup B}(g(e) - g(\emptyset)) + \sum_{e\in A\cap B}(g(e) - g(\emptyset)) + 2(g(\emptyset)).$$

□

Proof of the theorem: We consider only the submodular case. Let $X, Y \subseteq S$. Further let

$$f * g(X) = f(Z_X) + g(X - Z_X), f * g(Y) = f(Z_Y) + g(Y - Z_Y).$$

Then,

$$f * g(X) + f * g(Y) = f(Z_X) + g(X - Z_X) + f(Z_Y) + g(Y - Z_Y).$$

We observe that, since $Z_X \subseteq X, Z_Y \subseteq Y$,

$$(X - Z_X) \cup (Y - Z_Y) = (X \cup Y - (Z_X \cup Z_Y)) \cup (X \cap Y - (Z_X \cap Z_Y))$$

and

$$(X - Z_X) \cap (Y - Z_Y) = ((X \cup Y) - (Z_X \cup Z_Y)) \cap (X \cap Y - (Z_X \cap Z_Y)),$$

we must have, by Lemma 11.2.1,

$$g(X - Z_X) + g(Y - Z_Y) = g(X \cup Y - (Z_X \cup Z_Y)) + g(X \cap Y - (Z_X \cap Z_Y)).$$

Hence, $f * g(X) + f * g(Y)$
$\geq f(Z_X \cup Z_Y) + f(Z_X \cap Z_Y) + g(X \cup Y - (Z_X \cup Z_Y)) + g(X \cap Y - (Z_X \cap Z_Y)).$
Thus,

$$f * g(X) + f * g(Y) \geq f * g(X \cup Y) + f * g(X \cap Y),$$

which is the desired result.

□

Remark: It is clear that if $g(\cdot)$ is not modular, but only submodular, then $g(X - Z_X) + g(Y - Z_Y)$ need not be greater or equal to $g(X \cup Y - (Z_X \cup Z_Y)) + g(X \cap Y - (Z_X \cap Z_Y))$. Thus the above proof would not hold if $g(\cdot)$ is only submodular.

Henceforth we will **confine our attention to lower convolution of submodular functions with submodular or modular functions**. The results can be appropriately translated for upper convolution in the supermodular case.

Exercise 11.1 *(k) If $f(\cdot), g(\cdot)$ are both submodular, $f * g(\cdot)$ is not always submodular. Construct an example to illustrate this fact.*

Exercise 11.2 *(k) Show that in the following situations $f * g(\cdot)$ is submodular.*

i. $f(\cdot) - f(\emptyset) \leq g(\cdot) - g(\emptyset)$,

ii. $g(\cdot) = f(\cdot)$,

iii. $f(\cdot) - g(\cdot)$ *is monotonically decreasing (increasing).*

Exercise 11.3
Let $f_1(\cdot), f_2(\cdot)$ be increasing submodular functions and let $g(\cdot)$ be a non-negative weight function. Show that

$$((f_1 + f_2) * g)(\cdot) = ((f_1 * g + f_2 * g) * g)(\cdot)$$

Exercise 11.4 *(k) Show that $\mathcal{B}_{f,g}$ is closed under union and intersection, if $f(\cdot), g(\cdot)$ are both submodular.*

11.2.2 Examples

We now list, from the literature, a number of examples which are related to the notion of convolution.

i. Hall's Theorem(P.Hall [Hall35])
Hall's Theorem on systems of distinct representatives states the following in the language of bipartite matching :'Let $B \equiv (V_L, V_R, E)$ be a bipartite graph. There exists a matching meeting all the vertices in V_L iff for no subset X of V_L we have $|\Gamma(X)| < |X|$'. This condition is equivalent to saying '$\cdots$ iff $(|\Gamma| * |\cdot|)(V_L) = |V_L|$.'

ii. Dulmage - Mendelsohn decomposition of a bipartite graph (Dulmage and Mendelsohn [Dulmage+Mendelsohn59])
The above mentioned authors made a complete analysis of all min covers and max matchings in a bipartite graph through a unique decomposition into derived bipartite graphs. We present their decomposition using the language of convolution.

Let $B \equiv (V_L, V_R, E)$ be a bipartite graph. Let $\mathcal{B}_1$ denote the collection of subsets of V_L which minimize $h_1(X) \equiv |\Gamma_L|(X) + |V_L - X|$, with $\Gamma_L(X)$ denoting the set of vertices adjacent to vertices in X. Thus $min_{X \subseteq V_L} h_1(X) = (|\Gamma_L| * |\cdot|)(V_L)$. We have seen (Exercise 11.4) that $\mathcal{B}_1$ is closed under union and intersection. Let X_{min} and X_{max} be the minimal and maximal sets which are members of $\mathcal{B}_1$. Then $X_{max} - X_{min}$ can be partitioned into sets N_i such that for any given member of $\mathcal{B}_1$, each N_i is either contained in it or does not intersect it, and further, the partition is the coarsest with this property. Let Π be the partition whose blocks are X_{min}, all the N_i and $V_L - X_{max}$. Let us define a partial order $(\geq)$ on the blocks of Π as follows: $N_i \geq N_j$ iff N_j is present in a member of $\mathcal{B}_1$ whenever N_i is present. For all N_i, it is clear that $N_i \geq X_{min}$ and $V_L - X_{max} \geq N_i$. Next for each block K of Π we build the bipartite graph B^K as follows: Let I_K be the principal ideal of K (i.e., the collection of all elements (blocks of Π) that are 'less than or equal to' K) in the partial order. Let J_K be the union of all the elements in I_K. Then B^K is defined to be the subgraph of B on $K \uplus (\Gamma(J_K) - \Gamma(J_K - K))$. The partial order $(\geq)$ induces a partial order $(\geq_B)$ on the collection of bipartite graphs B^K, $K \in \Pi$. The Dulmage -Mendelsohn decomposition is the collection of all B^K together with the partial order $(\geq_B)$.

We now present the important properties of this decomposition.

- A set $(V_L - X) \uplus Y, X \subseteq V_L, Y \subseteq V_R$ is a minimum cover of B (**cover** $\equiv$ every edge of B is incident on some vertex of the set) iff X is the union of blocks which are in an ideal of the partial order $(\geq)$ and which are also contained in X_{max}, and $Y = \Gamma(X)$.
- Every maximum matching is incident on all the vertices in $\Gamma(X_{max})$ and $V_L - X_{min}$.
- A set of edges P is a maximum matching of B iff $P = \biguplus_{K \in \Pi} P^K$, where P^K is a maximum matching of B^K.

Exercise 11.5 *(k) Show that* $|\Gamma|(X) \geq | X | \quad \forall X \subseteq V_L$ *is equivalent to* $(|\Gamma| * | \cdot |)(V_L) = | V_L |$.

Exercise 11.6 *Prove*

Theorem 11.2.2 *[König36] (König) In a bipartite graph the sizes of a maximum matching and a min cover are equal.*

Exercise 11.7 *(k) Prove the properties of the Dulmage - Mendelsohn decomposition listed above.*

iii. Decomposition of a graph into minimum number of subforests (Tutte [Tutte61],Nash-williams [Nash-Williams61]).

Tutte and Nash-Williams characterized graphs which can be decomposed into k disjoint subforests as those which satisfy $kr(X) \geq |X|, \forall X \subseteq E(\mathcal{G})$. This condition can be shown to be equivalent to $(kr * |\cdot|)(E(\mathcal{G})) = |E(\mathcal{G})|$.

iv. The matroid intersection problem (Edmonds [Edmonds70], [Lawler76])
Given two matroids $\mathcal{M}_1, \mathcal{M}_2$ on S, find a maximum cardinality subset which is independent in both matroids.

The size of the maximum cardinality common independent set $=(r_1 * r_2)(S)$.To find this set one can either use Edmond's algorithm for this purpose or find bases b_1, b_2^* of $\mathcal{M}_1, \mathcal{M}_2^*$, which are maximally distant. (See the solution of Problem 10.15 and the matroid union algorithm in the next chapter).

v. The matroid union problem
Given two matroids $\mathcal{M}_1, \mathcal{M}_2$ find the maximum cardinality union of an independent set in $\mathcal{M}_1$ and an independent set in $\mathcal{M}_2$.

The collection of all unions of two sets, one independent in $\mathcal{M}_1$ and the other in $\mathcal{M}_2$ is also a matroid denoted $\mathcal{M}_1 \vee \mathcal{M}_2$. Thus the maximum cardinality union of an independent set of $\mathcal{M}_1$ and one of $\mathcal{M}_2$ is a base of $\mathcal{M}_1 \vee \mathcal{M}_2$. There is the well known ([Edmonds65a],[Edmonds68]) matroid union algorithm for constructing this set. The rank function of this matroid is $((r_1 + r_2) * |\cdot|)(\cdot)$. The union of all circuits of this matroid is the minimal set X which satisfies $((r_1 + r_2) * |\cdot|)(S) = (r_1 + r_2)(X) + |S - X|$.

vi. Representability of matroids (A.Horn [Horn55])
Horn showed that k independent sets of columns can cover the set of all columns of a matrix iff there exists no subset A of columns such that $|A| > kr(A)$. He conjectured that this might be correct only for representable matroids (i.e., for matroids which are associated with column sets of matrices over fields). If the conjecture had been true then there would have been a nice characterization of representability. However Edmonds [Edmonds65a] showed that this result is true for all matroids. He gave an algorithm for constructing k bases of a matroid whose union has the maximum cardinality. His results are equivalent to saying that k bases will cover the underlying set S of a matroid $\mathcal{M}$ iff $\mathcal{M}^k$ (the union of $\mathcal{M}$ with itself k times) has no circuits. The rank function of this matroid is $(kr * |\cdot|)(\cdot)$. So the result can be stated equivalently as 'covering is possible iff $(kr * |\cdot|)(S) = |S|$'.

11.2.3 Polyhedral interpretation for convolution

We now show that the convolution operation is naturally polyhedral in the sense that natural questions about submodular set polyhedra are related to it [Edmonds70].

Theorem 11.2.3 *i. Let $f(\cdot), g(\cdot)$ be set functions on subsets of S. Then*

$$P_f \cap P_g = P_{f*g}$$

ii. If $f(\cdot), g(\cdot)$ are submodular functions that take zero value on $\emptyset$ then $f * g(\cdot)$

is polyhedrally tight. Equivalently,

$$f * g(X) \equiv min_{Y \subseteq X}(f(Y) + g(X - Y)) = max(x(X)),$$

where $\mathbf{x}$ *is a vector satisfying* $x(Z) \leq f(Z), x(Z) \leq g(Z) \quad \forall Z \subseteq X$. *Further if* $f(\cdot), g(\cdot)$ *are integral, then* $\mathbf{x}$ *can be chosen to be integral.*

Proof:

i. A vector $\mathbf{x} \in P_{f*g}$ only if $x(X) \leq min_{Y \subseteq X}(f(Y) + g(X - Y)) \quad \forall X \subseteq S$, i.e., only if $x(X) \leq f(X)$ and $x(X) \leq g(X)$ for every subset X of S, i.e., only if $\mathbf{x} \in P_f \cap P_g$.

A vector $\mathbf{x} \in P_f \cap P_g$ only if $x(Y) \leq f(Y)$ and $x(X - Y) \leq g(X - Y) \quad \forall Y \subseteq X \subseteq S$, i.e., only if $x(X) \leq min_{Y \subseteq X \subseteq S}(f(Y) + g(X - Y)) \quad \forall X \subseteq S$, i.e., only if $\mathbf{x} \in P_{f*g}$.

ii. Any submodular (supermodular) function that takes zero value on the null set can be made into a polymatroid rank function by adding a large enough weight function $x(\cdot)$ which takes the same value on all singletons. In this case we know that

$$(f(\cdot) + x(\cdot)) * (g(\cdot) + x(\cdot)) = f * g(\cdot) + x(\cdot).$$

Further the polyhedron with the above function is $P_{f*g} + \mathbf{x}$. Hence, we need only to prove the required result for the case where $f(\cdot)$ and $g(\cdot)$ are polymatroid rank functions. This we assume henceforth in the proof.

Our proof follows Lovasz and Plummer [Lovász+Plummer86] and uses the Sandwich Theorem (Theorem 10.7.4). We will exhibit a vector $\mathbf{x}$ in P_{f*g} such that $x(Z) = f * g(Z)$ for a given $Z \subseteq S$. Since it is clear that $x(X) \leq f * g(X) \quad \forall X \subseteq S$, the result would follow.

Consider the (respectively) submodular and supermodular functions

$$f'(X) \equiv min(k, f(X)), \; g'(X) \equiv max(0, k - g(Z - X)) \quad \forall X \subseteq Z.$$

Choose $k = min_{X \subseteq Z}(f(X) + g(Z - X)) = f * g(Z)$.

We claim that $f'(X) \geq g'(X) \quad \forall X \subseteq Z$. If $f'(X) = k$ then this fact is immediate. Otherwise let $f'(X) = f(X)$. We know that $f(X) \geq 0$.Let us assume therefore that $g'(X) = k - g(Z - X)$. But $f(X) + g(Z - X) \geq k$.Hence $f(X) \geq k - g(Z - X)$.

Further, we observe that $f'(\emptyset) = g'(\emptyset) = 0$ and that $f'(Z) = g'(Z) = k = f * g(Z)$.

Therefore, by the Sandwich Theorem (Theorem 10.7.4), there exists a non-negative weight function $x(\cdot)$ on subsets of Z such that $f'(\cdot) \geq x(\cdot) \geq g'(\cdot)$ and $f'(Z) = x(Z) = g'(Z) = k$. We then have $x(X) \leq f(X)$ for all $X \subseteq Z$, and since $x(Z - X) \geq k - g(X)$ for all $X \subseteq Z$, we also have $x(X) = x(Z) - x(Z - X) \leq x(Z) - k + g(X) = g(X)$.

Thus the vector $\mathbf{x}$ corresponding to the weight function $x(\cdot)$ belongs to $P_f \cap P_g = P_{f*g}$ and further satisfies $x(Z) = f * g(Z)$.

Further the Sandwich Theorem assures us that the vector $\mathbf{x}$ can be chosen to be integral if $f(\cdot)$,$g(\cdot)$ are integral.

□

Exercise 11.8 *(k) Let $f(\cdot)$ be a submodular function and let $g(\cdot)$ be a weight function, i.e., $g(X) \equiv \sum_{e\in X} g(e)$. It is natural to ask whether the vector* $\mathbf{g}$ *whose components are $g(e_i), e_i \in S$, belongs to P_f. More generally one could ask for a set X on which $g(\cdot) - f(\cdot)$ reaches its maximum. (If this value is less than or equal to zero, we know that* $\mathbf{g}$ *belongs to P_f, otherwise we atleast know an inequality in the definition of P_f which* $\mathbf{g}$ *fails to satisfy in the worst possible way). This latter problem is called the* **membership problem of g over** *P_f. Show that the above mentioned sets X are precisely those in $\mathcal{B}_{f,g}$.*

Exercise 11.9 *(k) Let $\mu(\cdot)$ be a submodular function. Let $g(\cdot)$ be the weight function defined through $g(e) \equiv \mu(S-e) - \mu(S)$. Let $f(\cdot) \equiv \mu(\cdot) + g(\cdot)$. In Problem 10.5.2 we saw that $f(\cdot)$ is a polymatroid rank function. Show that $f(\cdot)$ reaches a minimum at $X \subseteq S$ iff $X \in \mathcal{B}_{f,g}$. Thus* **minimization of a submodular function** *is equivalent to solving the membership problem over a polymatroid.*

11.3 Matroids, Polymatroids and Convolution

In this section we relate matroids and polymatroids through the operation of convolution. The first result given below contains one of the most powerful ways of constructing matroid rank functions - by convolving a polymatroid rank function with a weight function which takes value 1 on singletons. The second result shows a way of regarding polymatroid rank functions as obtained through fusion of the underlying set of a matroid rank function. Once again convolution plays an important role.

Theorem 11.3.1 *Let $f(\cdot), g(\cdot)$ be arbitrary set functions on subsets of S.*

*i. Then $f * g(X \cup e) - f * g(X)$
$\leq min[max_{Y\subseteq X}(f(Y\cup e)-f(Y)), max_{Y\subseteq X}(g(Y\cup e)-g(Y))], \quad X \subseteq S, e \in S.$*

ii. Let $f(\cdot), g(\cdot)$ be increasing. Then $f * g(\cdot)$ is increasing.

iii. Let $f(\cdot), g(\cdot)$ be integral. Then so is $f * g(\cdot)$.

iv. (Edmonds [Edmonds70]) Let $f(\cdot)$ be an integral polymatroid rank function and let $g(\cdot) = |\cdot|$. Then $f * g(\cdot)$ is a matroid rank function.

Proof:
i. Let $f * g(X) = f(Z) + g(X - Z)$, where $Z \subseteq X$. Then

$$f * g(X \cup e) \leq min[f(Z \cup e) + g(X - Z), f(Z) + g((X - Z) \cup e)].$$

The proof is now immediate.

ii. Let, without loss of generality ,

$$f * g(X \cup e) = f(Z \cup e) + g(X - Z), Z \subseteq X, e \in (S - X).$$

But then

$$f * g(X) \leq f(Z) + g(X - Z) \leq f(Z \cup e) + g(X - Z).$$

iii. The proof is immediate from the definition of convolution.

iv. We need to show that $f * g(\cdot)$ is an integral polymatroid rank function that takes value atmost one on singletons. We have, $f(\cdot), g(\cdot)$ are increasing, integral, submodular, taking value zero on the null set and further $g(\cdot)$ is a weight function with $g(e) = 1 \ \forall e \in S$. From Theorem 11.2.1 it follows that $f * g(\cdot)$ is submodular. It is clear that $f * g(\emptyset) = 0$. The remaining properties for being a matroid rank function follow from the preceding sections of the present theorem.

□

Exercise 11.10 *(k) Let $\rho(\cdot)$ be an integral polymatroid rank function on subsets of S. Characterize the independent sets and circuits of the matroid rank function $\rho * |\cdot|$.*

Matroid expansion of a polymatroid

It is clear that we would get a polymatroid rank function if we fuse the elements of a matroid. Our next result shows that every polymatroid rank function is obtained by fusing the elements of an appropriate matroid.

Definition 11.3.1 *Let $f(\cdot)$ be an integral polymatroid rank function on subsets of a set $S \equiv \{e_1, e_2, \cdots, e_n\}$. Let $\hat{S}$ be another set with a partition $\{\hat{e}_1, \hat{e}_2 \cdots, \hat{e}_n\}$ and let $r(\cdot)$ be a matroid rank function on subsets of $\hat{S}$ such that whenever $X \subseteq \{e_1, e_2, \cdots, e_n\}$, we have,*

$$f(X) = r(\cup_{e_i \in X} \hat{e}_i).$$

Then $r(\cdot)$ is called a **matroid expansion** *of $f(\cdot)$.*

We will now show that every integral polymatroid rank function has a matroid expansion.

For each e_i, if $f(e_i) = k_i$, replace e_i by k_i copies of it, all in parallel, and call the resulting set $\hat{e}_i$. Thus $f(\cdot)$ is extended to a submodular function $\hat{f}(\cdot)$ on subsets of $\hat{S} \equiv \cup \hat{e}_i$ (equivalently, $f(\cdot)$ is a restriction of $\hat{f}(\cdot)$). Let $|\cdot|$ be the cardinality function on the subsets of $\hat{S}$. By Theorem 11.3.1 we know that $\hat{f}(\cdot) * |\cdot|$ is a matroid rank function, say $r(\cdot)$ on subsets of $\hat{S}$. We will now show that $r(\cdot)$ is a matroid expansion of $f(\cdot)$.

Let $X = \{e_{i_1}, \cdots, e_{i_t}\}$ and let $\hat{X} = \hat{e}_{i_1} \uplus \cdots \uplus \hat{e}_{i_t}$. We need to show that

$$r(\hat{X}) \equiv \hat{f} * |\cdot|(\hat{X}) = \hat{f}(\hat{X}) \equiv f(X).$$

Now let $\hat{f} * |\cdot|(\hat{X}) = \hat{f}(P) + |\hat{X} - P|$. The RHS of the above equation is the minimum possible among all such sums. Thus P and therefore $\hat{X} - P$ must be the union of sets of the form $\hat{e}_{i_j}$ (addition of parallel elements to P does not increase the value of $\hat{f}(\cdot)$ while decreasing the value of the $|\hat{X} - P|$ term). Further, (again since the RHS must be minimum), we must have $\hat{f}(\hat{X} - P) \geq |\hat{X} - P|$ as otherwise, since $\hat{f}(\cdot)$ is submodular and takes zero value on the null set, we must have

$$\hat{f}(P) + |\hat{X} - P| > \hat{f}(P) + \hat{f}(\hat{X} - P) \geq \hat{f}(\hat{X}),$$

which is a contradiction. On the other hand, by submodularity of $f(\cdot)$ and by reason of its taking zero value on the null set, we must have

$$\hat{f}(\hat{X} - P) \leq \sum_{\hat{e}_i \subseteq \hat{X} - P} \hat{f}(\hat{e}_i) = |\hat{X} - P|.$$

It follows that, $\hat{f}(\hat{X} - P) = |\hat{X} - P|$. Thus $\hat{f} * |\cdot|(\hat{X}) = \hat{f}(P) + \hat{f}(\hat{X} - P)$. The RHS of this equation must, using the above mentioned properties of $\hat{f}(\cdot)$, be greater or equal to $\hat{f}(\hat{X})$, and, by the definition of convolution be less than or equal to it. We conclude that

$$r(\hat{X}) \equiv \hat{f} * |\cdot|(\hat{X}) = \hat{f}(\hat{X}) \equiv f(X).$$

Exercise 11.11 *The matroid rank function $r(\cdot)$ has some unusual symmetry properties: Any subset of $\hat{e}_i$ behaves identically as any other subset of $\hat{e}_i$ that has the same cardinality. For instance if a circuit (base) of the matroid defined by $r(\cdot)$ contains a k element subset of $\hat{e}_i$ it can be replaced by any other such k element subset of $\hat{e}_i$ and the resulting subset would remain a circuit (base).*
Prove the above statements.

11.4 The Principal Partition

11.4.1 Introduction

The notion of principal partition (PP) is important because of the structural insight it provides in the case of many practical problems [Iri79b], [Iri+Fujishige81], [Iri83], [Iri84], [Fujishige91]. The literature on this subject is extensive. The idea began as the 'principal partition of a graph' [Kishi+Kajitani68] and was originally an offshoot of the work on maximally distant trees (summarized in Lemma 15.2.1). The extensions of this concept can be in two directions: towards making the partition finer or towards making the functions involved more general. Our present description favours the former approach and is based on the principal partition of a matroid ([Narayanan74], [Tomizawa76]). Thus, although we begin by defining the PP of $(f(\cdot), g(\cdot))$, where $f(\cdot), g(\cdot)$ are general submodular functions, most of the properties described are for the case where $g(\cdot)$ is a strictly increasing polymatroid rank function. For structural results (as in Sections 11.4.5 and 11.4.6) and algorithms (Section 11.6) we restrict ourselves to the important practical case of $g(\cdot)$ being a positive weight function.

11.4.2 Basic Properties of PP

Definition 11.4.1 *Let $f(\cdot), g(\cdot)$ be submodular functions on the subsets of a set S. The collection of all sets in $\mathcal{B}_{\lambda f,g}$ (i.e., the collection of sets which minimize $\lambda f(X) + g(S - X)$ over subsets of S), $\forall \lambda, \lambda \geq 0$, is called the* **principal partition (PP) of** $(f(\cdot), g(\cdot))$.
We denote $\mathcal{B}_{\lambda f,g}$ by $\mathcal{B}_{\lambda}$ when $f(\cdot), g(\cdot)$ are clear from the context. We denote the maximal and minimal members of $\mathcal{B}_{\lambda}$ by X^{λ}, X_{λ}, respectively.

We now list the important properties of the principal partition of $(f(\cdot), g(\cdot))$, where $f(\cdot)$ is a submodular function and $g(\cdot)$, a strictly increasing polymatroid rank function on subsets of S.

Property PP1
(This property is valid even if $g(\cdot)$ is not strictly increasing.)
The collection $\mathcal{B}_{\lambda f,g}, \lambda \geq 0$, is closed under union and intersection and thus has a unique maximal and a unique minimal element.

Remark: For the remaining properties we assume $f(\cdot)$ to be submodular and $g(\cdot)$ **to be a strictly increasing (i.e.,** $g(Y) < g(X) \quad \forall Y \subset X \subseteq S$**) polymatroid rank function.**

Property PP2
If $\lambda_1 > \lambda_2 \geq 0$, then $X^{\lambda_1} \subseteq X_{\lambda_2}$.

Definition 11.4.2 *A nonnegative value λ for which $\mathcal{B}_{\lambda}$ has more than one subset as a member is called a* **critical value of** $(f(\cdot), g(\cdot))$.

Property PP3
the number of critical values of $(f(\cdot), g(\cdot))$ is bounded by $|S|$.

Property PP4
Let $(\lambda_i), i = 1, \cdots, t$ be the decreasing sequence of critical values of $(f(\cdot), g(\cdot))$. Then, $X^{\lambda_i} = X_{\lambda_{i+1}}$ for $i = 1, \cdots, t-1$.

Property PP5
Let (λ_i) be the decreasing sequence of critical values. Let $\lambda_i > \sigma > \lambda_{i+1}$. Then $X^{\lambda_i} = X^{\sigma} = X_{\sigma} = X_{\lambda_{i+1}}$.

Definition 11.4.3 *Let $f(\cdot)$ be submodular and let $g(\cdot)$ be a strictly increasing polymatroid rank function on subsets of S. Let $(\lambda_i), i = 1, \cdots, t$ be the decreasing sequence of critical values of $(f(\cdot), g(\cdot))$. Then the sequence $\emptyset = X_{\lambda_1}, X_{\lambda_2}, \cdots, X_{\lambda_t}, X^{\lambda_t} = S$ is called the* **principal sequence of** $(f(\cdot), g(\cdot))$. *A member of $\mathcal{B}_{\lambda}$ would be alternatively referred to as a* **minimizing set corresponding to** λ *in the principal partition of $(f(\cdot), g(\cdot))$.*

Proof of the properties of the Principal Partition

i. PP1: Define $h(X) \equiv \lambda f(X) + g(S - X) \quad \forall X \subseteq S, \lambda \geq 0$. Observe that the function $g'(\cdot)$, defined through $g'(X) \equiv g(S - X) \quad \forall X \subseteq S$, is submodular. So it is clear that $h(\cdot)$ is a submodular function. The principal structure (i.e., the collection of subsets that minimize this function) has been shown in Problem 10.2 to be closed under union and intersection and thus to have a unique minimal and a unique maximal set. But then the principal structure of $h(\cdot)$ is precisely the same as $\mathcal{B}_\lambda$.

ii. PP2: Observe that minimizing $\lambda_i f(X) + g(S-X), \forall X \subseteq S, \lambda_i \geq 0, i = 1,2$, is equivalent to minimizing $f(X) + (\lambda_i)^{-1} g(S-X) \;\; \forall X \subseteq S, \lambda_i \geq 0, i = 1,2$. (Here $0 \times +\infty$ is treated as zero). So we may take the sets which minimize the latter expression to be the sets in $\mathcal{B}_{\lambda_i}, i = 1,2$. Define $p_i(X) \equiv f(X) + (\lambda_i)^{-1} g(S-X) \;\; \forall X \subseteq S, \lambda_i \geq 0, i = 1,2$. As in the case of $h_i(\cdot), p_i(\cdot), i = 1,2$ is also submodular. Let Z_1 minimize $p_1(\cdot)$. We will now show that $p_2(Z_1) < p_2(Y) \;\; \forall Y \subset Z_1$. By Theorem 10.4.1, it would then follow that Z_1 is a subset of every subset that minimizes p_2. In particular it would follow that $X^{\lambda_1} \subseteq X_{\lambda_2}$. Let $Y \subset Z_1$. We have,

$$p_2(Z_1) = p_1(Z_1) + ((\lambda_2)^{-1} - (\lambda_1)^{-1}) g(S - Z_1)$$

and

$$p_2(Y) = p_1(Y) + ((\lambda_2)^{-1} - (\lambda_1)^{-1}) g(S - Y).$$

Since $g(\cdot)$ is a strictly increasing submodular function, $S - Z_1 \subset S - Y$ and $((\lambda_2)^{-1} - (\lambda_1)^{-1}) > 0$, we must have $((\lambda_2)^{-1} - (\lambda_1)^{-1}) g(S - Z_1) < ((\lambda_2)^{-1} - (\lambda_1)^{-1}) g(S-Y)$. Since $p_1(Y) \geq p_1(Z_1)$, it follows that $p_2(Y) > p_2(Z_1)$.

iii. PP3: If $\mathcal{B}_\lambda$ has more than one set as a member then $|X^\lambda| > |X_\lambda|$. So if λ_1, λ_2 are critical values and $\lambda_1 > \lambda_2$, by Property PP2, we must have $|X_{\lambda_1}| < |X_{\lambda_2}|$. Thus the sequence X_{λ_i}, where (λ_i) is the decreasing sequence of critical values cannot have more than $|S|$ elements.

iv. PP4: We need the following lemma.

Lemma 11.4.1 *Let $\lambda > 0$. Then, for sufficiently small $\epsilon > 0$, the only set that minimizes $\lambda - \epsilon$ is X^λ.*

Proof of the Lemma: Since there are only a finite number of $(f(X), g(S-X))$ pairs, for sufficiently small $\epsilon > 0$ we must have the value of $(\lambda - \epsilon) f(X) + g(S-X)$ lower on the members of $\mathcal{B}_\lambda$ than on any other subset of S. We will now show that, among the members of $\mathcal{B}_\lambda$,X^λ takes the least value of $(\lambda - \epsilon) f(X) + g(S-X), \epsilon > 0$. This would prove the required result. If λ is not a critical value this is trivial. Let λ be a critical value and let X_1, X^λ be two distinct sets in $\mathcal{B}_\lambda$. Since $X_1 \subset X^\lambda$, we have, $g(S - X_1) > g(S - X^\lambda)$. But, $\lambda f(X_1) + g(S - X_1) = \lambda f(X^\lambda) + g(S - X^\lambda)$. So, $\lambda f(X_1) < \lambda f(X^\lambda)$. Since $\lambda > 0$, we must have, $-\epsilon f(X_1) > -\epsilon f(X^\lambda), \epsilon > 0$. It follows that, $(\lambda - \epsilon) f(X_1) + g(S - X_1) > (\lambda - \epsilon) f(X^\lambda) + g(S - X^\lambda)$.

□

Proof of PP4: By Lemma 11.4.1, for sufficiently small values of $\epsilon > 0$, X^{λ_i} would continue to minimize $(\lambda_i - \epsilon) f(X) + g(S - X)$. As ϵ increases, because there are only a finite number of $(f(X), g(S-X))$ pairs, there would be a least value ϵ_1 at which X^{λ_i} and atleast one other set minimize $(\lambda_i - \epsilon_1) f(X) + g(S - X)$. Clearly, the next critical value $\lambda_{i+1} = \lambda_i - \epsilon_1$. Since $\lambda_i > \lambda_i - \epsilon_1$, by Property PP2, we must have $X^{\lambda_i} \subseteq X_{\lambda_i - \epsilon_1}$. Hence we must have, $X^{\lambda_i} = X_{\lambda_i - \epsilon_1} = X_{\lambda_{i+1}}$, as desired.

v. **PP5:** This is clear from the above arguments.

Exercise 11.12 *(k) Show that the critical values have to be positive. What are* X^{λ}, X_{λ} *when* $\lambda = +\infty$*?*

Remark: All the properties hold in the case where $f(\cdot)$ is a strictly increasing polymatroid rank function while $g(\cdot)$ is merely submodular. Proofs are essentially the same except that while proving 'if $\lambda_1 > \lambda_2$, then $X^{\lambda_1} \subseteq X_{\lambda_2}$' we work with λ rather than $(\lambda)^{-1}$.

Exercise 11.13 *(k) Let* $f(\cdot)$ *be a submodular function on subsets of* S*. We say* (X, Y) *is a modular pair for* $f(\cdot)$ *iff*

$$f(X) + f(Y) = f(X \cup Y) + f(X \cap Y).$$

Let $g(\cdot)$ *be a positive weight function on* S *s.t.* $f(e) \leq g(e) \quad \forall e \in S$*. Show that if* X, Y *are in* $\mathcal{B}_{\lambda}$ *with respect to* $(f(\cdot), g(\cdot))$*, then*

i. (X, Y) *is a modular pair for* $f(\cdot)$*,*

ii. $(S - X, S - Y)$ *is a modular pair for* $f^*(\cdot)$ *(the dual of* $f(\cdot)$ *with respect to* $g(\cdot)$*).*

A characterization of principal partition would be useful for justifying algorithms for its construction. We will describe two such characterizations in Theorems 11.4.1 and Theorems 11.4.6 below. The first of these is a routine restatement of the properties.

Theorem 11.4.1 *Let* $f(\cdot)$ *be a submodular function on subsets of* S *and let* $g(\cdot)$ *be a strictly increasing polymatroid rank function on* S*. Let* $\mathcal{B}_{\lambda}$ *denote* $\mathcal{B}_{\lambda f,g}$*. Let* $\lambda_1, \cdots, \lambda_t$ *be a strictly decreasing sequence of numbers such that*

i. each $\mathcal{B}_{\lambda_i}, i = 1, \cdots, t$ *has atleast two members,*

ii. $\mathcal{B}_{\lambda_i}, \mathcal{B}_{\lambda_{i+1}}, i = 1, \cdots, t-1$ *have atleast one common member set,*

iii. $\emptyset$ *belongs to* $\mathcal{B}_{\lambda_1}$*, while* S *belongs to* $\mathcal{B}_{\lambda_t}$*.*

Then $\lambda_1, \cdots, \lambda_t$ *is the decreasing sequence of critical values of* $(f(\cdot), g(\cdot))$ *and therefore the collection of all the sets which are member sets in all the* $\mathcal{B}_{\lambda_i}, i = 1, \cdots, t$ *is the principal partition of* $(f(\cdot), g(\cdot))$*.*

Proof: Let $\lambda'_1, \cdots, \lambda'_k$ be the critical values and let $\emptyset = Y_0, \cdots, Y_k = S$ be the principal sequence of $(f(\cdot), g(\cdot))$. By Property PP2 of $(f(\cdot), g(\cdot))$, the only member set in $\mathcal{B}_{\lambda}$, when $\lambda > \lambda'_1$, is $\emptyset$. Further when $\lambda < \lambda'_1$, $\emptyset$ is not in $\mathcal{B}_{\lambda}$. Hence $\lambda_1 = \lambda'_1$. Next by Property PP5, when $\lambda'_1 > \lambda > \lambda'_2$, the only member in $\mathcal{B}_{\lambda}$ is Y_1 which is the maximal set in $\mathcal{B}_{\lambda'_1}$. Since $\mathcal{B}_{\lambda_2}$ has atleast two sets we conclude that $\lambda_2 \leq \lambda'_2$. We know that $\mathcal{B}_{\lambda_1}$ and $\mathcal{B}_{\lambda_2}$ have a common member which by Property PP2 can only be Y_1. But for $\lambda < \lambda'_2$, by Property PP5, Y_1 cannot be a member of $\mathcal{B}_{\lambda}$.

Hence $\lambda_2 = \lambda_2'$. By repeating this argument, we see that t must be equal to k and $\lambda_i = \lambda_i', i = 1, \cdots, t$.

$\square$

Exercise 11.14 *Let $g(\cdot)$ be a submodular function on subsets of S and let $f(\cdot)$ be a positive weight function on S. Describe the principal partition of $(f(\cdot), g(\cdot))$.*

Exercise 11.15 *What is the principal partition of a weight function $f(\cdot)$ with respect to another weight function $g(\cdot)$? How is it related to the principal partition of $g(\cdot)$ with respect to $f(\cdot)$?*

Exercise 11.16 *Let $f(\cdot), f_2(\cdot)$ be submodular functions on subsets of S and let $g(\cdot)$ be a positive weight function on S. Further let $f_2(e) \leq g(e) \;\; \forall e \in S$.*

i. Find the principal partition and critical values of

  (a) $(\beta f(\cdot), \alpha g(\cdot))$, where $\beta, \alpha > 0$,

  (b) $((f + \alpha g)(\cdot), g(\cdot))$, where $\alpha > 0$.

ii. Let $f_3(\cdot)$ be the comodular dual of $f_2(\cdot)$ with respect to $g(\cdot)$. Show that $\mathcal{B}_{\beta f + f_3, g} = \mathcal{B}_{\beta f, f_2}$.

Storing PP - Partial order representation of a distributive lattice

We have seen that $\mathcal{B}_\lambda$ is closed under union and intersection, equivalently, the elements of $\mathcal{B}_\lambda$ form a distributive lattice with **join** in place of union and **meet** in place of intersection. The number of elements in $\mathcal{B}_\lambda$ would be usually too large to store directly. Fortunately there is a very simple representation [Birkhoff67] available, by which a distributive lattice may be stored. We describe this below:

Let $\mathcal{C}$ be a collection of subsets of S closed under union and intersection. Let S be the union of all the sets in $\mathcal{C}$. Define a preorder '$\succeq_C$' on the elements of S through '$e_1 \succeq_C e_2 \;\; \forall e_1, e_2 \in S$ iff whenever e_1 belongs to a member of $\mathcal{C}$, e_2 also belongs to it'.

Definition 11.4.4 *The preorder '$\succeq_C$' and the partial order that it induces on its equivalence classes are referred to as the* **preorder and partial order associated with** *$\mathcal{C}$.*

Exercise 11.17 *(k) Verify that $\succeq_C$ is a preorder (and hence that its equivalent classes partition S). As discussed in Subsection 3.6.7, the preorder induces a partial order $\geq$ on this partition.*

We remind the reader that an **ideal** of a preorder is a subset $\mathcal{I}$ of the set over which the preorder is defined, with the property that $e_i \in \mathcal{I}, e_i \succeq_C e_j$ implies $e_j \in \mathcal{I}$. We now show that the ideals of '$\succeq_C$' are precisely the members of $\mathcal{C}$.

Let T be a member of $\mathcal{C}$. Let $e_1 \in T$. Suppose $e_1 \succeq_C e_2$. Then, we have that whenever e_1 belongs to a member of $\mathcal{C}$, e_2 will also belong to it. Hence $e_2 \in T$. Thus T is an ideal of $\mathcal{C}$. Next, let I be an ideal of '$\succeq_C$'. By the definition of the preorder, every member of I belongs to some set that is a member of $\mathcal{C}$. Since $\mathcal{C}$ is closed under intersection there is a unique minimal member say T_e that contains a given element $e \in I$. Now if e' is any other element in T_e we must have, by the definition of the preorder and the unique minimality of T_e, that $e \succeq_C e'$. Hence $T_e \subseteq I$. Since $\mathcal{C}$ is closed under union we have that $\bigcup_{e \in I} T_e$ is a set in $\mathcal{C}$. But this latter set is clearly the same as I.

We observe that, in general, $\mathcal{C}$ might have size exponential in the size of S. But the Hasse diagram contains atmost as many elements as in S and atmost $O(|S|^2)$ edges.

Example 11.4.1 *Let $\mathcal{M}$ be the collection* $\{1,2\}, \{2\}, \{1,2,3,6\}, \{2,3,4,6\}, \{1,2,3,4,6\}, \{2,3,6\}$. *This collection can be easily seen to be closed under union and intersection. The equivalence classes of the preorder are* $\{2\}, \{1\}, \{3,6\}, \{4\}$. *The Hasse diagram has* $\{1\} \geq \{2\}, \{3,6\} \geq \{2\}, \{4\} \geq \{3,6\}$.

Exercise 11.18 *Prove*

Lemma 11.4.2 *(k) Let $\mathcal{C}$ be a collection of subsets of S closed under union and intersection. Let $\mathcal{C}'$ be the collection of complements of sets in S. Then*

i. $\mathcal{C}'$ is closed under union and intersection.

ii. the preorders associated with $\mathcal{C}$ and $\mathcal{C}'$ are duals of each other,

iii. the equivalence classes of the preorders are identical and the induced partial orders are duals of each other.

The Partition - Partial Order Pair Associated with $(f(\cdot), g(\cdot))$

Each of the collections $\mathcal{B}_\lambda$, in the the principal partition of $(f(\cdot), g(\cdot))$ ($f(\cdot)$ submodular, $g(\cdot)$, a strictly increasing polymatroid rank function) is closed under union and intersection. Hence one can define preorders for each of them, so that the ideals of the preorders are identical to the members of the corresponding $\mathcal{B}_\lambda$. For a given λ, it is clear that X^λ is partitioned by the equivalence classes of the corresponding preorder. (One of the blocks of this partition is X_λ, provided this set is nonvoid). More conveniently,
let us denote by N_λ, the collection $\{X - X_\lambda : X \in \mathcal{B}_\lambda\}$.
There is a one to one correspondence between sets in $\mathcal{B}_\lambda$ and sets in N_λ, with union of the sets in the former being X^λ and of those in the latter being $X^\lambda - X_\lambda$. Further X_λ in the former corresponds to $\emptyset$ in the latter. Clearly N_λ is closed under union and intersection and induces a preorder on the elements of $X^\lambda - X_\lambda$.
Let us denote by $\Pi(\lambda)$, the partition of $X^\lambda - X_\lambda$, whose blocks are the equivalence classes of the preorder induced by N_λ.
This partition is identical to that of X^λ induced by the preorder of $\mathcal{B}_\lambda$ except that X_λ is omitted. Clearly the partitions induced in this manner by all the critical values, if put together, constitute a partition of S.

Let us denote by Π_{pp}, the union of all the partitions $\Pi(\lambda)$, λ a critical value. We denote by $\succeq_p$, the preorder on elements in S induced by all preorders of N_λ, λ a critical value, with the additional condition
$e_2 \succeq_p e_1$ whenever $e_1 \in X^{\lambda_1} - X_{\lambda_1}$,$e_2 \in X^{\lambda_2} - X_{\lambda_2}$, $\lambda_1 \geq \lambda_2$. The partial order induced on the blocks of Π_{pp} by $\succeq_p$ is denoted by $\geq_\pi$. The pair $(\Pi_{pp}, \geq_\pi)$ is referred to as the **partition-partial order pair associated with** $(f(\cdot), g(\cdot))$. This partial order is refined later.

11.4.3 Symmetry Properties of the Principal Partition of a Submodular Function

One of the attractive features of the principal partition is that, under fairly general conditions, it preserves the symmetries of the original structure. This fact is relatively easy to see if we use convolution as the basis for the development. But it is somewhat less obvious if we develop the idea of principal partition algorithmically (in terms of maximally distant forests and generalizations for instance). Below we formalize the notion of symmetry for a set function.

Definition 11.4.5 *Let $f(\cdot)$ be a real valued function on the subsets of S. An* **automorphism** *of $f(\cdot)$ is a bijection $\alpha : S \longrightarrow S$ such that $f(X) = f(\alpha(X)) \ \forall X \subseteq S$. A set X is* **invariant** *under $\alpha(\cdot)$ iff $\alpha(X) = X$. Let '$\succeq_Y$' be a preorder on the elements of $Y \subseteq S$. We say '$\succeq_Y$' is* **invariant** *under $\alpha(\cdot)$ iff*

- *Y is invariant under $\alpha(\cdot)$,*
- *$a \succeq_Y b$ iff $\alpha(a) \succeq_Y \alpha(b)$.*

A function $g(\cdot)$ is **symmetric** *with respect to $f(\cdot)$ iff every automorphism of $f(\cdot)$ is also an automorphism of $g(\cdot)$.*

Theorem 11.4.2 *Let $f(\cdot)$ be a submodular function and let $g(\cdot)$ be a strictly increasing polymatroid rank function on the subsets of S. If $g(\cdot)$ is symmetric with respect to $f(\cdot)$, then,*

i. the principal partition of $(f(\cdot), g(\cdot))$ is invariant under the automorphisms of $f(\cdot)$ and

ii. (equivalently) the partition-partial order pair of $(f(\cdot), g(\cdot))$ is invariant under such automorphisms.

Proof:
i. Let $\alpha(\cdot)$ be an automorphism of $f(\cdot)$. We need to show

- a set in $\mathcal{B}_\lambda$ moves to another such set under α
- the sets in the principal sequence remain invariant under $\alpha(\cdot)$.

We have that $\alpha(\cdot)$ is a bijection on S. It is then immediate that $X \subseteq Y$ iff $\alpha(X) \subseteq \alpha(Y)$. Let X_1 minimize $\lambda f(X) + g(S - X)$. Since $g(\cdot)$ is symmetric with respect to $f(\cdot)$, we must have that $g(S - \alpha(X_1)) = g(S - X_1)$. Hence $\alpha(X_1)$ also minimizes the expression $\lambda f(X) + g(S - X)$. Thus the image of a set in $\mathcal{B}_\lambda$ is also in $\mathcal{B}_\lambda$. Since $\alpha(\cdot)$ is a bijection, the sizes of these two sets must be the same. It then follows, from the fact that X_λ is the unique minimal set minimizing the above expression, that $X_\lambda = \alpha(X_\lambda)$. A similar argument shows that X^λ is invariant under α.

ii. To show that the partition - partial order pair of $(f(\cdot), g(\cdot))$ is invariant under $\alpha(\cdot)$, we need to show

- a block of $\Pi(\lambda)$ moves to another such block under $\alpha(\cdot)$ and
- the partial order on these blocks induced by $\mathcal{B}_\lambda$ is invariant under $\alpha(\cdot)$.

We saw that $\alpha(\cdot)$ is a bijection on S that leaves both X_λ and X^λ invariant and further simply permutes the sets in $\mathcal{B}_\lambda$ preserving the containment property. Now the blocks of $\Pi(\lambda)$ are determined by the member sets of $\mathcal{B}_\lambda$ (being the maximal subsets of X^λ which are not 'cut' by the member sets of $\mathcal{B}_\lambda$,i.e., which lie entirely inside or entirely outside these member sets). Hence a permutation of $\mathcal{B}_\lambda$ would determine a permutation of $\Pi(\lambda)$. Next, $(e_1) \geq_\pi (e_2)$ iff every set in $\mathcal{B}_\lambda$ containing (e_1) also contains (e_2) (denoting the equivalence class determined by an element e by (e)). Now X contains an element e iff $\alpha(X)$ contains $\alpha(e)$. Since α permutes the sets in $\mathcal{B}_\lambda$ it follows that every set in this collection which contains $\alpha((e_1))$ also contains $\alpha((e_2))$ and hence $\alpha((e_1)) \geq_\pi \alpha((e_2))$.

□

11.4.4 Principal Partition from the Point of View of Density of Sets

The principal partition gives information about which subsets are densely packed relative to $(f(\cdot), g(\cdot))$. For instance if $f(\cdot)$ is the rank function of a graph and $g(X) \equiv\mid X \mid$, the sets of the highest density (the sets in $\mathcal{B}_{\lambda^m}$, where λ^m is the highest critical value) correspond to subgraphs where we can pack the largest (fractional) number of disjoint forests.

Definition 11.4.6 *Let $f(\cdot)$ be a submodular function and $g(\cdot)$, a strictly increasing polymatroid rank function on subsets of S. The* **density** *of $X \subseteq S$ with respect to $(f(\cdot), g(\cdot))$ is the ratio $(g(S) - g(S - X))/(f(X) - f(\emptyset))$. The set S is said to be* **molecular with respect to $(f(\cdot), g(\cdot))$** *iff $(f(\cdot), g(\cdot))$ has only one critical value,* **equivalently**, *iff it has $\emptyset, S$ as the principal sequence. A set S that is molecular is said to be* **atomic with respect to** *$(f(\cdot), g(\cdot))$ iff S and $\emptyset$ are the only sets in $\mathcal{B}_\lambda$, λ being the only critical value. A set $X \subseteq S$ is said to be molecular (atomic) with respect to $(f(\cdot), g(\cdot))$ iff it is molecular (atomic) with respect to $(f/X(\cdot), g/X(\cdot))$.*

The problem of finding a subset T of S of highest density for a given $(g(S) - g(S-T))$ value would be NP hard even for very simple submodular functions.
Example: Let $f(\cdot) \equiv$ rank function of a graph, $g(X) \equiv\mid X \mid$. In this case $g(S) - g(S-T) =\mid T \mid$ and if we could find a set of branches of given size and highest density we can solve the problem of finding the maximal clique subgraph of a given graph. However, every set in the principal partition has the highest density for its $(g(S) - g(S-T))$ value (see Exercise 11.19) and further is easy to construct. This apparent contradiction is resolved when we note that there may be no set of the given value of $(g(S) - g(S-T))$ in the PP.

The idea of 'density' is natural for polymatroid rank functions. So, in this subsection, we confine ourselves to such functions even though the results can be generalized to arbitrary submodular functions.

Exercise 11.19
Let $f(\cdot), g(\cdot)$ be polymatroid rank functions on subsets of S with $g(\cdot)$ strictly increasing. Let T be a set in the principal partition of $(f(\cdot), g(\cdot))$. If $T' \subseteq S$ s.t. $g(S-T) = g(S-T')$ and T' not in the principal partition show that the density of $T \subseteq S$ is greater than that of T'.

Exercise 11.20 *Let $f(\cdot), g(\cdot)$ be polymatroid rank functions on subsets of S with $g(\cdot)$ strictly increasing.*

i. *Show that S is molecular with critical value λ iff $((\lambda f) * g)(S) = \lambda f(S) = g(S)$.*

ii. *When S is molecular show that the critical value is equal to $g(S)/f(S)$.*

iii. *Show that S is molecular (atomic) iff S has the highest density among all its subsets (has higher density than all its proper subsets).*

Remark: When the context makes $(f(\cdot), g(\cdot))$ clear we would simply say S is molecular (atomic) instead of S is molecular (atomic) with respect to $(f(\cdot), g(\cdot))$. Similarly while speaking of density.

* Alternative Development of PP

The following exercises constitute an alternative development of the principal partition from the density based point of view. They are included primarily to bring out an aspect of the analogy between principal partition and the principal lattice of partitions (to be introduced in the next chapter). They may therefore be omitted during a first reading.

We need a preliminary definition.

Definition 11.4.7 *Let $f(\cdot), g(\cdot)$ be polymatroid rank functions on subsets of S with $g(\cdot)$ strictly increasing.*
A set T satisfies the **λ-density gain** *(***λ-density loss***) condition with respect to*

$(f(\cdot), g(\cdot))$, *iff whenever* $T' \supseteq T$ ($T" \subseteq T$), *we have*

$$\frac{g(S-T)-g(S-T')}{f(T')-f(T)} \leq \lambda$$

$$\frac{g(S-T")-g(S-T)}{f(T)-f(T")} \geq \lambda.$$

We say that these conditions are satisfied **strictly** *if the inequalities above are strict.*

Exercise 11.21
Let $h_\lambda(X) \equiv g(S-X)+\lambda f(X), X \subseteq S, f(\cdot), g(\cdot)$ *as defined in Definition 11.4.7. Prove*

Theorem 11.4.3 *i. If* $T \subseteq S$ *satisfies the* $\lambda-$*density loss* ($\lambda-$*density gain) condition, then there exists a subset* $\hat{T}$ *of* S *such that* $\hat{T}$ *minimizes* $h_\lambda(\cdot)$ *over subsets of* S *and* $\hat{T} \supseteq T$ ($\hat{T} \subseteq T$).

ii. If $T \subseteq S$ *satisfies the* $\lambda-$*density loss* ($\lambda-$*density gain) condition strictly then whenever* $\hat{T}$ *minimizes* $h_\lambda(\cdot)$ *over subsets of* S *we must have* $\hat{T} \supseteq T$ ($\hat{T} \subseteq T$).

iii. A subset T *of* S *satisfies both the* $\lambda-$*density gain condition and* $\lambda-$*density loss condition iff it minimizes* $h_\lambda(\cdot)$ *over subsets of* S.

Exercise 11.22
Let $h_\lambda(X) \equiv g(S-X)+\lambda f(X), X \subseteq S$, *with* $f(\cdot), g(\cdot)$ *as defined in Definition 11.4.7. If* $T_1, T_2 \subseteq S$ *satisfy the* $\lambda-$*density gain* ($\lambda-$*density loss) property with respect to* $(f(\cdot), g(\cdot))$ *then*

$$h_\lambda(T_1 \cap T_2) \leq h_\lambda(T_i), i = 1, 2$$

$$(h_\lambda(T_1 \cup T_2) \leq h_\lambda(T_i), i = 1, 2)$$

Exercise 11.23
Let $h_\lambda(X) \equiv g(S-X)+\lambda f(X), X \subseteq S$, *with* $f(\cdot), g(\cdot)$ *as defined in Definition 11.4.7. Using the above exercises show that if* T_1, T_2 *minimize* $h_\lambda(\cdot)$, *then so do* $T_1 \cup T_2$ *and* $T_1 \cap T_2$.

Exercise 11.24
Let $h_\lambda(X) \equiv g(S-X)+\lambda f(X), X \subseteq S$, *with* $f(\cdot), g(\cdot)$ *as defined in Definition 11.4.7. Let* T_1 *minimize* $h_{\lambda_1}(\cdot)$, *and let* T_2 *minimize* $h_{\lambda_2}(\cdot)$ *with* $\lambda_1 > \lambda_2$. *Then* $T_2 \supseteq T_1$.

An alternative to the last problem:

Exercise 11.25
Let $h_\lambda(X) \equiv g(S-X)+\lambda f(X), X \subseteq S$. *Let* $f(\cdot), g(\cdot)$ *be as defined in Definition 11.4.7. Let* T_1, T_2 *minimize* $h_{\lambda_1}(\cdot), h_{\lambda_2}(\cdot)$ *respectively over subsets of* S, *with* $\lambda_1 > \lambda_2$. *Show that* $h_{\lambda_2}(T_1) < h_{\lambda_2}(T), T \subset T_1$ *and hence,* $T_1 \subseteq T_2$.

11.4.5 Principal Partition of $f^*(\cdot)$ and $f * g(\cdot)$

In this subsection we show that the principal partitions of $(f * g(\cdot), g(\cdot))$ and $(f^*(\cdot), g(\cdot))$ are closely related to that of $(f(\cdot), g(\cdot))$. In the case of $f^*(\cdot)$, the principal partition is 'oppositely directed' to that of $f(\cdot)$ while in the case of $f * g(\cdot)$, the principal partition remains identical to that of $f(\cdot)$ above a certain critical value and below it, becomes a set of coloops. Throughout, we take $f(\cdot)$ to be submodular on subsets of S and $g(\cdot)$ to be a positive weight function on S. While studying the case of $f * g(\cdot)$, we further assume that $f(\emptyset) = 0$.

We begin with some preliminary definitions.

Definition 11.4.8 *Let $f(\cdot)$ be a submodular function on subsets of S with $f(\emptyset) = 0$. The function $f(\cdot)$ is said to be* **modular over** $X \subseteq S$ *iff $(f/\mathbf{X})(\cdot)$ is a modular function. The* **modular part** *of $f(\cdot)$ is the union of all the separators* [1] *over which it is modular. Let $g(\cdot)$ be a positive weight function on S. If e belongs to the modular part of $f(\cdot)$ with $f(e) = 0$ ($f(e) \geq g(e)$) it is called a* **self loop** *(a* **coloop** *with respect to $g(\cdot)$).*

Remark:
i. When it is clear from the context, while speaking of coloops, we will omit reference to the function $g(\cdot)$.
ii. In the case of a general submodular function, the above definition, where we insist that $f(e) \geq g(e)$, seems necessary since single element separators, with value of $f(\cdot)$ greater or less than $g(\cdot)$, behave differently in the PP of $f * g(\cdot)$. In the case of a matroid, the $g(\cdot)$ function one would normally work with would satisfy $g(e) \leq 1 \forall e \in S$. So coloops of the matroid (singleton separator with rank 1) would turn out to be the same as the coloop with respect to $g(\cdot)$.

Lemma 11.4.3 *Let $f(\cdot)$ be a submodular function on subsets of S, taking zero value on the null set, and let $g(\cdot)$ be a positive weight function on S. Let $f(e) \leq g(e) \ \ \forall e \in S$. Let C be the set of coloops of $f(\cdot)$ with respect to $g(\cdot)$ and let L be the set of self loops of $f(\cdot)$. Then*

i. every subset of L is a minimizing set in the principal partition of $(f(\cdot), g(\cdot))$ corresponding to $\lambda = \infty$ and further L is the maximal such minimizing set.

ii. every set between S and $S - C$ (both sets inclusive) is a minimizing set corresponding to $\lambda = 1$ in the principal partition of $(f(\cdot), g(\cdot))$. Therefore no critical value is less than 1.

iii. $S - C$ is the minimal minimizing set corresponding to $\lambda = 1$ in the principal partition of $(f(\cdot), g(\cdot))$.

Proof:
i. This is immediate from the fact that $f(X) = 0$ iff $X \subseteq L$.

---

[1] K is a separator iff $f(K) + f(S - K) = f(S)$ (Section 10.6)

ii. Since $f(\emptyset) = 0$ we must have

$$f(X \cup Y) \leq f(X) + f(Y) \quad \forall X, Y \subseteq S, X \cap Y = \emptyset.$$

Now $f(e) \leq g(e) \quad \forall e \in S$. Hence

$$f(X) + g(S - X) \geq f(X) + f(S - X) \geq f(S).$$

Thus S is a minimizing set corresponding to $\lambda = 1$. Now by the definition of coloops and since $f(e) \leq g(e) \quad \forall e \in S$, it follows that if K is a set of coloops then $g(K) + f(S-K) = f(K) + f(S-K) = f(S)$. Therefore $S - K$ is a minimizing set corresponding to $\lambda = 1$. That no critical value is less than 1 follows from Property PP2.

iii. Let Z be any such minimizing set. We must have $f(Z) + g(S-Z) = f(S)$. But the LHS cannot be lower than $f(Z) + f(S - Z) \geq f(S)$. Thus the only way these conditions can be satisfied is to have $f(S - Z) = g(S - Z)$ and $f(Z) + f(S - Z) = f(S)$. Thus $S - Z$ must be a set of coloops. Hence $S - C$ is the minimal such minimizing set.

□

We now study the principal partition of $(f * g(\cdot), g(\cdot))$ through the following result.

Theorem 11.4.4 *Let $f(\cdot)$ be a submodular function on subsets of S with $f(\emptyset) = 0$ and let $g(\cdot)$ be a positive weight function on S. Let $p(X)$ denote $\lambda(f * g)(X) + g(S - X)$ and let $h(X)$ denote $\lambda f(X) + g(S - X) \quad \forall X \subseteq S$.*

i. When $\lambda \geq 1$

- *The minimum values of $p(\cdot)$ and $h(\cdot)$ over subsets of S are equal. If Y minimizes $p(\cdot)$ then it contains a subset Z that minimizes $h(\cdot)$.*
- *Any set that minimizes $h(\cdot)$ also minimizes $p(\cdot)$.*

ii. When $\lambda > 1$, Y minimizes $p(\cdot)$ iff it minimizes $h(\cdot)$.

iii. When $\lambda \geq 1$ there is a unique minimal set that minimizes both $p(\cdot)$ and $h(\cdot)$ and when $\lambda = 1$, this set is the complement of the set of coloops of $f * g(\cdot)$ with respect to $g(\cdot)$.

Proof:

i. By the definition of convolution,

$$(f * g)(X) \leq f(X) \quad \forall X \subseteq S.$$

Hence, since $\lambda \geq 0$, $p(X) \leq h(X) \quad \forall X \subseteq S$ and $min_{X \subseteq S} p(X) \leq min_{X \subseteq S} h(X)$. Next, for any subset X of S, when $\lambda \geq 1$ we have,

$$p(X) \equiv \lambda(f * g)(X) + g(S - X)$$

$= \lambda(f(Z) + g(X - Z)) + g(S - X)$ for some $Z \subseteq X$.

$$\geq \lambda f(Z) + g(S - Z) \equiv h(Z),$$

i.e., any subset X of S contains a subset Z such that $p(X) \geq h(Z)$ and $p(X)$ cannot be less than the minimum value of $h(\cdot)$. We conclude that

$$min_{X \subseteq S} p(X) = min_{X \subseteq S} h(X)$$

and that any set that minimizes $p(\cdot)$ contains a subset that minimizes $h(\cdot)$. Let m denote this minimum value. Suppose Y minimizes $h(\cdot)$. We then have,

$$m = \lambda f(Y) + g(S - Y) \geq \lambda(f * g)(Y) + g(S - Y) \geq m.$$

Thus Y must minimize $p(\cdot)$.

ii. $(\lambda > 1)$ We need to show that if Y minimizes $p(\cdot)$ it also minimizes $h(\cdot)$. We claim that in this case $f * g(Y) = f(Y)$, from which it would follow that the minimum value $m = h(Y)$. Suppose otherwise. Then, we must have

$$m = p(Y) \equiv \lambda(f * g(Y)) + g(S - Y)$$

$= \lambda(f(Z) + g(Y - Z)) + g(S - Y)$ for some $Z \subset Y$

$$> \lambda(f(Z)) + g(S - Z) \equiv h(Z) \geq m,$$

which is a contradiction. Thus we must have $f * g(Y) = f(Y)$ and hence Y minimizes $h(\cdot)$.

iii. Since $h(\cdot)$ is clearly submodular (it is the sum of the submodular function $\lambda f(Y)$ and the submodular function $g(S - Y)$), we must have the minimal minimizing set to be unique. From part (i) and (ii) above, this set is also the unique minimal minimizing set of $p(\cdot)$, when $\lambda \geq 1$. We observe that $f * g(e) \leq g(e) \ \forall e \in S$. Hence it follows from Lemma 11.4.3 that when $\lambda = 1$, the minimal minimizing set of $p(\cdot)$ is the complement of the set of coloops of $f * g(\cdot)$ with respect to $g(\cdot)$.

□

The following corollary is now obvious.

Corollary 11.4.1 *Let $f(\cdot)$ be a submodular function on subsets of S with $f(\emptyset) = 0$ and let $g(\cdot)$ be a positive weight function on S. Let $\lambda \geq 1$. Then,*

$$(\lambda(f * g) * g)(\cdot) = ((\lambda f) * g)(\cdot).$$

From Theorem 11.4.4 it is clear that

> *for $\lambda > 1$, the sets in the principal partition of $(f(\cdot), g(\cdot))$ ($f(\cdot)$ submodular with $f(\emptyset) = 0$, $g(\cdot)$, a positive weight function) and in that of $(f * g(\cdot), g(\cdot))$ are identical. The least critical value of $(f * g(\cdot), g(\cdot))$ is 1 and the minimizing sets for this value are the complements of subsets of coloops of $f * g(\cdot)$. For $\lambda = 1$, the minimal sets in the principal partition of $(f(\cdot), g(\cdot))$ and in that of $(f * g(\cdot), g(\cdot))$ coincide* (see Figure 11.1).

Figure 11.1: Comparison of PP of $(f(\cdot), g(\cdot)), (f * g(\cdot), g(\cdot)), (f^*(\cdot), g(\cdot))$

The principal partition of $(f(\cdot), g(\cdot))$ may have critical values lower than 1. But we lose this information when we construct the principal partition of $(f * g(\cdot), g(\cdot))$.

We next study the principal partition of the dual. We have the following result which summarizes the relation between the PP of $(f(\cdot), g(\cdot))$ and that of $(f^*(\cdot), g(\cdot))$ (see Figure 11.1).

Theorem 11.4.5 *Let $f(\cdot)$ be a submodular function on the subsets of S and let $g(\cdot)$ be a positive weight function on S. Let $\mathcal{B}_\lambda, \mathcal{B}^*_\lambda$ denote respectively the collection of minimizing sets corresponding to λ in the principal partitions of $(f(\cdot), g(\cdot)), (f^*(\cdot), g(\cdot))$, where $f^*(\cdot)$ denotes the dual of $f(\cdot)$ with respect to $g(\cdot)$. Let λ^* denote $(1-(\lambda)^{-1})^{-1}$ $\forall \lambda \in \Re$. Then*

i. A subset X of S is in $\mathcal{B}_\lambda$ iff $S - X$ is in $\mathcal{B}^*_{\lambda^*}$,

ii. if $\lambda_1, \cdots, \lambda_t$ is the decreasing sequence of critical values of $(f(\cdot), g(\cdot))$, then $\lambda^*_t, \cdots, \lambda^*_1$ is the decreasing sequence of critical values of $(f^*(\cdot), g(\cdot))$,

iii. if the principal sequence of $(f(\cdot), g(\cdot))$ is $\emptyset = X_0, \cdots, X_t = S$, then the principal sequence of $(f^*(\cdot), g(\cdot))$ is $\emptyset = S - X_t, \cdots, S - X_0 = S$.

iv. the partitions associated with the principal partitions of both $(f(\cdot), g(\cdot))$ and $(f^*(\cdot), g(\cdot))$ are identical but the partial orders are duals.

Proof:
i. We will show that Y minimizes $\lambda f(X) + g(S-X)$ iff $S-Y$ minimizes $\lambda^* f^*(X) + g(S-X)$. We have

$$\lambda^* f^*(X) + g(S-X) = \lambda^*[\sum_{e \in X} g(e) - (f(S) - f(S-X))] + g(S-X)$$

$$= \lambda^* f(S-X) + (\lambda^* - 1)g(X) - \lambda^* f(S) + g(S).$$

This is equivalent to minimizing the expression $\lambda^*(\lambda^*-1)^{-1} f(S-X) + g(X)$.Noting that $\lambda^*(\lambda^* - 1)^{-1} = \lambda$ we get the desired result.

The remaining sections of the theorem are now straightforward. For the last section however we need to use Lemma 11.4.2.

□

11.4.6 The Principal Partition associated with Special Minors

In general very little can be said about the relation between the principal partition of $f(\cdot)$ with respect to a positive weight function $g(\cdot)$, and the principal partitions of the contractions and restrictions of $f(\cdot)$ with respect to $g(\cdot)$. For certain special cases the situation is better. The following lemma allows us to study such cases.

Let $h_\lambda(X)$ denote $\lambda f(X) + g(S-X)$ $\forall X \subseteq S$,
$h_{\lambda K}(X)$ denote $\lambda f(X) + g(K-X)$ $\forall X \subseteq K$,

$h_{S-K,\lambda}(Y)$ denote $\lambda(f \diamond (\mathbf{S}-\mathbf{K}))(Y) + (g/(\mathbf{S}-\mathbf{K}))(S-K-Y) \quad \forall Y \subseteq (S-K)$. We then have

Lemma 11.4.4 *Let $f(\cdot)$ be a submodular function on subsets of S and let $g(\cdot)$ be a positive weight function on S. Let $K \subseteq S$.*

i. If $h_{\lambda K}(\cdot)$ minimizes at $Z \subseteq K$, then $h_\lambda(\cdot)$ minimizes at a superset of Z, i.e., X^λ in the principal partition of $(f(\cdot), g(\cdot))$, contains Z.

ii. $Y \subseteq (S-K)$ minimizes $h_{S-K,\lambda}(\cdot)$ iff $Y \cup K$ minimizes $h_\lambda(\cdot)$ among all supersets of K. Hence, if in addition K minimizes $h_\lambda(\cdot)$, then $Y \cup K$ minimizes $h_\lambda(\cdot)$.

Proof:
i. We observe that, $h_{\lambda K}(X) = h_\lambda(X) - g(S-K) \quad \forall X \subseteq K$. Hence $h_\lambda(Z) \leq h_\lambda(Z') \quad \forall Z' \subseteq Z$. We know that $h_\lambda(\cdot)$ is submodular (see proof of Property PP2). Hence by Theorem 10.4.1 the desired result follows.

ii. We have, $h_{S-K,\lambda}(X) = \lambda(f(K \cup X) - f(K)) + g(S-K-X) \quad \forall X \subseteq S-K$. So $Y \subseteq S-K$ minimizes $h_{S-K,\lambda}(\cdot)$ iff $Y \cup K$ minimizes $h_\lambda(\cdot)$ among all supersets of K.

□

The following corollary is immediate from the first part of the above lemma.

Corollary 11.4.2 *Let $f(\cdot)$ be a submodular function on subsets of S and let $g(\cdot)$ be a positive weight function on S.*
Let S_1 satisfy $(\lambda f) * g(S_1) = \lambda f(S_1)$. Then there exists a subset X^1 of S such that $X^1 \supseteq S_1$ and $X^1 \in \mathcal{B}_{\lambda f, g}$. Hence, X^λ, in the principal partition of $(f(\cdot), g(\cdot))$, contains S_1.

Exercise 11.26 *Prove*

Lemma 11.4.5 *Let $f(\cdot)$ be a submodular function on subsets of S and let $g(\cdot)$ be a positive weight function on S. Let $K \subseteq S$. We then have*

i. $(f * g)/\mathbf{K}(X) = (f/\mathbf{K} * g/\mathbf{K})(X) \quad \forall X \subseteq K$,

ii. if $f * g(S-K) = f(S-K)$, then $(f * g) \diamond \mathbf{K}(X) = (f \diamond \mathbf{K} * g \diamond \mathbf{K})(X) \quad \forall X \subseteq K$.

Let $f(\cdot)$ be a submodular function on the subsets of S and let $g(\cdot)$ be a positive weight function on S. Let (using the notation of Lemma 11.4.4) K minimize h_β and let P minimize h_θ, with $\beta \geq \theta$. Let $f_1(\cdot) \equiv (f \diamond (\mathbf{S}-\mathbf{K})/(\mathbf{P}-\mathbf{K}))(\cdot)$ and let $g_1(\cdot) \equiv (g/(\mathbf{P}-\mathbf{K}))(\cdot)$.

We now describe

i. the principal partition of $(f/\mathbf{K}(\cdot), g/\mathbf{K}(\cdot))$,

ii. the principal partition of $((f \diamond (\mathbf{S}-\mathbf{K}))(\cdot), (g/(\mathbf{S}-\mathbf{K}))(\cdot))$,

iii. the principal partition of $(f_1(\cdot), g_1(\cdot))$.

i. Let $\mathcal{B}'_\lambda$ denote $\mathcal{B}_{\lambda f/K, g/K}$, and let $\mathcal{B}_\lambda$ denote $\mathcal{B}_{\lambda f,g}$ as before. We have, $h_{\lambda K}(X) \equiv \lambda f/\mathbf{K}(X) + g(K - X) \quad \forall X \subseteq K$. So to determine the principal partition of $(f/\mathbf{K}(\cdot), g/\mathbf{K}(\cdot))$, we need only determine the subsets of K that minimize $h_{\lambda K}(\cdot)$. Now $h_{\lambda K}(X) = h_\lambda(X) - g(S - K) \quad \forall X \subseteq K$. Since K itself minimizes $h_\beta(\cdot)$, the sets that minimize $h_{\beta K}(\cdot)$ are precisely those that minimize $h_\beta(\cdot)$ and are contained in K. If $\lambda > \beta$ we know that $X^\lambda \subseteq K$ by Property PP2 of $(f(\cdot), g(\cdot))$. Hence the sets that minimize $h_{\lambda K}(\cdot)$ are the same as those that minimize $h_\lambda(\cdot)$. If, however $\lambda < \beta$, because K is in $\mathcal{B}'_\beta$ by Property PP2 of $(f/\mathbf{K}(\cdot), g/\mathbf{K}(\cdot))$, the minimal set that minimizes $h_{\lambda K}(X)$ must contain K, i.e., be equal to it. To summarize,

- $(\lambda = \beta)$. In this case $\mathcal{B}'_\lambda = \{$members of $\mathcal{B}_\lambda$ contained in $K\}$.
- $(\lambda > \beta)$. In this case $\mathcal{B}'_\lambda = \mathcal{B}_\lambda$
- $(\lambda < \beta)$. In this case $\mathcal{B}'_\lambda = \{K\}$.

ii. We are given that K is contained in $\mathcal{B}_\beta$ associated with the principal partition of $(f(\cdot), g(\cdot))$. Let us denote by $\mathcal{B}"_\lambda$ the collection of minimizing sets for λ in the principal partition of $((f \diamond (\mathbf{S} - \mathbf{K}))(\cdot), (g/(\mathbf{S} - \mathbf{K}))(\cdot))$. Now by Lemma 11.4.4, Y minimizes $h_{S-K,\lambda}(\cdot)$ iff among all supersets of K, $K \cup Y$ minimizes $h_\lambda(\cdot)$. Since K belongs to $\mathcal{B}_\beta$, when $\lambda = \beta$, these supersets are precisely those sets in $\mathcal{B}_\beta$ that contain K. Thus $\mathcal{B}"_\beta = \{Z - K, Z \in \mathcal{B}_\beta\}$. By Property PP2 of $(f(\cdot), g(\cdot))$ every set that minimizes $h_\lambda(\cdot)$, when $\lambda < \beta$, contains all sets in $\mathcal{B}_\beta$. So in this case the desired supersets are all the members of $\mathcal{B}_\lambda$, i.e., $\mathcal{B}"_\lambda = \{Z - K, Z \in \mathcal{B}_\lambda\}$. Again by Property PP2 of $((f \diamond (\mathbf{S} - \mathbf{K}))(\cdot), (g/(\mathbf{S} - \mathbf{K}))(\cdot))$, when $\lambda > \beta$, every set that is in $\mathcal{B}"_\lambda$ is contained in all sets in $\mathcal{B}"_\beta$. But the latter has $\emptyset$ as a member. To summarize

- $(\lambda = \beta)$. In this case $\mathcal{B}"_\lambda = \{Z - K, Z \in \mathcal{B}_\beta, Z \supseteq K\} = \{Z - K, Z \in \mathcal{B}_\beta\}$.
- $(\lambda < \beta)$. In this case $\mathcal{B}"_\lambda = \{Z - K, Z \in \mathcal{B}_\lambda\}$.
- $(\lambda > \beta)$. In this case $\mathcal{B}"_\lambda = \{\emptyset\}$.

iii. Observe that when $\theta < \beta$, P is a superset of K, and when $\theta = \beta$, $P \cup K$ is also a set in $\mathcal{B}_\beta$. So, without loss of generality, we need only consider the situation where $P \supseteq K$. Now by applying the ideas developed in the previous sections of the present problem we see that the principal partition of $(f_1(\cdot), g_1(\cdot))$ can be described as follows (see Figure 11.2):

- the critical values are those of the principal partition of $(f(\cdot), g(\cdot))$ that lie in the range β to θ, including both numbers,
- the minimizing sets corresponding to these critical values in the principal partition of $(f_1(\cdot), g_1(\cdot))$ are precisely those sets $(X \cap P) - K$, where X is a minimizing set corresponding to these critical values in the principal partition of $(f(\cdot), g(\cdot))$. Thus, if $\mathcal{B}^1_\lambda$ denotes $\mathcal{B}_{\lambda f_1, g_1}$ we have, when $\theta < \beta$

Figure 11.2: Comparison of PP of $(f(\cdot), g(\cdot)), (f_1(\cdot), g_1(\cdot))$

i. $(\beta \geq \lambda > \theta)$. $\mathcal{B}^1_\lambda = \{X - K, X \in \mathcal{B}_\lambda\}$

ii. $(\lambda = \theta)$. $\mathcal{B}^1_\lambda = \{(X \cap P) - K, X \in \mathcal{B}_\lambda\}$.

Further, when $\theta = \beta$, $\mathcal{B}^1_\lambda = \{(X \cap P) - K, X \in \mathcal{B}_\lambda\}$. Observe that in this case $P - K$ is molecular with respect to $(f_1(\cdot), g_1(\cdot))$. It would be atomic if no set lies strictly between P and $P \cap K$ in $\mathcal{B}_\beta (= \mathcal{B}_\theta)$, i.e., if $P - K$ is a block in $\Pi(\beta)$.

To describe the same situation in terms of the partition-partial order pair, denoting the partition for $(f_1(\cdot), g_1(\cdot))$ by Π^1 and the partial order by $\geq^1$, we have

- $\Pi^1 = \{Z, Z \in \Pi_{pp}, Z \subseteq P - K\}$
- a block in Π^1 corresponds to a critical value λ in the PP of $(f_1(\cdot), g_1(\cdot))$ iff it corresponds to λ in the PP of $(f(\cdot), g(\cdot))$
- the partial order $\geq^1$ is the restriction of the partial order of $(f(\cdot), g(\cdot))$ to Π^1.

The next theorem is a useful characterization of the principal partition using the foregoing ideas.

Theorem 11.4.6 (Uniqueness Theorem) *Let $f(\cdot)$ be a submodular function on subsets of S and let $g(\cdot)$ be a positive weight function on S. Let $\{S_1, \cdots, S_t\}$ be a partition of S. Let $S_1 \cup \cdots \cup S_k$ be denoted by E_k for $k = 1, \cdots, t$. Let*

$(f/\mathbf{E}_k \diamond (\mathbf{E}_k - \mathbf{E}_{k-1}))(\cdot), (g/(\mathbf{E}_k - \mathbf{E}_{k-1}))(\cdot)$ be denoted by $f_k(\cdot), g_k(\cdot)$ respectively for $k = 1, \cdots, t$, and let the collection of minimizing sets corresponding to λ in the principal partition of $(f_k(\cdot), g_k(\cdot))$ for $k = 1, \cdots, t$, by $\mathcal{B}^k_\lambda$.
Let S_k, for $k = 1, \cdots, t$, be molecular with respect to $(f_k(\cdot), g_k(\cdot))$ with critical value λ_k and let $\lambda_1 > \cdots > \lambda_t$. Then, for $(f(\cdot), g(\cdot))$,

i. the decreasing sequence of critical values is $\lambda_1, \cdots, \lambda_t$ and $\emptyset, E_1, \cdots, E_t$ is the principal sequence,

ii.

$$\mathcal{B}_{\lambda_k} = \{Z, Z = E_{k-1} \cup Y, Y \in \mathcal{B}^k_{\lambda_k}\}, k = 1, \cdots, t,$$

where $E_0 \equiv \emptyset$.

Proof:
We prove the theorem by induction on t. The result is obviously true for $t = 1$. Let it to be true for $t = n-1$. Let $f'(\cdot), g'(\cdot)$, denote $(f \diamond (\mathbf{S} - \mathbf{E}_1))(\cdot), (g/(\mathbf{S} - \mathbf{E}_1))(\cdot)$, respectively and let $\mathcal{B}'_\lambda$ denote the minimizing sets corresponding to λ in the principal partition of $(f'(\cdot), g'(\cdot))$. By the use of Theorem 10.3.1 we know that $f_k(\cdot) = (f'/(\mathbf{E}_k - \mathbf{E}_1) \diamond (\mathbf{E}_k - \mathbf{E}_{k-1}))(\cdot)$ for $k = 2, \cdots, t$ and $g_k(\cdot) = (g'/(\mathbf{E}_k - \mathbf{E}_{k-1}))(\cdot)$, for $k = 2, \cdots, t$.

By the induction assumption it follows that for $(f'(\cdot), g'(\cdot))$,

i. the decreasing sequence of critical values is $\lambda_2, \cdots, \lambda_t$ and $\emptyset, E_2 - E_1, \cdots, E_t - E_1$ is the principal sequence.

ii.

$$\mathcal{B}'_{\lambda_k} = \{Z, Z = (E_{k-1} - E_1) \cup Y, Y \in \mathcal{B}^k_{\lambda_k}\}, k = 2, \cdots, t.$$

We will use Lemma 11.4.4 and the notation adopted therein. Since by Property PP2 of $(f'(\cdot), g'(\cdot))$, $\emptyset$ is the only set that minimizes $h_{S-E_1,\lambda_1}(\cdot)$, we must have that $h_{\lambda_1}(\cdot)$ takes strictly lower value on E_1 than on all its proper supersets. But $E_1 \in \mathcal{B}^1_{\lambda_1}$. So by the lemma, E_1 is a subset of some set in $\mathcal{B}_{\lambda_1}$. We conclude that E_1 is the maximal set that minimizes $h_{\lambda_1}(\cdot)$. Now if $X \in \mathcal{B}^1_{\lambda_1}$, the value of $h_{\lambda_1 E_1}(\cdot)$ is the same on both X as well as E_1. Hence the value of $h_{\lambda_1}(\cdot)$ is the same on both these sets, i.e., $\mathcal{B}^1_{\lambda_1} = \mathcal{B}_{\lambda_1}$. Since $\emptyset$ is in $\mathcal{B}^1_{\lambda_1} (= \mathcal{B}_{\lambda_1})$, λ_1 must be the highest critical value of $(f(\cdot), g(\cdot))$.

When $\lambda < \lambda_1$, E_1 is the only set in $\mathcal{B}^1_\lambda$ and E_1 is contained in every set in $\mathcal{B}_\lambda$, by PropertyPP2 of $(f_1(\cdot), g_1(\cdot))$ and $(f(\cdot), g(\cdot))$. Once again by Lemma 11.4.4, $Y \cup E_1$ minimizes $h_\lambda(\cdot)$ iff Y minimizes $h_{S-E_1,\lambda}(\cdot)$. Thus $Y \cup E_1$ is in $\mathcal{B}_\lambda$ iff Y is in $\mathcal{B}'_\lambda$. We now see that $\lambda_1, \cdots, \lambda_t$ is a strictly decreasing sequence of numbers such that each of the $\mathcal{B}_{\lambda_i}$ has atleast two members, $\mathcal{B}_{\lambda_1}$ has $\emptyset$ as a member, $\mathcal{B}_{\lambda_t}$ has S as a member and further the maximal member of $\mathcal{B}_{\lambda_i}$ is the minimal member of $\mathcal{B}_{\lambda_{i+1}}$,$i = 1, \cdots, t-1$. Hence, by Theorem 11.4.1, we conclude that $\lambda_1, \cdots \lambda_t$ is the decreasing sequence of all critical values for $(f(\cdot), g(\cdot))$ and the $\mathcal{B}_{\lambda_i}$ together constitute the principal partition of $(f(\cdot), g(\cdot))$. Thus the proof is complete for $t = n$.

□

Problem 11.1 *([Fujishige80a], [Fujishige91]) Let* $f(\cdot)$ *be a submodular function on subsets of* $S \equiv \{e_1, \cdots, e_n\}$ *and let* $g(\cdot)$ *be a positive weight function on* S*. Let* $\emptyset = E_0, E_1, \cdots, E_k \equiv S$ *be the principal sequence and let* $\lambda_1, \cdots, \lambda_k$*, be the decreasing sequence of critical values of* $(f(\cdot), g(\cdot))$*. Let* $\mathbf{x}$ *be a vector defined by*

$$x(e_i) = g(e_i)/\lambda_j, e_i \in (E_j - E_{j-1}), j = 1, \cdots, k.$$

Show that

i. $\mathbf{x}(E_i) = f(E_i), i = 1, \cdots, k.$

ii. $\mathbf{x}$ *is a base of* P_f *(i.e.,* $\mathbf{x}(X) \leq f(X)\ \forall X \subseteq S$ and $\mathbf{x}(S) = f(S)$*).*

iii. $\mathbf{x}$ *is a* **F-lexicographically optimum base** *of* P_f *relative to* $g(\cdot)$*, i.e., if* $x(e_1)/g(e_1) \leq \cdots \leq x(e_n)/g(e_n)$*, whenever* $\mathbf{x}'$ *is a base of* P_f *with* $x'(e'_1)/g(e'_1) \leq \cdots \leq x'(e'_n)/g(e'_n)$ *and* t *is the first index for which* $x(e_t)/g(e_t) \neq x'(e'_t)/g(e'_t)$*, then* $x(e_t)/g(e_t) > x'(e'_t)/g(e'_t)$*.*

iv. the F-lexicographically optimum base is unique.

Remark: *The above definition of F-lexicographically optimum base is inconsistent (even as a generalization) with the one given in Section 4.6 for lexicographically optimum base of a matroid.*

Problem 11.2 *[Tomizawa+Fujishige82], [Fujishige91] Let* $f(\cdot), g(\cdot)$ *be submodular functions on subsets of* S *and further let* $g(\cdot)$ *be an increasing function. This problem describes the structure of the principal partition in this case.*

i. Let $\sigma \geq 0$*.*
$\mathcal{B}_{f,\sigma g}$ *is a distributive lattice. Hence, it has a unique maximal element* Y^σ *and a unique minimal element* Y_σ*.*

ii. If $0 \leq \sigma_1 < \sigma_2$*, then* $Y^{\sigma_1} \subseteq Y^{\sigma_2}$ *and* $Y_{\sigma_1} \subseteq Y_{\sigma_2}$*.*

iii. Let $g^d(X) \equiv g(S) - g(S - X)$ *and let* $\sigma < 0$*. Then* $\mathcal{B}_{f,\sigma g^d}$ *is a distributive lattice. Hence it has a unique maximal and a unique minimal element.*

iv. Let $\mathcal{B}_{\cdot\sigma}$ *refer to* $\mathcal{B}_{f,\sigma g}, \sigma \geq 0$ *and to* $\mathcal{B}_{f,\sigma g^d}, \sigma < 0$*. If* $X_1 \in \mathcal{B}_{\cdot\sigma_1}$ *and* $X_2 \in \mathcal{B}_{\cdot\sigma_2}, \sigma_1 < \sigma_2$*, then* $X_1 \cup X_2 \in B_{\cdot\sigma_2}$*,* $X_1 \cap X_2 \in B_{\cdot\sigma_1}$*. Hence,* $Y_{\sigma_1} \subseteq Y_{\sigma_2}$ *and* $Y^{\sigma_1} \subseteq Y^{\sigma_2}$*.*

11.5 *The Refined Partial Order of the Principal Partition

We spoke earlier about the partial order $(\geq_\pi)$ associated with $(f(\cdot), g(\cdot))$ ($f(\cdot)$ submodular, $g(\cdot)$, a positive weight function). The elements of this partial order were the blocks of Π_{pp} (maximal sets which are not cut by any of the members of the $\mathcal{B}_\lambda$). The ideals of the partial order correspond to members of all the $\mathcal{B}_\lambda$, i.e., correspond to the 'solutions' of $min_{X\subseteq S}\lambda f(X) + g(S-X)$ for all the λs. Suppose $\emptyset, E_1, \cdots, E_t$ is the principal sequence of $(f(\cdot), g(\cdot))$, with $E_1 = X^{\lambda_1}, \cdots, S = E_t = X^{\lambda_t}$. The partial order relationship between blocks that lie entirely within $E_{j+1} - E_j$ is determined by the family $\mathcal{B}_{\lambda_{j+1}}$ while if A is a block within $E_{j+1} - E_j$ and B a block within $E_{r+1} - E_r$ $(r > j)$ we take $A \geq B$. It follows by the Uniqueness Theorem (Theorem 11.4.6) that the principal partition and partial order would be unchanged even if we replace $f(\cdot)$ by $\oplus_k f_k(\cdot)$, where $f_k(\cdot)$ is as defined in the Uniqueness Theorem. Thus, the relationship (imposed by $f(\cdot)$) between the blocks corresponding to different critical values is not brought out by the partial order $(\geq_\pi)$.

Additional structure related to the principal partition is revealed if we refine the partial order (but use the same partition Π_{pp}) as described below. This new partial order $(\geq_R)$ contains all the ideals of $(\geq_\pi)$ and some more. It, therefore, has ideals which are not solutions of $min_{X\subseteq S}\lambda f(X) + g(S-X)$. But, as we will see, if X_1, X_2 are ideals in $(\geq_R)$ the principal partition of $f/\mathbf{X}_1 \diamond (\mathbf{X}_1 - \mathbf{X}_2)(\cdot)$ relative to $g/(\mathbf{X}_1 - \mathbf{X}_2)(\cdot)$ is easy to describe. As in the case of $(\geq_\pi)$ these ideals also are modular with respect to $f(\cdot)$. Further, the ideals behave well with respect to addition, dualization and convolution of functions.

Throughout this section we assume $f(\cdot)$ to be submodular and $g(\cdot)$ to be a positive weight function.

We now informally describe the construction of this refined partial order.

On the collection of blocks of Π_{pp} contained in $X^{\lambda_1} = X_{\lambda_2}$, both the partial orders $(\geq_\pi)$ and $(\geq_R)$ coincide. Suppose we have already built the partial order $(\geq_R)$ on the blocks contained in X_{λ_k}. We extend it to blocks in $X_{\lambda_{k+1}}$ as follows. Let X be a member of $\mathcal{B}_{\lambda_k}$. Then $X - X_{\lambda_k}$ is molecular with respect to $((f/\mathbf{X} \diamond (\mathbf{X} - \mathbf{X}_{\lambda_k}))(\cdot), (g/(\mathbf{X} - \mathbf{X}_{\lambda_k}))(\cdot))$. To reach this structure we restricted $f(\cdot)$ to X and contracted out X_{λ_k}. However, it may be possible to achieve this structure by restricting on a smaller set $Y \uplus (X - X_{\lambda_k})$, where $Y \subseteq X_{\lambda_k}$, and contracting out Y. But we would insist that Y be an ideal of the restriction of $(\geq_R)$ (already defined) on blocks of Π_{pp} contained in X_{λ_k}. It turns out that there is a unique smallest ideal of $(\geq_R)$ (restricted to the collection of blocks contained in X_{λ_k}) with the above property. We would take all the blocks contained in Y to be below the maximal blocks contained in X. (For technical reasons we insist on simpler but equivalent conditions which are brought out in the following exercise). For blocks of Π_{pp} within $X_{\lambda_{k+1}} - X_{\lambda_k}$, we retain the same relationship as in $(\geq_\pi)$.

Exercise 11.27
(k) Let $f(\cdot)$ be submodular on subsets of S. Let $T_1 \supseteq T_2 \supseteq T_3$.
If $f(T_2) - f(T_2 - T_3) = f(T_1) - f(T_1 - T_3)$,
then $(f/\mathbf{T}_1 \diamond \mathbf{T}_3)(X) = (f/\mathbf{T}_2 \diamond \mathbf{T}_3)(X), X \subseteq T_3$.

In the next couple of pages we give an inductive definition of the refined partial order $(\geq_R)$ associated with the principal partition of $(f(\cdot), g(\cdot))$ ($f(\cdot)$ submodular, $g(\cdot)$, a positive weight function). Before we can do so, however, we need a few preliminary definitions and a lemma.

Let $(\geq')$ be a partial order on the blocks of Π_{pp} contained in E_k. We say $(\geq')$ is a **modular refinement** of $(\geq_\pi)$ on the blocks of Π_{pp} contained in E_k iff

i. $A \geq' B \Rightarrow A \geq_\pi B$, and

ii. if Y_1, Y_2 are ideals of $(\geq')$ then

$$f(Y_1) + f(Y_2) = f(Y_1 \cup Y_2) + f(Y_1 \cap Y_2).$$

(Note that the second condition above is satisfied by $(\geq_\pi)$ (Exercise 11.13)).

Let $(\geq_k)$ be a modular refinement of $(\geq_\pi)$ over the blocks of Π_{pp} contained in $E_k (= X^{\lambda_k})$. Let X be a member of $\mathcal{B}_{\lambda_{k+1}}$. We say that $X - E_k$ is **contraction related** to a subset Y of E_k that is also a union of the blocks in an ideal of $(\geq_k)$ iff $f((X - E_k) \cup Y) - f(Y) = f(X \cup E_k) - f(E_k)$.

We then have the following lemma. (Henceforth we abuse the notation and say 'Y is an ideal of $(\geq)$' instead of 'Y is a union of blocks of Π_{pp} in an ideal of $(\geq)$'.)

Lemma 11.5.1 *Let $(\geq_k)$ be a modular refinement of $(\geq_\pi)$ over the blocks of Π_{pp} contained in E_k. Let X_1, X_2, X_3 be members of $\mathcal{B}_{\lambda_{k+1}}$ s.t. $X_1 \supseteq X_2$ and let $Y_1, Y_2, Y_3 \subseteq E_k$ be ideals of $(\geq_k)$ such that $Y_3 \supseteq Y_1$ and $X_1 - E_k, X_3 - E_k$ are contraction related to Y_1, Y_2.*
Then

i. $X_2 - E_k$ is contraction related to Y_1,

ii. $X_1 - E_k$ is contraction related to Y_3,

iii. $X_1 - E_k$ is contraction related to $Y_1 \cap Y_2$,

iv. $X_1 \cup X_3 - E_k$ is contraction related to Y_1.

Proof : We use the following notation:
For each set Z, $\hat{Z} \equiv Z - E_k$.
i. We have

$$f(\hat{X}_1 \cup Y_1) - f(Y_1) = f(\hat{X}_1 \cup E_k) - f(E_k).$$

Suppose

$$f(\hat{X}_2 \cup Y_1) - f(Y_1) > f(\hat{X}_2 \cup E_k) - f(E_k).$$

(Note that by submodularity, LHS $\geq$ RHS).
We then have

$$f(\hat{X}_1 \cup Y_1) - f(\hat{X}_2 \cup Y_1) < f(\hat{X}_1 \cup E_k) - f(\hat{X}_2 \cup E_k),$$

which contradicts the submodularity of $f(\cdot)$. We conclude that $\hat{X}_2$ is contraction related to Y_1.

ii. Suppose

$$f(\hat{X}_1 \cup Y_3) - f(Y_3) > f(\hat{X}_1 \cup E_k) - f(E_k).$$

We then have, using the fact that $\hat{X}_1$ is contraction related to Y_1,

$$f(\hat{X}_1 \cup Y_3) - f(\hat{X}_1 \cup Y_1) > f(Y_3) - f(Y_1),$$

which contradicts the submodularity of $f(\cdot)$. Thus, $\hat{X}_1$ is contraction related to Y_3.

iii. Since, over the blocks of Π_{pp} contained in E_k, $(\geq_k)$ is a modular refinement of $(\geq_\pi)$ and Y_1, Y_2 are ideals of $(\geq_k)$ we have

$$f(Y_1) + f(Y_2) = f(Y_1 \cup Y_2) + f(Y_1 \cap Y_2).$$

On the other hand

$$f(\hat{X}_1 \cup Y_1) + f(\hat{X}_1 \cup Y_2) \geq f(\hat{X}_1 \cup (Y_1 \cup Y_2)) + f(\hat{X}_1 \cup (Y_1 \cap Y_2)).$$

Hence, $(f(\hat{X}_1 \cup Y_1) - f(Y_1)) + (f(\hat{X}_1 \cup Y_2)) - f(Y_2))$

$$\begin{aligned} \geq \quad & (f(\hat{X}_1 \cup (Y_1 \cup Y_2)) - f(Y_1 \cup Y_2)) \\ & +(f(X_1 \cup (Y_1 \cap Y_2)) - f(Y_1 \cap Y_2)). \end{aligned}$$

In this inequality, each of the two terms on the LHS is equal to the first term on the RHS, all being equal to $(f(X_1 \cup E_k) - f(E_k))$, by the second part of the present lemma.
Since by submodularity of $f(\cdot)$,

$$f(\hat{X}_1 \cup Y_1) - f(Y_1) \leq f(\hat{X}_1 \cup (Y_1 \cap Y_2)) - f(Y_1 \cap Y_2),$$

we conclude that

$$f(\hat{X}_1 \cup (Y_1 \cap Y_2)) - f(Y_1 \cap Y_2) = f(\hat{X}_1 \cup Y_1) - f(Y_1) = f(\hat{X}_1 \cup E_k) - f(E_k).$$

iv. We have

$$f(\hat{X}_1 \cup Y_1) - f(Y_1) = f(\hat{X}_1 \cup E_k) - f(E_k) \qquad (*)$$

$$f(\hat{X}_3 \cup Y_1) - f(Y_1) = f(\hat{X}_3 \cup E_k) - f(E_k). \qquad (**)$$

Now $\hat{X}_1 \cup E_k, \hat{X}_3 \cup E_k$ are sets in $\mathcal{B}_{\lambda_{k+1}}$. Hence,

$$f(\hat{X}_1 \cup E_k) + f(\hat{X}_3 \cup E_k) = f(\hat{X}_1 \cup \hat{X}_3 \cup E_k) + f((\hat{X}_1 \cap \hat{X}_3) \cup E_k) \qquad (***)$$

By submodularity of $f(\cdot)$, we have

$$f(\hat{X}_1 \cup Y_1) + f(\hat{X}_3 \cup Y_1) \geq f(\hat{X}_1 \cup \hat{X}_3 \cup Y_1) + f((\hat{X}_1 \cap \hat{X}_3) \cup Y_1).$$

It follows using $(*), (**), (***)$, that

$$(f(\hat{X}_1 \cup \hat{X}_3 \cup Y_1) - f(Y_1)) + (f((\hat{X}_1 \cap \hat{X}_3) \cup Y_1) - f(Y_1))$$
$$\leq (f(\hat{X}_1 \cup \hat{X}_3 \cup E_k) - f(E_k)) + (f((\hat{X}_1 \cap \hat{X}_3) \cup E_k) - f(E_k)).$$

But $f(\cdot)$ is submodular. So the above inequality must be satisfied as an equality with the first and second terms on the LHS being respectively equal to the first and second terms on the RHS. Thus,

$$f(\hat{X}_1 \cup \hat{X}_3 \cup Y_1) - f(Y_1) = f(\hat{X}_1 \cup \hat{X}_3 \cup E_k) - f(E_k).$$

□

Corollary 11.5.1 *Let X be a member of $\mathcal{B}_{\lambda_{k+1}}$. Then there is a unique minimal ideal Y of the partial order $(\geq_k)$ s.t. $X - E_k$ is contraction related to Y.*

Proof : This is an immediate consequence of the third part of Lemma 11.5.1.

□

If Y is the minimal ideal of $(\geq_k)$ such that $X - E_k$ is contraction related to it, we say $X - E_k$ is **properly related** to Y. From the above lemma it is clear that there is a unique such subset Y.

We are now ready to present the inductive definition of $(\geq_R)$.
To begin with $(\geq_1)$ is defined to agree with $(\geq_\pi)$ on the blocks of Π_{pp} contained in $E_1 \equiv X_{\lambda_2}$. Let $(\geq_k)$ be a modular refinement of $(\geq_\pi)$ on the blocks contained in $E_k \equiv X_{\lambda_{k+1}}$. We now extend the partial order $(\geq_k)$ to the partial order $(\geq_{k+1})$, which is a modular refinement of $(\geq_\pi)$ on the blocks of Π_{pp} contained in $E_{k+1} \equiv X_{\lambda_{k+2}}$, as follows.
Let A, B be blocks of Π_{pp}.

i. If A, B are contained in E_k then $A \geq_{k+1} B$ iff $A \geq_k B$.

ii. If A, B are contained in $E_{k+1} - E_k$ then $A \geq_{k+1} B$ iff $A \geq_\pi B$.

iii. If A is contained in $E_{k+1} - E_k$ and B is contained in E_k, then $A \geq_{k+1} B$ iff $X_A - E_k$ is properly related to $Y_B \supseteq B$, where X_A is the minimal member of $\mathcal{B}_{\lambda_{k+1}}$ containing A and Y_B is a union of blocks of an ideal of $(\geq_k)$ defined over blocks contained in E_k.

Lemma 11.5.1 assures us that the above definition does yield a partial order on blocks of Π_{pp} contained in E_{k+1} and further that this partial order is a refinement of $(\geq_\pi)$ over blocks of Π_{pp} within E_{k+1}. It can be verified that (see Exercise 11.28 below), if X_1, X_2 are unions of blocks in an ideal of $(\geq_{k+1})$ then

$$f(X_1) + f(X_2) = f(X_1 \cup X_2) + f(X_1 \cap X_2).$$

The above procedure, therefore, extends to a unique partial order $(\geq_R)(\equiv(\geq_t))$ that is also a modular refinement of $(\geq_\pi)$, on all the blocks of Π_{pp}. We refer to this partial order as the **refined partial order** associated with the principal partition of $(f(\cdot), g(\cdot))$ ($f(\cdot)$ submodular, $g(\cdot)$, a positive weight function).

Exercise 11.28 *(k) Verify that if X_1, X_2 are unions of blocks in an ideal of $(\geq_{k+1})$ then*

$$f(X_1) + f(X_2) = f(X_1 \cup X_2) + f(X_1 \cap X_2).$$

We now give a simple characterization of the partial order $(\geq_R)$ associated with $(f(\cdot), g(\cdot))$.

Theorem 11.5.1 *Let $\lambda_1, \cdots, \lambda_t$ be the decreasing sequence of critical values of $(f(\cdot), g(\cdot))$ ($f(\cdot)$ submodular on subsets of S, $g(\cdot)$, a positive weight function on S). A set $X \subseteq S$ is an ideal of the refined partial order $(\geq_R)$ associated with $(f(\cdot), g(\cdot))$ iff it satisfies the following conditions:*

i. For each critical value $\lambda_j, (X \cap X_{\lambda_{j+1}}) \cup X_{\lambda_j}$ is a set in $\mathcal{B}_{\lambda_j}$

ii. Let $X \not\subseteq X_{\lambda_{j+1}}$. Then

$$f(X \cap X_{\lambda_{j+1}}) - f(X \cap X_{\lambda_j}) = f((X \cap X_{\lambda_{j+1}}) \cup X_{\lambda_j}) - f(X_{\lambda_j}).$$

Proof : We denote X^{λ_j} $(= X_{\lambda_{j+1}})$ by $E_j,\ j = 1, \cdots, t$.

only if

Let X be an ideal of $(\geq_R)$. Then $X \cap E_j$ is an ideal of $(\geq_R)$,since by definition E_j is an ideal of of $(\geq_\pi)$ and therefore of $(\geq_R)$ and the intersection of ideals yields another ideal. We observe that any two blocks of Π_{pp} that lie in $E_j - E_{j-1}$ have the same relationship in both $(\geq_\pi)$ and $(\geq_R)$. Let A be a block of Π_{pp} in $X \cap (E_j - E_{j-1})$. The blocks of Π_{pp} that are contained in $(E_j - E_{j-1})$ and that lie beneath A in $(\geq_R)$ must be identical to those beneath A in $(\geq_\pi)$.
The remaining blocks beneath A in $(\geq_\pi)$ are precisely those in E_{j-1}. Thus, $(X \cap (E_j - E_{j-1})) \cup E_{j-1}$ is an ideal of $(\geq_\pi)$ and therefore a member of $\mathcal{B}_{\lambda_j}$. This proves the first condition.
Next observe that E_j is an ideal of $(\geq_R)$. Hence $X \cap E_j$ is also an ideal of $(\geq_R)$. Since $X \cap E_j$ is an ideal of $(\geq_R)$, (by the inductive definition of $(\geq_R)$) it follows that each block A of Π_{pp} within $X \cap (E_j - E_{j-1})$ is contained in a minimal set X_A of E_j s.t. $X_A - E_{j-1}$ is properly related to an ideal Y of $(\geq_R)$ contained in $X \cap E_{j-1}$. Hence, by the first part of Lemma 11.5.1 it is clear that $(X_A - E_{j-1})$ is contraction related to $X \cap E_{j-1}$. The union of all such sets $(X_A - E_{j-1})$ is clearly equal to $X \cap (E_{j-1} - E_j)$. By the fourth part of Lemma 11.5.1 the latter set must be contraction related to $X \cap E_{j-1}$. This proves the second condition.

if

Conversely, suppose X satisfies conditions (i) and (ii). We will show that X is an ideal of $(\geq_R)$.

We proceed inductively. It is clear that $X \cap E_1$ is a set in $\mathcal{B}_{\lambda_1}$ and hence an ideal of $(\geq_\pi)$ as well as $(\geq_R)$. Suppose $X \cap E_{k-1}$ is an ideal of $(\geq_R)$. We are given that $(X \cap E_k) \cup E_{k-1}$ is a set in $\mathcal{B}_{\lambda_k}$ and also that $X \cap (E_k - E_{k-1})$ is contraction related to $X \cap E_{k-1}$. Hence, there exists an ideal $Y \subseteq X \cap E_{k-1}$ of $(\geq_R)$ s.t. $X \cap (E_k - E_{k-1})$ is properly related to Y. Now, if A is a block of Π_{pp} in $X \cap (E_k - E_{k-1})$, all blocks beneath A in $(\geq_R)$ which are in E_{k-1} are contained in Y. Further, since $(X \cap E_k) \cup E_{k-1}$ is an ideal of $(\geq_\pi)$, all blocks of Π_{pp} in $E_k - E_{k-1}$ which are beneath A in $(\geq_\pi)$ are contained in $X \cap (E_k - E_{k-1})$. We conclude that this is also true with respect to $(\geq_R)$ since the relationship between such blocks is identical both in $(\geq_\pi)$ and $(\geq_R)$. From the inductive construction of $(\geq_R)$ it is clear that $(X \cap (E_k - E_{k-1})) \cup Y$ is an ideal of $(\geq_R)$. Hence, $((X \cap (E_k - E_{k-1})) \cup Y) \cup (X \cap E_{k-1})$ is an ideal of $(\geq_R)$, i.e., $X \cap E_k$ is an ideal of $(\geq_R)$. This completes the proof.

□

Exercise 11.29 *(k) A* **base** *of a polymatroid rank function $f : 2^S \rightarrow \mathcal{R}$ is a vector* $\mathbf{x}$ *on S s.t.*

$$\mathbf{x}(X) \leq f(X) \quad \forall X \subseteq S,$$

$$\textit{and} \qquad \mathbf{x}(S) = f(S).$$

We say a base is consistent with a preorder $\succeq$ (and with the induced partial order on the equivalence classes of the preorder) iff $\mathbf{x}(X) = f(X)$, whenever X is an ideal of $\succeq$. Let $(\geq_\pi)$ be the partial order and $(\geq_R)$ be the refined partial order associated with the principal partition $(f(\cdot), g(\cdot))$ ($g(\cdot)$, a positive weight function). Let $\mathbf{x}$ be a base of $f(\cdot)$ consistent with $(\geq_\pi)$. Show that it is consistent with $(\geq_R)$.

Exercise 11.30 *Let $(\geq_R)$ be the refined partial order and let Π_{pp} be the partition of S associated with the principal partition of $(f(\cdot), g(\cdot))$ where $f(\cdot)$ is submodular on subsets of S and $g(\cdot)$ is a positive weight function on S. Let X be the union of blocks of Π_{pp} in an ideal of $(\geq_R)$. Describe the principal partition of $(f/\mathbf{X})(\cdot)$ and the partial orders associated with it.*

Exercise 11.31 *(k) Let $f(\cdot)$ be a submodular function on subsets of S and let $g(\cdot)$ be a positive weight function on S. Let $f^*(\cdot)$ be the dual of $f(\cdot)$ with respect to $g(\cdot)$. Then the refined partial order of $(f^*(\cdot), g(\cdot))$ is dual to the refined partial order of $(f(\cdot), g(\cdot))$.*

11.6 Algorithms for PP

11.6.1 Basic Algorithms

In this subsection we give a collection of algorithms which together would construct the principal partition of $(f(\cdot), g(\cdot))$, where $f(\cdot)$ is submodular and $g(\cdot)$ is a positive weight function. The submodular functions may be available in various ways. One common way is through a 'rank oracle' which when presented with a subset would return the value of the function on it. We assume that we have available an algorithm called $Convolve_K\ (f_1, f_2)$ which, given submodular functions $f_1(\cdot), f_2(\cdot)$ on subsets of K, would output the unique minimal and maximal sets (Minset(convolve) and Maxset(convolve) respectively) which minimize $f_1(X) + f_2(K - X)\ \ \forall X \subseteq K$. In general such an algorithm involves minimization of a submodular function. As we have remarked before, although polynomially solvable, we do not yet have truly practical algorithms for this problem. However, for the instances that are our primary concern in this book we do have very good algorithms.

Informally, Algorithm 11.1 proceeds as follows. We start with the set interval $(\emptyset, S)$. The subroutine, given below, breaks up the set interval $(\emptyset, S)$ into $(\emptyset, Z)$ and (Z, S), where Z minimizes the expression $\lambda f(X) + g(S - X), \lambda \equiv \frac{g(S)}{f(S)-f(\emptyset)}$ If for every set X between the endsets, the value of $\lambda f(X) + g(S - X)$, does not exceed its value at the endsets $(= g(S) + \lambda f(\emptyset))$, then we are done - the principal sequence is $\emptyset, S$ and the critical value is $\frac{g(S)}{f(S)-f(\emptyset)}$. Otherwise we find the minimal set, say T, that minimizes the above expression. Now we work with the intervals $(\emptyset, T), (T, S)$ and look for minimizing sets within the interval in question. In each case we use a value of λ for which $\lambda f(X) + g(T' - X)$, where T' is the right end of the interval, reaches the same value at both ends of the interval. When we are unable to subdivide the intervals any further we get a sequence of sets and a sequence of values which, the Uniqueness Theorem (Theorem 11.4.6) assures us, are respectively the principal sequence and the sequence of critical values of $(f(\cdot), g(\cdot))$.

$Subdivide_{f,g}(A, B)$

INPUT A submodular function $f(\cdot)$ and a positive weight function $g(\cdot)$ on subsets of S. Sets A, B s.t. $\emptyset \subseteq A \subseteq B \subseteq S$.

OUTPUT The unique minimal minimizing set (Minset) $A \uplus Z$ for $\lambda f(X) + g(B - X), A \subseteq X \subseteq B$, where $\lambda = \frac{g(B)-g(A)}{f(B)-f(A)}$.

STEP 1 $\lambda \leftarrow \frac{g(B)-g(A)}{f(B)-f(A)}$,
Let $f'(Y) \equiv f(A \uplus Y) - f(A)$,
$Convolve_{(B-A)}\ (\lambda f', g)$
Let Z be the Minset(convolve) of the output.
Output $A \uplus Z$ as the Minset.

STOP

ALGORITHM 11.1 Algorithm P Sequence

INPUT *A submodular function* $f(\cdot)$ *and a positive weight function* $g(\cdot)$ *on subsets of* S.

OUTPUT *The principal sequence of* $(f(\cdot), g(\cdot))$.

Initialize *Current Set Sequence* $\leftarrow (\emptyset, S)$
$\lambda_1 \leftarrow \frac{g(S)}{f(S)-f(\emptyset)}$
Current λ *Sequence* $\leftarrow (\lambda_1)$
$j \leftarrow 0$
$\emptyset$ *is unmarked.*

STEP 1 *Let Current Set Sequence be* $(S_1^j, \cdots, S_{r_j}^j)$ *and let Current* λ *Sequence be* $(\lambda_1^j, \cdots, \lambda_{r_j-1}^j)$.
If S_t^j $1 \leq t \leq r_j - 1$ *is unmarked,*
  then *Subdivide*$_{f,g}$ (S_t^j, S_{t+1}^j)
Else *GOTO STEP 3.*

STEP 2 *Let*

$$(S_1^j, \cdots, S_{r_j}^j) \equiv (T_1, \cdots, T_q)$$

$$(\lambda_1^j, \cdots, \lambda_{r_j-1}^j) \equiv (\lambda_1, \cdots, \lambda_{q-1})$$

Let T *be the Minset output by Subdivide*$_{f,g}$ (S_t^j, S_{t+1}^j).

If $T = S_t^j$,
  then
  $j \leftarrow j + 1$
  $r_j \leftarrow q$
  $(S_1^j, \cdots, S_{r_j}^j) \leftarrow (T_1, \cdots T_q)$
  $(\lambda_1^j, \cdots, \lambda_{r_j-1}^j) \leftarrow (\lambda_1, \cdots \lambda_{q-1})$.
  mark S_t^j
  GOTO STEP 1;

Else $(T \neq S_t^j)$
  $j \leftarrow j + 1$
  $r_j \leftarrow q + 1$
  $S_i^j \leftarrow T_i, i \leq t$
  $S_{t+1}^j \leftarrow T$
  $S_{i+1}^j \leftarrow T_i, t < i < r_j$
  $\lambda_i^j \leftarrow \lambda_i, i < t$
  $\lambda_t^j \leftarrow \lambda' = \frac{g(T)-g(T_t)}{f(T)-f(T_t)}$,

$\lambda^j_{t+1} \leftarrow \lambda" = \frac{g(T_{t+1})-g(T)}{f(T_{t+1})-f(T)}$,

$\lambda^j_{i+1} \leftarrow \lambda_i, t < i < r_j - 1$

The Current Set Sequence

$(S^j_1, \cdots, S^j_t, S^j_{t+1}, S^j_{t+2}, \cdots, S^j_{r_j}) \leftarrow (T_1, \cdots, T_t, T, T_{t+1}, \cdots, T_q)$.

The Current λ Sequence

$(\lambda^j_1, \cdots, \lambda^j_{t-1}, \lambda^j_t, \lambda^j_{t+1}, \lambda^j_{t+2}, \cdots, \lambda^j_{r_j-1}) \leftarrow (\lambda_1, \cdots, \lambda_{t-1}, \lambda', \lambda", \lambda_{t+1}, \cdots, \lambda_q)$

GOTO STEP 1.

STEP 3 *Output Current Set Sequence as the Principal Sequence and Current λ Sequence as the Critical Value Sequence.*

STOP

Next we consider the problem of construction of $\mathcal{B}_{\lambda f,g}$. Since the number of sets in this family is very large we would try to get a representation of it through a partial order whose ideals correspond to the members of the family. This is possible since $\mathcal{B}_{\lambda f,g}$ is closed under union and intersection (see page 449). We assume we have available an algorithm (easy to build) that, for a preorder $\succeq$ on S, if given the collection $\mathcal{F} \equiv \{(e, T_e), e \in S\}$, $T_e \equiv \{e_j, e_j \succeq e\}$, produces the Hasse Diagram of the induced partial order. We will call this algorithm **Hasse Diagram** ($\mathcal{F}$).

ALGORITHM 11.2 Algorithm $\mathcal{B}_{\lambda f,g}$

INPUT $S \equiv \{e_1, \cdots, e_n\}$. *Submodular function $f(\cdot)$ and positive weight function $g(\cdot)$ on subsets of S, $\lambda \geq 0$.*

OUTPUT *A preorder $\succeq_\lambda$ whose ideals are precisely the members of $\mathcal{B}_{\lambda f,g}$. The preorder is specified through the Hasse diagram of the induced partial order.*

STEP 1 *$Convolve_S(\lambda f, g)$*
Let Z be the Minset, Z' be the Maxset, $X_\lambda \leftarrow Z$, $X^\lambda \leftarrow Z'$.

STEP 2 *For each j, $e_j \in Z' - Z$,*

let $f^j(\cdot) \equiv f/(\mathbf{S} - \mathbf{e_j})(\cdot), g^j(\cdot) \equiv g/(\mathbf{S} - \mathbf{e_j})(\cdot)$.

$Convolve_{(S-e_j)}(\lambda f^j, g^j)$

Let Z^j be the Maxset.

$\{e, e \succeq_\lambda e_j,\} \equiv Z' - Z^j$.

STEP 3 *Let $\mathcal{F} \equiv \{(e_j, Y_j)$, $Y_j \equiv Z' - Z^j$ if $e_j \in Z' - Z$, $Y_j = Z'$ if $e_j \in Z$ $\}$.*
Hasse Diagram $\mathcal{F}$.

STOP

Construction of $(\Pi_{pp}, \geq_\pi)$ is as described in page 450.

Remark 11.6.1 *In Algorithm 11.2 we have found, for each element e in S, the set of all elements $\succeq_\lambda e$. From this the Hasse Diagram of the induced partial order* *has been built. An equivalent procedure would be to find for each e in S, the set of all elements $\preceq_\lambda e$. Clearly from this also the Hasse Diagram of the induced partial order can be constructed. The set of all elements $\preceq_\lambda e$ is obtained, when $e \in X^\lambda$, by finding the minimal minimizing set for the function $\lambda f(X) + g(S - X), e \in X \subseteq S$. But this is precisely what the subroutine $Subdivide_{f,g}(\{e\}, S)$ does.*

Justification for the PP algorithms

Justification for Algorithm P-sequence is directly by use of Uniqueness Theorem (Theorem 11.4.6). For, at the end of the algorithm, we have a sequence of sets and a sequence of critical values which satisfy the conditions of the theorem. Algorithm $\mathcal{B}_{\lambda_{f,g}}$ essentially involves, for each $e \in S$, finding the largest member Z^e of $\mathcal{B}_{\lambda_{f,g}}$, which does not contain e. By the definition of the preorder associated with $\mathcal{B}_{\lambda_{f,g}}$, the complement of Z^e contains precisely those elements which are present only in those members of $\mathcal{B}_{\lambda_{f,g}}$ which have e present. If $e \in X_\lambda$, there is no member of $\mathcal{B}_{\lambda_{f,g}}$ without e.

Complexity of the PP algorithms

The main subroutine is *Convolve*. So we will bound the number of calls to it. In Algorithm P-sequence, at each stage we have a nested sequence of sets. Hence the number of subdivisions is bounded by $\mid S \mid$. Each call to *Convolve* either creates a subdivision or marks a set. Marking a set S_j is equivalent to omitting (S_j, S_{j+1}) from further consideration. Now $S_{j+1} - S_j$ must have had atleast two elements - otherwise we could not have called *Convolve*. Thus, the total number of calls to *Convolve* cannot exceed $\mid S \mid$.
Algorithm $\mathcal{B}_{\lambda_{f,g}}$ requires $\mid Z' - Z \mid +1$ calls to *Convolve*. The total number of critical values is bounded by $\mid S \mid$. Hence, for all the critical values together we do not have to make more than $2 \mid S \mid$ calls to *Convolve*.
Thus, building the principal partition as well as the partition, partial order pair $(\Pi_{pp}, \geq_\pi)$ associated with the principal partition requires $0(\mid S \mid)$ calls to *Convolve*.

Remark: Note that we have assumed *Convolve* to be powerful since it produces Minset and Maxset instead of just any set minimising $\lambda f(X) + g(S - X)$. This however, appears valid for practical situations such as where $f(\cdot) \equiv$ rank function of a graph or a matroid and $g(\cdot) \equiv$ a positive weight function.

Exercise 11.32
Speeding up the PP algorithm *Let $f(\cdot)$ be a submodular function on subsets of S and let $g(\cdot)$ be a positive weight function. Let $X \subseteq S$ and let $e \in S - X$. Let T_{max}, T_{min} denote the maximal and minimal subsets that minimize $f(Y) + g(T - Y)$ over $Y \subseteq T$. Prove that the following hold.*

i. $(X \cup e)_{min} \supseteq X_{min}$,
$(X \cup e)_{max} \supseteq X_{max}$.

ii. *Let $f(\cdot)$ be increasing. Then $(f * g)(X \cup e) = (f * g)(X)$ iff $f(X_{max} \cup e) = f(X_{max})$.*
Further, if $(f * g)(X \cup e) = (f * g)(X)$, then $(X \cup e)_{max} = X_{max} \cup e$.
How would you use this fact for computing S_{max} efficiently when $f(\cdot)$ is integral and $f(S) \ll \mid S \mid$.

11.6.2 *Building the refined partial order given $(\Pi_{pp}, \geq_\pi)$.

We remind the reader that Π_{pp} is obtained by putting together the partitions $\Pi(\lambda)$ of $X^\lambda - X_\lambda$ for each critical value λ. Each block of $\Pi(\lambda)$ is a maximal subset of S which is not 'cut' by members of $\mathcal{B}_{\lambda f,g}$. Each $\mathcal{B}_{\lambda f,g}$ is equivalent to a partial order on blocks of $\Pi(\lambda)$. The overall partial order $(\geq_\pi)$ is built by putting together the partial orders $(\geq_\lambda)$ corresponding to the critical value λ and defining each block of $\Pi(\lambda_1)$ to be above every block of $\Pi(\lambda_2)$ whenever λ_2, λ_1 are critical values with $\lambda_2 < \lambda_1$. We use the same symbol for an ideal of $(\geq_\pi)((\geq_R))$ and the union of blocks of Π_{pp} contained in the ideal and use terms such as 'properly related' (defined in page 468) for the latter also.

Let $\lambda_1 > \cdots > \lambda_t$ be the decreasing sequence of critical values. The partial orders $(\geq_R)$ and $(\geq_\pi)$ agree with each other over $\Pi(\lambda_1)$. Suppose we have built $(\geq_R)$ for $\Pi(\lambda_1) \uplus \cdots \uplus \Pi(\lambda_k)$. For each block A_j of $\Pi(\lambda_{k+1})$, let I_j be the principal ideal determined by A_j in $(\geq_{\lambda_{k+1}})$. Let I_j be properly related to Y where Y is an ideal of $(\geq_R)$ (restricted to $\Pi(\lambda_1) \uplus \cdots \uplus \Pi(\lambda_k)$. Then the blocks beneath A_j in $(\geq_R)$ are the blocks in Y and the blocks in I_j.

We give an algorithm for computing Y below.

ALGORITHM 11.3 Algorithm Y

INPUT $(\geq_R)$ *over* $\Pi(\lambda_1) \uplus \cdots \uplus \Pi(\lambda_k), I_j$.

OUTPUT *Ideal Y of $(\geq_R)$ over $\Pi(\lambda_1) \uplus \cdots \uplus \Pi(\lambda_k)$ s.t.*
I_j is properly related to Y.

Initialize *Current ideal $X \leftarrow X_{\lambda_{k+1}}$.*

STEP 1 *For each B_j in $\Pi(\lambda_1) \uplus \cdots \uplus \Pi(\lambda_k)$, let*
$$U_j \equiv \{B_k, B_k \geq_R B_j, B_k \subseteq X_{\lambda_{k+1}}\}$$
and if I_j is contraction related to $X - U_j$,
$X \leftarrow (X - U_j)$.

STEP 2 *Output the current ideal as Y.*

STOP

Complexity of construction of refined partial order

To compute the principal ideal of A_j in the refined partial order, we need to compute atmost $\mid \Pi_{pp} \mid U_j$'s. For each U_j we need to compute ranks $f(X - U_j)$, $f(I_j \cup (X - U_j))$. Thus, $O(\mid \Pi_{pp} \mid)$ ranks have to be computed.

Hence, to compute the principal ideal for all the blocks Π_{pp}, $O(|\ \Pi_{pp}\ |^2)$ U_j's have to be computed and $O(|\ \Pi_{pp}\ |^2)$ ranks have to be computed. We note that each U_j is a principal ideal in an appropriate partial order.

11.6.3 Algorithm $Convolve_S(w_R(\Gamma_L), w_L)$

We must examine the algorithm $Convolve_S(f,g)$ for two important special cases:
i. $f(\cdot) \equiv w_R(\Gamma_L)(\cdot)$, where Γ_L is the left adjacency function of a bipartite graph (V_L, V_R, E), $w_R(\cdot)$ is a positive weight function on V_R, $g(\cdot) \equiv w_L(\cdot)$, a positive weight function on V_L.

ii. $f(\cdot) \equiv r(\cdot)$ rank function of a matroid, $g(\cdot) \equiv |\cdot|$.
We study the former case in this subsection and relegate the latter to the next chapter.

For the present instance

$$\lambda f(X) + g(S-X) = \lambda w_R(\Gamma_L)(X) + w_L(V_L - X). \text{ Let } \lambda > 0.$$

As we saw for the case $\lambda = 1$ in the proof of Theorem 11.2.2, minimizing the above function is equivalent to finding the min cut in the network $\mathcal{N}_\lambda \equiv F(B, \mathbf{w}_L, \lambda\mathbf{w}_R)$. For convenience we repeat the definition and sketch the discussion given in Subsection 3.6.10. The flow graph is built as follows: Each edge of the bipartite graph is directed from left to right with **capacity** ∞. There is a source vertex s and a sink vertex t. From s to each left vertex v_L there is a directed edge of **capacity** $w_L(v_L)$. From each right vertex v_R there is a directed edge to t with **capacity** $\lambda w_R(v_R)$. Using the facts that a mincut should not have infinite capacity and that $\lambda > 0$, we can show that a mincut must have the form

$$(s \uplus X \uplus \Gamma_L(X), t \uplus (V_L - X) \uplus (V_R - \Gamma_L(X)))$$

(see Figure 11.3).
The capacity of this cut is $w_L(V_L - X) + \lambda w_R(\Gamma_L)(X)$. On the other hand, given $\hat{X}$, we can always build a 'corresponding cut' of the above form whose capacity is given by the expression $w_L(V_L - \hat{X}) + \lambda w_R(\Gamma_L)(\hat{X})$. Thus, there is a one to one correspondence between min cuts and sets $\hat{X}$ which minimize the expression $w_L(V_L - X) + \lambda w_R(\Gamma_L)(X)$. Any standard max flow algorithm would yield a mincut. Finding the min cuts corresponding to the unique largest and unique smallest minimizing set $\hat{X}$ is also easy and does not cost additional complexity as we have shown in the above mentioned subsection.

Exercise 11.33 *The subroutine $Subdivide_{K,S}(\lambda w_R(\Gamma_L), w_L)$ involves minimizing $w_L(V_L - X) + \lambda w_R(\Gamma_L)(X), K \subseteq X$.*
Show that this minimization is equivalent to solving a max flow problem in which the capacity of the edges in the flow graph $F(B, \mathbf{w}_L, \lambda\mathbf{w}_R)$ are modified as follows: capacity of $(s,v), v \in K$ is changed from $w_L(v)$ to ∞.

Exercise 11.34
Let $B \equiv (V_L, V_R, E)$ and $w_L(\cdot), w_R(\cdot)$ be positive weight functions on V_L, V_R. Show that

Figure 11.3: Convolution through Flow Maximization

i. $Y \subseteq V_L$ *minimizes* $w_R(\Gamma_L(X)) + w_L(V_L - X)$
iff $(V_R - \Gamma_L(Y))$ *minimizes* $w_L(\Gamma_R(Z)) + w_R(V_R - Z)$

ii. $Y \subseteq V_L$ *is in* $\mathcal{B}_{(\lambda w_R \Gamma_L, w_L)}$
iff $V_R - \Gamma_L(Y) \subseteq V_R$ *is in* $\mathcal{B}_{(\frac{1}{\lambda} w_L \Gamma_R, w_R)}$ *(in the latter case the underlying set is* V_R*).*

11.6.4 Example: PP of $(|\Gamma_L|(\cdot), w_L(\cdot))$

We illustrate how to construct a bipartite graph with a desired principal partition of $(|\Gamma_L|(\cdot), w_L(\cdot))$ and a desired refined partial order say the one given in part (a) of Figure 11.4. Here $(|\Gamma_L|(\cdot), w_L(\cdot))$ are the adjacency function and a positive weight function on the left vertex set of a bipartite graph (V_L, V_R, E). We work with $|\cdot|$ but the same ideas work for any $w_L(\cdot)$.

We begin with a stock of '$(|\Gamma_L|(\cdot), |\cdot|)$ atomic bipartite graphs' (i.e., bipartite graphs for which the principal partition of $(|\Gamma_L|(\cdot), |\cdot|)$ has only two sets: $\emptyset$ and V_L) of the specified critical value. If a connected bipartite graph has a totally symmetric left vertex set, then it has to be atomic with respect to $(|\Gamma_L|(\cdot), |\cdot|)$. The critical value for such bipartite graphs equals $\frac{|V_L|}{|V_R|}$. These bipartite graphs are seen in part (b) of the figure if one ignores the dotted lines. Let us call these bipartite graphs $B_A, B_{B_1}, B_{B_2}, B_{B_3}, B_{C_1}, B_{C_2}, B_{D_1}, B_{D_2}$. We remind the reader that the bipartite graph $B_{\cdot LX}$ is defined to be the subgraph of B on $X \uplus \Gamma_L(X)$ whereas the bipartite

Figure 11.4: The Principal Partition and Refined Partial Order for $(\Gamma_L(\cdot), |\cdot|)$

graph $B_{\diamond LX}$ is defined to be the subgraph on $X \uplus (\Gamma_L(X \uplus (V_L - X))$ - $\Gamma_L(V_L - X))$. Given a bipartite graph $B \equiv (V_L, V_R, E)$ with left adjacency function $|\Gamma_L|(\cdot)$, the derived bipartite graphs $B_{\cdot LX}, B_{\diamond LX}$, $X \subseteq V_L$, have as left adjacency functions $|\Gamma_L|/\mathbf{X}(\cdot)$ and $|\Gamma_L|\diamond\mathbf{X}(\cdot)$ (see Exercise 10.7). Let S_i be one of the sets $A, B_1, \cdots, D_2$ and let I_{Si} be the set corresponding to the principal ideal of S_i relative to $(\geq_R)$. We want the structure on S_i to become atomic with respect to the left adjacency function and the $|\cdot|$ function, in the bipartite graph $(B_{\cdot LI_{Si}})_{\diamond LSi}$,. Thus, for instance, $(B_{\cdot L(C_1 \cup B_3 \cup B_1)})_{\diamond LC_1}$ must be the same as B_{C_1} (the bipartite graph on $C_1 \uplus \Gamma_L(C_1)$ when the dotted lines are removed). If the original bipartite graph has edges from the left vertex set C_1 to the right vertex set of B_{B_3}, then unless the ideal $B_3 \cup B_1$ is contracted we would be unable to get the desired atomic structure on C_1. We therefore force this, since we want $C_1 \geq_R B_1, B_3$, and attach such (dotted) edges. This procedure, if carried out for each of the sets $A, \cdots, D_2$, results in the bipartite graph with additional dotted edges shown in part (b) of the figure. Conversely the latter bipartite graph has the principal partition and refined partial order given in part (a) of the figure. It should be clear from this example that we can build a bipartite graph with **any desired refined partial order** by using the above procedure.
The Hasse Diagram for the partial order $(\geq_\pi)$ can be obtained from part (a) of the figure by adding the following additional lines:
$(D_1, C_2), (D_2, C_1), (C_1, B_2), (C_2, B_2), (B_1, A)$ and deleting (B_2, A).

Observe that the principal partition and refined partial order carry more information than the Dulmage-Mendelsohn decomposition. The latter would show the partial order on B_1, B_2, B_3 but lump all of D_1, D_2, C_1, C_2 and also lump all elements corresponding to $\lambda > 1$.

Exercise 11.35 *What is the significance of an atomic $(|\Gamma_L|, |\cdot|)$ structure for a critical value $\lambda \neq 1$?*

Exercise 11.36 *Verify the following simple rules for building atomic structures relative to $(|\Gamma_L|(\cdot), k|\cdot|)$ where $|\Gamma_L|(\cdot)$ is the adjacency function acting on the left vertex set of a bipartite graph (V_L, V_R, E) and k is any positive number. We say 'atomic bipartite graph' to be brief.*

i. *Any connected bipartite graph in which the left vertex set is totally symmetrical is atomic.*

ii. *Build two atomic bipartite graphs on the same left vertex set. Merge corresponding left vertices. The result is an atomic bipartite graph.*

iii. *Start with an atomic bipartite graph. Replace each left (right) vertex by m copies. The result is an atomic bipartite graph.*

iv. *Start with an atomic bipartite graph and interchange left and right vertex sets. This yields an atomic bipartite graph.*

11.7 *Aligned Polymatroid Rank Functions

In this section we deal with situations where certain submodular functions have strongly related principal partitions and these relations carry through under special operations. We show that if we perform these operations on a single polymatroid rank function the principal partition is relatively unaffected. Our ideas culminate in Theorem 11.7.5, where we show that a polymatroid obtained from another through a 'positive' or 'negative' expression is 'aligned ' to the original. Since the ideas of this section, though useful, are not standard, they are stated in the form of problems.

Definition 11.7.1 Aligned polymatroid rank functions with respect to a positive weight function *Let $f_0(\cdot), f_1(\cdot)$ be polymatroid rank functions on subsets of S and let $g(\cdot)$ be a positive weight function on S. Let $\mathcal{B}^i_\lambda$, $i = 0, 1$, denote the collections of sets in the principal partition of $(f_i(\cdot), g(\cdot)), i = 0, 1$, corresponding to λ. We say $f_0(\cdot), f_1(\cdot)$ are* **aligned** *with respect to $g(\cdot)$ iff*

i. The set of self loops of one of the polymatroid rank functions $f_0(\cdot), f_1(\cdot)$ is contained in that of the other polymatroid rank function. Similarly, the set of coloops relative to $g(\cdot)$ of one of the functions $f_0(\cdot), f_1(\cdot)$ is contained in that of the other.

ii. Every set in the principal sequence of one of the functions, which contains the set of selfloops of both the functions and does not intersect the coloops of either function is a set in the principal sequence of the other function.

iii. Whenever $X \subseteq S$

- *contains the set of self loops of one of the functions say $f_i(\cdot)$,*
- *does not contain any of its coloops,*
- *and $X \in \mathcal{B}^i_{\lambda_i}$ for some λ_i*

then $X \in \mathcal{B}^j_{\lambda_j}$ for some λ_j, where $j = i + 1$ mod 2.

In general the property of being aligned is not transitive.With some additional conditions however, as illustrated in the next problem,it is.

Problem 11.3 ***Definition 11.7.2*** *Let $f_0(\cdot), f_1(\cdot)$ be polymatroid rank functions on subsets of S and let $g(\cdot)$ be a positive weight function on S. If $f_0(\cdot), f_1(\cdot)$ are aligned with respect to $g(\cdot)$, and further every self loop of $f_0(\cdot)$ is a self loop of $f_1(\cdot)$ and every coloop relative to $g(\cdot)$ of $f_0(\cdot)$ is a coloop relative to $g(\cdot)$ of $f_1(\cdot)$ then we say that the principal partition of $(f_1(\cdot), g(\cdot))$ is* **coarser** *than that of $(f_0(\cdot), g(\cdot))$. and that the principal partition of $(f_0(\cdot), g(\cdot))$ is* **finer** *than that of $(f_1(\cdot), g(\cdot))$.*

Prove

Theorem 11.7.1 *Let $f_0(\cdot), f_1(\cdot), f_2(\cdot)$, be polymatroid rank functions on subsets of S and let $g(\cdot)$ be a positive weight function on S. If $f_0(\cdot), f_1(\cdot)$ and $f_1(\cdot), f_2(\cdot)$ are* *aligned with respect to $g(\cdot)$, and the principal partition of $(f_0(\cdot), g(\cdot))$ is finer than that of $(f_1(\cdot), g(\cdot))$ which in turn is finer than that of $(f_2(\cdot), g(\cdot))$ then $f_0(\cdot), f_2(\cdot)$ are aligned with respect to $g(\cdot)$ and further the principal partition of $(f_0(\cdot), g(\cdot))$ is finer than that of $(f_2(\cdot), g(\cdot))$.*

Solution: Proof of Theorem 11.7.1: We have,

- the set of self loops of $f_0(\cdot)$ is contained in that of $f_1(\cdot)$ which in turn is contained in the set of self loops of $f_2(\cdot)$ and
- the set of coloops of $f_0(\cdot)$ is contained in that of $f_1(\cdot)$ which in turn is contained in the set of coloops of $f_2(\cdot)$.

The alignedness of $f_0(\cdot), f_2(\cdot)$ now follows if the second and third parts of its definition is satisfied. It is easy to verify that the part of the definition of alignedness about principal sequences of the two functions is satisfied. We will therefore verify only the third part.

To see this, consider first the case where X is a set, corresponding to λ_2 in the principal partition of $(f_2(\cdot), g(\cdot))$, that contains all self loops of $f_2(\cdot)$ and none of its coloops. Since $f_1(\cdot), f_2(\cdot)$ are aligned, X must be a set in the principal partition of $f_1(\cdot)$ and further does not intersect any of the coloops of $f_1(\cdot)$ and contains all its self loops. Since $f_0(\cdot), f_1(\cdot)$ are aligned it follows that X must be a set in the principal partition of $(f_0(\cdot), g(\cdot))$.

Next, let X be a set corresponding to λ_0 in the principal partition of $(f_0(\cdot), g(\cdot))$, that contains all self loops of $f_0(\cdot)$ and none of its coloops. It must then be a set in the principal partition of $(f_1(\cdot), g(\cdot))$. We then have three possibilities, by Lemma 11.4.3.

- $(\infty > \lambda > 1$, i.e., X contains all selfloops of $f_1(\cdot)$ and none of its coloops). Clearly X must be a set in the principal partition of $(f_2(\cdot), g(\cdot))$ also, since $f_1(\cdot), f_2(\cdot)$ are aligned.
- $(\lambda = \infty$, i.e., X is a subset containing only self loops of $f_1(\cdot))$. Clearly X contains only selfloops of $f_2(\cdot)$ since the principal partition of $(f_2(\cdot), g(\cdot))$ is coarser than that of $(f_1(\cdot), g(\cdot))$.
- $(\lambda = 1$, i.e., X is a subset whose complement contains only coloops of $f_1(\cdot))$. Clearly the complement of the set X contains only coloops of $f_2(\cdot)$, since the principal partition of $(f_2(\cdot), g(\cdot))$ is coarser than that of $(f_1(\cdot), g(\cdot))$.

□

Problem 11.4 *Prove*

Lemma 11.7.1 *Let $f_0(\cdot), f_1(\cdot)$ be polymatroid rank functions on subsets of S and let $g(\cdot)$ be a positive weight function on S. Let $\mathcal{B}^3_\lambda$ denote the collection of sets corresponding to λ in the principal partition of $(f_0(\cdot) + f_1(\cdot), g(\cdot))$.*

i. Let X *be a set in* $\mathcal{B}^0_{\lambda_0}$ *as well as in* $\mathcal{B}^1_{\lambda_1}$ *. Then* $X \in \mathcal{B}^3_{\lambda_3}$*, where* $\lambda_3 = ((\lambda_0)^{-1} + (\lambda_1)^{-1})^{-1}$.

ii. Suppose, in addition, X *is a maximal (minimal) member of* $\mathcal{B}^0_{\lambda_0}$*, then* X *is a maximal (minimal) member of* $\mathcal{B}^3_{\lambda_3}$.

iii. If $\mathcal{B}^0_{\lambda_0} = \mathcal{B}^1_{\lambda_1}$ *then* $\mathcal{B}^3_{\lambda_3} = \mathcal{B}^1_{\lambda_1}$.

Solution: Proof of Lemma 11.7.1:

i. We have,

$$f_i(X) + ((\lambda_i)^{-1})g(S - X) \leq f_i(Y) + ((\lambda_i)^{-1})g(S - Y), i = 0, 1, Y \subseteq S.$$

Hence,

$$f_0(X) + f_1(X) + ((\lambda_0)^{-1} + (\lambda_1)^{-1})g(S - X) \leq$$
$$f_0(Y) + f_1(Y) + ((\lambda_0)^{-1} + (\lambda_1)^{-1})g(S - Y).$$

This proves the required result.

ii. In the above proof note that the final inequality reduces to an equality iff the former inequalities do so for $i = 0, 1$. The result now follows.

iii. Proof depends on the fact stated above.

□

The following theorem is an immediate consequence of the above lemma.

Theorem 11.7.2 *Let* $f_0(\cdot), f_1(\cdot)$ *be polymatroid rank functions on subsets of* S *and let* $g(\cdot)$ *be a positive weight function on* S*. Let* $(f_0(\cdot), g(\cdot))$*,* $(f_1(\cdot), g(\cdot))$ *have the same principal partition with decreasing sequence of critical values* $\lambda_{01}, \cdots, \lambda_{0t}$ *and* $\lambda_{11}, \cdots, \lambda_{1t}$*. Then* $((f_0 + f_1)(\cdot), g(\cdot))$ *has the same principal partition with decreasing sequence of critical values* $\lambda_{31}, \cdots, \lambda_{3t}$*, where* $\lambda_{3i} = ((\lambda_{0i})^{-1} + (\lambda_{1i})^{-1})^{-1}, i = 1, \cdots, t$.

Problem 11.5 *Prove*

Theorem 11.7.3 *Let* $f_1(\cdot), f_0(\cdot)$ *be aligned polymatroid rank functions on subsets of* S *relative to the positive weight function* $g(\cdot)$ *such that* $f_i(e) \leq g(e), i = 0, 1, \forall e \in S$*. Then*

i. The principal sequence of $(f_1(\cdot) + f_0(\cdot), g(\cdot))$ *is the coarsest common refinement of those of* $(f_0(\cdot), g(\cdot))$*,* $(f_1(\cdot), g(\cdot))$.

ii. $f_1(\cdot) + f_0(\cdot)$ *is aligned to both* $f_1(\cdot)$ *and* $f_0(\cdot)$ *with respect to* $g(\cdot)$.

iii. If both $f_1(\cdot)$ *and* $f_0(\cdot)$ *are aligned with* $f_3(\cdot)$ *with respect to* $g(\cdot)$ *and further if the principal partitions of* $(f_0(\cdot), g(\cdot))$ *and* $(f_1(\cdot), g(\cdot))$ *are coarser than that of* $(f_3(\cdot), g(\cdot))$*, then* $f_1(\cdot) + f_0(\cdot)$ *is aligned with* $f_3(\cdot)$ *and the principal partition of* $(f_1(\cdot) + f_0(\cdot), g(\cdot))$ *is coarser than that of* $(f_3(\cdot), g(\cdot))$.

Solution: The assumption $f_i(e) \leq g(e)$ is not essential. It has been made only to make the proof simpler and also because it is the only case of importance.
Proof of Theorem 11.7.3: Let $\mathcal{B}^i_\lambda, i = 0, 1, 2$ denote the collection of sets corresponding to λ in the principal partitions of $(f_0(\cdot), g(\cdot)), (f_1(\cdot), g(\cdot))$ and $(f_0(\cdot) + f_1(\cdot), g(\cdot))$ respectively.
Suppose the set of selfloops S^i of $f_i(\cdot)$ contains the set S^j of selfloops of $f_j(\cdot)$, where $j = i + 1 \ mod\ 2$. From the definition of alignedness it follows that the principal sequence of $(f_j(\cdot), g(\cdot))$ has the form $\emptyset, S^j, \cdots, S^i, \cdots$. Now by the use of Lemma 11.7.1 it is clear that

- the principal sequence of $(f_i(\cdot) + f_j(\cdot), g(\cdot))$ will be identical to the above sequence upto the set S^i and that
- if λ is a critical value of $(f_j(\cdot), g(\cdot))$ whose maximal minimizing set is contained in S^i it satisfies the same property with respect to $(f_i(\cdot) + f_j(\cdot), g(\cdot))$.

For these values of λ the collections of sets are identical both in$(f_j(\cdot), g(\cdot))$ and in $(f_i(\cdot) + f_j(\cdot), g(\cdot))$. From Lemma 11.4.3, if C_i is the set of coloops of $f_i(\cdot)$ with respect to $g(\cdot)$, we know that $S - C_i$ is the penultimate set in the principal sequence of $(f_i(\cdot), g(\cdot)), i = 0, 1$. Suppose next, without loss of generality, that the set of coloops C_0 of $f_0(\cdot)$ contains the set of coloops C_1 of $f_1(\cdot)$. We then have by Lemma 11.7.1

- the segment $S^i, \cdots, S - C_0$ appears in the principal sequence of $(f_0(\cdot) + f_1(\cdot), g(\cdot))$ since it appears in the principal sequences of both $(f_0(\cdot), g(\cdot))$ and $(f_1(\cdot), g(\cdot))$.
- if λ_0, λ_1 are the critical values of $(f_0(\cdot), g(\cdot))$, $(f_1(\cdot), g(\cdot))$ respectively coresponding to two successive sets in this segment then $\mathcal{B}^0_{\lambda_0} = \mathcal{B}^1_{\lambda_1} = \mathcal{B}^2_{\lambda_2}$, where $\lambda_2 = ((\lambda_0)^{-1} + (\lambda_1)^{-1})^{-1}$.
- since in the principal partition of $(f_0(\cdot), g(\cdot))$, $S - C_0, S$ are the minimal and maximal sets in $\mathcal{B}^0_{\lambda=1}$ and every set between these two is also in this collection, therefore the segment $S - C_0, \cdots, S$ would be identical in the principal sequences of $(f_1(\cdot), g(\cdot))$ and $(f_0(\cdot) + f_1(\cdot), g(\cdot))$. If λ^1 is a critical value of $(f_1(\cdot), g(\cdot))$ corresponding to two successive sets in the above segment then $\mathcal{B}^1_{\lambda^1} = \mathcal{B}^2_{\lambda^2}$, where $\lambda_2 = (1 + (\lambda^1)^{-1})^{-1}$.

All the parts of the theorem are now immediate

□

Problem 11.6 *Prove*

Lemma 11.7.2 *Let $f_0(\cdot), f_1(\cdot)$ be polymatroid rank functions on subsets of S and let $g(\cdot)$ be a positive weight function on S. Let $f_0(\cdot), f_1(\cdot)$ be aligned with respect to $g(\cdot)$. Then*

i. $f_0 * g(\cdot), f_1(\cdot)$ *are aligned with respect to* $g(\cdot)$

ii. *if the principal partition of* $(f_0(\cdot), g(\cdot))$ *is coarser than that of* $(f_1(\cdot), g(\cdot))$ *then the principal partition of* $(f_0 * g(\cdot), g(\cdot))$ *is coarser than that of* $(f_1(\cdot), g(\cdot))$.

Solution: Proof of Lemma 11.7.2: Let $\mathcal{B}^i_\lambda, i = 0, 1, 2$, denote the class of sets corresponding to λ in the respective principal partitions of
$(f_0(\cdot), g(\cdot)), (f_1(\cdot), g(\cdot)), (f_0 * g(\cdot), g(\cdot))$.
By Theorem11.4.4 it is clear that for $\lambda > 1, \mathcal{B}^0_\lambda = \mathcal{B}^2_\lambda$. In particular the maximal set in $\mathcal{B}^0_\lambda$ and $\mathcal{B}^2_\lambda$ are the same for any value of $\lambda > 1$. But then by Property PP5 of the principal partition, the minimal set say Z in $\mathcal{B}^0_\lambda$ and $\mathcal{B}^2_\lambda$ are the same for $\lambda \geq 1$. Again by Theorem11.4.4, $S - Z$ is the set of all coloops of $f_0 * g(\cdot)$ with respect to $g(\cdot)$. Thus we see that

- The principal sequence of $(f_0 * g(\cdot), g(\cdot))$ has the form

$$\emptyset, X_1, \cdots, X_r = Z, S,$$

  while that of $(f_0, g(\cdot))$ has the form

$$\emptyset, X_1, \cdots, X_r = Z, X_{r+1}, \cdots, S.$$

- $S - Z$ is the set of all coloops of $f_0 * g(\cdot)$ with respect to $g(\cdot)$. The sets Z and S are the minimal and maximal sets in $\mathcal{B}^2_\lambda$ when $\lambda = 1$.
- For $\lambda > 1, \mathcal{B}^0_\lambda = \mathcal{B}^2_\lambda$.

Thus $f_0(\cdot), f_0 * g(\cdot)$ are aligned with respect to $g(\cdot)$ and further the principal partition of $(f_0 * g(\cdot), g(\cdot))$ is coarser than that of $(f_0(\cdot), g(\cdot))$. The second part of the theorem now follows by Theorem 11.7.1.

□

Problem 11.7 *Prove*

Theorem 11.7.4 *Let* $f_0(\cdot), f_1(\cdot), f_3(\cdot)$ *be polymatroid rank functions on subsets of* S *and let* $g(\cdot)$ *be a positive weight function on* S. *Let* $f_0(\cdot), f_1(\cdot)$ *be aligned with respect to* $g(\cdot)$. *Let* $f_3(\cdot)$ *be aligned to both* $f_0(\cdot)$ *and* $f_1(\cdot)$ *with respect to* $g(\cdot)$ *and further let the principal partitions of* $(f_0(\cdot), g(\cdot)), (f_1(\cdot), g(\cdot))$ *be coarser than that of* $(f_3(\cdot), g(\cdot))$. *Then*

i. $f_0, (f_0 + f_1) * g(\cdot)$ *and* $f_1, (f_0 + f_1) * g(\cdot)$ *are aligned with respect to* $g(\cdot)$.

ii. *the principal partition of* $((f_0+f_1)*g(\cdot), g(\cdot))$ *is coarser than that of* $(f_3(\cdot), g(\cdot))$.

Solution: The result is immediate from Lemma 11.7.2 and Theorem 11.7.1.

Problem 11.8 *Prove*

Lemma 11.7.3 *Let* $f_i(\cdot), i = 0, 1$ *be polymatroid rank functions on the subsets of* S *and let* $g(\cdot)$ *be a positive weight function on* S *with* $g(e) \geq f_i(e), \forall e \in S, i = 0, 1$. *Then* *i.* $f_0(\cdot), f_1(\cdot)$ *are aligned with respect to* $g(\cdot)$ *iff* $f_0^*(\cdot), f_1^*(\cdot)$ *are so aligned. If in addition the principal partition of* $(f_0(\cdot), g(\cdot))$ *is coarser than that of* $(f_1(\cdot), g(\cdot))$ *then the principal partition of* $(f_0^*(\cdot), g(\cdot))$ *is coarser than that of* $(f_1^*(\cdot), g(\cdot))$.

ii. $f_0^*(\cdot), f_1(\cdot)$ *are aligned with respect to* $g(\cdot)$ *iff* $f_0(\cdot), f_1^*(\cdot)$ *are so aligned.*

iii. if $f_0(\cdot), f_1(\cdot)$ *are aligned with respect to* $g(\cdot)$ *then* $f_0^* * g(\cdot), f_1^*(\cdot)$ *are so aligned.*

Solution: The assumption $f_i(e) \leq g(e)$ is made only to make the duals into polymatroid rank functions.
Proof of Lemma 11.7.3:

i. This follows from Theorem11.4.5 and the definition of alignedness.

ii. This follows from the above result and the fact that $f_0^{**}(\cdot) = f_0(\cdot)$.

iii. This follows from the first part and Lemma 11.7.2.

□

Definition 11.7.3 *Let* $f_i(\cdot), i = 0, 1$ *be polymatroid rank functions on the subsets of* S *and let* $g(\cdot)$ *be a positive weight function on* S *with* $g(e) \geq f_i(e), \forall e \in S, i = 0, 1$. *We say that the polymatroid rank functions* $f_0(\cdot), f_1(\cdot)$ *are* **oppositely aligned** *with respect to* $g(\cdot)$ *iff* $f_0^*(\cdot), f_1(\cdot)$ *(equivalently* $f_1^*(\cdot), f_0(\cdot)$*) are aligned with respect to* $g(\cdot)$.

The results about alignedness presented thus far permit us to talk of the alignedness of polymatroid rank functions derived from simpler aligned polymatroid rank functions through certain formal expressions involving the operations of addition, convolution with $g(\cdot)$, and dualization. We know that addition of aligned polymatroid rank functions results in another such, convolution with $g(\cdot)$ results in a coarser aligned polymatroid rank function while dualization oppositely aligns the polymatroid rank function. It is therefore clear that if the formal expressions were constructed according to certain simple rules, then we would have complete knowledge of the principal partition associated with the resulting polymatroid rank function. The care that we have to take essentially lies in convolving with $g(\cdot)$ whenever the value of the polymatroid rank function can become greater than that of $g(\cdot)$ at any element - otherwise we cannot use dualization ideas freely.

Definition 11.7.4

- *'i' is a* **positive expression** *of* **length** 1.
- *'i^*' is a* **negative expression** *of* **length** 2.
- *If 'ω' is a positive expression of length $l-1$ then '$\omega * g$' is a positive expression of* **length** l.
- *If 'ω' is a negative expression of length $l-1$ then '$\omega * g$' is a negative expression of* **length** l.
- *If 'ω' is a positive expression of length $l-1$ then '$(\omega)^*$' is a negative expression of* **length** l.

- *If 'ω' is a negative expression of length $l-1$ then '$(\omega)^*$' is a positive expression of* **length** l.

- *If 'ω' is a positive expression (negative expression) of length $l-1$ then '$(\lambda\omega)*g$', $\lambda > 0$ is a positive expression (negative expression) of* **length** $l+1$.

- *If 'ω_0','ω_1' are positive expressions (negative expressions) of lengths k, l respectively, then '$(\omega_0 + \omega_1) * g$' is a positive expression (negative expression) of* **length** $k+l+1$.

Remark: If 'ω' is an expression (positive or negative) with respect to the weight function $g(\cdot)$ on S and $f(\cdot)$ is a polymatroid rank function on subsets of S then $\omega(f)(\cdot)$ denotes the polymatroid rank function obtained by replacing all the ocurrences of the symbol 'i' in 'ω' by $f(\cdot)$.

Example 11.7.1 *Consider the expression '$(i + i * g) * g$'. If this operates on the polymatroid rank function $f(\cdot)$ we get the polymatroid rank function $((f(\cdot) + f * g(\cdot)) * g)(\cdot)$. The expressions '$i$','$i * g$',are positive with lengths $1, 2$ respectively. The expressions '$i + i * g$', '$(i + i * g) * g$' are therefore positive with lengths $3, 4$ respectively. So the expression '$((i + i * g) * g)^* * g$' is negative with length 6 and the expression '$(((i + i * g) * g)^* * g + (3i * g)^*) * g$' is negative with length $6 + 4 + 1 = 11$.*

Remark: We will omit inverted commas henceforth while speaking of expressions.

Problem 11.9 *Prove*

Theorem 11.7.5 *Let ω be a positive (negative) expression with respect to $g(\cdot)$, a positive weight function on S. Let $f(\cdot)$ be a polymatroid rank function on subsets of S. Then the polymatroid rank function $\omega(f)(\cdot)$ is aligned to $f(\cdot)$ $(f^*(\cdot))$ with respect to $g(\cdot)$ and the principal partition of $(\omega(f)(\cdot), g(\cdot))$ is coarser than that of $(f(\cdot), g(\cdot))$ $((f^*(\cdot), g(\cdot)))$. Further $\omega(f)(e) \leq g(e) \ \ \forall e \in S$.*

Solution: Proof of Theorem 11.7.5: The proof is by induction using the following results:

- if $f_0(\cdot), f_1(\cdot)$ are aligned with the principal partition of the former coarser than that of the latter then $f_0(\cdot) * g(\cdot), f_1(\cdot)$ are aligned, with the former having a coarser principal partition than the latter (Lemma 11.7.2).

- if $f_0(\cdot), f_1(\cdot)$ are aligned with the principal partition of the former coarser than that of the latter then $f_0^*(\cdot), f_1^*(\cdot)$ are aligned, with the former having a coarser principal partition than the latter (Lemma 11.7.3).

- if $f_0(\cdot), f_1(\cdot)$ are aligned with the principal partition of the former coarser than that of the latter then $f_0^* * g(\cdot), f_1^*(\cdot)$ are aligned, with the former having a coarser principal partition than the latter (Lemma 11.7.3).

- if $f_0(\cdot), f_1(\cdot)$ are aligned with $f_2(\cdot)$ with their principal partitions coarser than that of the latter then $(f_0 + f_1) * g(\cdot)$ is aligned with $f_2(\cdot)$ and has a coarser principal partition than the latter has (Theorem 11.7.3, Lemma 11.7.2).

- if $f_0(\cdot), f_1(\cdot)$ are aligned with $f_2(\cdot)$ with their principal partitions coarser than that of the latter then $\lambda f_0(\cdot), \lambda f_1)(\cdot), \lambda > 0$ are aligned with $f_2(\cdot)$ and have coarser principal partitions than the latter has.

Clearly the theorem is true for expressions of length 1. Suppose it is true for expressions of length less or equal to k. Let ω be a positive (negative) expression of length $k+1$. Then ω could have been built up out of shorter expressions in one of the following ways, for each of which the results mentioned above enable us to show that ω satisfies the theorem.

- $\omega = \theta * g$, where θ is positive (negative) of length k. Here ω remains positive (negative) and the required result follows from Lemma 11.7.2.

- $\omega = (\theta)^*$, where θ is positive (negative) of length k. Here ω becomes negative (positive) and the required result follows from Lemma 11.7.3.

- $\omega = (\theta * g)^*$, where θ is positive (negative) of length $k-1$. Here ω becomes negative (positive) and the required result follows from Lemma 11.7.2 and Lemma 11.7.3.

- $\omega = \lambda\theta * g$,where θ is positive (negative) of length $k-1$ and $\lambda > 0$. Here ω remains positive (negative) and the required result follows from Lemma 11.7.2 and the fact that for any polymatroid rank function $f(\cdot)$, $(\lambda f(\cdot), g(\cdot))$, $\lambda > 0$ has the same principal partition (but different critical values) as $(f(\cdot), g(\cdot))$.

- $\omega = (\theta_1 + \theta_2) * g$,where $\theta_i, i = 1, 2$ are both positive (negative) of length $d, k-d$ respectively. Here ω remains positive (negative) and the required result follows from Theorem 11.7.3 and Lemma 11.7.2.

The fact that in each case the resulting polymatroid rank function has lower value on singletons than $g(\cdot)$ follows from the definition of convolution and the properties of the dual.

□

When $f(\cdot)$ is molecular with respect to $g(\cdot)$ we can make a stronger statement than in Theorem 11.7.5. In this case $\omega(f)(\cdot)$ is molecular even if ω is not positive or negative.

Problem 11.10 *Prove*
Let ω be an expression involving

- *convolution with $g(\cdot)$*

- *addition followed by convolution with $g(\cdot)$*

- *positive scalar multiplication followed by convolution with $g(\cdot)$*

- *dualization with respect to $g(\cdot)$*

Let $f(\cdot)$ be a molecular polymatroid rank function on subsets of S with respect to $g(\cdot)$. Then $\omega(f)(\cdot)$ is also a molecular polymatroid rank function on subsets of S.

Solution: If $f_0(\cdot), f_1(\cdot)$ are aligned with respect to $g(\cdot)$ we know that $(f_0 + f_1) * g(\cdot), \lambda f_0 * g(\cdot), \lambda > 0, f_0^*(\cdot)$ are all aligned to $f_1(\cdot)$ (by Theorem 11.7.3, Lemma 11.7.2, Lemma 11.7.3). Clearly by the definition of alignedness, if $f_0(\cdot), f_1(\cdot)$ are aligned and $f_1(\cdot)$ is molecular then so must $f_0(\cdot)$ be. It follows by induction on the number of elementary operations that $\omega(f(\cdot))$ must be molecular if $f(\cdot)$ is molecular. The next theorem shows that if we so wish we could generalize the notion of alignedness using different weight functions instead of a single one.

Problem 11.11 *Prove*

Theorem 11.7.6 *Let $f_0(\cdot), f_1(\cdot)$ be polymatroid rank functions on subsets of S which are molecular respectively with respect to the positive weight functions $g_0(\cdot), g_1(\cdot)$. Then $(\rho_0 f_0 + \rho_1 f_1)(\cdot)$ is molecular with respect to $(\sigma_0 g_0 + \sigma_1 g_1)(\cdot)$, where $\rho_i, \sigma_i, i = 0, 1$ are greater than zero.*

Solution: Proof of Theorem 11.7.6: Suppose not.Then there exists $X \subset S$ such that

$$\lambda(\rho_1 f_1 + \rho_2 f_2)(X) + (\sigma_1 g_1 + \sigma_2 g_2)(S - X) < \lambda(\rho_1 f_1 + \rho_2 f_2)(S).$$

Then,

$$\sigma_1[(\lambda/\sigma_1)\rho_1 f_1(X) + g_1(S - X)] + \sigma_2[(\lambda/\sigma_2)\rho_2 f_2(X) + g_2(S - X)]$$

$$< \sigma_1[(\lambda/\sigma_1)\rho_1 f_1(S)] + \sigma_2[(\lambda/\sigma_2)\rho_2 f_2(S)].$$

So we must have, for i=1 or 2,

$$(\lambda/\sigma_i)\rho_i f_i(X) + g_i(S - X) < (\lambda/\sigma_i)\rho_i f_i(S).$$

This contradicts the molecularity of $f_i(\cdot)$ with respect to $g_i(\cdot)$

□

Problem 11.12 *Let $f_0(\cdot), f_1(\cdot)$ be polymatroid rank functions on subsets of S and let $g(\cdot)$ be a positive weight function on S. Let $(f_0(\cdot), g(\cdot)), (f_1(\cdot), g(\cdot))$ have identical principal partitions. If the refined partial order with respect to $(f_0(\cdot), g(\cdot))$ and $(f_1(\cdot), g(\cdot))$ are identical, then it would also be identical to the refined partial order associated with $((f_0 + f_1)(\cdot), g(\cdot))$.*

Solution: Lemma 11.7.1 assures us that $((f_0 + f_1)(\cdot), g(\cdot))$ also has the same principal partition. Next the characterization of ideals of $\geq_R$ given in Theorem 11.5.1 is such that if the conditions hold for a particular set with respect to both $f_0(\cdot), f_1(\cdot)$ they would also hold for $(f_0 + f_1)(\cdot)$. This proves the required result.

The program that we carried out for aligned polymatroids can also be carried out for what could be called ‘strongly aligned polymatroids’ (defined below). We sketch these ideas in the following problems. However we omit the solutions.

Problem 11.13 *Let $f_0(\cdot), f_1(\cdot)$ be aligned polymatroid rank functions on subsets of S relative to $g(\cdot)$, a positive weight function on S. We say $f_0(\cdot), f_1(\cdot)$ are* **strongly aligned** *relative to $g(\cdot)$ iff whenever A, B are two blocks present in both the partitions associated $(f_0(\cdot), g(\cdot))$ and $(f_1(\cdot), g(\cdot))$, but which are not coloops or self loops in either partition the relationship between A and B is identical in both the refined partial orders.*
If $f_0(\cdot), f_1(\cdot)$ are strongly aligned polymatroid rank functions relative to the positive weight function $g(\cdot)$ show that $(f_0 + f_1)(\cdot)$ is strongly aligned to both $f_0(\cdot)$ as well as $f_1(\cdot)$ relative to $g(\cdot)$.

Problem 11.14 *Let $f(\cdot)$ be a polymatroid rank function on subsets of S and let $g(\cdot)$ be a positive weight function on S. Show that $f(\cdot)$ is strongly aligned with $\lambda f * g(\cdot), \lambda > 0$.*

Problem 11.15 *Let $f_1(\cdot), f_0(\cdot)$ be strongly aligned polymatroid rank functions on subsets of S relative to a positive weight function $g(\cdot)$. If both $f_1(\cdot)$ and $f_0(\cdot)$ are strongly aligned with $f_3(\cdot)$ with respect to $g(\cdot)$ and further if the principal partitions of $(f_0(\cdot), g(\cdot))$ and $(f_1(\cdot), g(\cdot))$ are coarser than that of $(f_3(\cdot), g(\cdot))$, then, show that $(f_1(\cdot) + f_0(\cdot)) * g(\cdot)$ is strongly aligned with $f_3(\cdot)$ relative to $g(\cdot)$ and the principal partition of $(f_1(\cdot) + f_0(\cdot)) * g(\cdot)$ is coarser than that of $f_3(\cdot)$.*

Problem 11.16 *Show that the statement obtained by replacing 'aligned' by 'strongly aligned' in Theorem 11.7.5 is true.*

Problem 11.17 *Let $f_1(\cdot), f_2(\cdot)$ be two polymatroid rank functions on subsets of S with identical principal partitions relative to the positive weight function $g(\cdot)$ and identical refined partial order given in Figure 11.4(a). The critical value sequences are given to be $\lambda_{11} = 5$, $\lambda_{12} = 4$, $\lambda_{13} = 3$, $\lambda_{14} = 2$ and $\lambda_{21} = 4$, $\lambda_{22} = 3$, $\lambda_{23} = \frac{3}{2}$, $\lambda_{24} = \frac{4}{3}$.*
Describe the principal partition and refined partial order of

i. $((f_1 + f_2) * g(\cdot), g(\cdot))$

ii. $(((f_1 + f_2) * g)^*(\cdot), g(\cdot))$

iii. $(((2f_1 * g)^* + {f_1}^*) * g)^*(\cdot), g(\cdot))$

iv. In the previous parts compute the value of the functions on S.

11.8 Notes

Convolution, as an operation on submodular functions, was probably first studied systematically by Edmonds [Edmonds70]. Principal partition began with graphs when Kishi and Kajitani [Kishi+Kajitani68] decomposed a graph into three parts - X_λ, X^λ and $S - X^\lambda$ for $\lambda = 2$. These ideas were generalized to matroids for integral λ by Bruno and Weinberg [Bruno+Weinberg71] and for rational (real) λ independently by Tomizawa and Narayanan [Tomizawa76], [Narayanan74]. For

about a decade and a half, from late sixties to middle eighties, extensive work was done in Japan on the principal partition, its extensions and applications. Good surveys of this work may be found in
[Iri79a], [Iri79b], [Iri+Fujishige81], [Tomizawa+Fujishige82], [Iri83] and in the comprehensive monograph due to Fujishige [Fujishige91].
Extensions of the basic ideas may be found for instance in
[Ozawa74], [Ozawa75], [Ozawa76], [Ozawa+Kajitani79], in several papers due to Tomizawa in Japanese
[Tomizawa80a], [Tomizawa80b], [Tomizawa80c], [Tomizawa80d] etc.,
in [Fujishige80a], [Fujishige80b], [Nakamura+Iri81], [Iri84], [Murota88] etc.
Applications may be found, apart from the above mentioned surveys and monograph, in the following very partial list of references:
[Iri71], [Tomi+Iri74], [Iri+Tomi76], [Ozawa76], [Fujishige78b], [Sugihara79], [Sugihara80], [Sugihara+Iri80], [Sugihara82], [Iri+Tsunekawa+Murota82], [Sugihara83], [Sugihara84], [Sugihara86], [Murota+Iri85], [Murota87], [Murota90].

The west, except for the notable case of Bruno and Weinberg, has been largely immune to the principal partition virus. Recently, however, there were some signs of activity (see for instance [Catlin+Grossman+Hobbs+Lai92]).

This chapter, as far as the discussion on principal partition theory goes, is in the main, a translation of the author's PhD thesis [Narayanan74], which used matroid union and partition as basic notions, to the language of convolution of polymatroids and submodular functions (Subsections 11.4.5, 11.4.6, Section11.7 etc. are very natural for matroids). We have adopted this approach because it is elementary and the extensions follow naturally. The readers interested in pursuing this subject further would do well to begin with the above mentioned survey papers of Iri. Those interested in studying these ideas and their extensions in the context of convex programming are referred to [Fujishige91].

11.9 Solutions of Exercises

E 11.1: Let B_1, B_2 be bipartite graphs on $V_L \equiv \{a, b, c\}, V_R \equiv \{1, 2, 3, 4\}$ with adjacency functions Γ_1, Γ_2 defined as follows:

$$\Gamma_1(a) = \{1, 2, 4\}, \Gamma_1(b) = \{1, 2, 4\}, \Gamma_1(c) = \{2, 3, 4\},$$

$$\Gamma_2(a) = \{1, 2, 3\}, \Gamma_2(b) = \{1, 2, 4\}, \Gamma_2(c) = \{2, 3\}.$$

It may be verified that

$$|\Gamma_1| * |\Gamma_2|(a) = 3, |\Gamma_1| * |\Gamma_2|(a, b) = 3, |\Gamma_1| * |\Gamma_2|(a, c) = 3, |\Gamma_1| * |\Gamma_2|(a, b, c) = 4.$$

Hence

$$|\Gamma_1| * |\Gamma_2|(a, b, c) - |\Gamma_1| * |\Gamma_2|(a, c) > |\Gamma_1| * |\Gamma_2|(a, b) - |\Gamma_1| * |\Gamma_2|(a).$$

This shows that $|\Gamma_1| * |\Gamma_2|(\cdot)$ is not submodular. But we do know that $|\Gamma_1(\cdot)|, |\Gamma_2(\cdot)|$ are submodular.

E 11.2:
i. We know that $f * g(\cdot) = (f(\cdot) - f(\emptyset)) * (g(\cdot) - g(\emptyset)) + f(\emptyset) + g(\emptyset)$. Thus $f * g(\cdot)$is submodular if $(f(\cdot) - f(\emptyset)) * (g(\cdot) - g(\emptyset)$ is submodular. Calling these new functions $f'(\cdot), g'(\cdot)$ respectively, we observe, using the fact that $f'(\emptyset) = 0$, by Theorem10.6.1 that

$$f' * g'(X) = min_{Y \subseteq X}(f'(Y) + g'(X-Y)) \geq min_{Y \subseteq X}((f'(X) - f'(X-Y)) + g'(X-Y))$$

$$\geq f'(X) + min_{Y \subseteq X}(g' - f')(X - Y).$$

But, $(g' - f')(X - Y) \geq 0$. Thus

$$f' * g'(X) = f'(X),$$

which latter is submodular. The next two instances are special cases of the above result.

E 11.3: Since $f_1(\cdot) \geq (f_1 * g)(\cdot)$ and $f_2(\cdot) \geq (f_2 * g)(\cdot)$, we have, LHS $\geq$ RHS. Now we prove the reverse inequality.
We have

$$((f_1 * g + f_2 * g) * g)(S) = \min_{X \subseteq S}((f_1 * g)(X) + (f_2 * g)(X) + g(S - X))$$

$$= \min_{X \subseteq S}(\min_{Y \subseteq X}(f_1(Y) + g(X - Y)) + \min_{Y \subseteq X}(f_2(Y) + g(X - Y)) + g(S - X))$$

$$= (f_1(Y_1) + g(Z - Y_1)) + (f_2(Y_2) + g(Z - Y_2)) + g(S - Z)$$

(for some Y_1, Y_2, Z with $Y_1 \subseteq Y_2 \subseteq Z \subseteq S$),

$$\geq f_1(Y_1 \cap Y_2) + f_2(Y_1 \cap Y_2) + g(Z - (Y_1 \cap Y_2)) + g(S - Z),$$

(since $f_1(\cdot), f_2(\cdot)$ are increasing and $g(\cdot)$ is a non-negative weight function)

$$\geq f_1(Y_1 \cap Y_2) + f_2(Y_1 \cap Y_2) + g(S - (Y_1 \cap Y_2))$$

$$\geq ((f_1 + f_2) * g)(S).$$

E 11.4: Suppose X_1, X_2 belong to $\mathcal{B}_{f,g}$. We then have,

$$f * g(S) = f(X_1) + g(S - X_1) = f(X_2) + g(S - X_2).$$

Thus

$$2f * g(S) = f(X_1) + g(S - X_1) + f(X_2) + g(S - X_2)$$

$$\geq f(X_1 \cup X_2) + g(S - (X_1 \cup X_2)) + f(X_1 \cap X_2) + g(S - (X_1 \cap X_2),$$

using the submodularity of $f(\cdot), g(\cdot)$. By the definition of convolution the only way the inequality can be satisfied is to have

$$f * g(S) = f(X_1 \cup X_2) + g(S - (X_1 \cup X_2) = f(X_1 \cap X_2) + g(S - (X_1 \cap X_2).$$

Thus $X_1 \cup X_2, X_1 \cap X_2$ belong to $\mathcal{B}_{f,g}$.

E 11.5: If $|\Gamma|(Y) <| Y |$ for some $Y \subseteq V_L$, then,

$$|\Gamma|(Y)+ | V_L - Y |<| V_L | .$$

Next if $|\Gamma|(X) \geq| X | \quad \forall X \subseteq V_L$, then

$$|\Gamma|(X)+ | V_L - X |\geq| V_L | \qquad \forall X \subseteq V_L.$$

But for $X = \emptyset$, the above inequality becomes equality. Hence, $(\Gamma * | \cdot |)(V_L) =| V_L |$.

E 11.6: Proof of Theorem 11.2.2:
Make each edge of the bipartite graph directed from left to right with a capacity equal to ∞. Join each left vertex to a source vertex s with edges directed away from s with capacity 1 and join each right vertex to a sink vertex t with edges directed towards t with capacity 1. It is easily seen that an integral maximum flow for this network corresponds to a maximum matching and vice versa (the edges from left vertices to right vertices carrying flow equal to 1 constitute the maximum matching) and has therefore value equal to cardinality of the maximum matching. On the other hand, since capacities of the edges are all integral, value of integral maximum flow = value of maximum flow = capacity of minimum cut.
Every minimum cut can be seen to be of the form

$$(s \uplus X \uplus \Gamma(X),\ (t \uplus (V_L - X) \uplus (V_R - \Gamma(X))).$$

The capacity of this min cut can be seen to be $| V_L - X | + | \Gamma(X) |$, since the edges of the cut go from s to $(V_L - X)$ or from $\Gamma(X)$ to t.

Every cover contains another of the form $\Gamma(X) \uplus (V_L - X), X \subseteq V_L$. Also corresponding to a cover of this form there exists a cut of the form described earlier whose capacity equals the size of the cover. Hence, size of the min cover equals the capacity of the min cut which in turn equals the size of the maximum matching.

□

E 11.7:
i. We note that a set of vertices of the form $\Gamma(X) \uplus (V_L - X)$ is a cover. Further any cover will contain a cover of the above form. If $(V_L - X) \uplus Y$, where $X \subseteq V_L$, $Y \subseteq V_R$, is a cover then $Y \supseteq \Gamma(X)$. For, otherwise, any edge with one endpoint in X and the other in $\Gamma(X)$ will not be incident on $(V_L - X) \uplus Y$. Thus, a set of vertices is a minimum cover iff it is a set of minimum cardinality of the form $\Gamma(X) \uplus (V_L - X)$,

i.e., iff it is a set of the form $\Gamma(X) \uplus (V_L - X)$, where $X \in \mathcal{B}_1, (\equiv \mathcal{B}_{f,g}$, where $f(\cdot) = |\Gamma|(\cdot)$ and $g(\cdot) = |\cdot|)$. Now a collection of blocks of Π_{pp} form an ideal of the partial order '$\geq$' iff their union is a member of $\mathcal{B}_1$ or if the collection is equal to Π_{pp}, in which case it includes $V_L - X_{max}$. Thus a set of vertices is a minimum cover iff it is a set of the form $\Gamma(X) \uplus (V_L - X)$, where X is the union of blocks of Π_{pp} contained in X_{max} and belonging to an ideal of the partial order $(\geq)$.

ii. Let $\Gamma(X) \uplus (V_L - X)$ be a min cover. Every edge of a matching will be incident on the vertices of this cover. By Theorem 11.2.2 the sizes of a maximum matching and a minimum cover are equal. Further, by definition, no two edges of a matching have a common vertex. Hence, precisely one edge of the maximum matching will be incident on each of the vertices of $\Gamma(X) \uplus (V_L - X)$. Each such edge will **either** have one end in X and the other in $\Gamma(X)$ **or** have one end in $(V_L - X)$ and the other end in $\Gamma(V_L - X) - \Gamma(X)$. Two possible min covers are $\Gamma(X_{min}) \uplus (V_L - X_{min})$ and $\Gamma(X_{max}) \uplus (V_L - X_{max})$. Thus it is clear that every maximum matching will meet every vertex of $(V_L - X_{min})$ and every vertex of $\Gamma(X_{max})$.

iii. The bipartite graphs B^K may be divided into three groups (changing the notation slightly for convenience) B', the subgraph on $X_{min} \uplus \Gamma(X_{min})$, B^f, the subgraph on $(V_L - X_{max}) \uplus (\Gamma(V_L - X_{max}) - \Gamma(X_{max}))$ and all the remaining bipartite graphs of the kind B^K, which we will call of the third kind.

By the arguments in part (ii) above, every maximum matching must have edges incident on each vertex of $\Gamma(X_{min})$ with the other end in X_{min} and must also have edges incident on each vertex of $(V_L - X_{max})$ with the other end in $\Gamma(V_L - X_{max}) - \Gamma(X_{max})$. Thus, every maximum matching intersects the edges of B' and B^f in maximum matchings for these bipartite graphs.
Next let $X_1, X_2 \in \mathcal{B}_1$ and let $X_2 \supseteq X_1$.
We have the min covers

$$\Gamma(X_2) \uplus (V_L - X_2) = \Gamma(X_1) \uplus (\Gamma(X_2) - \Gamma(X_1)) \uplus (V_L - X_2)$$

$$\Gamma(X_1) \uplus (V_L - X_1) = \Gamma(X_1) \uplus (X_2 - X_1) \uplus (V_L - X_2).$$

Any maximum matching must have edges which meet every vertex of $\Gamma(X_1)(\Gamma(X_2))$ and which have the other end in $X_1(X_2)$. It follows that it must have edges which meet every vertex of $(\Gamma(X_2) - \Gamma(X_1))$ and which have the other end in $(X_2 - X_1)$. Thus, every maximum matching intersects edges of the third kind of B^K in one of its maximum matchings.
Lastly, we observe that any minimum cover can be partitioned into vertex sets each of which is either the right vertex set or the left vertex set of the B^K (of all three kinds). Now, distinct B^K do not have common vertices. Since the maximum matching has the same size as the minimum cover, it follows that its intersection with the edge sets of the B^K partitions it.

E 11.8: $\mathcal{B}_{f,g}$ is the collection of subsets Z which minimize the expression $f(Y) + g(S - Y), Y \subseteq S$.This is precisely the collection of subsets which maximize the expression $g(S) - (f(Y) + g(S - Y)), Y \subseteq S$, i.e., maximize the expression $(g(Y) - f(Y)), Y \subseteq S$.

E 11.9: We have,

$$min_{X\subseteq S}\mu(X) = min_{X\subseteq S}(f(X) - g(X))$$

$$= min_{X\subseteq S}(f(X) + g(S - X) - g(S)) = -(max_{X\subseteq S}[g(X) - f(X)]).$$

The latter two expressions define the sets involved in $\mathcal{B}_{f,g}$ and the membership problem respectively.

E 11.10: Let $r(\cdot) \equiv \rho * \mid \cdot \mid$. A set $X \subseteq S$ is independent iff $r(X) = |X|$, i.e., iff $(\rho * \mid \cdot \mid)(X) =\mid X \mid$, i.e., iff

$$min_{Y\subseteq X}(\rho(Y)+ \mid X - Y \mid) =\mid X \mid .$$

Clearly this would happen iff $\rho(Y) \geq\mid Y \mid \quad \forall Y \subseteq X$. A set $X \subseteq S$ is a circuit of the matroid iff $r(X) =\mid X \mid -1$ and all its proper subsets are independent. Hence, in order that X is a circuit we must have $\rho(Y) \geq\mid Y \mid \quad \forall Y \subset X$ and $\rho(X) <\mid X \mid$.

E 11.11: The reader is referred to Definition 11.4.5. We use the following simple lemma.

Lemma 11.9.1 *Let $f(\cdot), g(\cdot)$ be real valued set functions on subsets of $\hat{S}$, with $g(\cdot)$ symmetric with respect to $f(\cdot)$. Then $f * g(\cdot)$ is symmetric with respect to $f(\cdot)$.*

Proof: Let α be an automorphism of $f(\cdot)$. Then α is also an automorphism of $g(\cdot)$. We will show that α is an automorphism of $f * g(\cdot)$.
Let $X \subseteq \hat{S}$. We need to show that $f * g(X) = f * g(\alpha(X))$. We have

$$\begin{aligned} f * g(X) &\equiv min_{Y\subseteq X}(f(Y) + g(X - Y)) \\ &= min_{Y\subseteq X}(f(\alpha(Y)) + g(\alpha(X - Y))) \\ &= min_{Y\subseteq X}(f(\alpha(Y)) + g(\alpha(X) - \alpha(Y))) \\ &= min_{\alpha(Y)\subseteq\alpha(X)}(f(\alpha(Y)) + g(\alpha(X) - \alpha(Y))) \\ &= f * g(\alpha(X)). \end{aligned}$$

□

Now let $r(\cdot) \equiv \hat{f} * g(\cdot)$ be a matroid rank function with $\hat{f}(\cdot)$, a polymatroid rank function and $g(\cdot) \equiv\mid \cdot \mid$. Let $X \subseteq \hat{S}$. Let $e \in X$ and let $Y \subseteq X$ be the set of all elements parallel to e in $\hat{f}(\cdot)$ and contained in X. Further let $Z \supseteq Y$ be the set of all elements of $\hat{S}$ parallel to e in $\hat{f}(\cdot)$. Clearly if $Y' \subseteq Z$ is such that $\mid Y \mid=\mid Y' \mid$, then there exists an automorphism α of $\hat{f}(\cdot)$ in which $\alpha(X) = (X - Y) \uplus Y'$. Since $g(\cdot)$ is the $\mid \cdot \mid$ function, α is an automorphism also for it. Hence, α is an automorphism also for $r(\cdot)$. Thus, $r(X) = r(\alpha(X))$. The required results now follow.

E 11.12: When $\lambda = 0$, the sets X in $\mathcal{B}_\lambda$ are those that minimize $g(S - X)$, i.e., the sets which are complements of sets on which $g(\cdot)$ takes zero value. Since $g(\cdot)$ is strictly increasing, the only possible such set is S. Thus all critical values have to be positive. When $\lambda = +\infty$ the sets X in $\mathcal{B}_\lambda$ are those that minimize $f(\cdot)$. These are the sets on which $f(\cdot)$ takes minimum value. Clearly, if $f(\cdot)$ is a polymatroid rank function $X_\lambda = \emptyset$, while X^λ is the maximal set on which $f(\cdot)$ takes zero value.

E 11.13:
i. We have, if $X, Y \in \mathcal{B}_\lambda$ so do $X \cup Y$ and $X \cap Y$ belong.

$$\begin{aligned}\lambda f * g(S) &= \lambda f(X) + g(S - X) \\ &= \lambda f(Y) + g(S - Y) \\ &= \lambda f(X \cup Y) + g(S - (X \cup Y)) \\ &= \lambda f(X \cap Y) + g(S - (X \cap Y))\end{aligned}$$

But $g(S - X) + g(S - Y) \geq g(S - (X \cup Y)) + g(S - (X \cap Y))$
and $\lambda f(X) + \lambda f(Y) \geq \lambda f(X \cup Y) + \lambda f(X \cap Y), \lambda \geq 0$.
We conclude that the inequalities must be satisfied as equalities. Since $\lambda \geq 0$, this proves the required result.

ii.

$$f^*(S - X) + f^*(S - Y)$$

$$= \sum_{e \in (S-X)} g(e) - (f(S) - f(X)) + \sum_{e \in (S-Y)} g(e) - (f(S) - f(Y))$$

$$= \sum_{e \in S-(X \cap Y)} g(e) - (f(S) - f(X \cap Y)) + \sum_{e \in S-(X \cup Y)} g(e) - (f(S) - f(X \cup Y))$$

$$= f^*(S - (X \cap Y)) + f^*(S - (X \cup Y)),$$

where we have used the facts that

$$\sum_{e \in S-X} g(e) + \sum_{e \in S-Y} g(e) = \sum_{e \in S-(X \cap Y)} g(e) + \sum_{e \in S-(X \cup Y)} g(e)$$

and

$$f(X) + f(Y) = f(X \cup Y) + f(X \cap Y).$$

E 11.14: All the properties derived for the earlier case also hold here. However in this case we can also speak of the minimization problem for $\lambda < 0$. This is because the function h_λ is submodular even for negative values of λ if $f(\cdot)$ is a positive weight function. Property PP2 then holds for all real λ. However for $\lambda = 0$ it is easily seen that S minimizes h_λ. Hence for all negative values of λ also S is the only minimizing set.

E 11.15: To obtain the principal partition of $f(\cdot)$ with respect to $g(\cdot)$: For each $e \in S$, compute the ratio $g(e)/f(e)$. The critical values are the various ratios obtained in this manner. For a particular critical value λ, $X_\lambda = \{e_i, g(e_i)/f(e_i) > \lambda\}$ and $X^\lambda = \{e_i, g(e_i)/f(e_i) \geq \lambda\}$. $\mathcal{B}_\lambda$ has as member every subset of X^λ that is also a superset of X_λ. It is easily seen that the critical values of $f(\cdot)$ with respect to $g(\cdot)$ are the reciprocals of those of $g(\cdot)$ with respect to $f(\cdot)$ while the minimal and maximal sets relative to a particular critical value of the former are complements of the maximal and minimal sets relative to the corresponding critical value of the latter.

E 11.16:
i(a). We have,

$$\lambda f(X) + g(S - X) = (1/\alpha)[(\lambda\alpha/\beta)\beta f(X) + \alpha g(S - X)].$$

Clearly therefore we must have,

$$\mathcal{B}_{\lambda f,g} = \mathcal{B}_{(\lambda\alpha/\beta)\beta f,\alpha g}.$$

Thus we see that the principal partitions of $(f(\cdot), g(\cdot))$ and $(\beta f(\cdot), \alpha g(\cdot))$ are the same when $\alpha, \beta > 0$. However, if the critical values of $(f(\cdot), g(\cdot))$ are $\lambda_1, \cdots, \lambda_t$, then the critical values of $(\beta f(\cdot), \alpha g(\cdot))$ are $(\alpha/\beta)\lambda_1, \cdots (\alpha/\beta)\lambda_t$.

i(b). We see that,

$$\lambda f(X) + g(S - X) = \lambda(f(X) + \alpha g(X)) + g(S - X) - \lambda\alpha(g(S) - g(S - X))$$
$$= \lambda(f(X) + \alpha g(X)) + (1 + \lambda\alpha)g(S - X) - \lambda\alpha g(S).$$

Clearly therefore we must have,

$$\mathcal{B}_{\lambda f,g} = \mathcal{B}_{\lambda(f+\alpha g),(1+\lambda\alpha)g} = \mathcal{B}_{(\lambda/(1+\lambda\alpha))(f+\alpha g),g}.$$

Thus the principal partitions of $(f(\cdot), g(\cdot))$ and $((f + \alpha g)(\cdot), g(\cdot))$, where $\alpha > 0$, are the same. If λ is a critical value of $(f(\cdot), g(\cdot))$ then $\lambda/(1 + \lambda\alpha)$ is the corresponding critical value of $((f + \alpha)(\cdot), g(\cdot))$, where $\alpha > 0$.

ii. We have,

$$(\beta f + f_3)(X) + g(S - X) = (\beta f(X) + [g(X) - (f_2(S) - f_2(S - X))]) + g(S - X)$$
$$= \beta f(X) + f_2(S - X) - f_2(S) + g(S).$$

Thus the desired result follows.

E 11.17: It is trivial that reflexivity holds. Next let $e_1 \succeq_C e_2$ and let $e_2 \succeq_C e_3$. To see that transitivity holds we need to verify that $e_1 \succeq_C e_3$. By the definition of the preorder we have that whenever e_1 belongs to a member of $\mathcal{C}$, e_2 will also belong to it and whenever e_2 belongs to a member of $\mathcal{C}$, e_3 will also belong to it. But this implies that whenever e_1 belongs to a member of $\mathcal{C}$, e_3 will also belong to it. Thus $e_1 \succeq_C e_3$, as required.

E 11.18: Proof of Lemma 11.4.2:
i. This follows from DeMorgan's rule, i.e.,

$$S - (X \cup Y) = (S - X) \cap (S - Y), S - (X \cap Y) = (S - X) \cup (S - Y).$$

ii. We have $e_i \succeq_C e_j$ iff whenever e_i belongs to a set in $\mathcal{C}$, e_j also belongs to it, i.e., iff whenever e_j does not belong to some set K in $\mathcal{C}$, e_i also does not belong to it, i.e., iff whenever e_j belongs to a set $S - K$ in $\mathcal{C}'$, e_i also belongs to it, i.e., iff $e_j \succeq_{C'} e_i$.

iii. This is immediate from the above.

□

E 11.19: Suppose otherwise. Let λ be the density of T. We must have $g(S) - g(S-T') - \lambda f(T') \geq 0 = g(S) - g(S-T) - \lambda f(T)$. Hence, $g(S-T) + \lambda f(T) \geq g(S-T') + \lambda f(T')$. But $g(S-T) = g(S-T')$. Hence, $f(T) \geq f(T')$, since $\lambda > 0$. Let $T \in \mathcal{B}_\sigma$. Then T minimizes the expression $g(S-X) + \sigma f(X) \ \ \forall X \subseteq S$. But since $\sigma > 0$, (see Exercise 11.12) $g(S-T) + \sigma f(T) \geq g(S-T') + \sigma f(T')$, a contradiction, since T' is given to be not a set in the principal partition (and therefore, not in B_σ).

E 11.20:
i. We have S molecular iff $\lambda f * g(S) \equiv min_{X \subseteq S}(\lambda f(X) + g(S-X))$
$= \lambda f(S) + g(\emptyset) = \lambda f(\emptyset) + g(S) = \lambda f(S) = g(S)$.

ii. Let λ be the only critical value. We then have, by the above argument, $\lambda = g(S)/f(S)$ (= the density of S).

iii. S is molecular iff there is only one critical value λ, i.e., iff $S, \emptyset$ minimize $\lambda f(X) + g(S-X)$ (and this minimum value is $g(S)$)
i.e., iff $S, \emptyset$ maximize $g(S) - g(S-X) - \lambda f(X)$ (and this maximum value is 0), i.e., iff $(g(S) - g(S-X))/f(X)$ reaches its maximum value (=λ) at S.
The argument for the atomic case is essentially the same.

E 11.21: Proof of Theorem 11.4.3:
i. Let T satisfy the $\lambda-$density loss condition, and let $\hat{T}$ minimize $h_\lambda(\cdot)$ over subsets of S. We have by submodularity of $h_\lambda(\cdot)$

$$h_\lambda(T) + h_\lambda(\hat{T}) \geq h_\lambda(T \cup \hat{T}) + h_\lambda(T \cap \hat{T}).$$

Since T satisfies the $\lambda-$density loss condition, it can be seen that

$$h_\lambda(T) \leq h_\lambda(T \cap \hat{T}).$$

Hence, $h_\lambda(\hat{T}) \geq h_\lambda(T \cup \hat{T})$, i.e., $T \cup \hat{T}$ minimizes $h_\lambda(\cdot)$ over subsets of S. The argument for the $\lambda-$density gain condition is similar.

ii. (Strict $\lambda-$density loss case)
Going through the argument of the λ - density loss case used above, here we get $h_\lambda(T) < h_\lambda(T \cap \hat{T})$ unless $T \cap \hat{T} = T$.
The former alternative implies $h_\lambda(T \cup \hat{T}) < h_\lambda(\hat{T})$, a contradiction. Hence, we must have $T \cap \hat{T} = T$. Hence, $\hat{T} \supseteq T$.

iii. Suppose T satisfies both the $\lambda-$ density loss and the $\lambda-$ density gain conditions. The former condition implies that there exists a set $\hat{T}$ with $T \subseteq \hat{T} \subseteq S$ s.t. $\hat{T}$ minimizes $h_\lambda(\cdot)$ over subsets of S. Suppose $h_\lambda(\hat{T}) < h_\lambda(T)$. Then, it can be seen that this violates the $\lambda-$density gain condition satisfied by T. We conclude that $h_\lambda(\hat{T}) = h_\lambda(T)$. Thus, T minimizes $h_\lambda(\cdot)$ over subsets of S.

□

E 11.22: We only prove the $\lambda-$density gain case. By the submodularity of $h_\lambda(\cdot)$

we have

$$h_\lambda(T_1) + h_\lambda(T_2) \geq h_\lambda(T_1 \cup T_2) + h_\lambda(T_1 \cap T_2).$$

By the $\lambda-$density gain property of T_1,

$$h_\lambda(T_1) \leq h_\lambda(T_1 \cup T_2).$$

Hence,

$$h_\lambda(T_2) \geq h_\lambda(T_1 \cap T_2).$$

Interchanging T_1, T_2 we get

$$h_\lambda(T_1) \geq h_\lambda(T_1 \cap T_2).$$

E 11.23: Using the result in Exercise 11.22 it follows that (since T_1, T_2 have the $\lambda-$density gain property),

$$h_\lambda(T_1) \geq h_\lambda(T_1 \cap T_2).$$

Hence, $T_1 \cap T_2$ minimizes $h_\lambda(\cdot)$.
Similarly, since T_1, T_2 have the $\lambda-$density loss property,

$$h_\lambda(T_1) \geq h_\lambda(T_1 \cup T_2).$$

The result follows.

E 11.24: We will show that T_1 satisfies the strict λ_2-density loss property. The result then follows from Theorem 11.4.3. Let $T_3 \subseteq T_1$. We have,

$$h_{\lambda_1}(T_3) \geq h_{\lambda_1}(T_1),$$

i.e., $g(S - T_3) + \lambda_1 f(T_3) \geq g(S - T_1) + \lambda_1 f(T_1)$.
Hence, $\frac{1}{\lambda_1} g(S - T_3) + f(T_3) \geq \frac{1}{\lambda_1} g(S - T_1) + f(T_1)$.
Hence,

$$\begin{aligned}(\frac{1}{\lambda_1} - \frac{1}{\lambda_2})\ g(S - T_3) + \frac{1}{\lambda_2}\ g(S - T_3) + f(T_3) \\ \geq (\frac{1}{\lambda_1} - \frac{1}{\lambda_2})\ g(S - T_1) + \frac{1}{\lambda_2}\ g(S - T_1) + f(T_1).\end{aligned}$$

Now $\frac{1}{\lambda_1} < \frac{1}{\lambda_2}$ and $g(S - T_1) < g(S - T_3)$.
Hence, $\frac{1}{\lambda_2}\ g(S - T_3) + f(T_3) > \frac{1}{\lambda_2}\ g(S - T_1) + f(T_1)$,
i.e., $h_{\lambda_2}(T_3) > h_{\lambda_2}(T_1)$,
which is the strict λ_2-density loss condition for T_1.

E 11.26: Proof of Lemma 11.4.5:
i. This is immediate from the definition.

ii. Let $X \subseteq K$. Then

$$(f * g) \diamond \mathbf{K}(X) \equiv (f * g)((S - K) \cup X) - (f * g)(S - K).$$

Since $f * g(S-K) = f(S-K)$, there must exist a subset Y' of $X \cup (S-K)$ such that $Y' \supseteq S-K$ and $(f*g)((S-K)\cup X) = f(Y') + g(((S-K)\cup X) - Y')$ (by taking $X \cup (S-K)$ as the underlying set in Corollary11.4.2). Hence

$$(f*g)\diamond \mathbf{K}(X) = f(Y') + g(((S-K)\cup X) - Y') - f(S-K)$$

$$= min_{(S-K)\cup X \supseteq Y \supseteq S-K}(f(Y) + g(((S-K)\cup X) - Y) - f(S-K)) \cdots (*).$$

Next

$$(f\diamond \mathbf{K} * g \diamond \mathbf{K})(X) = min_{Z\subseteq X}(f\diamond \mathbf{K}(Z) + g\diamond \mathbf{K}(X-Z))$$

$$= min_{Z\subseteq X}(f((S-K)\cup Z) - f(S-K) + g(X-Z)) \cdots \qquad (**).$$

Taking $Y = (S-K)\cup Z$ we see that $(*)$ and $(**)$ are identical minimization problems. The result follows.

□

E 11.27: We have, for $X \subseteq T_3$,

$$(f/\mathbf{T}_1 \diamond \mathbf{T}_3)(X) \equiv f(X \cup (T_1 - T_3)) - f(T_1 - T_3),$$

$$(f/\mathbf{T}_2 \diamond \mathbf{T}_3)(X) \equiv f(X \cup (T_2 - T_3)) - f(T_2 - T_3),$$

$$f(T_1) - f(T_1 - T_3) = f(T_2) - f(T_2 - T_3),$$

$$f(X\cup(T_1 - T_3)) - f(T_1 - T_3) \leq f(X \cup (T_2 - T_3)) - f(T_2 - T_3).$$

Suppose the above inequality is strict. Then

$$f(T_1) - f(X\cup (T_1 - T_3)) > f(T_2) - f(X \cup (T_2 - T_3)),$$

which contradicts the submodularity of $f(\cdot)$. We conclude that the inequality must be satisfied as an equality which proves the required result.

E 11.28: Each of $X_1 \cap E_k$, $X_2 \cap E_k$, $(X_1 \cup X_2)\cap E_k$, $(X_1 \cap X_2)\cap E_k$, is a union of blocks of Π_{pp} in an ideal of $\geq_k$. By the manner in which $\geq_k$ was extended to blocks in E_{k+1} and by use of Lemma 11.5.1, we know that $X_1 - E_k$ is contraction related to $X_1 \cap E_k$, $X_2 - E_k$ to $X_2 \cap E_k$, $X_1 \cup X_2 - E_k$ to $(X_1 \cup X_2) \cap E_k$ and $X_1 \cap X_2 - E_k$ to $(X_1 \cap X_2) \cap E_k$. Since we assume that $\geq_k$ is a modular refinement of $\geq_\pi$ over blocks contained in E_k and the concerned sets correspond to ideals of $\geq_k$, we have,

$$f(X_1 \cap E_k) + f(X_2 \cap E_k) = f(X_1 \cup X_2) \cap E_k + f((X_1 \cap X_2)\cap E_k) \qquad (*)$$

Further we have,

$$f(X_1 \cup E_k) + f(X_2 \cup E_k) = f(X_1 \cup X_2 \cup E_k) + f((X_1 \cap X_2)\cup E_k),$$

since these are all sets in $\mathcal{B}_{\lambda_{k+1}}$. Hence,

$$(f(X_1 \cup E_k) - f(E_k)) + (f(X_2 \cup E_k) - f(E_k)) = (f(X_1 \cup X_2 \cup E_k) - f(E_k)) +$$

$$(f((X_1 \cap X_2) \cup E_k) - f(E_k)).$$

Hence, using the facts that $X_1 - E_k$ is contraction related to $X_1 \cap E_k$ etc.,

$$(f(X_1) - f(X_1 \cap E_k)) + (f(X_2) - f(X_2 \cap E_k)) = (f(X_1 \cup X_2) - f((X_1 \cup X_2) \cap E_k))$$
$$+(f(X_1 \cap X_2) - f((X_1 \cap X_2) \cap E_k)).$$

Using $(*)$ this reduces to

$$f(X_1) + f(X_2) = f(X_1 \cup X_2) + f(X_1 \cap X_2)$$

as desired.

E 11.29: Clearly $\mathbf{x}/E_1$ is a base for $(f/\mathbf{E}_1)(\cdot)$. On the blocks of Π_{pp} contained in E_1 both the partial orders agree. So the statement is true in this case. Next $\mathbf{x}/E_k$ is a base for $(f/\mathbf{E}_k)(\cdot)$. Suppose the statement is true for this case. We will show that it is also true for the base $\mathbf{x}/E_{k+1}$ of $(f/\mathbf{E}_{k+1})(\cdot)$. Now $\mathbf{x}/E_{k+1}$ is consistent with respect to $\geq_\pi$. Let $X \subseteq E_{k+1}$ be the union of blocks of an ideal of $\geq_R$. Then, $X \cap E_k$ corresponds to an ideal of $\geq_R$ and by the induction assumption we must have

$$\mathbf{x}(X \cap E_k) = f(X \cap E_k). \qquad (*)$$

Further, $X \cup E_k, E_k$ correspond to ideals of $\geq_\pi$.
Hence, $\mathbf{x}(X \cup E_k) = f(X \cup E_k)$
and $\mathbf{x}(E_k) = f(E_k)$.
Hence, $\mathbf{x}(X - E_k) = f(X \cup E_k) - f(E_k) = f(X) - f(X \cap E_k)$,
since X is contraction related to $X \cap E_k$. Hence, using $(*)$, $\mathbf{x}(X) = f(X)$.

E 11.30: Let $E_1, \cdots, E_t$ be the principal sequence of $(f(\cdot), g(\cdot))$. Let $X_{i_1}, \cdots, X_{i_r}$ be the earliest distinct nonvoid terms (in the same order) in the sequence $E_1 \cap X, \cdots, E_t \cap X$ and $\lambda_{i_1}, \cdots, \lambda_{i_r}$ be the corresponding critical values. Then $\emptyset, X_{i_1}, \cdots, X_{ir}$ is the principal sequence of $(f/\mathbf{X})(\cdot), (g/\mathbf{X})(\cdot))$ and $\lambda_{i_1}, \cdots, \lambda_{i_r}$ is the decreasing sequence of critical values. The partition Π' associated would be the collection of blocks of Π_{pp} which are contained in X. The partial order associated with the principal partition would be the restriction of $\geq_\pi$ to Π' and the refined partial order would be the restriction of $\geq_R$ to Π'. We only sketch the proof.
It can be seen that $E_{i_1} \cap X$ is molecular with critical value λ_{i_1} and that if $E_{i_1} \cap X$ is contracted $((E_{i_2} - E_{i_1}) \cap X)$ is molecular with critical value λ_{i_2} and so on. (By the definition of $\geq_R$, using Exercise 11.27, $(f/(\mathbf{E}_{i_1} \cap \mathbf{X}) \cup \mathbf{E}_{i_1-1}) \diamond (\mathbf{E}_{i_1} \cap \mathbf{X}))(\cdot) = (f/\mathbf{E}_{i_1} \cap \mathbf{X})(\cdot)$, noting that $E_{i_1-1} \cap X = \emptyset$. Further

$$(f/((\mathbf{E}_{i_2} \cap \mathbf{X}) \cup \mathbf{E}_{i_2-1}) \diamond ((\mathbf{E}_{i_2} - \mathbf{E}_{i_1}) \cap \mathbf{X}))(\cdot) = (f/(\mathbf{E}_{i_2} \cap \mathbf{X}) \diamond ((\mathbf{E}_{i_2} - \mathbf{E}_{i_1}) \cap \mathbf{X}))(\cdot)).$$

Also if Y is an ideal of $\geq_\pi$, $Y \cap X \cap E_{i_1}$, if nonnull, would be molecular with critical value λ_{i_1} and when $X \cap E_{i_1}$ (or $Y \cap E_{i_1}$) is contracted, $Y \cap X \cap (E_{i_2} - E_{i_1})$, if nonnull, would be molecular with critical value λ_{i_2} etc. Thus, using Theorem 11.4.6, the partial order associated with the principal partition of $(f/\mathbf{X})(\cdot), (g/\mathbf{X})(\cdot))$ is the restriction of $\geq_\pi$ to Π'. Further if Y is the union of blocks of Π_{pp} in an ideal of $\geq_R$ contained in X, $Y \cap E_{i_1}$ would similarly correspond to an ideal of $\geq_R$. Thus, in the principal partition of $(f(\cdot), g(\cdot))$, $Y \cap (E_{i_2} - E_{i_1})$ would be properly related to $Y \cap E_{i_1}$ which is contained in X. Using this argument inductively it follows that the ideals of the refined partial order of $(f/\mathbf{X})(\cdot), (g/\mathbf{X})(\cdot))$ are precisely the ideals of $\geq_R$ of $(f(\cdot), g(\cdot))$ contained in X, as required.

E 11.31: We will show that the collection of all complements of ideals of $\geq_R$ with respect to $(f(\cdot), g(\cdot))$ satisfy the characteristic properties (given in Theorem 11.5.1) of the ideals of the refined partial order with respect to $(f^*(\cdot), g(\cdot))$. We claim that the ideals of this partial order are precisely the complements of ideals of $\geq_R$. (We note that this statement is true for the partial orders associated with the principal partitions of $(f(\cdot), g(\cdot))$, $(f^*(\cdot), g(\cdot))$ respectively). Let Z_σ, Z^σ denote respectively the minimal and maximal members of $\mathcal{B}^*_\sigma$, the collection of subsets which minimize the expression $\sigma f^*(X') + g(S - X')$. We need to verify two conditions:
When Z is a complement of an ideal X of $\geq_R$,

- $(Z \cap Z_{\sigma_{j+1}}) \cup Z_{\sigma_j}$ is a set in $\mathcal{B}^*_{\sigma_j}$.
- If $Z \not\subseteq Z_{\sigma_{j+1}}$, then

$$f^*(Z \cap Z_{\sigma_{j+1}}) - f^*(Z \cap Z_{\sigma_j}) = f^*((Z \cap Z_{\sigma_{j+1}}) \cup Z_{\sigma_j}) - f(Z_{\sigma_j}).$$

i. We know that (by Theorem 11.4.5), for some λ_r,

$$Z_{\sigma_{j+1}} = S - X_{\lambda_r}$$
$$Z_{\sigma_j} = S - X_{\lambda_{r+1}},$$

where $X_{\lambda_r}, X_{\lambda_{r+1}}$ are the minimal members respectively of $\mathcal{B}_{\lambda_r}, \mathcal{B}_{\lambda_{r+1}}$. Next,

$$\begin{aligned}
(Z \cap Z_{\sigma_{j+1}}) \cup Z_{\sigma_j} &= ((S - X) \cap (S - X_{\lambda_r})) \cup (S - X_{\lambda_{r+1}}) \\
&= (S - (X \cup X_{\lambda_r})) \cup (S - X_{\lambda_{r+1}}) \\
&= S - ((X \cup X_{\lambda_r}) \cap X_{\lambda_{r+1}}) \\
&= S - [(X \cap X_{\lambda_{r+1}}) \cup X_{\lambda_r}]
\end{aligned}$$

Thus the LHS is the complement of a member of $\mathcal{B}_{\lambda_r}$ (whose maximal member is $X_{\lambda_{r+1}}$). It is therefore, a member of $\mathcal{B}^*_{\sigma_j}$ (whose minimal member is $Z_{\sigma_j} = S - X_{\lambda_{r+1}}$). This proves the first condition.

ii. This follows directly by using the definition of $f^*(\cdot)$, that Z is a complement of an ideal X in $\geq_R$, that $Z_{\sigma_{j+1}}, Z_{\sigma_j}$ are complements of $X_{\lambda_r}, X_{\lambda_{r+1}}$ respectively and that any two ideals of $\geq_R$ form a modular pair for $f(\cdot)$.

E 11.32:
i. Let $h_1(Y), h_2(Y)$ denote respectively $f(Y) + g(X - Y)$ and $f(Y) + g(X \cup e - Y), e \in (S - X)$. It is easily verified that these functions are submodular. Now $h_2(Y) = h_1(Y) + g(e),\ Y \subseteq X$. Hence, $h_2(Y) > h_2(X_{min})\quad \forall Y \subset X_{min}$ and $h_2(Y) \geq h_2(X_{max})\quad \forall Y \subseteq X_{max}$. So using the submodular inequality for $h_2(\cdot)$ on

X_{min} and $(X \cup e)_{min}$, we conclude that $(X \cup e)_{min} \supseteq X_{min}$. Similarly using it on X_{max} and $(X \cup e)_{max}$ we conclude that $(X \cup e)_{max} \supseteq X_{max}$.

ii. If $f(X_{max} \cup e) = f(X_{max})$, then
$(f * g)(X)$

$$= f(X_{max}) + g(X - X_{max}) = f(X_{max} \cup e) + g(X - X_{max}) \geq (f * g)(X \cup e).$$

However, we know that (since $f(\cdot), g(\cdot)$ are increasing) $(f * g)(X \cup e) \geq (f * g)(X)$. Hence, $(f * g)(X) = (f * g)(X \cup e)$ (by Theorem 11.3.1).

Next let $(f * g)(X) = (f * g)(X \cup e)$. We know $(X \cup e)_{max} \supseteq X_{max}$. Suppose $e \notin (X \cup e)_{max}$. Then

$$\begin{aligned} (f * g)(X \cup e) &= f((X \cup e)_{max}) + g((X - (X \cup e)_{max}) \cup e) \\ &> f((X \cup e)_{max}) + g(X - (X \cup e)_{max}), \end{aligned}$$

since $g(\cdot)$ is a positive weight function. But the last expression on the RHS $\geq (f * g)(X)$, which is a contradiction. Hence,

$$(X \cup e)_{max} \supseteq X_{max} \cup e.$$

We then have

$$\begin{aligned} (f * g)(X \cup e) &= f((X \cup e)_{max}) + g(X - (X \cup e)_{max}) \\ &\geq f((X \cup e)_{max} - e) + g(X - (X \cup e)_{max}) \\ &\geq (f * g)(X) \end{aligned}$$

The only way these inequalities can be satisfied is through equalities. Hence,

$$(X \cup e)_{max} - e \subseteq X_{max},$$

Hence, $(X \cup e)_{max} \subseteq X_{max} \cup e$, and therefore,

$$(X \cup e)_{max} = X_{max} \cup e.$$

Now we have

$$\begin{aligned} (f * g)(X \cup e) &= f(X_{max} \cup e) + g(X - X_{max}) \\ &= (f * g)(X). \\ &= f(X_{max}) + g(X - X_{max}) \end{aligned}$$

Hence, $f(X_{max}) = f(X_{max} \cup e)$,and therefore,

$$f(X_{max}) = f((X \cup e)_{max}).$$

To compute S_{max} efficiently we start from $e_1 \in S$ and grow it to S in the following manner. Suppose we have reached the set $X = \{e_1, \cdots, e_k\}$ in this process and know X_{max}. For each $e \in S - X$ we check if $f(X_{max}) = f(X_{max} \cup e)$. If so we discard e. If $f(X_{max}) \neq f(X_{max} \cup e')$ we update X to the set $X \cup e'$. We compute $(X \cup e')_{max}$. We continue this process until we reach a set Y with $(f * g)(Y) = (f * g)(S)$. Observe that $\mid Y \mid \ll \mid S \mid$. At this stage S_{max} can be computed since

$$S_{max} = Y_{max} \cup (S - Y).$$

(If $e \in S - Y$, we know that for some $X' \subseteq Y_{max},\ f(X') = f(X' \cup e)$. So if $e \in S - Y$, and $Y' \supseteq Y_{max}, then\ f(Y') = f(Y' \cup e)$, by using submodularity of $f(\cdot)$ and the fact that it is increasing. By induction we conclude

$$S_{max} = Y_{max} \cup (S - Y).)$$

Thus, we performed $\mid Y \mid$ computations of X_{max} on sets which are much smaller than S instead of directly computing S_{max}. Suppose direct computation of X_{max} is $O(\mid X \mid^{\alpha})$ where $\alpha > 1$. Then

$$\mid Y \mid \mid Y \mid^{\alpha} = \mid Y \mid^{\alpha+1} < \mid S \mid^{\alpha},$$

provided $\mid Y \mid < \mid S \mid^{\alpha/\alpha+1}$ e.g. if $\alpha = 2, \mid S \mid = 1000$, we need $\mid Y \mid < (1000)^{2/3} = 100$. Often there are additional advantages in following this procedure. Whenever X_{max} is not a singleton, we could fuse it and work with the set $X - X_{max} \cup \{X_{max}\}$ and the corresponding fusion of the function. Or, when X_{max} is nonnull, we could contract X_{max} out and work with $f \diamond (\mathbf{S} - \mathbf{X}_{max})(\cdot)$.

E 11.33: Let $w'_L(\cdot)$ be the modified weight function. By Theorem 3.6.2, in order to minimize the expression $w'_L(V_L - X) + \lambda w_R(\Gamma)(X)$, we only need look for a min cut in the flow graph $F(B, \mathbf{w'}_L, \lambda \mathbf{w}_R)$. Since $w'_L(\cdot)$ and $w_R(\cdot)$ are strictly positive weight functions, this cut will 'correspond' to a subset $\hat{X} \subseteq V_L$. Now $w'_L(v) = \infty, v \in K$. It is then clear that $\hat{X}$ must contain K as otherwise the above expression would reach infinite value on $\hat{X}$. But if $\hat{X} \supseteq K$, $w'_L(V_L - \hat{X}) + \lambda w_R(\Gamma)(\hat{X}) = w_L(V_L - \hat{X}) + \lambda(w_R(\Gamma)(\hat{X})$. So such an $\hat{X}$ minimizes the above expression over all supersets of K.

E 11.34:
i. We construct the flow graph associated with B with $w_L(v_L)$ the capacity of the edge $(s, v_L), w_R(v_R)$ the capacity of the edge (v_R, t) and the edges of the bipartite graph directed from left to right with capacity ∞. A min cut will have the form

$$(s \uplus Y \uplus \Gamma_L(Y), t \uplus V_L - Y \uplus V_R - \Gamma_L(Y))$$

with capacity $w_R \Gamma_L(Y) + w_L(V_L - Y)$. This cut can have no forward arc of capacity ∞. Also every arc $(s, v), v \in V_L - X$ must be saturated and therefore carry nonzero flow. Hence there is an arc carrying nonzero flow from v into V_R. This arc cannot lead into $\Gamma_L(X)$ (otherwise it would be a backward arc carrying zero flow). So the arc goes from v into $V_R - \Gamma_L(X))$. Hence, $\Gamma_R(V_R - \Gamma_L(Y)) = V_L - Y$. So the capacity can also be written as

$$w_L(\Gamma_R(V_R - \Gamma_L(Y))) + w_R(V_R - (V_R - \Gamma_L(Y)))$$

and the result follows.

ii. This is an immediate consequence of the previous part.

E 11.35: Let $\lambda = p/q$. If we make q copies of each vertex v in V_L (making the copies adjacent to the right side vertices to which v is adjacent) and p copies of each vertex in V_R, then the resulting bipartite graph would have a matching which meets all the left vertices.

11.10 Solutions of Problems

P 11.1:
i. By the definition of the vector $\mathbf{x}$, we have

$$\frac{x(E_1)}{g(E_1)} = \frac{1}{\lambda_1}, \; i.e., \; \frac{x(E_1) - x(\emptyset)}{g(E_1) - g(\emptyset)} = \frac{1}{\lambda_1}.$$

By induction let us assume $x(E_i) = f(E_i),\ i < q$. We then have

$$\frac{1}{\lambda_q} = \frac{x(E_q) - x(E_{q-1})}{g(E_q) - g(E_{q-1})} = \frac{x(E_q) - f(E_{q-1})}{g(E_q) - g(E_{q-1})}.$$

But since $E_1, \cdots, E_k$ is the principal sequence of $(f(\cdot), g(\cdot))$ we know that,

$$g(E_q) - \lambda_q f(E_q) = g(E_{q-1}) - \lambda_q f(E_{q-1})$$

$$i.e., \qquad \frac{1}{\lambda_q} = \frac{f(E_q) - f(E_{q-1})}{g(E_q) - g(E_{q-1})} = \frac{x(E_q) - f(E_{q-1})}{g(E_q) - g(E_{q-1})}.$$

Hence, $x(E_q) = f(E_q)$.
So the result is true for all i.

ii. Let $S_j \equiv \{e_1, \cdots, e_j\}, j = 1, \cdots, n$. We have already seen that $x(E_i) = f(E_i), i = 1, \cdots, k$. In particular $x(E_k) = x(S) = f(S)$. We will now show that $x(S_j) \leq f(S_j), j = 1, \cdots, n$. By Theorem 10.7.2, it would then follow that $x(X) \leq f(X)$
$\forall X \subseteq S$ and, therefore, since $x(E_k) = x(S) = f(S)$, that $\mathbf{x}$ is a base of P_f.
Suppose $x(S_j) > f(S_j)$ for some j. Let r be the least index for which $S_j \not\supseteq E_r$.
We have $f(S_j) + f(E_r) \geq f(S_j \cap E_r) + f(S_j \cup E_r)$,
while $x(S_j) + x(E_r) = x(S_j \cap E_r) + x(S_j \cup E_r)$.
We conclude that either $x(S_j \cap E_r) > f(S_j \cap E_r)$ or $x(S_j \cup E_r) > f(S_j \cup E_r)$. In the former eventuality we pick the set $S_j \cap E_r$ for our subsequent arguments. Otherwise we repeat the process with $S_j \cup E_r$. Thus without loss of generality we may assume that $E_r \supset S_j \supseteq E_{r-1}$ for some r.
Now we have, by the definition of $\mathbf{x}$,

$$\frac{x(S_j) - x(E_{r-1})}{g(S_j) - g(E_{r-1})} = \frac{1}{\lambda_r}.$$

But $x(S_j) > f(S_j)$ while $x(E_{r-1}) = f(E_{r-1})$. Hence,

$$\frac{f(S_j) - f(E_{r-1})}{g(S_j) - g(E_{r-1})} < \frac{1}{\lambda_r},$$

i.e., $\lambda_r f(S_j) - g(S_j) < \lambda_r f(E_{r-1}) - g(E_{r-1})$. This contradicts the fact that E_{r-1} minimizes $\lambda_r f(X) + g(S - X)$. We conclude that $x(S_j) \leq f(S_j), j = 1, \cdots, n$, and hence, by Theorem 10.7.2, $\mathbf{x}$ is a base of P_f.

iii. Let $\mathbf{x}'$ be any other base of P_f with

$$\frac{x'(e'_1)}{g(e'_1)} \leq \cdots \leq \frac{x'(e'_n)}{g(e'_n)}.$$

Define $S'_j \equiv \{e'_1, \cdots, e'_j\}, j = 1, \cdots, n$. Let us say that the set E_r is broken by a sequence $e'_1, \cdots, e'_n$ iff for some i, $e'_i \notin E_r$ but $e'_j \in E_r$ for some $j > i$. Let t be the least index for which $x'(e'_t)/g(e'_t) \neq x(e_t)/g(e_t)$. We need to show that $x'(e'_t)/g(e'_t) < x(e_t)/g(e_t)$. We will prove this by contradiction. Let $x'(e'_t)/g(e'_t) > x(e_t)/g(e_t)$. We proceed by first proving the following claim.

Claim: Sets $E_1, \cdots, E_k$ are not broken by $e'_1, \cdots, e'_{t-1}$.

Suppose not. Let m be the least index for which E_m is broken by $e'_1, \cdots, e'_{t-1}$. Now $e_1, \cdots, e_t$ does not break $E_1, \cdots, E_k$. So if $e_t \in E_j - E_{j-1}$ it is clear, since $E_1 \subset E_2 \subset \cdots \subset E_n$, that $m \leq j$ and therefore, $\lambda_m \geq \lambda_j$. Hence,

$$x(e_t)/g(e_t) \geq \frac{1}{\lambda_m}.$$

Let q be the last index for which $e'_q \in E_m$. Since $e_1, \cdots, e_q$ does not break E_m, it follows that $e_q \in E_s - E_m, s > m$. Hence, if $q \leq t - 1$, we have

$$\frac{x'(e'_q)}{g(e'_q)} = \frac{x(e_q)}{g(e_q)} > \frac{1}{\lambda_m},$$

and if $q > t - 1$, we have

$$\frac{x'(e'_q)}{g(e'_q)} \geq \frac{x'(e'_t)}{g(e'_t)} > \frac{x(e_t)}{g(e_t)} \geq \frac{1}{\lambda_m}.$$

Thus it is clear that in every case

$$\frac{x'(e'_q)}{g(e'_q)} > \frac{1}{\lambda_m}.$$

Now let $\{e'_{i1}, \cdots, e'_{iw}\} = E_m - E_{m-1}$, where $i1 < \cdots < iw$. If $i1 < t$, then

$$\frac{x'(e'_{i1})}{g(e'_{i1})} = \frac{x(e_{i1})}{g(e_{i1})} \geq \frac{1}{\lambda_m}$$

(since E_{m-1} is not broken by $e'_1, \cdots, e'_n$, and, therefore, $e_{i1} \notin E_{m-1}$). If $i1 \geq t$, then

$$\frac{x'(e'_{i1})}{g(e'_{i1})} \geq \frac{x'(e'_t)}{g(e'_t)} > \frac{x(e_t)}{g(e_t)} \geq \frac{1}{\lambda_m},$$

We thus see that for each $e' \in E_m - E_{m-1}$, $\dfrac{x'(e')}{g(e')} \geq \dfrac{1}{\lambda_m}$.

In addition we have already seen that $\dfrac{x'(e'_q)}{g(e'_q)} > \dfrac{1}{\lambda_m}$.

It follows that

$$\frac{x'(e'_{i1}) + \cdots + x'(e'_{iw})}{g(e'_{i1}) + \cdots + g(e'_{iw})} > \frac{1}{\lambda_m},$$

$$i.e., \ \frac{x'(E_m) - x'(E_{m-1})}{g(E_m) - g(E_{m-1})} > \frac{1}{\lambda_m}.$$

Now $E_1, \cdots, E_{m-1}$ are not broken by $e'_1, \cdots, e'_{t-1}$. They are also not broken by $e_1, \cdots, e_n$. Further, $e_t \in E_j$ with $j \geq m$. So $E_{m-1} \subseteq \{e'_1, \cdots, e'_{t-1}\}$. It follows that for each $e'_x \in E_i - E_{i-1}, i \leq m-1$, we must have

$$\frac{x'(e'_x)}{g(e'_x)} = \frac{x(e_x)}{g(e_x)} = \frac{1}{\lambda_i}.$$

Hence, $x'(E_i) = x(E_i), i = 1, \cdots, m-1$. But $x(E_i) = f(E_i),\ i = 1, \cdots, k$. Hence, $x'(E_{m-1}) = f(E_{m-1})$. Hence,

$$\frac{f(E_m) - f(E_{m-1})}{g(E_m) - g(E_{m-1})} \geq \frac{x'(E_m) - x'(E_{m-1})}{g(E_m) - g(E_{m-1})} > \frac{1}{\lambda_m}.$$

This is a contradiction as we have shown in the previous section. This contradiction can be avoided only if $E_1, \cdots, E_k$ is not broken by $e'_1, \cdots, e'_{t-1}$. Thus, the claim is justified.
Now let $E_r \supseteq S'_{t-1} \supset E_{r-1}$. We consider two cases.
Case 1: $E_r \supseteq S'_t \supset E_{r-1}$.
Suppose $S'_{t-1} - E_{r-1} = \{e'_{j1}, \cdots, e'_{jy}\}, j1 < \cdots < jy$. Then $S_{t-1} - E_{r-1} = \{e_{j1}, \cdots, e_{jy}\}$. Now

$$\begin{aligned} \frac{x'(e'_{j1})}{g(e'_{j1})} &= \frac{x(e_{j1})}{g(e_{j1})} = \frac{1}{\lambda_r}, \\ &\vdots \\ \frac{x'(e'_{jy})}{g(e'_{jy})} &= \frac{x(e_{jy})}{g(e_{jy})} = \frac{1}{\lambda_r}. \end{aligned}$$

Further,

$$\frac{x'(e'_t)}{g(e'_t)} > \frac{x(e_t)}{g(e_t)} \geq \frac{1}{\lambda_r}.$$

Hence, for each $e' \in E_r - S'_{t-1}$,

$$\frac{x'(e')}{g(e')} > \frac{1}{\lambda_r}.$$

It follows that

$$\frac{x'(E_r) - x'(E_{r-1})}{g(E_r) - g(E_{r-1})} > \frac{1}{\lambda_r}.$$

By arguments used in the previous sections of this problem

$$x'(E_{r-1}) = x(E_{r-1}) = f(E_{r-1}).$$

Further, $x'(E_r) \leq f(E_r)$. We therefore have

$$\frac{f(E_r) - f(E_{r-1})}{g(E_r) - g(E_{r-1})} > \frac{1}{\lambda_r}, \text{ a contradiction.}$$

Case 2: $E_r = S'_{t-1}$
In this case

$$\frac{x'(E_{r+1}) - x'(E_r)}{g(E_{r+1}) - g(E_r)} \geq \frac{x'(e'_t)}{g(e'_t)} > \frac{x(e_t)}{g(e_t)} = \frac{1}{\lambda_{r+1}}.$$

Since $x'(E_r) = x(E_r) = f(E_r)$, and $x'(E_{r+1}) \leq f(E_{r+1})$, it follows that

$$\frac{f(E_{r+1}) - f(E_r)}{g(E_{r+1}) - g(E_r)} > \frac{1}{\lambda_{r+1}}, \text{ a contradiction.}$$

Since the assumption that

$$\frac{x'(e'_t)}{g(e'_t)} > \frac{x(e_t)}{g(e_t)},$$

leads us to a contradiction in every case, and since the two sides are not equal, we conclude that

$$\frac{x'(e'_t)}{g(e'_t)} < \frac{x(e_t)}{g(e_t)}.$$

(iv) From the arguments of the previous section, the only F-lexicographically optimum base is the one defined by

$$\frac{x(e)}{g(e)} = \frac{1}{\lambda_j}, e \in E_j - E_{j-1}, j = 1, \cdots, k.$$

This definition yields a unique base. So the F-lexicographically optimum base is unique.

P 11.2: [Fujishige91]
i. This has been already shown for the case where $g(\cdot)$ is a general submodular function (in the proof of PP1).

ii. We proceed as in the proof of PP2 slightly modifying the notation of that proof.
Let $p_1(X) \equiv f(X) + \sigma_1 g(S - X)$
$p_2(X) \equiv f(X) + \sigma_2 g(S - X)$.
If Z_1 minimizes $p_1(\cdot)$ and $Y \subset Z_1$ we have

$$p_2(Y) = p_1(Y) + (\sigma_2 - \sigma_1)g(S - Y)$$

$$p_2(Z_1) = p_1(Z_1) + (\sigma_2 - \sigma_1)g(S - Z_1)$$

Since $g(\cdot)$ is increasing, $Y \subset Z_1$ and $\sigma_2 > \sigma_1$, $p_2(Y) \geq p_2(Z_1)$. Hence, there is a set that minimizes $p_2(\cdot)$ and contains Z_1 (by Theorem 10.4.1). Hence, $Y^{\sigma_1} \subseteq Y^{\sigma_2}$. If Z_1 is the unique set that minimizes $p_1(\cdot)$ we have $p_1(Y) > p_1(Z_1)$, $\forall Y \subset Z_1$. Hence, $p_2(Y) > p_2(Z_1)$, $\forall Y \subset Z_1$.
Thus, in this case every set that minimizes $p_2(\cdot)$ would contain Z_1. Thus, $Y_{\sigma_2} \supseteq Y_{\sigma_1}$.

iii. The proof is similar to the case where $\sigma \geq 0$. Note that $\sigma g^d(\cdot)$ is a submodular function if $g(\cdot)$ is submodular and σ is negative.

iv. We have three cases:
a. $0 \leq \sigma_1 < \sigma_2$.
b. $\sigma_1 < 0 < \sigma_2$.
c. $\sigma_1 < \sigma_2 \leq 0$.
Cases (a) and (c) have already been considered in the previous sections of the present problem. So we consider only Case (b).
Case (b): Let $X_1 \in \mathcal{B}_{\cdot\sigma_1}, X_2 \in \mathcal{B}_{\cdot\sigma_2}$. We use the following facts:

$$g(S - (X_1 \cap X_2)) - [g(S - X_1) - g(S - (X_1 \cup X_2))] \leq g(S - X_2),$$
$$\sigma_2(g(S - (X_1 \cap X_2)) - g(S - X_1)) \geq 0,$$
$$\sigma_1(g(X_1 \cap X_2) - g(X_1)) \geq 0.$$

We have

$$\begin{aligned} f(X_1) &+ \sigma_1(g(S) - g(X_1)) + f(X_2) + \sigma_2 g(S - X_2) \\ &\geq f(X_1 \cup X_2) + f(X_1 \cap X_2) + \sigma_2 g(S - (X_1 \cup X_2)) \\ &\quad +\sigma_1(g(S) - g(X_1 \cap X_2)) + \sigma_2(g(S - (X_1 \cap X_2)) - g(S - X_1)) \\ &\quad +\sigma_1(g(S) - g(X_1)) - \sigma_1(g(S) - g(X_1 \cap X_2)) \\ &\geq f(X_1 \cup X_2) + \sigma_2 g(S - (X_1 \cup X_2)) \\ &\quad +f(X_1 \cap X_2) + \sigma_1(g(S) - g(X_1 \cap X_2)). \end{aligned}$$

The only way the final inequality can be satisfied is to have
$f(X_1) + \sigma_1(g(S) - g(X_1)) = f(X_1 \cap X_2) + \sigma_1(g(S) - g(X_1 \cap X_2))$ and
$f(X_2) + \sigma_2 g(S - X_2) = f(X_1 \cup X_2) + \sigma_2 g(S - (X_1 \cup X_2))$.
The required result now follows.

P 11.17: Familiarity with Theorems 11.4.4, 11.4.5,11.7.2 and Problems 11.12 and 11.10 is assumed in the following solution. A polymatroid rank function $f(\cdot)$ and another obtained from it by taking direct sum of the structures

$f/\mathbf{X}_{\lambda_{i+1}} \diamond (\mathbf{X}_{\lambda_{i+1}} - \mathbf{X}_{\lambda_i})(\cdot)$ have identical principal partitions and would continue to have identical principal partitions even if both are operated on by positive or negative expressions. This fact can be proved similar to the way in which Theorem 11.7.5 is proved and is also used below.

i. $(f_1 + f_2)(\cdot)$ would have identical principal partition and partial order as $f_1(\cdot)$ and $f_2(\cdot)$. The critical values would change as follows:

$$\lambda_{+1} = (\lambda_{11}^{-1} + \lambda_{21}^{-1})^{-1} = (\frac{1}{5} + \frac{1}{4})^{-1} = \frac{20}{9},$$

$$\lambda_{+2} = (\frac{1}{4} + \frac{1}{3})^{-1} = \frac{12}{7},$$

$$\lambda_{+3} = (\frac{1}{3} + \frac{2}{3})^{-1} = 1$$

$$\lambda_{+4} = (\frac{1}{2} + \frac{3}{4})^{-1} = \frac{4}{5}$$

$(f_1+f_2)*g$ would have $D_1 \cup D_2 \cup C_1 \cup C_2$ as coloops. The refined partial order would not change as far as A, B_1, B_2, B_3 are concerned. The elements in $D_1 \cup D_2 \cup C_1 \cup C_2$ would remain as isolated vertices in the Hasse Diagram of $(\geq_R)$.

ii. The function $((f_1 + f_2) * g)^*(\cdot)$ would have $D_1 \cup D_2 \cup C_1 \cup C_2$ as self loops. The refined partial order would be the dual of that of $((f_1 + f_2) * g)(\cdot)$. The critical values of $(((f_1 + f_2) * g)(\cdot)$ are $20/9, 12/7$ and 1, where the value 1 corresponds to $D_1 \cup D_2 \cup C_1 \cup C_2$. So the critical values of $((f_1 + f_2) * g)^*(\cdot)$ are

$$\infty, (1 - \frac{7}{12})^{-1}, (1 - \frac{9}{20})^{-1}$$

where the critical value ∞ corresponds to $D_1 \cup D_2 \cup C_1 \cup C_2$, $\frac{12}{5} = (1 - \frac{7}{12})^{-1}$ corresponds to $B_1 \cup B_2 \cup B_3$ and $\frac{20}{11} = (1 - \frac{9}{20})^{-1}$ corresponds to A.

A quick way of computing the principal partition in such cases is to examine what happens to each of the molecular structures $f/\mathbf{X}_{\lambda_{i+1}} \diamond (\mathbf{X}_{\lambda_{i+1}} - \mathbf{X}_{\lambda_i})(\cdot)$ corresponding to the different critical values. In the following discussion, for notational convenience, the function which is the direct sum of the above functions on the sets $(X_{\lambda_{i+1}} - X_{\lambda_i})$ will be denoted by $f(\cdot)$. Let us examine what would happen to the molecular structure on $B_1 \cup B_2 \cup B_3$ in $((f_1 + f_2) * g)^*(\cdot)$.
The corresponding critical value in $f_1(\cdot)$ is 4 and in $f_2(\cdot)$ it is 3. When two functions have molecular $(f_i(\cdot), g(\cdot))$ structure, adding them corresponds to addition of the reciprocal of the critical values. So the critical value for $(f_1 + f_2)(\cdot)$ is $(\frac{1}{4} + \frac{1}{3})^{-1} = \frac{12}{7}$. This value is above 1. So convolution with $g(\cdot)$ will not affect the critical value. Dualization would replace λ by $(1 - \frac{1}{\lambda})^{-1}$. So we get $\frac{12}{5}$ as the critical value for $((f_1 + f_2) * g)^*(\cdot)$. This does not correspond to coloops. So we know the refined partial order to be as in the case of $f_1(\cdot)$ and $f_2(\cdot)$ as far as B_1, B_2, B_3 and elements below these are concerned. If, however, for an intermediate expression $\omega(f_1, f_2)$ the critical value for $E_{i+1} - E_i$ (where E_i are the sets in the principal

sequence) falls ≤ 1, convolving with $g(\cdot)$ will make that region into coloops for the resulting function. In the present case no such intermediate expression occurs.

iii. Let us use the above technique with critical values to study the principal partition of $(((2f_1 * g)^* + f_1{}^*) * g)^*(\cdot)$. Critical value λ_{11} for $f_1(\cdot) \Rightarrow \frac{1}{2}\lambda_{11}$ for $2f_1(\cdot)$. Since $\frac{1}{2}\lambda_{11} > 1$, therefore, the corresponding critical value continues to be $\frac{1}{2}\lambda_{11}$ for $2f_1 * g(\cdot)$.
Next, $\frac{1}{2}\lambda_{11}$ for $2f_1 * g(\cdot) \Rightarrow (1 - \frac{2}{\lambda_{11}})^{-1}$ for $(2f_1 * g)^*(\cdot)$.
So $\lambda_{11} = 5$ for $f_1(\cdot)$ becomes the critical value $(1-\frac{2}{\lambda_{11}})^{-1} = \frac{5}{3}$ for $(2f_1 * g)^*(\cdot)$. Next, corresponding to λ_{11}, $f_1{}^*(\cdot)$ has critical value $(1-\frac{1}{5})^{-1} = \frac{5}{4}$. So $((2f_1 * g)^* + f_1{}^*)(\cdot)$ has critical value $(\frac{4}{5} + \frac{3}{5})^{-1} = \frac{5}{7}$ corresponding to λ_{11}. This value is less than 1. So convolution with $g(\cdot)$ will convert this set A into coloops and dualization would make these into selfloops. Repeating this computation with $\lambda_{12}, \lambda_{13}$ we find that $B_1 \cup B_2 \cup B_3$ and $C_1 \cup C_2$ also become sets of self loops in $((2f_1 * g)^* + f_1{}^*)(\cdot)$. However, in the case $\lambda_{14} = 2$, $2f_1(\cdot)$ has critical value 1. So $(2f_1 * g(\cdot))(\cdot)$ has only coloops and its dual has only selfloops in $D_1 \cup D_2$. Hence, $((2f_1 * g)^* + f_1{}^*)(\cdot)$ would coincide with $f_1{}^*(\cdot)$ on $D_1 \cup D_2$. Now $f_1{}^*(\cdot)$ has critical value 2. So does $f_1{}^* * g(\cdot)$ and also $(f_1{}^* * g)^*(\cdot)$. Thus the function $(((2f_1 * g)^* + f_1{}^*) * g)^*(\cdot)$ has the value $\frac{g(D_1 \cup D_2)}{2}$ on $D_1 \cup D_2$ and, since $S - (D_1 \cup D_2)$ is made of selfloops, the same value also on S.

Calculations in the case of the other functions are similar.

Chapter 12

Matroid Union

12.1 Introduction

In this chapter we study the important operation of matroid union. We first prove that the matroid union operation yields a matroid if we start with matroids. There are many ways of proving this result. We have chosen a route which has the merit of displaying, along the way, some of the deepest results in matroid theory (e.g. Rado's Theorem). However, space considerations have prevented us from giving the results the motivation they deserve. The key notion in our development is the idea of a submodular function induced through a bipartite graph. Next we present an algorithm for matroid union that is an immediate extension of Edmonds' famous algorithm for matroid partition [Edmonds65a]. We study this algorithm in detail and, as a consequence, the structure of the union matroid. Finally we use this algorithm to construct the principal partition of the rank function of a matroid with respect to the $|\cdot|$ function.

12.2 Submodular Functions induced through a Bipartite Graph

In this section we study the effect that a submodular function, defined on one side of a bipartite graph, has on the other side. Results of this nature include some of the deepest (e.g. Rado's Theorem on independent transversals) and some of the most practically useful (e.g. Matroid Union Theorem) in matroid theory. The treatment in this section largely follows that of Welsh [Welsh76]. However, we choose to work in the context of bipartite graphs rather than in that of families of sets.

Remark: In this chapter and in subsequent chapters, maximal independent sets would be denoted invariably by b and not by B.

We begin with a simple result which is a restatement of the one found in Problem 10.3 (second part).

Theorem 12.2.1 *Let $B \equiv (V_L, V_R, E)$ be a bipartite graph. Let $f(\cdot)$ be an increasing submodular function on subsets of V_R. Then $f(\Gamma_L(\cdot))$ (where $\Gamma_L(X), X \subseteq V_L$ is the set of vertices adjacent to X in B), is an increasing submodular function on subsets of V_L.*

(For proof see solution of the above mentioned problem.)
It is convenient at this stage to introduce terms which are commonly used in 'transversal theory'. Let $B \equiv (V_L, V_R, E)$ be a bipartite graph. A family $(x_i : i \in I)$ of vertices in V_L is a **system of representatives (SR)** of a subset $Y \equiv \{y_i : i \in I\}$ of V_R iff there exists a bijection $\tau : I \rightarrow I$ s.t. $x_i \in \Gamma_R(y_{\tau(i)})$ (where $\Gamma_R(X), X \subseteq V_R$ is the set of vertices adjacent to X in B). (We remind the reader that the definition of family (see page 16) permits $x_i = x_j$ even if $i \neq j$).
The system of representatives of Y becomes a **system of distinct representatives (SDR)** or a **transversal** of Y iff $x_i \neq x_j, i \neq j$. Alternatively, a set $T \subseteq V_L$ is a **transversal** of Y iff there is a bijection $\tau : T \rightarrow I$ s.t. $x \in \Gamma_R(y_{\tau(x)})\ \ \forall x \in T$. (A convenient way of defining this bijection when $Y \equiv V_R$ is to use the same index set for both the family of left vertices in an SR and the right vertex set V_R and the bijection to be the identity mapping). A **matching** in B is a set of edges of B no two of which have a vertex in common. Thus, $T \subseteq V_L$ is a transversal of Y iff there exists a matching in B with T as the set of left vertices and Y as the set of right vertices. Throughout this section, index sets (such as I) are taken to be finite. The set underlying a family $(x_i : i \in I)$ is denoted by $\{x_i : i \in I\}$.

Theorem 12.2.2 *(Welsh [Welsh76]) Let $B \equiv (V_L, V_R, E)$ be a bipartite graph with no parallel edges. Let $V_R \equiv \{y_i : i \in I\}$.*
Let $f(\cdot)$ be an increasing submodular function on subsets of V_L. Then V_R has a system of representatives $(x_i : i \in I)$ such that

$$f(\{x_i : i \in J\}) \geq \mid J \mid \quad \forall J \subseteq I \qquad (*)$$

iff

$$f(\Gamma_R(Y)) \geq \mid Y \mid \quad \forall Y \subseteq V_R. \qquad (**)$$

Proof : Only if: Let $(x_i : i \in I)$ be an SR of V_R. Let $Y \equiv \{y_i : i \in J\}$. (We note that by definition $y_i \neq y_j, i \neq j$). Then $\Gamma_R(Y) = \bigcup_{i \in J} \Gamma_R(y_i) \supseteq \{x_i : i \in J\}$, since $x_i \in \Gamma_R(y_i)\ \ \forall i \in I$.
Further $f(\cdot)$ is increasing. Hence, $f(\Gamma_R(Y)) \geq f(\{x_i : i \in J\}) \geq \mid J \mid = \mid Y \mid$. So $(*)$ is satisfied.

If: Let $(**)$ be satisfied. The proof is by induction on the number of edges in the bipartite graph. Clearly the result is true if there is only one edge. Let us assume that the result is true for all bipartite graphs with k edges and let B have $k+1$ edges. The result is in fact trivially true (there can be no SR) unless each vertex in V_R has degree at least 1. If each vertex in V_R has degree 1 the result is immediately true (the SR is $(\Gamma_R(y_i) : i \in I)$). So we assume that each vertex of V_R has degree atleast 1 and without loss of generality that vertex $y_1 \in V_R$ has degree atleast 2.

Let edges (x_1, y_1), (x_2, y_1) be incident on y_1. Let $B_1(B_2)$ be the bipartite graph obtained by deleting (x_1, y_1) $((x_2, y_1))$ from B. We claim that one of B_1, B_2 must satisfy $(**)$ (with $\Gamma_R(\cdot)$ replaced by their right adjacency functions ${\Gamma_R}^1(\cdot), {\Gamma_R}^2(\cdot)$ respectively). Suppose not. Then there must exist subsets Y_1, Y_2 of $V_R - \{y_1\}$ s.t.

$$f(({\Gamma_R}^1(y_1) \cup \Gamma_R(Y_1)) \leq \mid Y_1 \mid,$$

$$f(({\Gamma_R}^2(y_1) \cup \Gamma_R(Y_2)) \leq \mid Y_2 \mid .$$

Now since $f(\cdot)$ is submodular, increasing and since
$\Gamma_R(P \cap Q) \subseteq \Gamma_R(P) \cap \Gamma_R(Q)\ \forall P, Q \subseteq V_R$,

$$f({\Gamma_R}^1(y_1) \cup \Gamma_R(Y_1)) + f({\Gamma_R}^2(y_1) \cup \Gamma_R(Y_2))$$

$$\geq f(\Gamma_R(y_1) \cup \Gamma_R(Y_1 \cup Y_2)) + f(\Gamma_R(Y_1 \cap Y_2) \cup ({\Gamma_R}^1(y_1) \cap {\Gamma_R}^2(y_1)))$$

$$\geq f(\Gamma_R(y_1) \cup \Gamma_R(Y_1 \cup Y_2)) + f(\Gamma_R(Y_1 \cap Y_2)).$$

Hence,

$$\begin{aligned} \mid Y_1 \mid + \mid Y_2 \mid +1 &= \mid Y_1 \cup Y_2 \mid + \mid Y_1 \cap Y_2 \mid +1 \\ &> f((\Gamma_R(y_1) \cup \Gamma_R(Y_1 \cup Y_2)) + f(\Gamma_R(Y_1 \cap Y_2)). \end{aligned}$$

Now

$$f(\Gamma_R(Y_1 \cap Y_2)) \geq \mid Y_1 \cap Y_2 \mid$$

by $(**)$. But then $f(\Gamma_R(y_1) \cup \Gamma_R(Y_1 \cup Y_2)) < \mid Y_1 \cup Y_2 \mid +1$. This violates $(**)$ for B. We conclude that one of B_1, B_2, say B_1, must satisfy $(**)$. But B_1 has k edges and by the induction assumption has an SR satisfying $(*)$. This is also an SR satisfying $(*)$ for B.

□

We can now prove

Theorem 12.2.3 *(Rado's Theorem [Rado42])*
Let $B \equiv (V_L, V_R, E)$ be a bipartite graph. Let $\mathcal{M}$ be a matroid on V_L with rank function $f(\cdot)$. Then V_R has a transversal that is independent in $\mathcal{M}$ iff

$$f(\Gamma_R(Y)) \geq \mid Y \mid \quad \forall Y \subseteq V_R \cdots (**).$$

Proof : Necessity of the condition is trivial. So we only consider sufficiency. Let $(**)$ hold. By Theorem 12.2.2, V_R has an SR $(x_i : i \in I)$ s.t.

$$f(\{x_i : i \in J\}) \geq \mid J \mid \quad \forall J \subseteq I.$$

Since $f(\cdot)$ is the rank function of a matroid, the only way this inequality can be satisfied is for $\mid \{x_i : i \in J\} \mid$ to be equal to $\mid J \mid$ and $f(\{x_i : i \in J\})$ to be also equal to $\mid J \mid$. The former of these two conditions makes the SR into an SDR and the latter ensures that $\{x_i : i \in I\}$ is independent in $\mathcal{M}$. Thus $\{x_i : i \in I\}$ is an independent transversal of V_R.

□

Hall's Theorem is an easy corollary by taking $\mathcal{M}$ to be the free matroid (in which every set is independent).

Theorem 12.2.4 *(Hall's Theorem [Hall35])*
Let $B \equiv (V_L, V_R, E)$ be a bipartite graph. Then V_R has a transversal (there exists a matching which meets every vertex in V_R) iff $|\Gamma_R(Y)| \geq |Y| \quad \forall Y \subseteq V_R$.

We are now in a position to prove the following useful result due to Perfect [Perfect69]

Theorem 12.2.5 *Let $B \equiv (V_L, V_R, E)$ be a bipartite graph. Let $\mathcal{M}_L$ be a matroid on V_L with rank function $f(\cdot)$. Then*

i. the collection of subsets of V_R which have independent (in $\mathcal{M}_L$) transversals is the family of independent sets of a matroid $\mathcal{M}_R$

ii. the rank of a set $A \subseteq V_R$ in $\mathcal{M}_R$ is $\min_{X\subseteq A}(f(\Gamma_R(X))+ | A - X |)$.

We need the following preliminary lemma for the proof of the theorem (see also Theorem 11.3.1).

Lemma 12.2.1 *Let $\mu(\cdot)$ be an integral polymatroid rank function on subsets of S (an increasing integral submodular function with $\mu(\emptyset) = 0$). Then the collection of subsets P with $\mu(K) \geq |K| \quad \forall K \subseteq P$, forms the collection of independent sets of a matroid with rank function $(\mu * |\cdot|)(\cdot)$.*

Proof of the Lemma: The function $(\mu * |\cdot|)(\cdot)$ is easily seen to be a matroid rank function since it is submodular (Theorem 11.2.1),

$$(\mu * |\cdot|)(X \cup e) - (\mu * |\cdot|)(X) = 1 \textit{ or } 0$$

and $(\mu * |\cdot|)(\cdot)$ is increasing, integral and takes zero value on the null set. Further $\mu(K) \geq |K| \quad \forall K \subseteq P$ is equivalent to saying that $(\mu * |\cdot|)(P) = |P|$, i.e., equivalent to saying that P is independent in the matroid whose rank function is $(\mu * |\cdot|)(\cdot)$.

□

Proof of the Theorem:
i. We define the function $f_R(\cdot)$ on subsets of V_R by

$$f_R(X) \equiv f(\Gamma_R(X)) \quad \forall X \subseteq V_R.$$

By Theorem 12.2.1, $f_R(\cdot)$ is an increasing submodular function with $f_R(\emptyset) = 0$. It is also integral. Hence $(f_R * |\cdot|)(\cdot)$ is a matroid rank function and a set P is independent with respect to it iff $f_R(K) \geq |K| \quad \forall K \subseteq P$, i.e., iff $f(\Gamma_R(K)) \geq |K| \quad \forall K \subseteq P$.

But by Theorem 12.2.3, using the bipartite graph $(\Gamma_R(P), P, E_P)$, (where E_P is the set of edges of B incident on P), this happens iff P has a transversal T that is independent relative to $f(\cdot)$.

ii. We saw above that the rank function of $\mathcal{M}_R$ is $(f_R * \mid \cdot \mid)(\cdot)$. Since $f_R(\cdot) \equiv f(\Gamma_R(\cdot))$ the result follows.

□

The following corollary is due to Nash-Williams [Nash-Williams67].

Corollary 12.2.1 *Let $\Gamma_L : V_L \rightarrow V_R$ be a function and let $\mathcal{M}_L$ be a matroid on V_L with rank function $f(\cdot)$.*

i. The collection of images of independent sets of $\mathcal{M}_L$ under $\Gamma_L(\cdot)$ are the independent sets of a matroid $\mathcal{M}_R$ on V_R.

ii. The rank of a subset $A \subseteq V_R$ in $\mathcal{M}_R$ is given by

$$min_{X \subseteq A}(f({\Gamma_L}^{-1}(X)) + \mid A - X \mid).$$

Proof : Build a bipartite graph (V_L, V_R, E) where $e = (a, b)$ belongs to E iff $\Gamma_L(a) = b$. The result now follows from Theorem 12.2.5 when we observe that $\Gamma_R(\cdot) = {\Gamma_L}^{-1}(\cdot)$.

□

We can now prove the Matroid Union Theorem.

Theorem 12.2.6 *Let $\mathcal{M}_a, \mathcal{M}_b$ be matroids on S. Let $I_a \vee I_b \equiv \{T : T = A \cup B$, A independent in $\mathcal{M}_a, B$ independent in $\mathcal{M}_b\}$.*
Then $I_a \vee I_b$ is the family of independent sets of a matroid on S. The rank function of this matroid is $((r_a + r_b) * \mid \cdot \mid)(\cdot)$, where $r_a(\cdot), r_b(\cdot)$ are the rank functions of $\mathcal{M}_a, \mathcal{M}_b$ respectively.

Proof : : We build copies S_1, S_2 of S and build a matroid $\mathcal{M}_{a1}$ on S_1 that is a copy of $\mathcal{M}_a$ and a matroid $\mathcal{M}_{b2}$ on S_2 that is a copy of $\mathcal{M}_b$. We thus have the matroid $\mathcal{M}_{a1} \oplus \mathcal{M}_{b2}$ on $S_1 \uplus S_2$. Define the function $\Gamma : S_1 \uplus S_2 \rightarrow S$ by $\Gamma(e') \equiv e$ whenever e' is a copy of e. The rank function of $\mathcal{M}_{a1} \oplus \mathcal{M}_{b2}$ on $S_1 \uplus S_2$ is $(r_{a1} \oplus r_{b1})(\cdot)$ where $r_{a1}(\cdot), r_{b1}(\cdot)$ are copies of $r_a(\cdot), r_b(\cdot)$. Using Corollary 12.2.1, let $\mathcal{M}_a \vee \mathcal{M}_b$ denote the matroid on S whose independent sets are images under $\Gamma(\cdot)$ of independent sets of $\mathcal{M}_{a1} \oplus \mathcal{M}_{b2}$. Now a set $A_1 \uplus B_2, A_1 \subseteq S_1, B_2 \subseteq S_2$ is independent in $\mathcal{M}_{a1} \oplus \mathcal{M}_{b2}$ iff A_1 is independent in $\mathcal{M}_{a1}$ and B_2 independent in $\mathcal{M}_{b2}$. The image of this set under $\Gamma(\cdot)$ is $A \cup B$, where A_1 is a copy of A and B_2, a copy of B. (i.e., A is independent in $\mathcal{M}_a$ and B in $\mathcal{M}_b$). Hence, a set is independent in $\mathcal{M}_a \vee \mathcal{M}_b$ iff it is a union of an independent set of $\mathcal{M}_a$ and an independent set of $\mathcal{M}_b$. Let $r_{ab}(\cdot)$ be the rank function of $\mathcal{M}_a \vee \mathcal{M}_b$. By Corollary 12.2.1,

$$r_{ab}(A) = min_{X \subseteq A}((r_{a1} \oplus r_{b2})(\Gamma^{-1}(X)) + \mid A - X \mid)$$

Now $\Gamma^{-1}(X) = X_1 \uplus X_2$ where X_1 is a copy of X within S_1 and X_2 is a copy of X

within S_2 and

$$\begin{aligned}(r_{a1} \oplus r_{b2})(X_1 \uplus X_2) &= r_{a1}(X_1) + r_{b2}(X_2)\\ &= r_a(X) + r_b(X)\\ &= (r_a + r_b)(X).\end{aligned}$$

Hence,

$$r_{ab}(A) = min_{X \subseteq A}((r_a + r_b)(X) + \mid A - X \mid),$$

i.e., $r_{ab}(\cdot) = ((r_a + r_b) * \mid \cdot \mid)(\cdot)$.

□

The corollary given below follows by routine induction.

Corollary 12.2.2 *(Nash-Williams' Rank Formula) Let $\mathcal{M}_1, \cdots, \mathcal{M}_k$ be matroids on S with rank function $r_1(\cdot), \cdots, r_k(\cdot)$. Then $I \equiv \{T : T = T_1 \cup \cdots \cup T_k, T_i$ independent in $\mathcal{M}_i \quad \forall i\}$ is the collection of independent sets of a matroid whose rank function is given by $((r_1 + \cdots + r_k) * \mid \cdot \mid)(\cdot)$.*

Definition 12.2.1 *Given matroids $\mathcal{M}_1, \mathcal{M}_2$ on S, the matroid each of whose independent sets is a union of an independent set of $\mathcal{M}_1$ and an independent set of $\mathcal{M}_2$ is called* **union of $\mathcal{M}_1$ and $\mathcal{M}_2$** *and is denoted by $\mathcal{M}_1 \vee \mathcal{M}_2$.*

The construction of the base of the union of matroids is a very common occurrence in combinatorial optimization problems, e.g. the problem of finding maximally distant trees, the hybrid rank problem, the maximum rank-minimum term rank problem etc. (see Section 15.2). Very often the problem is disguised as the 'Matroid intersection problem: Given two matroids $\mathcal{M}_1, \mathcal{M}_2$ find the largest set that is independent in $\mathcal{M}_1$ and $\mathcal{M}_2$'.

Exercise 12.1
(k) Let $\mathcal{M}_1, \mathcal{M}_2$ be matroids on S. Let b_{12} be the largest set independent in $\mathcal{M}_1$ as well as in $\mathcal{M}_2$. Show that

i. b_{12} can be represented as $b_{12*} - b_2^*$ where b_{12*} is a base of $\mathcal{M}_1 \vee \mathcal{M}_2^*$ which is the union of a base b_1 of $\mathcal{M}_1$ and a base b_2^* of $\mathcal{M}_2^*$,

ii. $\mid b_{12} \mid = r(\mathcal{M}_1 \vee \mathcal{M}_2^*) - r(\mathcal{M}_2^*)$,

iii. every set of the form $b_{12*} - b_2^*$ is a common independent set of $\mathcal{M}_1, \mathcal{M}_2$ of maximum size.

Exercise 12.2
(k) Let $\mathcal{M}_1, \cdots, \mathcal{M}_k$ be matroids on S and let $T \subseteq S$. Let $r(\cdot)$ be the rank function of $\mathcal{M}_v = \mathcal{M}_1 \vee \cdots \vee \mathcal{M}_k$. Then

i. $\mathcal{M}_1 \vee \cdots \vee \mathcal{M}_k \cdot T = \mathcal{M}_1 \cdot T \vee \cdots \vee \mathcal{M}_k \cdot T$,

ii. if $r(S - T) = \sum_{i=1}^{k} r_i(S - T)$ then

$$(\mathcal{M}_1 \vee \cdots \vee \mathcal{M}_k) \times T = \mathcal{M}_1 \times T \vee \cdots \vee \mathcal{M}_k \times T.$$

Exercise 12.3
(k) Show that there exist bases $b_1, \cdots, b_k$ of matroids $\mathcal{M}_1, \cdots, \mathcal{M}_k$ on S s.t. $\bigcup b_i = S$ iff there exist no subsets T of S s.t. $\sum_{i=1}^{k} r_i(T) <\mid T \mid$.

Exercise 12.4 *Let $\mathcal{M}_1, \mathcal{M}_2$ be two matroids on S. Let $\mathcal{M}_3 = \mathcal{M}_1 \vee \mathcal{M}_2$. Let $d_i(Q) =\mid Q \mid /r_i(\mathcal{M}_i \cdot Q), \emptyset \subset Q \subseteq S$. where $r_i(\cdot)$ is the rank function of $\mathcal{M}_i, i = 1, 2, 3$. If $d_1(\cdot), d_2(\cdot)$ reach their maximum value on $P \neq \emptyset$, show that $d_3(\cdot)$ also reaches its maximum on P.*

Exercise 12.5
(k) Let $B \equiv (V_L, V_R, E)$ be a bipartite graph. Use the matroid union theorem to show that

i. *[Edmonds+Fulkerson65] the collection of transversals of subsets of V_R form the independent sets of a matroid on V_L (the* **transversal matroid on** V_L **defined by** *B);*

ii. *the union of rank one matroids is a transversal matroid;*

iii. *if $\mathcal{M}$ is a transversal matroid on V_L and $K \subseteq V_L$ then $\mathcal{M} \cdot K$ is a transversal matroid on K; if $\mathcal{M} \cdot K$ contains no coloops then $\mathcal{M} \times (V_L - K)$ is a transversal matroid.*

iv. *size of the maximum matching $= (|\Gamma_L| * \mid \cdot \mid)(V_L)$.*

Exercise 12.6
[Pym+Perfect70] Let $f_1(\cdot), f_2(\cdot)$ be non-negative, increasing, integral submodular functions and let $\mathcal{M}(f)$ denote the matroid whose rank function is $(f * \mid \cdot \mid)(\cdot)$. Show that

$$\mathcal{M}(f_1 + f_2) = \mathcal{M}(f_1) \vee \mathcal{M}(f_2).$$

Exercise 12.7
Let $\mathcal{M}_1, \mathcal{M}_2$ be matroids on S and let $\mathcal{M}_1^*, \mathcal{M}_2^*$ be their duals. Let $r_1(\cdot), r_2(\cdot), {r_1}^*(\cdot)$, ${r_2}^*(\cdot)$ be the rank of functions of $\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_1^*, \mathcal{M}_2^*$ respectively. Let R, R^* be the minimal sets that minimize $(r_1 + r_2)(X) + |S - X|, X \subseteq S$ and $({r_1}^* + {r_2}^*)(X)+ \mid S - X \mid, X \subseteq S$ respectively. Show that

i. *A set $K \subseteq S$ minimizes $(r_1 + r_2)(X)+ \mid S - X \mid, X \subseteq S$ iff $S - K$ minimizes $({r_1}^* + {r_2}^*)(X)+ \mid S - X \mid, X \subseteq S$.*

ii. *$S - R^*, S - R$ are the maximal sets that minimize*

$$(r_1 + r_2)(X)+ \mid S - X \mid, X \subseteq S,$$

$$({r_1}^* + {r_2}^*)(X)+ \mid S - X \mid, X \subseteq S$$

respectively.

iii. *the collection of non-coloops of $\mathcal{M}_1 \vee \mathcal{M}_2$ is disjoint from the collection of non-coloops of $\mathcal{M}_1^* \vee \mathcal{M}_2^*$.*

Exercise 12.8
Let $\mathcal{M}_1, \mathcal{M}_2$ be matroids on S with rank functions $r_1(\cdot), r_2(\cdot)$ respectively. Show that

i. [Pym+Perfect70]

$$\max_{X\subseteq S}(r_1(X) + r_2(S-X)) = \min_{X\subseteq S}((r_1+r_2)(X) + \mid S-X \mid).$$

ii. [Edmonds70] maximum cardinality of a common independent set = $\min_{X\subseteq S}(r_1(X) + r_2(S-X))$.

iii.

$$max \mid b_1 - b_2 \mid = \min_{X\subseteq S}(r_1(X) + {r_2}^*(S-X)),$$

where b_1, b_2 are bases of $\mathcal{M}_1, \mathcal{M}_2$ respectively.

Exercise 12.9 *[Narayanan74] Let us call the PP of $(r, \mid\cdot\mid)$ where $r(\cdot)$ is the rank function of matroid $\mathcal{M}$, the PP of $\mathcal{M}$. Given the PP of $\mathcal{M}$, how would you compute the PP of $\mathcal{M}\vee\mathcal{M}, (\mathcal{M}\vee\mathcal{M})^*\vee\mathcal{M}^*$, in general of 'positive' or 'negative' expressions of $\mathcal{M}$*
($\mathcal{M}$ is positive, $\mathcal{M}^*$ negative,
$f(\mathcal{M}), g(\mathcal{M})$ positive $\Rightarrow f(\mathcal{M})\vee g(\mathcal{M})$ positive
$f(\mathcal{M}), g(\mathcal{M})$ negative $\Rightarrow f(\mathcal{M})\vee g(\mathcal{M})$ negative
$f(\mathcal{M})$ positive (negative) $\Rightarrow (f(\mathcal{M}))^*$ negative (positive)).

12.3 Matroid Union: Algorithm and Structure

12.3.1 Introduction

In this section we present the well known algorithm for the construction of a base of the union of matroids. The algorithm is due to Edmonds [Edmonds65a]. It can be easily modified to give the maximum size common independent set of two matroids. It also allows us to discuss the structure of various standard 'objects' associated with the matroid union viz. f-circuit, the set of coloops etc.

12.3.2 The Algorithm

In the **Algorithm Matroid Union** that we describe below, we make use of a directed graph, $\mathcal{G}(b_1, \cdots, b_k)$, associated with bases $b_1, \cdots, b_k$ of matroids $\mathcal{M}_1, \cdots, \mathcal{M}_k$ respectively defined on S. The **graph** $\mathcal{G}(b_1, \cdots, b_k)$ **is built as follows:** S is the vertex set of the directed graph. Let v_1, v_2 be vertices. Then there is an edge (v_1, v_2, i) directed from v_1 to v_2 iff $v_2 \in L_i(v_1, b_i)$, i.e., iff v_2 lies in the fundamental circuit of v_1 with respect to b_i in the matroid $\mathcal{M}_i$. If $v_1 \in b_i$ there is no edge of the kind (v_1, v_2, i).

ALGORITHM 12.1 Algorithm Matroid Union

INPUT *Matroids* $\mathcal{M}_1, \cdots, \mathcal{M}_k$ *on* S. *Bases* $b_1, \cdots, b_k$ *of* $\mathcal{M}_1, \cdots, \mathcal{M}_k$ *respectively.*

OUTPUT *i. Bases* $b_{1f}, \cdots, b_{kf}$ *of* $\mathcal{M}_1, \cdots, \mathcal{M}_k$ *respectively such that* $\bigcup_{i=1}^k b_{if}$ *is a base of* $\mathcal{M}_1 \vee \cdots \vee \mathcal{M}_k$.

ii. The set R *of all element reachable from* $S - \bigcup_{i=1}^k b_{if}$ *in* $\mathcal{G}(b_{1f}, \cdots, b_{kf})$.

Initialize $j \leftarrow 0$

(COMMENT: j describes the current index of the base set.)

$b_{1j} \leftarrow b_1, \cdots, b_{kj} \leftarrow b_k$.

STEP 1 *Construct* $\mathcal{G}(b_{1j}, \cdots, b_{kj})$. *If* $S = \bigcup_{i=1}^k b_{ij}$, *GOTO STEP 7.*

STEP 2 *Mark all vertices which belong to more than one of the* b_{ij}.

For each $v \in S - \bigcup_{i=1}^k b_{ij}$ *in* $\mathcal{G}(b_{1j}, \cdots, b_{kj})$, *do*

Starting from v *do a* bfs *and find the set of all vertices reachable through directed paths from* v.

(COMMENT: The directed edges (v_a, v_b, p), (v_c, v_d, q), $p \neq q$ *may be in the same directed path.)*

If no marked vertex is reachable from v *in* $\mathcal{G}(b_{1j}, \cdots, b_{kj})$ *call* v **good**. *Otherwise* v *is* **bad.**

STEP 3 *If all* $v \in S - \bigcup_{i=1}^k b_{ij}$ *are* **good**, *GOTO STEP 7.*

STEP 4 *Let* v *be a bad vertex of* $\mathcal{G}(b_{1j}, \cdots, b_{kj})$ *and let* v_m *be a marked vertex reachable from* v. *Let* $v = v_o, e_1, v_1, \cdots, e_m, v_m$ *be the shortest directed path from* v *to* v_m *(where* e_i *is the directed edge from* v_{i-1} *to* v_i*).*

For $i = 0$ *to* $m - 1$, *do*

If $e_{m-i} \equiv (v_{m-i-1}, v_{m-i}, q)$

$b_{qj} \leftarrow (b_{qj} \cup v_{m-i-1}) - v_{m-i}$

(COMMENT: The union of the updated bases has size one more than the union of the original bases since v_o *has moved into the union by pushing* v_m *out of one of the bases to which it belonged.)*

STEP 5 *For* $i = 1, \cdots, k$, *do*

$b_{i(j+1)} \leftarrow b_{ij}$

STEP 6 $j \leftarrow j + 1$. *GOTO STEP 1*

STEP 7 *Declare:* $b_{1f} = b_{1j}, \cdots, b_{kf} = b_{kj}$ *and* R *to be the set of all vertices reachable in* $\mathcal{G}(b_{1f}, \cdots, b_{kf})$ *from* $S - \bigcup_{i=1}^k b_{if}$.

STOP

12.3.3 Justification and complexity of Algorithm Matroid Union

In order to justify Algorithm Matroid Union we need to verify two facts.

i. In STEP 4, the bases are updated according to a rule – the new objects actually are bases of the corresponding matroids.

ii. $\bigcup b_{if}$ that we obtain at the end of the algorithm cannot be enlarged.

The following lemmas do the needful.

Lemma 12.3.1 *Let v be a bad vertex of $\mathcal{G}(b_{1j}, \cdots, b_{kj})$ and let $v = v_o, e_1, v_1, \cdots, e_m, v_m$ be the shortest directed path from v to v_m with e_i the directed edge from v_{i-1} to v_i. Let $e_n \equiv (v_{n-1}, v_n, q)$ and $e_p \equiv (v_{p-1}, v_p, q)$ with $n < p$. Then $v_p \notin L_q(v_{n-1}, b_{qj})$*

Proof : Suppose $v_p \in L_q(v_{n-1}, b_{qj})$. Then there is a directed edge $e \equiv (v_{n-1}, v_p, q)$ in $\mathcal{G}(b_{1j}, \cdots, b_{kj})$. The path that replaces the segment $v_{n-1}, e_n, \cdots, v_p$ by v_{n-1}, e, v_p would be shorter than the original path from v to v_m. This is a contradiction. We conclude $v_p \notin L_q(v_{n-1}, b_{qj})$.

□

As a consequence of this lemma, if we update the base b_{qj} to $\hat{b}_{qj} \equiv ((v_{p-1} \cup b_{qj}) - v_p)$

$$L_q(v_{n-1}, \hat{b}_{qj}) = L_q(v_{n-1}, b_{qj}).$$

Hence, $((\hat{b}_{qj} \cup v_{n-1}) - v_n)$ is a base of the matroid $\mathcal{M}_q$. This proves that the set of updated subsets $b_{1j}, \cdots, b_{kj}$ that we obtain at the end of STEP 4 are bases of $\mathcal{M}_1, \cdots, \mathcal{M}_k$ respectively.

Lemma 12.3.2 *Let $b_{1f}, \cdots, b_{kf}$ be the bases output by the Algorithm Matroid Union. Let $\bigcup b_{if} \neq S$. Let $R_v \subseteq S$ be the set of all elements reachable from $v \in S - \bigcup_{i-1}^{k} b_{if}$ in $\mathcal{G}(b_{1f}, \cdots, b_{kf})$ and let R be the union of all such R_v. Then*

i. $R \cap b_{if} \cap b_{jf} = \emptyset, i \neq j$.

ii. $R \cap b_{if}$ is a base of $\mathcal{M}_i \cdot R, i = 1, \cdots, k$.

Proof :

i. Since $\bigcup_{i=1}^{k} b_{if} \neq S$ at the termination of the algorithm, no element of $b_{if} \cap b_{jf}, i \neq j$ can be reached from any element in $S - \bigcup b_{if}$.

ii. We need to prove that if $v' \in R - b_{if}$ then $L_i(v', b_{if}) \subseteq R$. If $v_j \in L_i(v', b_{if})$, since v' is reachable in $\mathcal{G}(b_{1f}, \cdots, b_{kf})$ from some $v \in S - \bigcup_{i=1}^{k} b_{if}$, we must have that v_j is reachable from v. Hence, $v_j \in R$.

□

We can now see why $\bigcup b_{if}$ cannot be enlarged. If $\bigcup b_{if} = S$ we are done. Otherwise let $\bigcup b'_i \supseteq \bigcup b_{if}$, where b'_i are independent in $\mathcal{M}_i$. Now

$$| (\bigcup b'_i) \cap R | \leq \sum_i r(\mathcal{M}_i . R) = | (\bigcup b_{if}) \cap R |$$

Hence, $(\bigcup b'_i) \cap R = (\bigcup b_{if}) \cap R$.

Further by definition of R, $S - R \subseteq \bigcup b_{if}$. Hence, $\bigcup b'_i = \bigcup b_{if}$. We thus see that $\bigcup_{i=1}^{k} b_{if}$ is a base of $\mathcal{M}_1 \vee \cdots \vee \mathcal{M}_k$. This completes the justification of Algorithm Matroid Union.

Complexity of Algorithm Matroid Union

We can express the complexity in terms of calls to standard oracles we define below

i. Rank oracle gives, once per call, the rank of a specified subset of S in any desired one of the matroids $\mathcal{M}_1, \cdots, \mathcal{M}_k$.

ii. Independence oracle would declare, once per call, whether a particular subset of S is independent in the specified matroid $\mathcal{M}_i, i = 1, \cdots, k$.

Independence oracle is clearly weaker and we will express the complexity in terms of calls to it. In STEP 1, we build $\mathcal{G}(b_{1j}, \cdots, b_{kj})$. This requires the knowledge of the f-circuits of an element outside b_{ij} with respect to it in the matroid $\mathcal{M}_i$. To build $L_i(v, b_{ij})$ we check for each $v' \in b_{ij}$ whether $v \cup b_{ij} - v'$ is independent in the matroid $\mathcal{M}_i$. This requires atmost $r(\mathcal{M}_i)$ calls to the independence oracle. Thus the total number of calls to the independence oracle to build $\mathcal{G}_j \equiv \mathcal{G}(b_{ij}, \cdots, b_{kj})$ is atmost $\sum_{i=1}^{k}(| S - b_{ij} || b_{ij} |)$. Finding the reachable set from $S - \cup_{i=1}^{k} b_{ij}$ requires $O(| E(\mathcal{G}_j) |)$ elementary steps, where $| E(\mathcal{G}_j) |$ is the number of edges in $\mathcal{G}_j$ treating parallel edges as a single edge. We may have started with (in the worst case) $b_{ij} = \cdots = b_{kj}$ and end with a base of the union of size atmost $\sum_{i=1}^{k} | b_{ij} |$. Each time STEP 4 is used the bases are updated and the size of their union increases by one. Hence, $\mathcal{G}(b_{ij}, \cdots b_{kj})$ has to be built at most $\sum_{i=1}^{k-1} | b_{ij} |$ times. So the overall complexity is $O(\sum_{i=1}^{k-1} | b_{ij} | \sum_{i=1}^{k}(| S - b_{ij} || b_{ij} |))$ calls to the independence oracle. If r, r' are the maximum and minimum of the ranks of the matroids, $\mathcal{M}_1, \cdots, \mathcal{M}_k$, the above expression for **Complexity of Algorithm Matroid Union** can be simplified to
$O(k(k-1)r^2(| S | - r'))$ calls to the independence oracle.
In addition there are $O((k-1)r | E(\mathcal{G}_j) |)$ **elementary steps** involved in building the reachable set for each $\mathcal{G}_j$.
If we wish to suppress the factor k, we may replace kr as well as $(k-1)r$ by $| S |$ and $| E(\mathcal{G}_j) |$ by $| S |^2$. In this case the complexity of Algorithm Matroid Union is $O(| S |^2 (| S | - r'))$ calls to the independence oracle $+O(| S |^3)$ elementary operations.

The space requirement is that of storing the updated version of the graph $\mathcal{G}(b_1, \cdots b_k)$. This has atmost $\mid S \mid^2$ edges. So
the **space complexity of Algorithm Matroid Union is** $O(\mid S \mid^2)$.

Exercise 12.10
Let $b_1, \cdots, b_k$ be bases of $\mathcal{M}_1, \cdots, \mathcal{M}_k$ respectively and let $v_o \notin \bigcup_{i=1}^k b_i$ and $v_m \in b_i \cap b_j, i \neq j$. Let v_m be reachable from v_o in $\mathcal{G}(b_1, \cdots, b_k)$. Show that there exist bases $b'_1 \cdots, b'_k$ and $v'_m \in b'_i \cap b'_j, i \neq j$ s.t. $\bigcup_{i=1}^k b'_i = \bigcup_{i=1}^k b_i$, and v'_m can be reached from v_o in one step in $\mathcal{G}(b'_1, \cdots, b'_k)$.

Exercise 12.11
Let $b_1, \cdots, b_k$ be bases of the matroid $\mathcal{M}$ s.t. $\bigcup b_i$ is base of $\mathcal{M}^k$.

i. *In the graph $\mathcal{G}(b_1, \cdots, b_k)$ show that the shortest path between two vertices cannot exceed length r.*

ii. *Let T be the set of all vertices in $\mathcal{G}(b_1, \cdots, b_k)$ from which no element common to two bases can be reached. Build the preorder on $T - R$ by the following rule: $v_1 \geq v_2$ iff v_2 can be approached from v_1. If (v) is an equivalence class of this preorder show that $(v) \cap b_i \neq \emptyset$ for each i.*

Exercise 12.12
(k) Convert Algorithm Matroid Union to an algorithm for finding the maximum size common independent set of two matroids $\mathcal{M}_1, \mathcal{M}_2^*$.

12.3.4 Structure of the Matroid Union

The Algorithm Matroid Union gives some insight into the structure of the union of matroids. Specifically, using the algorithm, we present simple results on the set of non-coloops, circuits and f-circuits for this matroid. Finally we discuss the idea of approachability relative to a base of the union and show that this notion is helpful in the construction of the principal partition of a matroid.

We define bases $b_1, \cdots, b_k$ of $\mathcal{M}_1, \cdots, \mathcal{M}_k$ respectively to be **maximally distant** iff $\bigcup_{i=1}^k b_i$ is a base of $\mathcal{M}_1 \vee \cdots \vee \mathcal{M}_k$ (i.e., iff $\bigcup_{i=1}^k b_i$ cannot be enlarged).

Lemma 12.3.3 *Let $b_1, \cdots, b_k$ be maximally distant bases of matroids $\mathcal{M}_1, \cdots, \mathcal{M}_k$ respectively on S. Let $b_\vee \equiv \bigcup_{i=1}^k b_i$ and let R be the set of all elements reachable from $S - \bigcup_{i=1}^k b_i$ in the graph $\mathcal{G}(b_1, \cdots, b_k)$. Then*

i. *$b_1 \cap R, \cdots, b_k \cap R$ are pairwise disjoint bases of $\mathcal{M}_1 \cdot R, \cdots, \mathcal{M}_k \cdot R$,*

ii. *$\mathcal{M}_1 \cdot R, \cdots, \mathcal{M}_k \cdot R$ do not contain any coloops within R,*

iii. *R is the unique minimal set that minimizes $\sum_{i=1}^k r_i(X) + \mid S - X \mid, X \subseteq S$, equivalently, minimizes $\sum_{i=1}^k r_i(X) - \mid X \mid, X \subseteq S$,*

iv. *$S - R$ is the set of coloops of $\mathcal{M}_1 \vee \cdots \vee \mathcal{M}_k$.*

Proof :

i. By definition, $R \supseteq S - b_\vee$ and the set of all elements reachable from elements in $S - b_\vee$ in $\mathcal{G}(b_1, \cdots, b_k)$. Further $R \cap b_i$ are pairwise disjoint. Otherwise an element $v_c \in b_i \cap b_j$ is reachable from some $v_o \in S - b_\vee$ in $\mathcal{G}(b_i \cdots, b_k)$. Use of STEP 4 of Algorithm Matroid Union will allow us to enlarge $\bigcup b_i$, a contradiction.
Next, if $R \cap b_j$ is not a base of $\mathcal{M}_j \cdot R$, then for some $v \in R - b_j$, $L_j(v, b_j)$ has an element $v' \notin R$. But then v is reachable from some $v_o \in S - b_\vee$ and so must v' be. Thus, $v' \in R$, a contradiction.

ii. We have to show that the base $b_j \cap R$ of $\mathcal{M}_j \cdot R$ contains no coloop of $\mathcal{M}_j \cdot R$. Let $v_m \in b_j \cap R$. Then v_m is reachable from $v_o \in S - b_\vee$ in $\mathcal{G}(b_1, \cdots, b_k)$. But this means there is $v_{m-1} \in R - b_j$, s.t. $L_j(v_{m-1}, b_j)$ contains v_m. So v_m is not a coloop of $\mathcal{M}_j \cdot R$.

iii. A set minimizes $\sum_{i=1}^k r_i(X)+ \mid S - X \mid, X \subseteq S$
iff it minimizes $\sum_{i=1}^k r_i(X)- \mid X \mid, X \subseteq S$,
i.e., iff it maximizes $\mid X \mid - \sum r_i(X),\ X \subseteq S$.
We will work with the latter function. We have

$$\mid X \mid - \sum r_i(X) \leq \mid X \mid - r_\vee(X),$$

where $r_\vee(\cdot)$ is the rank function of the union of the matroids.
Now $\mid X \mid - r_\vee(X)$ is a supermodular function, takes value zero on the null set and 1 or 0 on singletons. It is therefore an increasing function and reaches its maximum on S.
Next $\mid S \mid - r_\vee(S) = \mid S - b_\vee \mid$. We know that $R \supseteq S - b_\vee$. Further $R \cap b_i$ are pairwise disjoint bases of $\mathcal{M}_i \cdot R, i = 1, \cdots, k$. Hence,

$$\mid R \mid - r_\vee(R) \ = \mid R \mid - \sum_{i=1}^k r_i(R) \ = \mid S - b_\vee \mid = \mid S \mid - r_\vee(S).$$

Thus, $\mid X \mid - \sum r_i(X)$ reaches a maximum at R. This function is supermodular and use of the supermodular inequality reveals that it has a unique minimal set maximizing it.
So it suffices to show that no proper subset of R maximizes the function.
Suppose $R' \subseteq R$ is such that

$$\mid R' \mid - \sum_{i=1}^k r_i(R') \ = \mid R \mid - \sum_{i=1}^k r_i(R),$$

i.e.,

$$\mid R' \mid - \sum_{i=1}^k r_i(R') \ = \mid S - b_\vee \mid \geq \mid R' - b_\vee \cap R' \mid .$$

On the other hand

$$\mid R' \mid - \mid b_\vee \cap R' \mid \geq \mid R' \mid - r_\vee(R') \geq \mid R' \mid - \sum_{i=1}^k r_i(R').$$

We conclude that $| S - b_\vee |=| R' - b_\vee \cap R' |$ and $| b_\vee \cap R' | = \sum_{i=1}^k r_i(R')$. Since $S - b_\vee \supseteq R' - b_\vee \cap R'$, we have $S - b_\vee = R' - b_\vee \cap R'$. Since $| b_\vee \cap R' | = \sum_{i=1}^k r_i(R')$, $b_\vee \cap b_i$ are pairwise disjoint bases of $\mathcal{M}_i \cdot R', i = 1, \cdots, k$. Hence, from the elements in $S - b_\vee$ it is impossible to reach outside R' in $\mathcal{G}(b_1, \cdots, b_k)$, i.e., $R' \supseteq R$. Thus, $R = R'$, i.e., R is the minimal set that maximizes $| X | - \sum r_i(X)$.

iv. This part follows by Lemma 11.4.3.
We give an alternative proof because the technique of the proof is useful.
We need to show that $S - R$ is contained in every base of the union and for each $v \in R$ there is some base of the union that does not contain it.
If we initialize Algorithm Matroid Union on $(b_1, \cdots, b_k)$ the algorithm must output the same set of bases. Hence,

$$S - R \subseteq \bigcup_{i=1}^{k} b_i = b_\vee$$

Now if $b'_1, \cdots, b'_k$ are bases of $\mathcal{M}_1, \cdots, \mathcal{M}_k$ s.t. $\bigcup_{i=1}^k b'_i$ is a base of the union, we must have

$$| (\bigcup_{i=1}^{k} b'_i) \cap R | \leq \sum_{i=1}^{k} r(\mathcal{M}_i \cdot R) = | (\bigcup_{i=1}^{k} b_i) \cap R | .$$

Hence, $\bigcup_{i=1}^k b'_i \supseteq S - R$, as otherwise its size would be less than that of $\bigcup_{i=1}^k b_i$. Next let $v_m \in R$. If $v_m \notin \bigcup b_i$ we already have a base of the union to which v_m does not belong. So let $v_m \in \bigcup b_i$. There exists $v \in S - \bigcup b_i$ from which v_m can be reached in $\mathcal{G}(b_1, \cdots, b_k)$. Using a shortest path from v to v_m we apply STEP 4 of Algorithm Matroid Union. This would result in a new set of bases $b'_1, \cdots, b'_k$, the cardinality of whose union would be the same as that of $\bigcup b_i$. However, $v \in \bigcup b'_i$ but $v_m \notin \bigcup b'_i$. Thus, we have found a base of the union to which v_m does not belong. So we conclude that v_m is not a coloop.

□

Using arguments similar to the proof of the above lemma we can prove the following two results.

Lemma 12.3.4 *Let $b_1, \cdots, b_k$ be bases of matroids $\mathcal{M}_1, \cdots, \mathcal{M}_k$ respectively on S s.t. $\bigcup_{i=1}^k b_i$ is a base $b_\vee$ of $\mathcal{M}_1 \vee \cdots \vee \mathcal{M}_k$. Let $v \in S - \bigcup_{i=1}^k b_i$. Let R_v be the set of all elements reachable from v in $\mathcal{G}(b_1, \cdots, b_k)$. Then the f-circuit $L_\vee(v, b_\vee) = R_v$.*

We thus see that R_v does not change in $\mathcal{G}(b'_1, \cdots, b'_k)$ provided $\bigcup b_i = \bigcup b'_i$ and this subset is a base of the matroid union.

Lemma 12.3.5 *Let C be a circuit of $\mathcal{M}_1 \vee \cdots \vee \mathcal{M}_k$. Then C can be expressed as $v \uplus b_1 \uplus \cdots \uplus b_k$, where b_i is a base of $\mathcal{M}_i \cdot C, i = 1, \cdots, k$, the b_i are pairwise disjoint and every element of $\bigcup_{i=1}^k b_i$ can be reached from v in $\mathcal{G}(b_1, \cdots, b_k)$.*

Approachability
Let $b_1, \cdots, b_k$ be maximally distant bases of matroids $\mathcal{M}_1, \cdots, \mathcal{M}_k$ respectively on S. Let $b_\vee \equiv \bigcup_{i=1}^k b_i$. Let D be the set of all vertices of $\mathcal{G}(b_1, \cdots, b_k)$ contained in $b_\vee$

from which no element common to more than one b_i can be reached. It is clear that $D \cap b_i$ are bases of $\mathcal{M}_i \cdot D, i = 1, \cdots, k$, and further are disjoint. Let $v_p, v_q \in D$. We say v_q is **approachable** from v_p relative to $(b_1, \cdots, b_k)$ iff v_q is reachable from v_p in $\mathcal{G}(b_1, \cdots, b_k)$. Approachability depends on the base $b_\vee$ of $\mathcal{M}_1 \vee \cdots \vee \mathcal{M}_k$ and the matroids $\mathcal{M}_1, \cdots, \mathcal{M}_k$ but not on the individual b_i. However it cannot be defined knowing only $\mathcal{M}_1 \vee \cdots \vee \mathcal{M}_k$ (without knowledge of $\mathcal{M}_1, \cdots, \mathcal{M}_k$). This idea is algorithmically useful in building the principal partition of a matroid rank function. We collect all the useful properties of approachability in the following lemma. We remind the reader that minimizing $\sum r_i(X)- \mid X \mid$ is equivalent to minimizing $\sum r_i(X)+ \mid S - X \mid$.

Lemma 12.3.6 *Let $b_1, \cdots, b_k$ be maximally distant bases of matroids $\mathcal{M}_1, \cdots, \mathcal{M}_k$ respectively on S. Let $b_\vee \equiv \bigcup b_i$ and let $D \subseteq b_\vee$ be the set of all vertices of $\mathcal{G}(b_1, \cdots, b_k)$ contained in $b_\vee$ from which no element common to more than one b_i can be reached. Let $f(\cdot) \equiv (\sum r_i(\cdot)- \mid \cdot \mid)$ where $r_i(\cdot)$ is the rank function of $\mathcal{M}_i, i = 1, \cdots, k$. Then,*

i. $f(\cdot)$ reaches a minimum on D among all subsets of $b_\vee$ and D is the unique maximal such set,

ii. if $v_p \in D$, then there is a unique minimal subset A_p of D on which $f(\cdot)$ reaches a minimum among all subsets of $b_\vee$ s.t. $v_p \in A_p$; further, A_p is the set of all elements approachable from v_p relative to $(b_1, \cdots, b_k)$,

iii. if R is the set of all elements reachable from $S - b_\vee$ in the graph $\mathcal{G}(b_1, \cdots, b_k)$, then $R \cup D$ is the maximal subset of S that minimizes $f(\cdot)$ over subsets of S,

iv. if $v_p \in D - R$ then $R \cup A_p$ minimizes $f(\cdot)$ over subsets of S and is the minimal set containing v_p with this property.

Proof:

i. The function $f(\cdot)$ is obtained by summing modular functions and subtracting a submodular function and so is submodular. Let $X \subseteq b_\vee$. Then, since $\bigcup_{i=1}^k (b_i \cap X) = X$, it is clear that $f(X) \geq 0$. Next let $b_\vee \supseteq D' \supset D$. If $b_i \cap D'$ are all pairwise disjoint bases of $\mathcal{M}_i \cdot D', i = 1, \cdots, k$, then D' does not contain any element common to more than one b_i and it is impossible to reach outside D' from $v \in D'$ in $\mathcal{G}(b_1, \cdots, b_k)$. Hence, $D' \subseteq D$, a contradiction. So we must have that either the $b_i \cap D'$ have nonvoid intersections or they do not all form bases of $\mathcal{M}_i \cdot D'$. In either case $f(D') > 0$. On the other hand, $f(D) = 0$, since $D \cap b_i$ are bases of $\mathcal{M}_i \cdot D, i = 1, \cdots, k$ and pairwise disjoint. So the minimum value of $f(\cdot)$ over subsets of $b_\vee$ is reached at D and it is the maximal such set. The unique maximality of D follows from the submodularity of $f(\cdot)$.

ii. If there are two distinct minimal sets which minimize $f(\cdot)$ among subsets of $b_\vee$ containing v_p, by using the submodular inequality it would follow that their intersection is also a set of the same kind. This would contradict the minimality of the sets. So there must be a unique minimal set A_p which minimizes $f(\cdot)$ among subsets of $b_\vee$ containing v_p. Also $v_p \in D$. So $f(D) = f(A_p) = 0$.Thus A_p minimizes $f(\cdot)$ among subsets of $b_\vee$ and further is the unique minimal such set containing v_p.

Since $f(A_p) = 0$, $\sum_{i=1}^{k} r_i(A_p) =\mid A_p \mid$. Further, $\bigcup_{i=1}^{k}(b_i \cap A_p) = A_p$. We conclude that $b_i \cap A_p$ are pairwise disjoint bases of $\mathcal{M}_i.A_p, i = 1, \cdots, k$. Since $v_p \in A_p$, the set A'_p of all elements approachable from v_p relative to $b_1, \cdots, b_k$ cannot have element from $S - A_p$. Hence, $A'_p \subseteq A_p$. On the other hand, it is easy to see that A'_p has the property that $b_i \cap A'_p$ are pairwise disjoint bases of $\mathcal{M}_i \cdot A'_p, i = 1, \cdots, k$. Hence,

$$f(A'_p) = \sum_{i=1}^{k} r_i(A'_p) - \mid A'_p \mid = 0.$$

Also $v_p \in A'_p$ and hence A'_p must contain the minimal set A_p that contains v_p and minimizes $f(\cdot)$ among subsets of $b_\vee$. So $A'_p \supseteq A_p$. Thus $A'_p = A_p$.

iii. By Lemma 12.3.3 we know that R minimizes $f(\cdot)$. Consider the submodular inequality:

$$f(R) + f(D) \geq f(R \cup D) + f(R \cap D).$$

We know that $f(D) \leq f(R \cap D)$. Hence, $f(R) \geq f(R \cup D)$, i.e., $R \cup D$ minimizes $f(\cdot)$.
Now $R = (S - b_\vee) \uplus (R \cap b_\vee)$. Further $R \cap b_i$ is a base of $\mathcal{M}_i \cdot R, i = 1, \cdots, k$, and these bases are pairwise disjoint. So $f(R \cap b_\vee) = 0$ and hence $R \cap b_\vee \subseteq D$. Therefore,

$$\begin{aligned} f(R) &= (\sum r_i(R) - \mid R \cap b_\vee \mid) - \mid R - b_\vee \mid \\ &= (\sum r_i(R \cap b_\vee) - \mid R \cap b_\vee \mid) - \mid S - b_\vee \mid \\ &= f(R \cap b_\vee) - \mid S - b_\vee \mid . \\ &= - \mid S - b_\vee \mid . \end{aligned}$$

Since $S - R \subseteq b_\vee$, in order to show that $R \cup D$ is the maximal subset of S that minimizes $f(\cdot)$ over subsets of S, it suffices to prove that $f(R \cup D') > f(R \cup D)$ whenever $b_\vee \supseteq D' \supset D$. We have

$$\begin{aligned} f(R \cup D') &= \sum r_i(R \cup D') - \mid R \cup D' \mid \\ &= (\sum_{i=1}^{k} r_i(R \cup D') - \mid D' \mid) - \mid R - D' \mid \\ &= (\sum_{i=1}^{k} r_i(D') - \mid D' \mid) - \mid R - b_\vee \mid \end{aligned}$$

(since R is spanned by $R \cap b_\vee$ in each $\mathcal{M}_i$ and $R \cap b_\vee \subseteq D \subseteq D'$). We saw earlier that $D' \cap b_i$ are either not all bases of $\mathcal{M}_i, i = 1, \cdots, k$ or are not pairwise disjoint. Hence, $f(D') > 0$. Thus, $f(R \cup D') = f(D') - |R - b_\vee| > 0 - \mid S - b_\vee \mid$. We have already seen that $f(R \cup D) = f(R) = - \mid S - b_\vee \mid$. Thus, $f(R \cup D') > f(R \cup D)$ as needed.

iv. We have,

$$\begin{aligned} f(R \cup A_p) &= \sum_{i=1}^{k} r_i(R \cup A_p) - \mid R \cup A_p \mid) \\ &= (\sum_{i=1}^{k} r_i(R \cup A_p) - \mid (R \cup A_p) \cap D \mid) - \mid R - b_\vee \mid, \\ &= (\sum_{i=1}^{k} r_i(R \cup A_p) - \mid (R \cup A_p) \cap D \mid) - \mid S - b_\vee \mid . \end{aligned}$$

Now $A_p \cap b_i$ are bases of $\mathcal{M}_i \cdot A_p, i = 1, \cdots, k$, and $R \cap b_i$ are bases of $\mathcal{M}_i \cdot R, i = 1, \cdots, k$. So $(R \cup A_p) \cap b_i$ span $R \cup A_p$ in $\mathcal{M}_i, i = 1, \cdots, k$. Further, b_i are independent in $\mathcal{M}_i$ and $b_i \cap D$ are pairwise disjoint. Also $R \cap b_\vee, A_p \cap b_\vee$ are subsets of D. Hence, $(R \cup A_p) \cap b_i$ are pairwise disjoint bases of $\mathcal{M}_i \cdot (R \cup A_p)$, $i = 1, \cdots, k$. Hence,

$$(\sum_{i=1}^{k} r_i((R \cup A_p) \cap D) - \mid (R \cup A_p) \cap D \mid) = 0.$$

Hence,

$$\begin{aligned} f(R \cup A_p) &= (\sum_{i=1}^{k} r_i((R \cup A_p) \cap D) - \mid (R \cup A_p) \cap D \mid) - \mid S - b_\vee \mid . \\ &= 0 - \mid S - b_\vee \mid \end{aligned}$$

Thus, $R \cup A_p$ minimizes $f(\cdot)$.
Next suppose $v_p \in R \cup A"_p$ and $R \cup A"_p$ minimizes $f(\cdot)$. We must have,

$$- \mid S - b_\vee \mid = f(R \cup A"_p) = (\sum_{i=1}^{k} r_i(R \cup A"_p) - \mid (R \cup A"_p) \cap b_\vee \mid) - \mid (R \cup A"_p) - b_\vee \mid .$$

Since $S - b_\vee = R - b_\vee$ we must have $S - b_\vee = (R \cup A"_p) - b_\vee$ and therefore,

$$\sum_{i=1}^{k} r_i(R \cup A"_p) = \mid (R \cup A"_p) \cap b_\vee \mid .$$

This can happen only if $(R \cup A"_p) \cap b_i$ span $\mathcal{M}_i \cdot (R \cup A"_p), i = 1, \cdots, k$, and are pairwise disjoint. Thus $f((R \cup A"_p) \cap b_\vee) = 0$. So $(R \cup A"_p) \cap b_\vee$ minimizes $f(\cdot)$ among subsets of $b_\vee$ containing v_p and therefore $(R \cup A"_p) \cap b_\vee \supseteq A_p$, by the definition of A_p. Hence,

$$\begin{aligned} R \cup A_p &= (R - b_\vee) \uplus ((R \cup A_p) \cap b_\vee) \\ &\subseteq (R - b_\vee) \uplus ((R \cup A"_p) \cap b_\vee) \\ &\subseteq R \cup A"_p \; as\ required. \end{aligned}$$

□

12.4 PP of the Rank Function of a Matroid

12.4.1 Constructing $\mathcal{B}_{\lambda r,|\cdot|}$

Algorithm Matroid Union can be used as a basic subroutine in building the principal partition of $(r, |\cdot|)$, where $r(\cdot)$ is the rank function of a matroid.

Consider the problem of building $\mathcal{B}_{\lambda r,|\cdot|}$. Our way of constructing this family would be as in the Remark 11.6.1 of page 473, i.e., we first find the minimal and maximal sets, X_λ, X^λ respectively, minimizing the function $\lambda r(X) + \mid S - X \mid$. Next we find, for each $e \in X^\lambda$, the minimal minimizing set containing e. From this we can build the Hasse Diagram of the preorder $(\succeq_\lambda)$ whose ideals are the members of $\mathcal{B}_{\lambda r,|\cdot|}$. Finding sets that minimize $\lambda r(X) + \mid S - X \mid$, where $\lambda = p/q, p, q$ positive integers,$X \subseteq S$, is equivalent to that of computing minimizing sets of $pr(X) + q \mid S - X \mid$. (Observe that the λ's that come up in this case when Algorithm P-sequence is used are all rational. Further, $p \leq \mid S \mid$ and $q \leq r(S)$. Hence one can use a technique called 'balanced bisection' which is described in page 599). The following lemma is useful in solving this latter problem.

We begin with some preliminary notation.
Let $f(\cdot)$ be a polymatroid rank function on subsets of S and let $g(\cdot)$ be an integral positive weight function on S. Let $\hat{S}$ be obtained from S by replacing each $e \in S$ by $g(e)$ copies of it. We take S to be a subset of $\hat{S}$. For $X \subseteq S$, let **hat(X)** denote the subset of $\hat{S}$ which contains X and all the copies of elements in X. For $\hat{X} \subseteq \hat{S}$ let **floor**$(\hat{X})$ denote the subset of S s.t. $hat(floor(\hat{X})) \supseteq \hat{X} \supseteq floor(\hat{X})$, i.e., $floor(\hat{X})$ contains all those elements of S some of whose copies are contained in $\hat{X}$. Define $\hat{f}(\hat{X}) \equiv f(floor(\hat{X})), \hat{X} \subseteq \hat{S}$. Thus, $\hat{f}(\cdot)$ is a polymatroid rank function on subsets of $\hat{S}$ being obtained by making all the copies of $e \in S$ parallel (see Definition 10.5.3).

Lemma 12.4.1 *i. A set $\hat{Y} \subseteq S$ minimizes $\hat{f}(\hat{X}) + \mid \hat{S} - \hat{X} \mid, \hat{X} \subseteq \hat{S}$, only if $\hat{Y} = hat(Y)$ for some $Y \subseteq S$.*

ii. A set $Y \subseteq S$ minimizes $f(X) + g(S - X), X \subseteq S$, iff $hat(Y)$ minimizes $\hat{f}(\hat{X}) + \mid \hat{S} - \hat{X} \mid, \hat{X} \subseteq \hat{S}$.

Proof : i. We have $\hat{f}(\hat{X}) = \hat{f}(hat(floor(\hat{X})))$ and $\hat{X} \subseteq hat(floor(\hat{X}))$. The result follows since $|\cdot|$ is a strictly increasing function.

ii. We have

$$f(Y) + g(S - Y) = \hat{f}(hat(Y)) + \mid \hat{S} - hat(Y) \mid$$

The result now follows using the previous part of the present lemma.

□

Thus minimizing $pr(X) + q \mid S - X \mid$ is equivalent to minimizing

$$p\hat{r}(\hat{X}) + \mid \hat{S} - \hat{X} \mid, \text{ where } g(e) \equiv q|e|.$$

If $\mathcal{M}$ is a matroid on S with rank function $r(\cdot)$, the matroid $\hat{\mathcal{M}}$ on $\hat{S}$ which has rank function $\hat{r}(\cdot)$ is obtained by replacing each $e \in S$ by q parallel edges. By Lemma 12.3.3, the minimal set that minimizes $p\hat{r}(\hat{X})+ \mid \hat{S}-\hat{X} \mid$ is the set of non-coloops of the matroid $(\hat{\mathcal{M}})^p \equiv \hat{\mathcal{M}} \vee \cdots \vee \hat{\mathcal{M}}$, where the union operation is performed on $\hat{\mathcal{M}}, p$ times $(\hat{\mathcal{M}}^1 \equiv \hat{\mathcal{M}}, (\hat{\mathcal{M}})^2 \equiv \hat{\mathcal{M}} \vee \hat{\mathcal{M}}, \cdots)$.

Use of Algorithm Matroid Union gives us bases $\hat{b}_1 \cdots \hat{b}_p$ of $\hat{\mathcal{M}}$ s.t. $\bigcup_{i=1}^p \hat{b}_i$ is a base of $\hat{\mathcal{M}}^p$. It also gives us the set $\hat{R}$ of all vertices reachable from $\hat{S} - \bigcup_{i=1}^p \hat{b}_i$ in the graph $\mathcal{G}(\hat{b}_1, \cdots, \hat{b}_p)$. Now $\hat{R}$ is the set of noncoloops of the matroid $\hat{\mathcal{M}}^p$ and is the minimal set that minimizes $p\hat{r}(\hat{X})+ \mid \hat{S}-\hat{X} \mid$ (equivalently, minimizes $p\hat{r}(\hat{X})- \mid \hat{X} \mid$) among subsets of $\hat{S}$. The set $(floor(\hat{R}))$ would be the minimal set X_λ that minimizes $pr(X) + q \mid S - X \mid$ (equivalently, minimizes $\lambda r(X) + \mid S - X \mid, X \subseteq S, \lambda = \frac{p}{q}$).Let $\hat{D}$ be the set of all elements in $\bigcup_{i=1}^p \hat{b}_i$ from which no element common to more than one $\hat{b}_i$ can be reached in $\mathcal{G}(\hat{b}_1, \cdots, \hat{b}_p)$. By Lemma 12.3.6, $\hat{R} \cup \hat{D}$ is the maximal set that minimizes $p\hat{r}(\hat{X})+ \mid \hat{S}-\hat{X} \mid, \hat{X} \subseteq \hat{S}$. The set $floor(\hat{R} \cup \hat{D}))$ would be the maximal set X^λ that minimizes $pr(X) + q|S - X|$ (equivalently, $\lambda r(X)+ \mid S - X \mid, X \subseteq S$).

To build $\mathcal{B}_{\lambda r, |\cdot|}$ we need to be able to find the minimal set, containing a specified element in $floor((\hat{R} \cup \hat{D}) - \hat{R})$, that minimizes $\lambda r(X)+ \mid S-X \mid$. This is equivalent to finding the minimal set, containing a specified element in $\hat{D} - \hat{R}$, that minimizes $p\hat{r}(\hat{X})+ \mid \hat{S}-\hat{X} \mid$. For this purpose, Lemma 12.3.6 shows that, we need to find for each $\hat{v}_p \in \hat{D} - \hat{R}$ the set $\hat{A}_p$ of all elements approachable from it relative to $(\hat{b}_1, \cdots, \hat{b}_p)$. Actually it can be shown that it is sufficient to do this for each $\hat{v}_p \in \hat{b}_1 - \hat{R}$. The set $floor(\hat{A}_p)$ would be the minimal set that minimizes $\lambda r(X)+ \mid S - X \mid$ under the condition that it contains the element in S that is parallel to $\hat{v}_p$. Repeating the process for each $\hat{v}_p \in \hat{b}_1 - \hat{R}$ yields the preorder defining $\mathcal{B}_{\lambda r, |\cdot|}$.

12.4.2 Complexity of constructing $\mathcal{B}_{\lambda r, |\cdot|}$

i. Algorithm Matroid Union

Some care is needed in applying the Algorithm Matroid Union to compute $\mathcal{B}_{\lambda r, |\cdot|}$. The set $\hat{S}$ has size $q \mid S \mid$ but is derived by adding $(q-1)$ elements in parallel to each element in S. So the construction of $\mathcal{G}(\hat{b}_1, \cdots, \hat{b}_p)$ is somewhat simpler than appears at first sight. The following points need to be noted.

i. If two elements outside a base are in parallel, their f-circuits with respect to a base would have identical intersections with the base.

ii. The shortest path between two vertices of $\mathcal{G}(\hat{b}_1, \cdots, \hat{b}_p)$ cannot exceed $r(\hat{\mathcal{M}})$ in length. So in STEP 4 of Algorithm Matroid Union no more than $r(\hat{\mathcal{M}})$ bases need to be tampered with.

Initial work to set up $\mathcal{G}(\hat{b}_{o1}, \cdots, \hat{b}_{op})$
Here $\hat{b}_{o1}, \cdots, \hat{b}_{op}$ are the bases to initialize Algorithm Matroid Union. They may be obtained conveniently by taking q parallel copies of a base of $\mathcal{M}$ and repeating one of the bases an additional $(p-q)$ times. We need only one vertex per element v in S (with no additional vertices for parallel elements of v in $\hat{S}$). Let us call this graph $\mathcal{G}_{red}(\hat{b}_{o1}, \cdots, \hat{b}_{op})$. Here f-circuit computation has to be done only for $\mid S \mid$ elements. As we have shown earlier each such computation requires at most r calls to the independence oracle. Hence building $\mathcal{G}_{red}(\hat{b}_{o1}, \cdots, \hat{b}_{op})$ requires atmost $r \mid S \mid$ calls to the independence oracle. Storage requirement is that of storing the edges of this graph. In general, the space complexity (if $b_{o1}, \cdots, b_{op}$ are unrelated) is $O(rp \mid S \mid)$ since an arrow might go from each element outside a base to each element inside and there are p bases each of size r.

Effort to find the set of all reachable elements from $\hat{S} - \bigcup_{i=1}^{p} \hat{b}_i$
Here we replace parallel (and same direction) edges of $\mathcal{G}_{red}$ by a single edge. The number of resulting reduced edges cannot exceed the square of the number of vertices in $\mathcal{G}_{red}(\hat{b}_1, \cdots, \hat{b}_p)$. Thus the number of reduced edges is $O(\mid S \mid^2)$. Reachability computations therefore take $O(\mid S \mid^2)$ elementary steps.

Effort to update $\mathcal{G}_{red}(b_1, \cdots, b_p)$
At each update no more than r of the bases have to be altered. So we need to perform atmost $r|S|$ f-circuit computations, i.e., at most $r^2|S|$ calls to the independence oracle.

The number of updates of $\mathcal{G}_{red}(b_1, \cdots, b_p)$
This number cannot exceed $q(\mid S \mid -r)$ since that is an upper bound on the number of elements in $\hat{S}$ outside $\bigcup_{i=1}^{k} b_{oi}$. It cannot also exceed $(p-1)(r)$, since pr is the maximum number of elements in a base of $\hat{\mathcal{M}}^p$.

Thus, the **overall complexity of Algorithm Matroid Union** with each element parallel to atleast $(q-1)$ others is as follows.

Calls to the independence oracle:

$$O(min(q(\mid S \mid -r)r^2 \mid S \mid, (p-1)r^3 \mid S \mid) + rp \mid S \mid)$$

Since $\lambda = p/q, p \leq \mid S \mid$ and $q \leq r$, we can simplify this to $O(\mid S \mid (\mid S \mid -r)r^3)$

Elementary Steps:

$$O(q \mid S \mid^2 (\mid S \mid -r)) = O(\mid S \mid^2 (\mid S \mid -r)r)$$

Space complexity: $O(rp \mid S \mid)$.

ii. Computing the preorder for $\mathcal{B}_{\lambda r, |\cdot|}$

Once the final $\mathcal{G}_{red}(\hat{b}_1, \cdots, \hat{b}_p)$ is obtained from the Algorithm Matroid Union the remaining work for computing the preorder defining $\mathcal{B}_{\lambda r, |\cdot|}$ involves only the finding of the set $\hat{D}$ ($\equiv$ set of all elements in $\bigcup_{i=1}^{k} \hat{b}_i$, from which no element common to more than one $\hat{b}_i$ can be reached in $\mathcal{G}(\hat{b}_1, \cdots, \hat{b}_p)$), and for each element

$\hat{v}_p \in \hat{D} - \hat{R}$, finding the set of all elements approachable from $\hat{v}_p$. This can be done by finding for each $\hat{b}_i - \hat{R}$ the set of all elements reachable from it in $\mathcal{G}_{red}(\hat{b}_1, \cdots, \hat{b}_p)$. If the set includes elements in more than one $\hat{b}_i$, $\hat{v}_p$ is outside $\hat{D}$, otherwise inside. So these computations take no more than $r \mid S \mid^2$ elementary steps which therefore can be absorbed in the complexity of Algorithm Matroid Union. Thus the overall complexity for computing $\mathcal{B}_{\lambda r, |\cdot|}$ is the same as that for Algorithm Matroid Union when there are $p \leq \mid S \mid$ bases (whose union is being taken) and each element of $\hat{S}$ belongs to a set of parallel elements of size $q \leq r$.

iii. Computation of the PP of $(r, \mid \cdot \mid)$
In the case of a matroid it is easy to see that Algorithm P-sequence would call subdivide no more than $r(\mathcal{M})$ times. So the **complexity for construction of PP can be taken to be**
$O(\mid S \mid (\mid S \mid -r)r^4)$ calls to the independence oracle+
$+ \ O(\mid S \mid^2 (\mid S \mid -r)r^2)$ elementary steps.
However, somewhat more careful techniques would replace the extra r factor with a log $\mid S \mid$ factor. This requires the use of the balanced bisection technique which is described in page 599.

Space complexity is $O(r \mid S \mid^2)$.

12.4.3 Example

Principal partition of $(r(\cdot), \mid \cdot \mid)$ where $r(\cdot)$ is the rank function of a graph
Consider the graph $\mathcal{G}$ in Figure 12.1. We have, $E(\mathcal{G}) \equiv \{1, \cdots, 20\}$. The principal sequence of $(r(\cdot), \mid \cdot \mid)$ is

$$E_0 \equiv \emptyset = X_{\lambda_1}, E_1 = X_{\lambda_2}, E_2 = X_{\lambda_3}, E_3 = X_{\lambda_4}, E_4 = X_{\lambda^4} \equiv E(\mathcal{G}),$$

where $E_0 = \emptyset, E_1 = A \equiv \{1,2,3\}$, $E_2 \equiv \{4, \cdots, 13\} \uplus E_1$
$E_3 \equiv \{14, 15, 16\} \uplus E_2$, $E_4 \equiv \{17, \cdots, 20\} \uplus E_3$.
The critical values are

$$\lambda_1 = 3, \lambda_2 = 2, \lambda_3 = 3/2, \lambda_4 = 4/3.$$

The partition $\Pi_{pp} \equiv \{A, B_1, B_2, B_3, C, D\}$
where $A \equiv \{1,2,3\}$, $B_1 \equiv \{4,5,6,7,8,9\}$, $B_2 \equiv \{10,11\}$, $B_3 \equiv \{12,13\}$, $C \equiv \{14,15,16\}$, $D \equiv \{17,18,19,20\}$.
The partial order $\geq_\pi$ on Π_{pp} can be built as follows:
Consider $\mathcal{B}_{\lambda_1}$. This has only two sets $\emptyset$ and E_1. The partial order $\geq_{\lambda_1}$ therefore, has only the element $E_1 \equiv A$. $\mathcal{B}_{\lambda_2}$ has the sets $A, A \uplus B_1$, $A \uplus B_1 \uplus B_2$, $A \uplus B_1 \uplus B_3$, $A \uplus B_1 \uplus B_2 \uplus B_3$.
The partial order $\geq_{\lambda_2}$ therefore has the elements B_1, B_2, B_3. The Hasse diagram of $(\geq_{\lambda_2})$ is shown in part (c) of the figure between lines '$\lambda_1 = 3$' and '$\lambda_2 = 2$.' In the partial order $(\geq_\pi)$ all elements of the partial order $(\geq_{\lambda_1})$ are taken to be below all the elements of the partial order $(\geq_{\lambda_2})$. This is achieved in the Hasse diagram by

Figure 12.1: The Principal Partition and Refined Partial Order for a Graph

drawing from each minimal element of $(\geq_{\lambda_2})$ an arrow to each maximal element of $(\geq_{\lambda_1})$. $\mathcal{B}_{\lambda_3}$ has the two sets $E_2 = A \uplus B_1 \uplus B_2 \uplus B_3$ and $E_3 = A \uplus B_1 \uplus B_2 \uplus B_3 \uplus C$. The partial order $(\geq_{\lambda_3})$ has only the element C. $\mathcal{B}_{\lambda_4}$ has the two sets E_3 and E_4. The partial order $(\geq_{\lambda_4})$ has only the element D.
The part (d) of the figure refers to the refined partial order $(\geq_R)$ and is discussed in Section 11.5.

The partial order $(\geq_R)$ agrees with $(\geq_\pi)$ over the elements of $(\geq_{\lambda_1})$, $(\geq_{\lambda_2})$, $(\geq_{\lambda_3})$ and $(\geq_{\lambda_4})$. However, the relationship between elements in partial orders corresponding to different critical values is now changed.
We see that B_1 is properly related to A, $B_2 \cup B_1$ to A, $B_3 \cup B_1$ to A. So $(\geq_R)$ agrees with $(\geq_\pi)$ over $\{A, B_1, B_2, B_3\}$. However, C is properly related to $B_1 \cup A$, i.e., $r(C \cup B_1 \cup A)$ - $r(B_1 \cup A) = r(C \cup E_2) - r(E_2)$ where $E_2 = A \cup B_1 \cup B_2 \cup B_3$ and no ideal of the restriction of $(\geq_R)$ to $\{A, B_1, B_2, B_3\}$, properly contained in $\{B_1, A\}$, would satisfy the above relationship. This can also be seen directly by inspection. When the edges of E_2 are contracted, the graph on C is a 'triangle' with $14, 15, 16$ appearing in series. This effect is also achieved simply by contracting the branches of $B_1 \cup A$. Thus, we take only B_1, A to be below C in $(\geq_R)$. In the Hasse diagram there is, therefore, a directed edge from C to B_1. A similar argument shows that there should be an arrow from D to B_1. The Hasse Diagram of $(\geq_R)$ is shown in (d).

Now some informal remarks on how to construct examples with a desired refined partial order. (Not every refined partial order may be 'realizable' in this manner). We first build atomic structures corresponding to A, B_1, B_2, B_3, C, D, with appropriate critical values. This is shown in part (b) of Figure 12.1. There are many ways in which atomic structures can be built. It is easily seen that any totally edge symmetric matroid that is connected has to be atomic if $g(\cdot)$ takes the same value on all edges. For graphs this translates to 'totally symmetric 2-connected'. Once this is available we build the refined partial order starting from the highest critical value and going down. In the present example the sequence is as follows: On A we build the graph of three parallel edges. On B_1 we build K_4. In this graph $9, 4, 5$ are incident at a node. We split this node and attach the two halves across A. The result would be that when A is contracted we see K_4 on B_1. An arrow goes from B_2 to B_1. So we ensure that unless B_1 is contracted we would not see the atomic structure of two parallel branches on B_2.

This procedure is continued until we exhaust all the elements of $\geq_R$ (i.e., the blocks of Π_{pp}). We are ensuring that on E_1 there is a molecular structure of critical value λ_1, and on $E_2 - E_1$, when E_1 is contracted, one of critical value λ_2, and on $E_3 - E_2$, when E_2 contracted, one of critical value λ_3, and on $E_4 - E_3$, when E_3 contracted, one of critical value λ_4. Within each molecular structure the relationship between the atomic structures is maintained as in the partial order $(\geq_{\lambda_i})$. Further, $\lambda_1 > \lambda_2 > \lambda_3 > \lambda_4$. These facts are sufficient, by the use of Uniqueness Theorem (Theorem 11.4.6), to ensure that the graph that is built as described has the ideals of $\geq_\pi$ as principal partition. **Once this holds**, since we have built the structure consistent with $(\geq_R)$, the refined partial order of the graph would have to be $(\geq_R)$.

The Uniqueness Theorem also ensures that parts of the principal partition remain unaffected when changes are made in certain other parts of the graph.
Suppose we **contract** a branch in B_1. The principal partition would remain unaffected as far as all elements above B_1 in $(\geq_\pi)$ are concerned. This also is true of $(\geq_R)$. If we **delete** a branch in C say, the principal partition would remain unaffected as far as all elements below C are concerned.
The above facts are true for all $(f(\cdot), g(\cdot))$, where $f(\cdot)$ is submodular and $g(\cdot)$, a positive weight function. Using the Uniqueness Theorem, we can also show that increasing $g(e), e \in B_1$, is analogous to contraction of e (i.e., principal partition unaffected as far as elements above B in the partial order $(\geq_\pi)$ or $(\geq_R)$ are concerned), and decreasing $g(e)$ is analogous to deletion of e.

Exercise 12.13 *Verify the following simple rules for building atomic graphs (i.e., $E(\mathcal{G})$ is atomic with respect to $(r(\cdot), k \mid \cdot \mid)$, where k is any positive number).*

i. Any graph that is 2-connected and totally edge symmetric is atomic.

ii. Build two graphs, one atomic and the other molecular, on the same set of vertices. Now put the two graphs together (merge corresponding vertices). The result is an atomic graph.

iii. Start from an atomic graph. Replace each edge by m parallel copies (m series edges). The result is an atomic graph.

iv. If $(f(\cdot), g(\cdot))$ is atomic so is $(f^*(\cdot), g(\cdot))$ where the dual is taken with respect to $g(\cdot)$. Hence, if an atomic graph is planar its planar dual is atomic.

v. Start with an atomic (molecular) graph with density $\frac{|E|}{r(E)} = d$. Suppose $d < d(\frac{r(E)}{r(E)+1}) + 1$. Then adding a new vertex and joining it to each of the original nodes would result in an atomic graph.

12.5 Notes

The problem of matroid intersection (find the maximum size common independent set of two given matroids) and its solution has received more attention in the literature than matroid union. This is probably because Lawler based his well known book [Lawler76] on matroid intersection. A good way of approaching either of these ideas is through submodular functions induced through a bipartite graph [Welsh76]. Related material may be found in the survey paper by Brualdi [Brualdi74]. Detailed information on transversal theory may be found in [Mirsky71]. The principal partition of the rank function of a matroid can be generalized to polymatroids defined on two sides of a bipartite graph. Such results have the flavour of the Dulmage-Mendelsohn decomposition ([Dulmage+Mendelsohn59]) of a bipartite graph [Iri79a] and have good practical applications. In this book, since we wish to be 'device independent', we have steered away from this important class of applications of the matroid union theorem and its generalizations, namely, structural solvability of systems [Murota87], [Recski89].

12.6 Solutions of Exercises

E 12.1:
i. Let b_1, b_2 be bases of $\mathcal{M}_1, \mathcal{M}_2$ s.t. $b_1 \cap b_2 = b_{12}$. Let $b_2{}^* = S - b_2$. Then $b_{12} = b_1 \cup b_2{}^* - b_2{}^*$. Next $b_1 \cup b_2{}^*$ is independent in $\mathcal{M}_1 \vee \mathcal{M}_2^*$. Let this be contained in the base $b_{1n} \cup b_{2n}^*$ of $\mathcal{M}_1 \vee \mathcal{M}_2^*$. Now $b_{1n} \cup b_{2n}^* - b_{2n}^*$ is independent in $\mathcal{M}_1$ and $\mathcal{M}_2$. Further,

$$\mid b_{1n} \cup b_{2n}^* - b_{2n}^* \mid \geq \mid b_1 \cup b_2{}^* - b_2{}^* \mid = \mid b_{12} \mid .$$

But b_{12} is the largest common independent set of $\mathcal{M}_1$ and $\mathcal{M}_2$. We conclude that $b_{12} = b_{1n} \cup b_{2n}^* - b_{2n}^*$, where $b_{1n} \cup b_{2n}^*$ is a base of $\mathcal{M}_1 \vee \mathcal{M}_2^*$.

ii. It is clear from the above that $\mid b_{12} \mid = r(\mathcal{M}_1 \vee \mathcal{M}_2^*) - r(\mathcal{M}_2^*)$.

iii. If b_{12}^* is a base of $\mathcal{M}_1 \vee \mathcal{M}_2^*$ with $b_{12}^* = b_1 \cup b_2{}^*$, where $b_1, b_2{}^*$ are bases of $\mathcal{M}_1, \mathcal{M}_2^*$ respectively, then $b_{12}^* - b_2{}^*$ is independent in $\mathcal{M}_1$ as well as in $\mathcal{M}_2$ and further its size equals $r(\mathcal{M}_1 \vee \mathcal{M}_2^*) - r(\mathcal{M}_2^*)$.

E 12.2:
i. An independent set of the LHS matroid is a union of independent sets of the matroids $\mathcal{M}_i$ which are each contained in T. Clearly an independent set of the RHS matroid has precisely the same property.

ii. The rank function of the LHS matroid is $r \diamond \mathbf{T}(\cdot)$. Now $r(\cdot) \equiv ((\sum r_i) * \mid \cdot \mid)(\cdot)$. We are given that

$$((\sum r_i) * \mid \cdot \mid)(S - T) = \sum_{i=1}^{k} r_i(S - T).$$

So by Lemma 11.4.5,

$$r \diamond \mathbf{T}(\cdot) = (((\sum r_i) * \mid \cdot \mid) \diamond \mathbf{T})(\cdot) = (((\sum r_i) \diamond \mathbf{T}) * \mid \cdot \mid)(\cdot).$$

The RHS is clearly the rank function of the matroid $\mathcal{M}_1 \times T \vee \cdots \vee \mathcal{M}_k \times T$.

E 12.3: Let $r_\vee(\cdot)$ be the rank function of $\mathcal{M}_1 \vee \cdots \vee \mathcal{M}_k$. The problem reduces to characterizing the case where $r_\vee(S) = \mid S \mid$. By Nash-William's rank formula (Corollary 12.2.2)

$$r_\vee(S) = \min_{X \subseteq S} (\sum_{i=1}^{k} r_i(X) + \mid S - X \mid).$$

Clearly (see for instance Exercise 11.5) the RHS equals $\mid S \mid$ iff there exists no subset T of S s.t.

$$\sum_{i=1}^{k} r_i(T) <\mid T \mid .$$

E 12.4: Consequence of Theorem 11.7.4.

E 12.5:

i. For each $\Gamma_R(a), a \in V_R$, build the matroid $\mathcal{M}_a$ on V_L by taking all elements in $\Gamma_R(a)$ in parallel and those in $V_L - \Gamma_R(a)$ to be selfloops. Consider the matroid $\mathcal{M}_L$ on V_L defined by $\mathcal{M}_L \equiv \vee_{a \in V_R} \mathcal{M}_a$. We will show that $T \subseteq V_L$ is a transversal of some subset of V_R iff it is independent in the above matroid.
Let T be a transversal of $Y \subseteq V_R$. Then there is a bijection $\tau : T \rightarrow Y$ s.t. $(e, \tau(e))$ is an edge in B. Thus for each $e \in T, \mathcal{M}_{\tau(e)}$ has $\{e\}$ as an independent set. Hence, $\vee_{e \in T} \mathcal{M}_{\tau(e)}$ has T as an independent set. On the other hand suppose T is an independent set of $\vee_{a \in V_R} \mathcal{M}_a$. Then for each $e \in T$ there is a matroid $\mathcal{M}_{\tau(e)}$ in which $\{e\}$ is independent, i.e., there is an edge $(e, \tau(e))$ in B. Also $\tau(e_i) \neq \tau(e_j),\ i \neq j$. This shows that T is a transversal of $\{\tau(e), e \in T\}$.

ii. Let $\mathcal{M} = \mathcal{M}_1 \vee \cdots \vee \mathcal{M}_k$, where $\mathcal{M}_1, \cdots, \mathcal{M}_k$ are rank one matroids on S. Let $B \equiv (V_L, V_R, E)$ where $V_L \equiv S$, $V_R \equiv \{1, \cdots, k\}$ with edge (v, i) present iff v is not a selfloop in $\mathcal{M}_i$. ($\Gamma_R(i) \equiv$ the set of non-selfloops of $\mathcal{M}_i$, also these are in parallel in $\mathcal{M}_i$). It is easy to verify that $\mathcal{M}$ is the matroid on V_L whose bases are transversals of V_R.

iii. If $\mathcal{M}$ is a transversal matroid on V_L then $\mathcal{M} = \mathcal{M}_1 \vee \cdots \vee \mathcal{M}_k$, where $\mathcal{M}_i$ are rank one matroids on V_L. Now, $\mathcal{M} \cdot K = \vee_{i=1}^{k} \mathcal{M}_i \cdot K$. If $\mathcal{M} \cdot K$ has no coloops then by Lemma 12.3.3, $r(\mathcal{M} \cdot K) = \sum_{i=1}^{k} r(\mathcal{M} \cdot K)$. This means (see Exercise 12.2) $\mathcal{M} \times (V_L - K) = \vee_{i=1}^{k} \mathcal{M}_i \times (V_L - K)$. Each matroid $\mathcal{M}_i \times (V_L - K)$ has rank atmost one (if zero, the matroid has only selfloops). So $\mathcal{M} \times (V_L - K)$ is either of rank zero or can be expressed as a union of rank one matroids, (if $\mathcal{M}_2$ has rank zero $\mathcal{M}_1 \vee \mathcal{M}_2 = \mathcal{M}_1$). It is therefore a transversal matroid (perhaps a trivial one).

iv. By Nash-Williams Rank formula (Corollary 12.2.2),

$$r(\vee_{a \in V_R} \mathcal{M}_a) = \min_{X \subseteq V_L} (\sum_{a \in V_R} r_a(X) + \mid V_L - X \mid), \qquad (*)$$

where $r_a(\cdot)$ is the rank function of $\mathcal{M}_a$. Now for each $X \subseteq V_L, r_a(X)$ is one if $a \in \Gamma_L(X)$ and zero otherwise. Hence, $\sum_{a \in V_R} r_a(X) =\mid \Gamma_L(X) \mid$. Thus the RHS of $(*)$ is $\min_{X \subseteq V_L}(\mid \Gamma_L(X) \mid + \mid V_L - X \mid)$. Now the size of the maximum matching is the size of the maximum transversal of a subset of V_R and hence the result follows.

E 12.6: The rank function of $\mathcal{M}(f_1) \vee \mathcal{M}(f_2)$ (by Nash-Williams' rank formula (Corollary 12.2.2)) is $(f_1 * \mid \cdot \mid + f_2 * \mid \cdot \mid)(\cdot)$, while that of $\mathcal{M}(f_1 + f_2)$ is
$((f_1 + f_2) * \mid \cdot \mid)(\cdot)$. The result now follows from the one in Exercise 11.3.

E 12.7:

i. We have

$$(r_1{}^* + r_2{}^*)(X) + \mid S - X \mid$$
$$= (2 \mid X \mid -(r_1(S) + r_2(S) - r_1(S-X) - r_2(S-X)) + \mid S - X \mid$$
$$= ((r_1 + r_2)(S-X) + \mid S - (S-X) \mid) + (\mid S \mid -(r_1+r_2)(S)).$$

It is thus clear that K minimizes $(r_1+r_2)(Y)+|S-Y|, Y \subseteq S$ iff $(S-K)$ minimizes $(r_1{}^* + r_2)^*(Y) + \mid S - Y \mid, Y \subseteq S$.

ii. This is an immediate consequence of the above result.

iii. We observe that (by Lemma 12.3.3) the collection of non-coloops of $\mathcal{M}_1 \vee \mathcal{M}_2$ ($\mathcal{M}_1^* \vee \mathcal{M}_2^*$) is the minimal set R (R^*) that minimizes $(r_1+r_2)(X) + \mid S - X \mid, X \subseteq S$ ($(r_1{}^* + r_2{}^*)(X) + \mid S - X \mid, X \subseteq S$). However, the second part (above) of this exercise shows that $S - R \supseteq R^*$.

E 12.8:

i. Let b_1 be a base of $\mathcal{M}_1 \cdot X$ and b_2', a base of $\mathcal{M}_2 \cdot (S-X)$. Clearly $b_1 \cup b_2'$ is independent in $\mathcal{M}_1 \vee \mathcal{M}_2$. Thus,

$$\max_{X \subseteq S}(r_1(X) + r_2(S-X)) \leq r(\mathcal{M}_1 \vee \mathcal{M}_2).$$

On the other hand let $b_\vee$ be a base $\mathcal{M}_1 \vee \mathcal{M}_2$ and let b_1, b_2 be bases of $\mathcal{M}_1, \mathcal{M}_2$ respectively s.t. $b_\vee = b_1 \cup b_2$. Clearly

$$r_1(b_1) + r_2(S - b_1) = \mid b_1 \mid + \mid b_2 - b_1 \mid = \mid b_\vee \mid .$$

But Nash-Williams' rank formula (Corollary 12.2.2) gives

$$b_\vee = r(\mathcal{M}_1 \vee \mathcal{M}_2) = \min_{X \subseteq S}((r_1 + r_2)(X) + \mid S - X \mid).$$

This proves the result.

ii. Let $b_\vee^*$ be a base of $\mathcal{M}_1 \vee \mathcal{M}_2^*$ and let $b_\vee^* = b_1 \cup b_2^*$ where b_1, b_2^* are bases of $\mathcal{M}_1, \mathcal{M}_2^*$ respectively. We saw in Exercise 12.1 that $b_\vee^* - b_2^*$ is a maximum size common independent set of $\mathcal{M}_1, \mathcal{M}_2$. Now if $r_2^*(\cdot)$ denotes the rank function of $\mathcal{M}_2^*$,

$$\begin{aligned}
\mid b_\vee^* - b_2^* \mid &= \min_{X \subseteq S}((r_1 + r_2{}^*)(X) + \mid S - X \mid) - r_2^*(S) \\
&= \min_{X \subseteq S}(r_1(X) + \mid X \mid - r_2(S) + r_2(S-X) + \mid S - X \mid - \mid S \mid + r_2(S)) \\
&= \min_{X \subseteq S}(r_1(X) + r_2(S-X)).
\end{aligned}$$

This proves the result.

iii. This follows from the above when we recognize that $(b_1 - b_2)$ is a common independent set between $\mathcal{M}_1$ and $\mathcal{M}_2^*$ (also every common independent set can be so written) and so $max \mid b_1 - b_2 \mid =$ maximum size of common independent set of $\mathcal{M}_1$ and $\mathcal{M}_2^* = \min_{X \subseteq S}(r_1(X) + r_2{}^*(S-X))$.

E 12.9: Observe that rank function of $\mathcal{M} \vee \mathcal{M}$ is $(2r * \mid \cdot \mid)(\cdot)$,
$(\mathcal{M} \vee \mathcal{M})^* \vee \mathcal{M}^*$ is $(((2r * \mid \cdot \mid)^* + r^*) * \mid \cdot \mid)(\cdot)$ etc.
So we can use the ideas developed in Section 11.7.

E 12.10: Let the shortest path from v_o in $\mathcal{G}(b_1, \cdots, b_k)$ be $v_o, e_1, \cdots, e_m, v_m$. In this path let v_m be the first to belong to more than one of the b_i. Now apply STEP 4 of Algorithm Matroid Union to $v_1, e_2, v_2, \cdots, e_m, v_m$. The result would be that v_1 belongs to two of the updated b_i''s and v_o, e_1, v_1 would be a valid path in $\mathcal{G}(b_1', \cdots, b_k')$ and $\bigcup b_i' = \bigcup b_i$.

E 12.11:
i. Start from any vertex v and do a bfs. We need to show that in r steps or less we would have covered all vertices reachable from v. Let R_v^i denote the set of vertices reachable in not more than i steps from v. We will show that $r(R_v^{i+1}) \geq r(R_v^i) + 1$ or R_v^{i+1} is the set of all elements reachable from v. It is clear that $r(R_v^{i+1}) \geq r(R_v^i)$. If the inequality is an equality then $R_v^{i+1} \cap b_j$ must span R_v^i for each j and therefore must span R_v^{i+1} for each j. Hence, from inside R_v^{i+1} it is impossible to reach outside it in $\mathcal{G}(b_1, \cdots, b_k)$. Thus we see that if the first equality is reached at the $(k+1)^{th}$ step, i.e., $r(R_v^k) = r(R_v^{k+1})$, then $r(R_v^{k+1}) \geq k+1$ (if $r(R_v^o) = r(v) = 1$). Hence, $k+1$ cannot exceed $r(\mathcal{M})$. (If $r(v) = 0$, then v is the only element reachable from v.)

ii. Let K be any subset of T with the property
$(*)$ from no element in K we can approach outside K.
Then it is clear that $b_i \cap K$ form pairwise disjoint bases of $\mathcal{M} \cdot K$. Let T_v denote the sets of all elements approachable from v and let T_v' denote the set of all elements approachable from $v \in T - R$ but from which v cannot be approached. Both T_v and T_v' have the property $(*)$. Further $T_v - T_v'$ is the equivalence class (v) containing v. Now $T_v \cap b_i$ form pairwise disjoint bases of $\mathcal{M}_i \cdot T_v$ and $T_v' \cap b_i$ form pairwise disjoint bases of $\mathcal{M}_i \cdot T_v'$. For at least one of $i \in \{1, \cdots, k\}$ we must have $\mid T_v' \cap b_i \mid < \mid T_v \cap b_i \mid$. But then $r_i(T_v') < r_i(T_v)$ and $T_v' \cap b_i \subset T_v \cap b_i$, for each $i \in \{1, \cdots, k\}$. Hence, $(v) \cap b_i \neq \emptyset, i = 1, \cdots, k$.

E 12.12: We begin with two bases of b_{01}, b_{02}^* of matroids $\mathcal{M}_1, \mathcal{M}_2^*$ respectively on S. Let $b_{02} = S - b_{02}^*$. We now try to push updated versions of b_{01}, b_{02} apart. However, we would like to work with f-circuits of $\mathcal{M}_2^*$ rather than with f-circuits of $\mathcal{M}_2$. For this it suffices to observe that $v_p \in L_2^*(v_q, b_2^*)$ iff $v_q \in L_2(v_p, S - b_2^*)$, where $L_2^*(\cdot,\cdot), L_2(\cdot,\cdot)$ denote f-circuits of $\mathcal{M}_2^*, \mathcal{M}_2$ respectively. So while constructing $\mathcal{G}(b_1, S - b_2^*)$ it is convenient to build edges of the type $(v_p, v_q, 2)$ at the node v_q directed into v_q (rather than at v_p directed away from v_p). If $b_1, S - b_1^*$ are maximally distant bases of $\mathcal{M}_1, \mathcal{M}_2$, then $b_1 \cap b_2^*$ is a common independent set of $\mathcal{M}_1, \mathcal{M}_2^*$ of maximum size (see Exercise 12.1).

Chapter 13

Dilworth Truncation of Submodular Functions

13.1 Introduction

In this chapter we study the **Dilworth truncation** (**truncation** for short) operation on submodular functions and the related notion of **principal lattice of partitions**. The theory of Dilworth truncation bears a strong resemblance to that of convolution. This chapter has been written in a manner that emphasizes this resemblance. The truncation operation was first used by Dilworth [Dilworth44] to build a new matroid with certain natural properties on the collection of flats of specified rank in a given matroid. The definition of the operation involves special partitions of the set over which a submodular function is defined. These partitions are of theoretical and practical significance. So one could, in addition to studying the truncation of a submodular function, also study the partitions used to define the operation. This leads us to the principal lattice of partitions of the submodular function.

We begin with formal properties of the truncation operation and present a number of examples from the literature, including Dilworth's own, relevant to the operation. We then study the principal lattice of partitions (PLP) of a submodular function and point out the similarity in its properties with those of the principal partition (PP) - indeed the characteristic properties of the PLP can be obtained by 'translating' those of the PP from sets to partitions. Keeping in mind applications of the PLP we have presented an alternative development of the theory from a 'cost of partition' point of view. We have also described a technique for building approximation algorithms for 'optimum partitioning' problems (including vertex partitioning for a graph, minimizing number of cut edges). After this we study the relation of the PLP of a given submodular function to the PLP of naturally derived functions such as truncations and fusions of the original. We relegate the study of algorithms for the construction of the PLP to the next chapter.

13.2 Dilworth Truncation

13.2.1 Formal Properties

We remind the reader that a **partition** of S is a collection of pairwise disjoint subsets whose union is S. The members of the partition are called **blocks**. A partition of an underlying set, clear from the context but usually S, which has N as a block and the remaining blocks, if any, as singletons, is denoted by Π_N. The partition which has all blocks as singletons is denoted by Π_0. The collection of all partitions of S is denoted by $\mathcal{P}_S$. We say $\Pi_1 \geq \Pi_2$, $\Pi_1, \Pi_2 \in \mathcal{P}_S$ (equivalently, Π_1 is **coarser** than Π_2 or Π_2 is **finer** than Π_1) iff every block of Π_2 is contained in some block of Π_1. The partition $\Pi_1 \vee \Pi_2$ ($\Pi_1 \wedge \Pi_2$) is the least upper bound (greatest lower bound) of the partitions Π_1 and Π_2 in the partial order ($\geq$). For a description of $\Pi_1 \vee \Pi_2$ and $\Pi_1 \wedge \Pi_2$ see Subsection 3.6.8.

Definition 13.2.1 *Let $f(\cdot)$ be a real set function on the subsets of S. The* **partition associate** *of $f(\cdot)$, defined on the collection $\mathcal{P}_S$ of all partitions of S, is denoted by $\overline{f}(\cdot)$ and is defined by $\overline{f}(\Pi) \equiv \sum_{N_i \in \Pi} f(N_i)$. The* **lower (upper) Dilworth truncation** *of $f(\cdot)$ is denoted by $f_t(\cdot)$ ($f^t(\cdot)$) and is defined by*

$$f_t(\emptyset) \equiv 0, f_t(X) \equiv min_{\Pi \in \mathcal{P}_X}(\sum_{X_i \in \Pi} f(X_i))$$

$$\left(f^t(\emptyset) \equiv 0, f^t(X) \equiv max_{\Pi \in \mathcal{P}_X}(\sum_{X_i \in \Pi} f(X_i)) \right).$$

Remark: Let $f_T(\cdot)$ be the restriction of $f(\cdot)$ on T. If there is no possibility of confusion we would say 'the function $\overline{f}(\cdot)$ over $\mathcal{P}_T$' instead of '$\overline{f}_T(\cdot)$'.

Exercise 13.1 *Let $f(\cdot), g(\cdot)$ be real set functions on subsets of S and let $\lambda, \beta \geq 0$. Show that,*

i.
$$(\lambda f)_t(\cdot) = \lambda(f_t(\cdot)),$$
$$(\lambda f)^t(\cdot) = \lambda(f^t(\cdot)),$$

ii.
$$(\lambda f + \beta g)_t(\cdot) \geq \lambda f_t(\cdot) + \beta g_t(\cdot)\ ,$$
$$(\lambda f + \beta g)^t(\cdot) \leq \lambda f^t(\cdot) + \beta g^t(\cdot),$$

If $g(\cdot)$ is a weight function, the inequalities may be replaced by equalities.

Definition 13.2.2 *Let $f(\cdot)$ be a real set function on the subsets of S. Then $f(\cdot)$ is said to be* **intersecting submodular** *iff*

$$f(X) + f(Y) \geq f(X \cup Y) + f(X \cap Y) \quad \forall X, Y \subseteq S, X \cap Y \neq \emptyset.$$

Intersecting supermodular functions *are defined similarly with the inequality reversed.*

When $f(\cdot)$ is intersecting submodular (intersecting supermodular) $\overline{f}(\cdot)$ has the following attractive property. It is the main cause of the strong analogies that exist between structural properties of convolution and those of Dilworth truncation.

Theorem 13.2.1 *[Narayanan91] Let $f(\cdot)$ be a real set function on the subsets of S and let $N \subseteq S$. Then $f(\cdot)$ is intersecting submodular (intersecting supermodular) iff*

$$\overline{f}(\Pi) + \overline{f}(\Pi_N) \geq \overline{f}(\Pi_N \vee \Pi) + \overline{f}(\Pi_N \wedge \Pi)$$
$$(\overline{f}(\Pi) + \overline{f}(\Pi_N) \leq \overline{f}(\Pi_N \vee \Pi) + \overline{f}(\Pi_N \wedge \Pi)) \quad \forall \Pi \in \mathcal{P}_S.$$

We need the following lemma in the proof of Theorem 13.2.1. A similar result is true for intersecting supermodular functions.

Lemma 13.2.1 *Let $f(\cdot)$ be an intersecting submodular function on the subsets of S. Let $M_1, \cdots, M_r, N$ be subsets of S such that $M_i \cap N \neq \emptyset \ \forall i$, $M_i \cap M_j = \emptyset$, $i \neq j$. Then,*

$$\sum_{i=1}^{r} f(M_i) + f(N) \geq \sum_{i=1}^{r} f(N \cap M_i) + f((\bigcup_{i=1}^{r} M_i) \cup N).$$

Proof of the lemma: By the intersecting submodularity of $f(\cdot)$, the result is true for r=1. Suppose it to be true for $r = k-1$. Then

$$\begin{aligned}
\sum_{i=1}^{k} f(M_i) + f(N) &= \sum_{i=2}^{k} f(M_i) + f(N) + f(M_1) \\
&\geq \sum_{i=2}^{k} f(M_i) + f(N \cup M_1) + f(N \cap M_1) \\
&\geq \sum_{i=2}^{k} f((N \cup M_1) \cap M_i) + f((\bigcup_{i=2}^{k} M_i) \cup (N \cup M_1)) \\
&\quad + f(N \cap M_1), \\
&\quad \textit{by the induction assumption for } r = k-1.
\end{aligned}$$

But, since $M_i \cap M_j = \emptyset$, $i \neq j$, we must have

$$f((N \cup M_1) \cap M_i) = f(N \cap M_i), \ i = 2, \cdots, k.$$

The required inequality is now immediate for $r = k$.

□

Proof of Theorem 13.2.1: We consider only the intersecting **submodular** case.

Only if

Let M_i be the blocks of Π. We observe that

$$\bar{f}(\Pi) + \bar{f}(\Pi_N) = \sum_{M_i \cap N = \emptyset} f(M_i) + \sum_{M_i \cap N \neq \emptyset} f(M_i) + f(N) + \sum_{e \in S-N} f(e),$$

$$\bar{f}(\Pi \vee \Pi_N) + \bar{f}(\Pi \wedge \Pi_N) = \sum_{M_i \cap N = \emptyset} f(M_i) + f(\bigcup_{M_i \cap N \neq \emptyset} M_i) + \sum_{M_i \cap N \neq \emptyset} f(M_i \cap N) + \sum_{e \in S-N} f(e).$$

The result now follows from Lemma 13.2.1 when we observe that

$$\bigcup_{M_i \cap N \neq \emptyset} M_i \supseteq N.$$

if
Let $\bar{f}(\cdot)$ satisfy the inequality in the statement of the theorem. Replacing Π by say Π_T, it is immediate from the definition of $\bar{f}(\cdot)$ that
$f(T) + f(N) \geq f(N \cup T) + f(N \cap T) \ \ \forall\ N, T \ s.t.\ N \cap T \neq \emptyset.$

□

Remark: This result is not true for two arbitrary partitions Π_1, Π_2. Consider for example the modular function $f(X) = 1, X \subseteq \{1,2,3,4\}$. Let $\Pi_1 \equiv \{\{1,2\},\{3,4\}\}$, $\Pi_2 \equiv \{\{1,3\},\{2,4\}\}$. Then $\Pi_1 \vee \Pi_2 \equiv \{1,2,3,4\}$ and $\Pi_1 \wedge \Pi_2 \equiv \{\{1\},\{2\},\{3\},\{4\}\}$. We then see that

$$\bar{f}(\Pi_1) + \bar{f}(\Pi_2) = 4 < 5 = \bar{f}(\Pi_1 \vee \Pi_2) + \bar{f}(\Pi_1 \wedge \Pi_2).$$

The following simple lemma is used frequently in this chapter.

Lemma 13.2.2 *Let $f(\cdot)$ be a real set function on subsets of S. Let $\bar{f}(\hat{\Pi}) = min_{\Pi \in \mathcal{P}_S} \bar{f}(\Pi)$ and let $N_1, \cdots, N_k$ be some of the blocks of $\hat{\Pi}$. Let $N = \bigcup_i N_i$. Then*

i. $\bar{f}(\{N_1, \cdots, N_k\}) \leq \bar{f}(\Pi')$ for each partition Π' of N.

ii. If, further, $\hat{\Pi}$ is minimal s.t. $\bar{f}(\hat{\Pi}) = min_{\Pi \in \mathcal{P}_S} \bar{f}(\Pi)$, and Π' is a nontrivial partition of a block N_1 of $\hat{\Pi}$, then $f(N_1) < \bar{f}(\Pi')$.

Proof : i. If $\bar{f}(\{N_1, \cdots, N_k\}) > \bar{f}(\Pi')$, then in $\hat{\Pi}$ we could replace $N_1, \cdots, N_k$ by the blocks of Π' and get a new partition Π_{new} s.t. $\bar{f}(\Pi_{new}) < \bar{f}(\hat{\Pi})$.

ii. Proof similar to the above.

□

Using Theorem 13.2.1 and Lemma 13.2.2 we can prove the following results.

Theorem 13.2.2 *[Narayanan91] Let $f(\cdot)$ be intersecting submodular (intersecting supermodular) over subsets of S and let Π_1 ,Π_2 minimize (maximize) $\overline{f}(\cdot)$ over $\mathcal{P}_S$. Then*

i. $\Pi_1 \vee \Pi_2$,$\Pi_1 \wedge \Pi_2$ also minimize (maximize) $\overline{f}(\cdot)$.

ii. if $N_1, \cdots, N_k$ are some of the blocks of Π_1 and $M_1, \cdots, M_r$ are some of the blocks of Π_2 such that $N_i \bigcap M_j = \emptyset \ \ \forall i, j$ and $(\bigcup N_i) \cup (\bigcup M_j) = S$, then the partition $\{N_1, \cdots, N_k, M_1, \cdots, M_r\}$ minimizes (maximizes) $\overline{f}(\cdot)$.

Proof:
We consider only the intersecting **submodular** case.
i.(a) $\Pi_1 \vee \Pi_2$ **minimizes** $\bar{f}(\cdot)$**:** Let N_1 be a block of Π_2. By Theorem 13.2.1,

$$\bar{f}(\Pi_1) + \bar{f}(\Pi_{N_1}) \geq \bar{f}(\Pi_1 \vee \Pi_{N_1}) + \bar{f}(\Pi_1 \wedge \Pi_{N_1}). \qquad (*)$$

By Lemma 13.2.2,

$$\bar{f}(\Pi_{N_1}) \leq \bar{f}(\Pi_1 \wedge \Pi_{N_1}).$$

Hence,

$$\bar{f}(\Pi_1) \geq \bar{f}(\Pi_1 \vee \Pi_{N_1}).$$

Hence, $\bar{f}(\cdot)$ reaches a minimum also at $\Pi_1 \vee \Pi_{N_1}$. Repeating the argument with $\Pi_1 \vee \Pi_{N_1} \vee \cdots \vee \Pi_{N_j}$ and $\Pi_{N_{j+1}}$ for $j = 1$ *to* $r-1$, where r is the number of blocks of Π_2, we have the required result.

i.(b) $\Pi_1 \wedge \Pi_2$ **minimizes** $\bar{f}(\cdot)$**:** Let N_i be a block of Π_2. By the argument used above,

$$\bar{f}(\Pi_1 \vee \Pi_{N_i}) = \bar{f}(\Pi_1).$$

But using this in the inequality $(*)$ and using Lemma 13.2.2, we see that

$$\bar{f}(\Pi_{N_i}) = \bar{f}(\Pi_1 \wedge \Pi_{N_i}).$$

Let Π_2 have r blocks. Then by the definition of Π_{N_i}, we have

$$\bar{f}(\Pi_2) = \sum_{i=1}^{r} \bar{f}(\Pi_{N_i}) - (r-1)\sum_{e \in S} f(e)$$

and

$$\bar{f}(\Pi_1 \wedge \Pi_2) = \sum_{i=1}^{r} \bar{f}(\Pi_1 \wedge \Pi_{N_i}) - (r-1)\sum_{e \in S} f(e).$$

So

$$\bar{f}(\Pi_2) = \bar{f}(\Pi_1 \wedge \Pi_2).$$

ii. The result follows directly by the use of Lemma 13.2.2.

□

Theorem 13.2.3 *[Narayanan95b] Let $f(\cdot)$ be intersecting submodular (intersecting supermodular) over subsets of S and let $X \subseteq Y \subseteq S$. Let Π minimize (maximize) $\overline{f}(\cdot)$ over $\mathcal{P}_X$. Then there exists a Π' in $\mathcal{P}_Y$ such that the blocks of Π are contained in the blocks of Π' and Π' minimizes (maximizes) $\overline{f}(\cdot)$ over $\mathcal{P}_Y$.*

Proof :
We handle only the intersecting **submodular** case. Let Π, Π" minimize $\bar{f}(\cdot)$ over $\mathcal{P}_X$, $\bar{f}(\cdot)$ over $\mathcal{P}_Y$ respectively. Suppose Π has a block N that is not contained in any block of Π". We then have by Theorem 13.2.1, taking Π_N to be a partition of Y,

$$\bar{f}(\Pi_N) + \bar{f}(\Pi") \geq \bar{f}(\Pi_N \wedge \Pi") + \bar{f}(\Pi_N \vee \Pi").$$

Now $\bar{f}(\Pi_N) \leq \bar{f}(\Pi_N \wedge \Pi")$, using Lemma 13.2.2. Hence,

$$\bar{f}(\Pi") \geq \bar{f}(\Pi_N \vee \Pi").$$

Thus, $\Pi_N \vee \Pi"$ minimizes $\bar{f}(\cdot)$ over $\mathcal{P}_Y$ and has a block containing N. Repeating this process yields a partition Π' of Y such that its blocks contain the blocks of Π.

□

Theorem 13.2.4 *[Lovász83] If $f(\cdot)$ is intersecting submodular (intersecting supermodular) on subsets of S then $f_t(\cdot)$ ($f^t(\cdot)$) is submodular (supermodular).*

Proof:
Select minimizing partitions $\Pi(X), \Pi(X \cup a), \Pi(Y), \Pi(Y \cup a)$ respectively for $\bar{f}(\cdot)$ over $\mathcal{P}_X$, $\bar{f}(\cdot)$ over $\mathcal{P}_{X\cup a}$, $\bar{f}(\cdot)$ over $\mathcal{P}_Y$, $\bar{f}(\cdot)$ over $\mathcal{P}_{Y\cup a}$ s.t. each block of $\Pi(X)$ is contained in some block of $\Pi(X \cup a)$ as well as some block of $\Pi(Y)$ and each block $\Pi(Y)$ is contained in some block of $\Pi(Y \cup a)$. This is possible by Theorem 13.2.3. Let N_a, M_a be the blocks of $\Pi(X \cup a)$, $\Pi(Y \cup a)$ respectively that have a as a member. By Lemma 13.2.2, there is no loss of generality in assuming that the blocks outside N_a are identical in $\Pi(X \cup a)$ and $\Pi(X)$ and those outside M_a are identical in $\Pi(Y \cup a)$ and $\Pi(Y)$. We need to show that

$$\bar{f}(\Pi(X \cup a)) - \bar{f}(\Pi(X)) \geq \bar{f}(\Pi(Y \cup a)) - \bar{f}(\Pi(Y)).$$

We can cancel terms involving common blocks between $\Pi(X \cup a)$ and $\Pi(X)$ and those involving common blocks between $\Pi(Y \cup a)$ and $\Pi(Y)$. Let $N_1, \cdots, N_k$ be the blocks of $\Pi(X)$ contained in N_a and $M_1, \cdots, M_r$ be the blocks of $\Pi(Y)$ contained in M_a. (Observe that each N_i is contained in some M_j). Thus, we need to show that

$$f(N_a) - \bar{f}(\{N_1, \cdots, N_k\}) \geq f(M_a) - \bar{f}(\{M_1, \cdots, M_r\})$$

Let Π_{+M} denote the partition of M_a that has blocks $M_1, \cdots, M_r$ and $\{a\}$. Let Π_{N_a} denote the partition of M_a that has N_a as a block and all the rest as singletons. We then have by Theorem 13.2.1,

$$\bar{f}(\Pi_{N_a}) + \bar{f}(\Pi_{+M}) \geq \bar{f}(\Pi_{N_a} \wedge \Pi_{+M}) + \bar{f}(\Pi_{N_a} \vee \Pi_{+M}).$$

On both sides there are singleton terms corresponding to elements in $M_a - N_a$ and a. If we cancel these and shift terms appropriately we have

$$f(N_a) - \bar{f}(\{N_1', \cdots, N_t'\}) \geq \bar{f}(\Pi_{N_a} \vee \Pi_{+M}) - \bar{f}(\{M_1, \cdots, M_r\}),$$

where $N_1', \cdots, N_t'$ is a partition of N induced by $\Pi_{N_a} \wedge \Pi_{+M}$.

The required result now follows, since we must have by Lemma 13.2.2 that

$\bar{f}(\{N_1', \cdots, N_t'\}) \geq \bar{f}(\{N_1, \cdots, N_k\})$ and $\bar{f}(\Pi_{N_a} \vee \Pi_{+M}) \geq f(M_a)$.

□

Exercise 13.2 *Let $f(\cdot)$ be submodular on subsets of S. If $f(\cdot)$ is increasing, show that so is $f_t(\cdot)$.*

Exercise 13.3 *[Lovász83] Let $f(\cdot), g(\cdot)$ be submodular functions on subsets of S with $g(\cdot) \leq f(\cdot)$. If $g(\emptyset) = 0$, show that*

i. $g_t(\cdot) = g(\cdot)$.

ii. $g(\cdot) \leq f_t(\cdot)$.

13.2.2 Examples

We now list a number of examples from the literature relevant to the Dilworth truncation operation.

i. Truncation of matroids(Dilworth [Dilworth44], see also [Mason81])
Let $\mathcal{M}$ be a matroid on S. Let S_k be the collection of $k-$rank flats of $\mathcal{M}$. Build a matroid $\mathcal{M}_k$ on S_k such that

- each element of S_k has rank 1
- if A is a flat of $\mathcal{M}$ with rank $p > k$ then $\hat{A}$, the collection of all $k-$rank flats of $\mathcal{M}$ contained in A, is a flat of $\mathcal{M}_k$ with rank $p - (k-1)$.

Solution Let $\mathcal{P}_X$ denote the collection of all partitions of $X \subseteq S_k$. Define the rank function $r_k(\cdot)$ on subsets of S_k as follows:

$$r_k(\emptyset) \equiv 0, r_k(X) \equiv min_{\Pi \in \mathcal{P}_X} \sum_{X_i \in \Pi} (r' - (k-1))(X_i).$$

Here $r'(\cdot)$ is the function induced by $r(\cdot)$ on subsets of S_k (i.e., $r'(X) \equiv r(\bigcup Y_i), Y_i \in X$). Clearly, $r_k(\cdot) = (r' - (k-1))_t(\cdot)$.

It can be shown that

- $r_k(\cdot)$ is a matroid rank function
- if A is a flat of $\mathcal{M}$ with rank $p > k$ then $\hat{A}$ is a flat of $\mathcal{M}_k$ with rank $p-(k-1)$.

ii. Hybrid rank relative to a partition of the edges of a graph([Narayanan90])
The problem described below arises when we attempt to solve an electrical network by decomposing it. First we define two operations on graphs. A **node pair fusion** means fusing two specified vertices v_1, v_2 into a single vertex v_{12} while a

node fission means splitting a node v_1 into v_{11}, v_{12}, making some of the edges incident at v_1 now incident at v_{11} and the remaining at v_{12}. We are given a partition Π of the edge set $E(\mathcal{G})$ of a graph $\mathcal{G}$ such that the subgraph on each block of the partition is connected. Find a sequence of fusion and fission operations least in number such that the resulting graph has no circuit intersecting more than one block of Π.

Solution (This problem is handled in detail in Section 15.4). It is easy to see that one cannot lose if one performs fusion operations first and then fission operations. Let $I(X)$, $X \subseteq V(\mathcal{G})$ be the set of blocks of Π whose member branches are incident on vertices in X. Let Π^V be a partition that minimizes $\overline{|I| - 2}(\cdot)$. The best sequence is the following: Fuse each block of Π^V into a single node. (If k nodes are in a single block this involves $k-1$ operations). In the resulting graph, which we shall call $\mathcal{G}'$, perform the minimum number of node fissions required to destroy all circuits intersecting more than one block of Π. This is relatively easy to do and the number of such fission operations is $\sum_{N_i \in \Pi} r'(N_i) - r'(E(\mathcal{G}))$,where $r'(\cdot)$ is the rank function of $\mathcal{G}'$.

iii. New matroids

A simple method (see Exercise 13.4 below) for generating new matroids from polymatroid rank functions is the following (see for example [Patkar92], [Patkar+Narayanan92a]). Let $f(\cdot)$ be an integral polymatroid rank function with $f(e) = k, e \in S$. Let $pk - q = 1$. Then $(pf - q)_t(\cdot)$ is a matroid rank function.

Example Let $|V|(\cdot)$ be the polymatroid rank function on the subsets of $E(\mathcal{G})$ (where $\mathcal{G}$ is a selfloop free graph) such that $|V|(X) \equiv$ number of vertices incident on edges in X. Clearly $|V|(e) = 2$. Then $(k|V|(\cdot) - (2k-1))_t$ is a matroid rank function. In particular $(|V|(\cdot) - 1)_t$ is the rank function of the graph and $(2|V|(\cdot) - 3)_t$ is the rank function of the **rigidity matroid** associated with the graph [Laman70], [Asimow+Roth78], [Asimow+Roth79].

Exercise 13.4 *Let $f(\cdot)$ be an integral polymatroid rank function on subsets of S with $f(e) = k \;\; \forall e \in S$, k an integer. Let p, q be integers s.t. $pk - q = 1$. Prove that $(pf - q)_t(\cdot)$ is a matroid rank function.*

Exercise 13.5 *Prove the statements about the function $r_k(\cdot)$ given in the solution to the Dilworth truncation of matroids problem.*

iv. Posing convolution problems as truncation problems

([Narayanan90], [Narayanan91], [Patkar+Narayanan92b] [Narayanan+Roy+Patkar92]) We give an example. Consider the convolution problem: 'Find $min_{X \subseteq E(\mathcal{G})} \lambda r(X) + w(E(\mathcal{G}) - X), \lambda \geq 0$, where $E(\mathcal{G})$ is the edge set and $r(\cdot)$ is the rank function of the graph $\mathcal{G}$ and $w(\cdot)$ is a nonnegative weight function on $E(\mathcal{G})$.'

Let $I(Y) \equiv$ set of edges incident on vertices in $Y \subseteq V(\mathcal{G})$, let $E(Y) \equiv$ set of edges incident only on vertices in $Y \subseteq V(\mathcal{G})$ and let $w(I(Y)), w(E(Y))$ denote the sum of the weights of edges in the corresponding sets. Then one can show that $X \subseteq E(\mathcal{G})$ solves the above convolution problem iff $X = \bigcup_{N_i \in \Pi'} E(N_i)$, where Π' solves the truncation problem: 'Find $min_{\Pi \in P_{V(\mathcal{G})}} \overline{w(I(\cdot)) - \lambda}(\Pi)$ or **equivalently**

find $max_{\Pi \in P_{V(\mathcal{G})}} \overline{w(E(\cdot)) + \lambda}(\Pi)$.'
Thus the principal partition of the rank function of a graph can be determined by solving either of the above mentioned truncation problems for appropriate values of λ. Indeed, this approach yields the fastest algorithm currently known for this principal partition problem - ($O(|E||V|^2 log^2(|V|))$ for the unweighted case and $O(|E||V|^3 log(|V|))$ for the weighted case).

Polyhedral Interpretation for Truncation

The polyhedral interpretation for Dilworth truncation appears to be less important than is the case for convolution. We however have the following simple result.

Theorem 13.2.5 *Let $f(\cdot)$ be a real set function on subsets of S.*

i. $P_f = P_{f_t}$ *and* $P^d_f = P^d_{f^t}$

ii. If $f(\cdot)$ is polyhedrally tight then $f_t(\cdot) = f(\cdot)$. If $f(\cdot)$ is dually polyhedrally tight then $f^t(\cdot) = f(\cdot)$.

iii. If $f(\cdot)$ is submodular then $f_t(\cdot)$ is polyhedrally tight and is the greatest polyhedrally tight function below $f(\cdot)$. If $f(\cdot)$ is supermodular then $f^t(\cdot)$ is dually polyhedrally tight and is the least dually polyhedrally tight function above $f(\cdot)$.

Proof : We will handle only the 'polyhedral' (as opposed to the 'dually polyhedral') case.
i. If $x(X) \leq f(X) \;\; \forall \; X \subseteq S$ then

$$\sum_{i=1}^{k} x(X_i) \leq \sum_{i=1}^{k} f(X_i)$$

for all partitions $\{X_1, \cdots, X_k\}$ of $X \subseteq S$. Hence $P_f \subseteq P_{f_t}$. The result follows since $f_t(X) \leq f(X), X \neq \emptyset$ and therefore $P_f \supseteq P_{f_t}$.

ii. We have $P_{f_t} = P_f$. For each $X \subseteq S$ there exists $\mathbf{x} \in P_f$ s.t. $x(X) = f(X)$. But $x(X) \leq f_t(X) \leq f(X)$. Hence, $f_t(X) = f(X) \;\; \forall X \subseteq S$.

iii. If $f(\cdot)$ is submodular $f_t(\cdot)$ is submodular and $f_t(\emptyset) = 0$. We have seen (Corollary 10.7.1) that every submodular function that takes zero value on the empty set is polyhedrally tight. The remaining part of the statement follows from Exercise 13.3.

□

13.3 The Principal Lattice of Partitions

13.3.1 Basic Properties of the PLP

The principal lattice of partitions has its roots in the variation of the hybrid rank problem described in page 545. It is curious that the principal partition is strongly linked to the original hybrid rank problem as we indicate in Chapter 15. There are also formal analogies between principal partition and principal lattice of partitions which we will indicate at appropriate places. We begin by suggesting that the reader compare Subsection 11.4.2 with the present subsection.

Definition 13.3.1 *Let $f(\cdot)$ be submodular on the subsets of S. Let $\mathcal{L}_{\lambda f}$ ($\mathcal{L}_\lambda$ when $f(\cdot)$ is clear from the context) denote the collection of partitions of S that minimize $(f-\lambda)(\cdot)$. The collection of all partitions of S which belong to some $\mathcal{L}_\lambda, \lambda \in \Re$ is called the* **principal lattice of partitions of** $f(\cdot)$.

As in the case of the principal partition, one of the interesting features of the principal lattice of partitions is that one need only examine a few (not more than $|S|$) λs in order to solve the optimization problems for all the λs.

We list below the main properties of the principal lattice of partitions. The reader might like to compare them with those of the principal partition (Subsection 11.4.2).

i. Property PLP1
The collection $\mathcal{L}_\lambda$ is closed under join ($\vee$) and meet ($\wedge$) operations and thus has a unique maximal and a unique minimal element.

ii. Property PLP2
If $\lambda_1 > \lambda_2$, then $\Pi^{\lambda_1} \leq \Pi_{\lambda_2}$,
where Π^λ, Π_λ ,respectively denote the maximal and minimal elements of $\mathcal{L}_\lambda$.

iii.

Definition 13.3.2 *A number λ for which $\mathcal{L}_\lambda$ has more than one partition as a member is called a* **critical PLP value of** $f(\cdot)$ *(***critical value** *for short).*

Property PLP3
The number of critical PLP values of $f(\cdot)$ is bounded by $|S|$.

iv. Property PLP4
Let $\lambda_1, \cdots, \lambda_t$ be the decreasing sequence of critical PLP values of $f(\cdot)$. Then, $\Pi^{\lambda_i} = \Pi_{\lambda_{i+1}}$ for $i = 1, \cdots, t-1$.

v. Property PLP5
Let $\lambda_1, \cdots, \lambda_t$ be the decreasing sequence of critical PLP values. Let $\lambda_i > \sigma > \lambda_{i+1}$. Then $\Pi^{\lambda_i} = \Pi^\sigma = \Pi_\sigma = \Pi_{\lambda_{i+1}}$.

Definition 13.3.3 *Let $f(\cdot)$ be submodular on subsets of S. Let $(\lambda_i), i = 1, \cdots, t$ be the decreasing sequence of critical PLP values of $f(\cdot)$. Then the sequence $\Pi_0 = \Pi_{\lambda_1}, \Pi_{\lambda_2}, \cdots, \Pi_{\lambda_t}, \Pi^{\lambda_t} = \{S\}$ is called the* **principal sequence of partitions of** $f(\cdot)$. *A member of $\mathcal{L}_\lambda$ would be alternatively referred to as a* **minimizing partition corresponding to** λ *in the principal lattice of partitions of $f(\cdot)$.*

Proof of the properties of the Principal Lattice of Partitions
i. PLP1: This follows directly from Theorem 13.2.2.

ii. PLP2: The following lemma and theorem are needed for the proof of PLP2.

Lemma 13.3.1 *Let $f(\cdot)$ be a submodular function on subsets of S and let $N \subseteq S$.*

Let Π be any partition of S. Then,

$$\begin{aligned}\overline{(f-\lambda_2)}(\Pi)+\overline{(f-\lambda_1)}(\Pi_N) \;\geq\; & \overline{(f-\lambda_2)}(\Pi\vee\Pi_N)+\overline{(f-\lambda_1)}(\Pi\wedge\Pi_N)\\ & -(\lambda_2-\lambda_1)(\mid\Pi\wedge\Pi_N\mid - \mid\Pi_N\mid).\end{aligned}$$

Proof : We have, by the definition of $\overline{(f-\lambda_i)}(\cdot)$,

$$\begin{aligned}\overline{(f-\lambda_2)}(\Pi\vee\Pi_N)+\overline{(f-\lambda_1)}(\Pi\wedge\Pi_N) \;=\; & \overline{(f-\lambda_2)}(\Pi\vee\Pi_N)+\overline{(f-\lambda_2)}(\Pi\wedge\Pi_N)\\ & +(\lambda_2-\lambda_1)\mid\Pi\wedge\Pi_N\mid .\end{aligned}$$

By Theorem 13.2.1, the RHS

$$\leq \overline{(f-\lambda_2)}(\Pi)+\overline{(f-\lambda_2)}(\Pi_N)+(\lambda_2-\lambda_1)\mid\Pi\wedge\Pi_N\mid .$$

$$\leq \overline{(f-\lambda_2)}(\Pi)+\overline{(f-\lambda_1)}(\Pi_N)+(\lambda_2-\lambda_1)(\mid\Pi\wedge\Pi_N\mid - \mid\Pi_N\mid).$$

The required result now follows immediately.

□

Theorem 13.3.1 *Let $f(\cdot)$ be a submodular function on the subsets of S. Let $\Pi_i, i=1,2$, be a partition at which $\overline{(f-\lambda_i)}(\cdot)$, $i=1,2$ reaches a minimum. If $\lambda_1>\lambda_2$ then $\Pi_2\geq\Pi_1$.*

Proof : Let N be any block of Π_1. By the definition of Π_2, we have

$$\overline{(f-\lambda_2)}(\Pi_2\vee\Pi_N)\geq\overline{(f-\lambda_2)}(\Pi_2)$$

Hence, by Lemma 13.3.1, we have

$$\overline{(f-\lambda_1)}(\Pi_2\wedge\Pi_N)\leq\overline{(f-\lambda_1)}(\Pi_N)+(\lambda_2-\lambda_1)(\mid\Pi_2\wedge\Pi_N\mid - \mid\Pi_N\mid)$$

Since $\lambda_1>\lambda_2$, using Lemma 13.2.2, we must have $\mid\Pi_2\wedge\Pi_N\mid=\mid\Pi_N\mid$. Thus, N is contained in a block of Π_2. Hence, $\Pi_2\geq\Pi_1$.

□

iii. PLP3: If $\mathcal{L}_\lambda$ has more than one partition as a member, then $\mid\Pi_\lambda\mid>\mid\Pi^\lambda\mid$. So if λ_1,λ_2 are critical values and $\lambda_1>\lambda_2$, by the property PLP2, we must have $\mid\Pi_{\lambda_1}\mid>\mid\Pi_{\lambda_2}\mid$. Now the maximum number of blocks a partition of S can have cannot exceed $\mid S\mid$. Hence, the number of critical values cannot exceed $\mid S\mid$.

iv. PLP4: We need the following lemma.

Lemma 13.3.2 *Let λ be a real number. Then for sufficiently small $\epsilon>0$, the only partition that minimizes $\overline{(f-(\lambda-\epsilon))}(\cdot)$ over $\mathcal{P}_S$ is Π^λ.*

Proof : Since there is only a finite number of partitions of S, for sufficiently small $\epsilon>0$, we must have the value of $\overline{(f-(\lambda-\epsilon))}(\cdot)$ lower on the members of $\mathcal{L}_\lambda$ than on any other partition of S. But if $\Pi\in\mathcal{L}_\lambda$ and $\Pi\neq\Pi^\lambda$

$$\begin{aligned}\overline{(f-(\lambda-\epsilon))}(\Pi) &= \overline{(f-\lambda)}(\Pi)+\epsilon\mid\Pi\mid\\ &> \overline{(f-\lambda)}(\Pi^\lambda)+\epsilon\mid\Pi^\lambda\mid\end{aligned}$$

The result follows.

□

Proof of PLP4: For sufficiently small values of $\epsilon > 0, \Pi^{\lambda_i}$ would continue to minimize $\overline{(f - (\lambda_i - \epsilon))}(\cdot)$ over partitions of S. As ϵ increases, because there are only a finite number of partitions of S, there would be a least value ϵ_1 at which Π^{λ_i} and at least one other partition of S minimize $\overline{(f - (\lambda_i - \epsilon_1))}(\cdot)$. Clearly $(\lambda_i - \epsilon_1)$ is the next critical value λ_{i+1}. Since $\lambda_i > \lambda_i - \epsilon_1$ by property PLP2 we must have $\Pi^{\lambda_i} \leq \Pi_{\lambda_{i+1}}$. Hence, we must have

$$\Pi^{\lambda_i} = \Pi_{\lambda_{i+1}} \qquad \text{as desired.}$$

v. PLP5: This is clear from the above arguments.

Exercise 13.6 *Let $g(\cdot)$ be a modular function on the subsets of S. Describe the PLP of $g(\cdot)$.*

Exercise 13.7 *Show that*

i. the PLP of $\beta f(\cdot)$, where $\beta > 0, f(\cdot)$ submodular on subsets of S, is the same as that of $f(\cdot)$. λ is a critical value of the PLP of $f(\cdot)$ iff $\lambda\beta$ is a critical value of the PLP of $\beta f(\cdot)$.

ii. the PLP of $(f + g)(\cdot)$ is the same as that of $f(\cdot)$ if $g(\cdot)$ is a weight function. The critical values of the PLP of $f(\cdot)$ and the PLP of $(f+g)(\cdot)$ are identical.

iii. the PLP of $f(\cdot)$ and $(f + k)(\cdot)$, where k is a real number, are identical. The critical values of $(f + k)(\cdot)$ are $\lambda_i + k$, where λ_i are the critical values of $f(\cdot)$.

Just as in the case of the principal partition we give two characterizations for the principal lattice of partitions also. The first of these is presented below. Both the result and the proof are line by line translations of the statement and proof of Theorem 11.4.1.

Lemma 13.3.3 *Let $f(\cdot)$ be a submodular function on subsets of S. Let $\lambda_1, \cdots, \lambda_t$ be a strictly decreasing sequence of numbers such that*

i. each $\mathcal{L}_{\lambda_i}, i = 1, \cdots, t$ has atleast two member partitions,

ii. $\mathcal{L}_{\lambda_i}, \mathcal{L}_{\lambda_{i+1}}, i = 1, \cdots, t-1$ have at least one common member partition.

iii. Π_0 belongs to $\mathcal{L}_{\lambda_1}$ while Π_S belongs to $\mathcal{L}_{\lambda_t}$.

Then $\lambda_1, \cdots, \lambda_t$ is the decreasing sequence of critical values of the PLP of $f(\cdot)$ and therefore the collection of all partitions which are member partitions in all the $\mathcal{L}_{\lambda_i}, i = 1, \cdots, t$ is the principal lattice of partitions of $f(\cdot)$.

Proof : Let $\lambda_1', \cdots, \lambda_k'$ be the critical values and let $\Pi_0, \Pi_1, \cdots, \Pi_k = \Pi_S$ be the principal sequence of partitions of $f(\cdot)$. By Property PLP2 the only member partition in $\mathcal{L}_\lambda$, when $\lambda > \lambda_1'$ is Π_0. Further, when $\lambda < \lambda_1', \Pi_0$ is not in $\mathcal{L}_\lambda$. Hence, $\lambda_1 = \lambda_1'$. Next by Property PLP5 when $\lambda_1' > \lambda > \lambda_2'$, the only member in $\mathcal{L}_\lambda$ is Π_1 which is the maximal partition in $\mathcal{L}_{\lambda_1'}$. Since $\mathcal{L}_{\lambda_2}$ has at least two member partitions we conclude that $\lambda_2 \leq \lambda_2'$. We know that $\mathcal{L}_{\lambda_1}$ and $\mathcal{L}_{\lambda_2}$ have a common member which by Property PLP2 can only be Π_1. But for $\lambda < \lambda_2'$, by Property PLP5, Π_1 cannot be a member of $\mathcal{L}_\lambda$. Hence, $\lambda_2 = \lambda_2'$. By repeating this argument, we see that $t = k$ and $\lambda_i = \lambda_i', i = 1, \cdots, t$.

□

The next result is the main characterization of PLP. Later for convenience we restate it as Theorem 13.6.1. The latter result could be viewed as a 'translation' of Uniqueness Theorem (PP) (Theorem 11.4.6).

Theorem 13.3.2 (Uniqueness Theorem (PLP))*[Narayanan91] Let $f(\cdot)$ be a submodular function on subsets of S. Let $\Pi_0 < \Pi_1 < \cdots < \Pi_t = \Pi_S$ be a strictly increasing sequence of partitions of S and let $\lambda_1, \cdots, \lambda_t$ be a strictly decreasing sequence of real numbers such that*

i. $\overline{(f - \lambda_{i+1})}(\Pi_i) = \overline{(f - \lambda_{i+1})}(\Pi_{i+1}), i = 0, \cdots, t-1$

ii. $\overline{(f - \lambda_{i+1})}(\Pi) \geq \overline{(f - \lambda_{i+1})}(\Pi_i), \Pi_i \leq \Pi \leq \Pi_{i+1}.$

Then $\Pi_0, \cdots, \Pi_t$ is the principal sequence of partitions of $f(\cdot)$ and $\lambda_1, \cdots, \lambda_t$ is its decreasing sequence of critical values. Further,

$$\mathcal{L}_{\lambda_{i+1}} = \left\{\Pi : \Pi_{i+1} \geq \Pi \geq \Pi_i \text{ and } \overline{(f - \lambda_{i+1})}(\Pi) = \overline{(f - \lambda_{i+1})}(\Pi_i)\right\}, i = 0, \cdots, t-1.$$

Proof : The proof is by induction on t, where $t + 1$ is the length of the principal sequence. The theorem is obviously true for $t = 1$ when the principal sequence is Π_0, Π_S. Let the theorem be true for $t < k$. Let $t = k$.
Let $\hat{\Pi}_i$ be the minimal partition of S at which $\overline{(f - \lambda_{i+1})}(\cdot)$ reaches a minimum.
We will first show that $\hat{\Pi}_i \leq \Pi_i$ for $i = 0, \cdots, k$.
We know that $\hat{\Pi}_0 = \Pi_0$. Suppose for some $i > 0$, $\hat{\Pi}_i \not\leq \Pi_i$. Then there exists a block N of $\hat{\Pi}_i$ that is not contained in any block of Π_i. It follows that $\Pi_i \wedge \Pi_N$ is distinct from Π_i. Use of Lemma 13.2.2 shows that

$$\overline{(f - \lambda_{i+1})}(\Pi_N) < \overline{(f - \lambda_{i+1})}(\Pi_i \wedge \Pi_N)$$

By Theorem 13.2.1,

$$\overline{(f - \lambda_{i+1})}(\Pi_i) + \overline{(f - \lambda_{i+1})}(\Pi_N) \geq \overline{(f - \lambda_{i+1})}(\Pi_i \vee \Pi_N) + \overline{(f - \lambda_{i+1})}(\Pi_i \wedge \Pi_N).$$

Hence, $\overline{(f - \lambda_{i+1})}(\Pi_i) > \overline{(f - \lambda_{i+1})}(\Pi_i \vee \Pi_N)$.
Now consider the function $f_{fus \cdot \Pi_i}(\cdot)$. Clearly it follows that

$$\overline{(f_{fus \cdot \Pi_i} - \lambda_{i+1})}((\Pi_i \vee \Pi_N)_{fus \cdot \Pi_i}) < \overline{(f_{fus \cdot \Pi_i} - \lambda_{i+1})}((\Pi_i)_{fus \cdot \Pi_i}). \qquad (*)$$

However, on the sequence of partitions $(\Pi_i)_{fus\cdot\Pi_i}, \cdots, (\Pi_k)_{fus\cdot\Pi_i} = (\Pi_S)_{fus\cdot\Pi_i}$ (which are partitions of Π_i), and on the sequence of values $\lambda_{i+1}, \cdots, \lambda_k$, the function $f_{fus\cdot\Pi_i}(\cdot)$ satisfies the conditions of the theorem. Therefore, by the induction assumption, $(\Pi_i)_{fus\cdot\Pi_i}, \cdots, (\Pi_k)_{fus\cdot\Pi_i}$ is the principal sequence and $\lambda_{i+1}, \cdots, \lambda_k$ the critical values of $f_{fus\cdot\Pi_i}(\cdot)$. But then $(*)$ cannot be true, a contradiction. We therefore conclude $\hat{\Pi}_i \leq \Pi_i,\ i = 0, \cdots, k$.
We will now prove that $\hat{\Pi}_i = \Pi_i,\ i = 0, \cdots, k$ by induction on k.
Clearly $\hat{\Pi}_0 = \Pi_0$. Let $\hat{\Pi}_i = \Pi_i$ for $i = 0, \cdots, k-1$. Let $\Pi_{i(max)}$ be the maximal partition at which $\overline{(f - \lambda_{i+1})}(\cdot)$ reaches a minimum for $i = 0, \cdots, k$. Suppose $\hat{\Pi}_k < \Pi_k$. By Property PLP2 we have $\Pi_{k-1(max)} \leq \hat{\Pi}_k$. Hence, $\Pi_{k-1(max)} < \Pi_k$, and by the definition of $\Pi_{k-1(max)}$, $\overline{(f - \lambda_k)}\ (\Pi_{k-1(max)}) < \overline{(f - \lambda_k)}(\Pi_k)$. Therefore, $\overline{(f - \lambda_k)}(\hat{\Pi}_{k-1}) < \overline{(f - \lambda_k)}(\Pi_k)$. Since, by the conditions of the theorem, in the interval (Π_{k-1}, Π_k), $\overline{(f - \lambda_k)}(\cdot)$ reaches the same minimum value at Π_{k-1} and Π_k, this implies that $\overline{(f - \lambda_k)}(\hat{\Pi}_{k-1}) < \overline{(f - \lambda_k)}(\Pi_{k-1})$, i.e., $\hat{\Pi}_{k-1} \neq \Pi_{k-1}$, which is a contradiction. We therefore, conclude that $\hat{\Pi}_k = \Pi_k$.
Further, by Property PLP2, $\Pi_{k-1} \leq \Pi_{k-1(max)} \leq \hat{\Pi}_k$. By the above mentioned conditions of the theorem, this implies that $\Pi_{k-1(max)} = \Pi_k$. For each $\lambda_{i+1}, i = 0, \cdots, k-1$, it is clear that Π_i, Π_{i+1} are the unique minimal and maximal partitions which minimize $\overline{(f - \lambda_{i+1})}(\cdot)$. Hence, $\mathcal{L}_{\lambda_{i+1}} = \{\Pi : \Pi_{i+1} \geq \Pi \geq \Pi_i$ and $\overline{(f - \lambda_{i+1})}(\Pi) = \overline{(f - \lambda_{i+1})}(\Pi_i)\}$. We now see that the sequence $\lambda_1, \cdots, \lambda_t$ and the sequence $\mathcal{L}_{\lambda_1}, \cdots, \mathcal{L}_{\lambda_t}$, together satisfy the conditions of Lemma 13.3.3. Hence, $\Pi_0, \cdots, \Pi_t$ is the principal sequence of partitions of $f(\cdot)$, $\lambda_1, \cdots, \lambda_t$ its decreasing sequence of critical values and $\mathcal{L}_{\lambda_i}$, the collections indicated above.

$\square$

Range of Critical Values of the PLP

It is of algorithmic importance to have some idea of the range of critical values of a given submodular function. The following simple result [Narayanan91] is useful in this regard.

Theorem 13.3.3 *Let $f(\cdot)$ be a submodular function on subsets of S. Let λ be a critical value of $f(\cdot)$. Let $max_{e_i \in S} f(e_i) = p$ and let $min_{e_i \in S} f(e_i) = q$. Then,*

i. $\lambda \geq f(\emptyset)$

ii. $\lambda \leq 2p$ *if $f(\cdot)$ is non-negative.*

iii. $\lambda \leq 2p - q$ *if $f(\cdot)$ is a nonnegative increasing set function.*

iv. if $f(\cdot)$ is the left adjacency function of a bipartite graph, we have $0 \leq \lambda \leq 2$*(max degree of a left vertex) - min degree of a left vertex.*

v. if $f(\cdot)$ is the rank function of a selfloop free matroid, we have $0 \leq \lambda \leq 1$.

Proof :

i. A partition Π of S minimizes $\overline{(f-\lambda)}(\cdot)$ iff it minimizes $\overline{(f-k)-(\lambda-k)}(\cdot)$. Hence, λ is a critical value of $f(\cdot)$ iff $(\lambda - k)$ is a critical value of $(f-k)(\cdot)$. Let $g(\cdot) \equiv f(\cdot) - f(\emptyset)$. Then $g(\emptyset) = 0$ and by repeated use of the submodular inequality we have

$$\bar{g}(\Pi) \geq g(S), \ i.e., \ \bar{g}(\Pi) \geq \bar{g}(\Pi_S) \quad \forall \Pi \in \mathcal{P}_S.$$

By Property PLP2 we conclude that the critical values of $g(\cdot)$ are greater than or equal to zero, i.e., the critical values of $f(\cdot)$ are greater than or equal to $f(\emptyset)$.

ii. If λ is the highest critical value of $f(\cdot)$, then $\overline{(f-\lambda)}(\cdot)$ reaches its minimum on two partitions Π_0, Π_1 of S, where Π_0 is the partition which has only singleton blocks. Let N be a non-singleton block of Π_1. Clearly

$$(f-\lambda)(N) = \sum_{e \in N} (f-\lambda)(e).$$

Hence,

$$\begin{aligned} \lambda(1- \mid N \mid) + \sum_{e \in N} f(e) = f(N) &\geq 0, \qquad (*) \\ i.e., \qquad \lambda(1- \mid N \mid) + p(\mid N \mid) &\geq 0 \\ i.e., \qquad p(\mid N \mid) &\geq \lambda(\mid N \mid -1) \\ i.e., \qquad \lambda &\leq \frac{p(\mid N \mid)}{(\mid N \mid -1)} \\ i.e., \qquad \lambda &\leq 2p. \end{aligned}$$

iii. If $f(\cdot)$ is increasing

$$\begin{aligned} f(N) &\geq f(e) \qquad \forall e \in N \\ &\geq q. \end{aligned}$$

Hence,if $f(\cdot)$ is also nonnegative in addition, we may replace $(*)$ above by

$$\lambda(1- \mid N \mid) + p \mid N \mid -q \geq 0,$$

which gives

$$\begin{aligned} \lambda &\leq \frac{p(\mid N \mid) - q}{\mid N \mid -1} \\ &\leq \frac{2p(\mid N \mid -1) - q - p(\mid N \mid) + 2p}{\mid N \mid -1} \\ &\leq 2p - q \end{aligned}$$

(since $p(\mid N \mid -2) + q \geq q(\mid N \mid -1)$).

The last two parts of theorem follow by direct application of the above results.

□

Storing the lattice $\mathcal{L}_\lambda$

We saw in the case of the principal partition that $\mathcal{B}_\lambda$ is a distributive lattice which can be stored in the form of a Hasse diagram. It is a bit more difficult to store $\mathcal{L}_\lambda$. The following result shows that we only need to store $\mid \Pi_\lambda \mid$ distributive lattices. (In the next chapter we present an alternative way of storing $\mathcal{L}_\lambda$).

Lemma 13.3.4 *Let $f(\cdot)$ be a submodular function on subsets of S. Let $N_i, i = 0, \cdots, k$, be the blocks of Π_λ, the unique minimal partition that is a member of $\mathcal{L}_\lambda$. Let $\mathcal{D}_{N_i}$ be the collection of all blocks of partitions in $\mathcal{L}_\lambda$ which contain N_i. Then*

i. $\mathcal{D}_{N_i}$ is a distributive lattice.

ii. A partition $\Pi \equiv \{M_1, \cdots, M_t\}$ of S belongs to $\mathcal{L}_\lambda$ iff each M_j belongs to one of the $\mathcal{D}_{N_i}$.

Proof :
i. Let $P, Q \in \mathcal{D}_{N_i}$.
Since P, Q are unions of blocks of Π_λ so must $P \cup Q$ as well as $P \cap Q$ be. Let $\Pi_p, \Pi_q, \Pi_{pq}, \Pi_{p\cup q}$ denote respectively the partitions of S which have P as a block and remaining blocks from Π_λ, Q as a block and remaining blocks from $\Pi_\lambda, P \cap Q$ as a block and remaining blocks from $\Pi_\lambda, P \cup Q$ as a block and remaining blocks from Π_λ. By Theorem 13.2.2, Π_p, Π_q belong to L_λ and so do $\Pi_p \vee \Pi_q$ and $\Pi_p \wedge \Pi_q$. But $\Pi_p \vee \Pi_q = \Pi_{p\cup q}$ and $\Pi_p \wedge \Pi_q = \Pi_{pq}$ (note that $P \cap Q \supseteq N_i$). The result follows.

ii. If each block of Π belongs to one of the $\mathcal{D}_{N_i}$, then by Theorem 13.2.2, $\Pi \in \mathcal{L}_\lambda$. On the other hand if $\Pi \in \mathcal{L}_\lambda$ we must have $\Pi \geq \Pi_\lambda$ which implies that each block of Π is a union of blocks of Π_λ and therefore belongs to one of the $\mathcal{D}_{N_i}$.

□

Symmetry Properties of the PLP

The PLP of a submodular function exhibits the expected symmetry properties. However, unlike the PP it does not induce a partial order among the elements since the basic object here is a partition.
The reader might like to review the definitions preceding Theorem 11.4.2 in page 451. If $\alpha : S \rightarrow S$ is a bijection and $\Pi \equiv \{N_1, \cdots, N_k\}$ is a partition of S then $\alpha(\Pi)$ denotes $\{\alpha(N_1), \cdots, \alpha(N_k)\}$. We then have the following elementary but useful result.

Theorem 13.3.4 *Let $f(\cdot)$ be a submodular function on subsets of S and let $\alpha(\cdot)$ be an automorphism of $f(\cdot)$.*

i. $\overline{(f-\lambda)}(\cdot)$ reaches a minimum at Π among all partitions of S iff it reaches a minimum at $\alpha(\Pi)$.

ii. For each λ the partitions Π_λ and Π^λ are invariant under $\alpha(\cdot)$. Hence, if

N_1, N_2 are blocks of different sizes in some Π_λ and $x \in N_1, y \in N_2$ then no automorphism of $f(\cdot)$ can map x to y.

iii. *If N_1, N_2 are blocks of Π_λ s.t. $\alpha(N_1) = N_2$, then $\mathcal{D}_{N_1}$ (the collection of all blocks of partitions in $\mathcal{L}_\lambda$ which contain N_1) is isomorphic to $\mathcal{D}_{N_2}$ under $\alpha(\cdot)$.*

Proof : i. Immediate from the definition of an automorphism of $f(\cdot)$ and the definition of $\alpha(\Pi)$.

ii. We have $\alpha(\Pi_\lambda)$ also as a member of $\mathcal{L}_\lambda$. Further, it has the same number of blocks as Π_λ. But Π_λ is the unique minimal partition that is a member of $\mathcal{L}_\lambda$. Hence, $\alpha(\Pi_\lambda) \geq \Pi_\lambda$ and we conclude that $\alpha(\Pi_\lambda) = \Pi_\lambda$. The case of Π^λ is similar. If $\alpha(\cdot)$ is an automorphism of $f(\cdot)$ and $\Pi_\lambda \equiv \{N_1, \cdots, N_t\}$ then $\Pi_\lambda = \alpha(\Pi_\lambda) = \{\alpha(N_1), \cdots, \alpha(N_k)\}$ clearly $N_i, \alpha(N_i)$ have the same size since $\alpha(\cdot)$ is a bijection.

iii. If M is a block containing $N_1, \alpha(M)$ must contain $\alpha(N_1) = N_2$. The converse must be true since α^{-1} is also a bijection. The result follows.

□

13.3.2 PLP from the Point of View of Cost of Partitioning

The principal partition gives information about which subsets are densely packed relative to $(f(\cdot), g(\cdot))$ while the principal lattice of partitions gives information about weak links between different subsets relative to $f(\cdot)$. The latter thus appears to be a better way of examining natural partitions relative to $f(\cdot)$. Below we give an informal description of the principal lattice of partitions from this point of view.

Let us define the **cost** of a partition Π of S relative to $f(\cdot)$ to be $\bar{f}(\Pi) - f(S)$. (Note that the cost of the single block partition Π_S is zero). Let **cost rate** (**gain rate**) of a partition $\Pi_1 (\Pi_2)$ with respect to a coarser partition Π_2 (finer partition Π_1) be defined to be

$$\frac{\bar{f}(\Pi_2) - \bar{f}(\Pi_1)}{\mid \Pi_2 \mid - \mid \Pi_1 \mid}.$$

Let $\lambda_1, \cdots, \lambda_t$ be the decreasing sequence of critical values and let
$\Pi_0 = \Pi_{\lambda_1}, \cdots, \Pi_{\lambda_t}, \Pi^{\lambda_t} = S$ be the principal sequence of partitions of $f(\cdot)$.

The partition Π_{λ_t} would have the cost $\lambda_t(\mid \Pi_{\lambda_t} \mid -1)$ (i.e., cost rate λ_t relative to Π_S). We may imagine that we are attempting to break S but for lower cost rate no partition occurs. Note that even at the above mentioned cost rate we may not be able to reach from Π_S to Π_{λ_t} through partitions whose number of blocks increases one at a time. (To reach Π_{λ_t} from Π^{λ_t} we can use partitions in $\mathcal{L}_{\lambda_t}$ but these do not necessarily have a given number of blocks between $|\Pi_{\lambda_t}|$ and $|\Pi^{\lambda_t}|$). To further break up Π_{λ_t} we have to pay a higher cost rate λ_{t-1} and so on, with the cost rate increasing each time we reach a partition in the principal sequence of partitions, until we reach the partition Π_0 in which each block is a singleton.

Every partition in the PLP has least cost relative to its number of blocks (see Exercise 13.8 below) and further is easy to construct (has a polynomial algorithm which is often quite fast). However, the problem of finding the partition of least cost relative to a given number of blocks is NP Hard even for the simplest submodular functions (example: $f(\cdot) \equiv (wI)(\cdot)$, where $(wI)(X) \equiv$ weighted sum of edges incident on the vertex set $X \subseteq V(\mathcal{G}), \mathcal{G}$ a graph [Saran+Vazirani91]). This apparent contradiction is resolved when we remember that there may be no partition of the given number of blocks in the PLP. Even in this case, however, good approximation algorithms can be given, as we will indicate later (see Section 13.4).

Exercise 13.8 *(Compare Exercise 11.21) Let Π be a partition of S in the PLP of a submodular function $f(\cdot)$. If Π' is any other partition of S with the same number of blocks as Π then $\bar{f}(\Pi) \geq \bar{f}(\Pi')$, the equality holding only if Π' is also a partition in the PLP of $f(\cdot)$.*

The following exercises constitute an alternative development of PLP from the cost point of view [Roy93],[Roy+Narayanan93b]. We need a preliminary definition and a basic result.

A partition Π satisfies the **λ-gain rate (λ-cost rate)** condition with respect to $f(\cdot)$ iff whenever $\Pi' \geq \Pi$ ($\Pi" \leq \Pi$) we have

$$\frac{\bar{f}(\Pi) - \bar{f}(\Pi')}{\mid \Pi \mid - \mid \Pi' \mid} \leq \lambda \text{ equivalently } (\overline{f - \lambda})(\Pi) \leq (\overline{f - \lambda})(\Pi')$$

$$\left(\frac{\bar{f}(\Pi") - \bar{f}(\Pi)}{\mid \Pi" \mid - \mid \Pi \mid} \geq \lambda \text{ equivalently } (\overline{f - \lambda})(\Pi) \leq (\overline{f - \lambda})(\Pi") \right).$$

We say that these conditions are satisfied **strictly** if the inequalities above are strict.

Theorem 13.3.5 (The Optimum Cost Rate Theorem) *Let $f(\cdot)$ be a submodular function on subsets of S.*

i. If a partition Π of S satisfies the $\lambda -$ cost rate ($\lambda -$ gain rate) condition, then there exists a partition $\hat{\Pi}$ of S such that $\hat{\Pi}$ minimizes $(\overline{f - \lambda})(\cdot)$ and $\hat{\Pi} \geq \Pi$ ($\hat{\Pi} \leq \Pi$).

ii. If a partition Π of S satisfies the λ-cost rate (λ-gain rate) condition strictly, then whenever $\hat{\Pi}$ minimizes $(\overline{f - \lambda})(\cdot)$ we must have $\hat{\Pi} \geq \Pi$ ($\hat{\Pi} \leq \Pi$).

iii. A partition Π of S satisfies both the $\lambda-$gain rate condition and $\lambda-$cost rate condition iff it minimizes $(\overline{f - \lambda})(\cdot)$.

Proof :
i. Let Π satisfy the $\lambda-$cost rate condition, let $\hat{\Pi}$ minimize $(\overline{f - \lambda})(\cdot)$ and let N be a block of Π. We have by Theorem 13.2.1

$$(\overline{f - \lambda})(\Pi_N) + (\overline{f - \lambda})(\hat{\Pi}) \geq (\overline{f - \lambda})(\Pi_N \vee \hat{\Pi}) + (\overline{f - \lambda})(\Pi_N \wedge \hat{\Pi}).$$

Let $\{N_1, \cdots, N_k\}$ be the partition of N in $\Pi_N \wedge \hat{\Pi}$.
Suppose $(\overline{f-\lambda})(\Pi_N \wedge \hat{\Pi}) < (\overline{f-\lambda})(\Pi_N)$, i.e.,

$$\sum_{i=1}^{k} f(N_i) - k\lambda < f(N) - \lambda.$$

We now replace N by $\{N_1, \cdots, N_k\}$ in Π. Let this new partition be called Π". We then have

$$\bar{f}(\Pi") - \bar{f}(\Pi) < (k-1)\lambda = (\mid \Pi" \mid - \mid \Pi \mid)\lambda,$$

which violates the λ-cost rate condition. Hence,

$$(\overline{f-\lambda})(\Pi_N \wedge \hat{\Pi}) \geq (\overline{f-\lambda})(\Pi_N).$$

Hence,

$$(\overline{f-\lambda})(\Pi_N \vee \hat{\Pi}) \leq (\overline{f-\lambda})(\hat{\Pi}).$$

Thus, $(\Pi_N \vee \hat{\Pi})$ minimizes $(\overline{f-\lambda})(\cdot)$ and one of its blocks contains N. Repeating this procedure gives us a partition that minimizes $(\overline{f-\lambda})(\cdot)$ and is coarser than Π. The argument for the '$\lambda-$gain rate' case is similar. (We use a block N of $\hat{\Pi}$ and use Theorem 13.2.1 on $\Pi, \hat{\Pi}_N$).

ii. (strict $\lambda-$cost rate case)
Going through the argument of the $\lambda-$ cost rate case used above, here we get

$$(\overline{f-\lambda})(\Pi_N \wedge \hat{\Pi}) > (\overline{f-\lambda})(\Pi_N)$$

unless $\Pi_N \wedge \hat{\Pi} = \Pi_N$.
The former alternative implies

$$(\overline{f-\lambda})(\Pi_N \vee \hat{\Pi}) < (\overline{f-\lambda})(\hat{\Pi}),$$

a contradiction. Hence, we must have

$$\Pi_N \wedge \hat{\Pi} = \Pi_N.$$

Hence, $\hat{\Pi}$ must be coarser than Π.

iii. Suppose Π satisfies both the $\lambda-$ cost rate and the $\lambda-$gain rate conditions. Then, by the former condition, there exists a partition $\hat{\Pi} \geq \Pi$ s.t. $\hat{\Pi}$ minimizes $(\overline{f-\lambda})(\cdot)$. Suppose $(\overline{f-\lambda})(\hat{\Pi}) < (\overline{f-\lambda})(\Pi)$. It is easily seen that this violates the $\lambda-$gain rate condition satisfied by Π. We conclude that $(\overline{f-\lambda})(\hat{\Pi}) = (\overline{f-\lambda})(\Pi)$. Thus, Π minimizes $(\overline{f-\lambda})(\cdot)$.
On the other hand let Π minimize $(\overline{f-\lambda})(\cdot)$. For any Π' we then have

$$(\overline{f-\lambda})(\Pi) \leq (\overline{f-\lambda})(\Pi').$$

By taking Π' to be coarser (finer) than Π, it follows that Π satisfies the $\lambda-$gain rate ($\lambda-$cost rate) condition.

□

Exercise 13.9 *If two partitions* Π_1, Π_2 *have the* $\lambda-$*gain rate* ($\lambda-$*cost rate*) *property with respect to the submodular function* $f(\cdot)$ *on subsets of* S *then show that*

$$\overline{(f-\lambda)}(\Pi_1 \wedge \Pi_2) \leq \overline{(f-\lambda)}(\Pi_i), i=1,2,$$

$$(\overline{(f-\lambda)}(\Pi_1 \vee \Pi_2) \leq \overline{(f-\lambda)}(\Pi_i), i=1,2).$$

Exercise 13.10 *Let* Π_1, Π_2 *minimize* $\overline{(f-\lambda)}$. *Then, using the above exercise, show that so do* $\Pi_1 \vee \Pi_2, \Pi_1 \wedge \Pi_2$.

Exercise 13.11 *Let* Π_1 *minimize* $\overline{(f-\lambda_1)}(\cdot)$, *and let* Π_2 *minimize* $\overline{(f-\lambda_2)}(\cdot)$ *with* $\lambda_1 > \lambda_2$. *Then* Π_1 *satisfies the strict* λ_2 *cost rate property and hence,* $\Pi_2 \geq \Pi_1$.

The following result is useful for creating approximation algorithms for finding partitions of least cost with given number of blocks. The result is easiest to apply when the principal sequence is simple. We therefore, introduce some appropriate preliminary definitions.

Let $f(\cdot)$ be a submodular function on subsets of S. We say S is $\Pi-$**molecular** relative to $f(\cdot)$ iff Π_0, Π_S is the principal sequence of $f(\cdot)$. If in addition the only partitions in the PLP are Π_0, Π_S we say S is $\Pi-$**atomic**. (The reader might like to compare these notions with 'molecular' and 'atomic' in the case of PP).

Theorem 13.3.6 *Let* Π_j, Π_{j+1} *both minimize* $\overline{(f-\lambda)}(\cdot)$ *for some* λ.

i. If $\mid \Pi_j \mid \geq k \geq \mid \Pi_{j+1} \mid$, *then for any* $\Pi \in \mathcal{P}_S$ *with* $|\Pi| = k$,

$$\bar{f}(\Pi) \geq \bar{f}(\Pi_{j+1}) + \frac{k- \mid \Pi_{j+1} \mid}{\mid \Pi_j \mid - \mid \Pi_{j+1} \mid}(\bar{f}(\Pi_j) - \bar{f}(\Pi_{j+1})).$$

Further the equality holds only if Π *is a member of the PLP of* $f(\cdot)$.

ii. If S *is* Π- *molecular with respect to* $f(\cdot)$ *and* Π *is a partition of* S *then,*

$$cost\ of\ \Pi\ \geq \frac{k-1}{\mid S \mid -1}(cost\ of\ \Pi_0).$$

Proof : i. Suppose $\bar{f}(\Pi) <$ RHS. It follows that

$$\bar{f}(\Pi) - \lambda_{j+1} \mid \Pi \mid < \bar{f}(\Pi_{j+1}) - \lambda_{j+1} \mid \Pi_{j+1} \mid, \qquad (*)$$

where

$$\lambda_{j+1} = \frac{\bar{f}(\Pi_j) - \bar{f}(\Pi_{j+1})}{\mid \Pi_j \mid - \mid \Pi_{j+1} \mid}.$$

Now Π_j, Π_{j+1} minimize $\overline{(f-\lambda)}(\cdot)$ for a certain value of λ. But the only value of λ for which $\overline{(f-\lambda)}(\Pi_j) = \overline{(f-\lambda)}(\Pi_{j+1})$ is clearly λ_{j+1}. Hence, $\lambda = \lambda_{j+1}$ and therefore,$(*)$ is a contradiction. If there is equality then

$$\overline{(f-\lambda_{j+1})}(\Pi) = \overline{(f-\lambda_{j+1})}(\Pi_{j+1}).$$

Hence, Π also minimizes $\overline{(f-\lambda_{j+1})}$ and is therefore a member of the PLP of $f(\cdot)$.

ii. This is a restatement of the above for the molecular case.

□

We give some simple examples of submodular functions and examine the cost of a partition in each case.

Examples

i. Let $\mathcal{G}$ be a graph and let $w(\cdot)$ be a positive weight function on the edge set of $\mathcal{G}$. Let $(wI)(X) \equiv$ sum of the weights of edges incident on $X \subseteq V(\mathcal{G})$.
In this case $\overline{(wI)}(\Pi) - (wI)(V(\mathcal{G}))$, which is the cost of Π relative to $(wI)(\cdot)$, is the sum of the weights of the edges whose end points are in different blocks of Π.
($\overline{(wI)}(\Pi)$ counts the weights of such edges twice and that of other edges once while $(wI)(V(\mathcal{G}))$ counts the weights of all edges once).

ii. Let $(wE)(X) \equiv$ sum of weights of edges with both end points in X. Clearly $-(wE)(\cdot)$ is a submodular function and $(wE)(V(\mathcal{G})) - \overline{(wE)}(\Pi)$, which is the cost of Π relative to $-(wE)(\cdot)$, is again the sum of the weights of edges whose end points are in different blocks of Π. ($(wE)(V(\mathcal{G}))$ is the sum of weights of all the edges in the graph, $\overline{(wE)}(\Pi)$ counts the weight of all edges with both end points within a block once).

iii. Let $B \equiv (V_L, V_R, E)$ be a bipartite graph and let $w(\cdot)$ be a positive weight function on V_R. Let $(w\Gamma_L)(X) \equiv$ sum of the weights of vertices adjacent to $X \subseteq V_L$ and let $(wE_L)(X) \equiv$ sum of the weights of the vertices adjacent to vertices in X but not to vertices in $V_L - X$. Here cost of Π relative to $(w\Gamma_L)(\cdot)$ is

$$\overline{(w\Gamma_L)}(\Pi) - (w\Gamma_L)(V_L) = \sum_{v_i \in V_R} (k_i - 1)(w(v_i)),$$

where k_i is the number of blocks of Π to whose vertices v_i is adjacent, and the cost of Π relative to $-(wE_L)(\cdot)$ is

$$(wE_L)(V_L) - \overline{(wE_L)}(\Pi) \quad = \quad \textit{sum of weights of vertices which are adjacent to vertices in more than one block of } \Pi.$$

Observe that the cost of the partition in both cases, $(w\Gamma_L)(\cdot)$ as well as $(wE_L)(\cdot)$, is related to the 'overlap' between blocks of Π as reflected by the vertex sets in V_R adjacent to these blocks. However, in the above two cases, the overlap is measured in very different ways. In the case of $(w\Gamma_L)(\cdot)$, if a vertex is adjacent to k blocks its weights is counted $(k-1)$ times. (In particular, a vertex adjacent to only one block does not contribute to the cost.) In the case of $(wE_L)(\cdot)$, however, each vertex which is adjacent to more than one block of Π has its weight counted exactly once and a vertex adjacent to only one block is not counted.

A graph $\mathcal{G}$ can be associated with a bipartite graph $B_{\mathcal{G}}$ with $V_L \equiv V(\mathcal{G})$, $V_R \equiv E(\mathcal{G})$ with $e \in V_R$ adjacent to $v \in V_L$ iff in $\mathcal{G}$ edge e is incident on v. However, in this case, the cost of a partition Π of $V(\mathcal{G})$ would be the same relative to $(w\Gamma_L)(\cdot)$ as well as $(wE_L)(\cdot)$ since here each vertex $e \in V_R$ is adjacent to exactly two vertices in V_L.

Problem 13.1 *This problem is analogous to Problem 11.2. To complete the analogy we remind the reader that if $g'(\cdot)$ is a submodular function on subsets of S then $g(X) \equiv g'(S - X) \;\; \forall X \subseteq S$ is also a submodular function.*
Let $f(\cdot), g(\cdot)$ be submodular on subsets of S and let $g(\emptyset) = 0$.

i. *Let $\mathcal{L}_{\lambda_{g,f}}, \lambda \geq 0$, denote the collection of all partitions that minimize*

$$\bar{f}(\Pi) + \lambda \bar{g}(\Pi), \Pi \in \mathcal{P}_S.$$

If $\Pi_1, \Pi_2 \in \mathcal{L}_{\lambda_{g,f}}$, then $\Pi_1 \vee \Pi_2, \Pi_1 \wedge \Pi_2 \in \mathcal{L}_{\lambda_{g,f}}$.

ii. *If $\lambda_2 > \lambda_1 \geq 0, \Pi_1 \in \mathcal{L}_{\lambda_{1g,f}}$ and $\Pi_2 \in \mathcal{L}_{\lambda_{2g,f}}$ then $\Pi_1 \vee \Pi_2 \in \mathcal{L}_{\lambda_{2g,f}}$ and $\Pi_1 \wedge \Pi_2 \in \mathcal{L}_{\lambda_{1g,f}}$.*

iii. *If instead of $g(\emptyset) = 0$, we have the condition that $g(X) < \bar{g}(\Pi)$ for all partitions Π of X not equal to Π_X, then every partition in $\mathcal{L}_{\lambda_{1g,f}}$ is below every partition in $\mathcal{L}_{\lambda_{2g,f}}$ when $\lambda_2 > \lambda_1 \geq 0$.*

Remark: The idea in the above problem can be used to modify the PLP of $f(\cdot)$. Instead of minimizing $\overline{(f - \lambda)}(\cdot)$ we could minimize $\overline{(f - \lambda) + \sigma g}(\cdot)$. In practice this could allow us to tamper with the size of blocks. The cost of a partition in the PLP of this new function would deviate from the optimum $\bar{f}(.)$ (for its size of blocks) at most by $\overline{\sigma g}(\Pi)$.

13.4 *Approximation Algorithms through PLP for the Min Cost Partition Problem

We present a simple scheme for constructing approximation algorithms for partitions of minimum cost which appears to do well when the cost is in terms of a polymatroid rank function. For greater simplicity we first consider the case where the underlying set S is $\Pi-$molecular with respect to the submodular function $f(\cdot)$. The generalization to the non-molecular case is easy and will be presented afterwards.

ALGORITHM 13.1 Algorithm Min cost Partition Approximation Molecular
INPUT *A polymatroid rank function $f(\cdot)$ whose value is available at each subset $X \subseteq S$ (through a rank oracle) and $k \equiv$ the number of blocks for the desired partition. S is $\Pi-$molecular relative to $f(\cdot)$.*

OUTPUT *A partition Π of S with k blocks whose cost*

$$\leq \frac{\bar{f}(\Pi_0)}{\bar{f}(\Pi_0) - f(S)} * (\textit{cost of optimum partition}),$$

where $n = \mid S \mid$.

STEP 1 *Sort the elements of S according to decreasing $f(\cdot)$ value. Let N be the block composed of the first $(n-k+1)$ elements in this sequence. Output Π_N as desired partition.*

STOP

Justification
Let $\hat{\Pi}$ be the min cost partition of k blocks. We have

$$\bar{f}(\hat{\Pi}) - f(S) \geq \frac{k-1}{n-1}(\bar{f}(\Pi_0) - f(S)),$$

using Theorem 13.3.6. Since N is composed of the largest valued first $(n-k+1)$ elements of S, we must have

$$\begin{aligned} \bar{f}(\Pi_N) &\leq f(N) + \frac{k-1}{n}\bar{f}(\Pi_0) \\ &\leq f(S) + \frac{k-1}{n}\bar{f}(\Pi_0), \\ &\quad \text{since } f(\cdot) \text{ is a polymatroid rank function.} \end{aligned}$$

Hence,

$$\frac{\bar{f}(\Pi_N) - f(S)}{\bar{f}(\hat{\Pi}) - f(S)} \leq \frac{\bar{f}(\Pi_0)}{\bar{f}(\Pi_0) - f(S)}(\frac{n-1}{n}).$$

□

Let us now examine the ratio $\frac{\bar{f}(\Pi_0)}{\bar{f}(\Pi_0)-f(S)}$. If we were to subtract a positive weight function $g(\cdot)$ from $f(\cdot)$, the above ratio would reduce even though the cost of a partition remains the same. The largest weight function that we can subtract from $f(\cdot)$, still retaining it as a polymatroid rank function, is the function $g(e) \equiv f(S) - f(S-e) \;\; \forall \, e \in S$. (Since $f((X-e)\cup e) - f(X-e) \geq f(S) - f(S-e)$, it is clear that $(f-g)(\cdot)$ is a polymatroid rank function). So the above algorithm must use as its input, the function $(f-g)(\cdot)$.
Let us, therefore, assume without loss of generality that

$$f(S) - f(S-e) = 0,$$

and bound the ratio

$$\frac{\bar{f}(\Pi_0)}{\bar{f}(\Pi_0) - f(S)}.$$

In this case

$$f(S) = f(S-e) \leq \bar{f}(\Pi_0) - f(e) \qquad \forall \, e \in S.$$

Hence,

$$\begin{aligned} \frac{\bar{f}(\Pi_0)}{\bar{f}(\Pi_0) - f(S)} &\leq \frac{\bar{f}(\Pi_0)}{f(e)} \qquad \forall \, e \in S \\ &\leq n = \mid S \mid . \end{aligned}$$

In the above discussion we only used the fact that Π_0 and Π_S minimize $\overline{(f-\lambda)}(\cdot)$. The '$\Pi-$molecularity' was not otherwise used. In general, let $\Pi_0, \Pi_1, \cdots, \Pi_t = \Pi_S$ be the principal sequence of partitions of $f(\cdot)$. As before let it be required that we find an optimum partition of k blocks. If $k = \mid \Pi_j \mid$ for some j, then Π_j is the optimum partition. So let $\mid \Pi_j \mid > k > \mid \Pi_{j+1} \mid$.
In what follows, in place of successive terms of the principal sequence, we can take any two partitions that minimize $\overline{(f-\lambda)}(\cdot)$ for the same λ and whose numbers of blocks are on either side of k. Let $\Pi_{j+1} \equiv \{S_1, S_2, \cdots, S_m\}$ and let $\Pi_j \equiv \{N_{11}, \cdots, N_{1p_1}, \cdots, N_{m1}, \cdots, N_{mp_m}\}$, where $\{N_{j1}, \cdots, N_{jp_j}\}$ is a partition of S_j.
By Theorem 13.2.2 any partition of S that is obtained by taking some blocks from Π_{j+1} and others from Π_j would also minimize $\overline{(f-\lambda_{j+1})}(\cdot)$. Thus, we could obtain two partitions

$$\Pi'_{j+1} \equiv \{S_1, S_2, \cdots, S_{r-1}, N_{r1}, \cdots N_{rp_r}, \cdots, N_{m1}, \cdots, N_{mp_m}\},$$

$$\Pi'_j \equiv \{N_{11}, \cdots, N_{1p_1}, S_2, \cdots, S_{r-1}, N_{r_1}, N_{rp_r}, \cdots, N_{m_1}, \cdots, N_{mp_m}\}$$

which both minimize $\overline{(f-\lambda_{j+1})}(\cdot)$ and differ from each other only in that Π'_{j+1} has S_1 as a block while Π'_j has in its place $N_{11}, \cdots, N_{1p_1}$ are blocks. Further, $\mid \Pi'_{j+1} \mid > k > \mid \Pi'_j \mid$.

ALGORITHM 13.2 Algorithm Min cost Partition Approximation

INPUT *i. A polymatroid rank function $f(\cdot)$ whose value is available at each $X \subseteq S$ through a rank oracle,*

ii. $k \equiv$ the number of blocks for the desired partition,

iii. partitions $\Pi' \equiv \{N_{11}, \cdots, N_{1p_1}, S_2, \cdots, S_m\}$, $\Pi" \equiv \{S_1, S_2, \cdots, S_m\}$ which both minimize $\overline{(f-\lambda)}(\cdot)$ for some λ and s.t. $\mid \Pi' \mid > k > \mid \Pi" \mid$ (equivalently, $(p_1 - 1) > k - m > 0$).

OUTPUT *A partition Π of S with k blocks whose cost $\leq \alpha$ (cost of optimum partition), (α to be defined at the end of the algorithm), in general $\alpha \leq (p_1 - 1)$.*

STEP 1 *Let $f'(\cdot) = f/\mathbf{S}_1(\cdot)$ and let $f^2(\cdot) \equiv (f'_{fus \cdot \Pi(S_1)})(\cdot)$, where $\Pi(S_1) \equiv \{N_{11}, \cdots, N_{1p_1}\}$. Let $\omega(\{N_{ij}\}) = f^2(\Pi(S_1)) - f^2(\Pi(S_1) - \{N_{ij}\})$ $(= f(S_1) - f(S_1 - N_{ij}))$ Let $f^3(\cdot) = f^2(\cdot) - \omega(\cdot)$. Sort $\{N_{11}\}, \cdots, \{N_{1p_1}\}$ in order of decreasing value of $f^3(\cdot)$.*

STEP 2 *Lump the first $(p_1 - k + m)$ of the sorted list into a single block M. Let $\{N'_{11}, \cdots, N'_{k-m}\}$ be the remaining blocks of $\Pi(S_1)$. Let $\Pi \equiv \{N'_{11}, \cdots, N'_{k-m}, M, S_2, \cdots, S_m\}$. Let*

$$\beta \equiv \sum_{j=1}^{p_1} f^3(\{N_{1j}\})$$

Define

$$\alpha \equiv \frac{\beta}{\beta - f^3(\Pi(S_1))}\left(\frac{p_1 - 1}{p_1}\right).$$

Output Π *as the desired partition.*

STOP

Justification
For simplicity of notation, we replace $f^3(\{N_{ij}\})$ by $f^3(N_{ij})$. We have

$$\begin{aligned}\bar{f}(\Pi) - \bar{f}(\Pi") &= \sum_{j=1}^{k-m} f^3(N'_{1j}) + f^3(M) - f^3(\Pi(S_1)) \leq \sum_{j=1}^{k-m} f^3(N'_{1j}) \\ &\leq \frac{k-m}{p_1}\beta \qquad (*).\end{aligned}$$

If $\hat{\Pi}$ is the optimum k-block partition we must have

$$\begin{aligned}\bar{f}(\hat{\Pi}) - \bar{f}(\Pi") &\geq \frac{k - \mid \Pi" \mid}{\mid \Pi' \mid - \mid \Pi" \mid}\left(\bar{f}(\Pi') - \bar{f}(\Pi")\right) \\ &\geq \frac{k-m}{p_1 - 1}\left(\sum_{j=1}^{p_1} f(N_{1j}) - f(S_1)\right) \\ &\geq \frac{k-m}{p_1 - 1}\left(\sum_{j=1}^{p_1} f^3(N_{1j}) - f^3(\Pi(S_1))\right), \\ \textit{since} \sum_{j=1}^{p_1} f^3(N_{1j}) - f^3(\Pi(S_1)) &= \sum_{j=1}^{p_1} f(N_{1j}) - f(S_1).\end{aligned}$$

Hence,

$$\bar{f}(\hat{\Pi}) - \bar{f}(\Pi") \geq \frac{k-m}{p_1 - 1}(\beta - f^3(\Pi(S_1))) \qquad (**).$$

Hence, using $(*)$ and $(**)$

$$\frac{\bar{f}(\Pi) - \bar{f}(\Pi")}{\bar{f}(\hat{\Pi}) - \bar{f}(\Pi")} \leq \frac{p_1 - 1}{p_1}\frac{\beta}{\beta - f^3(\Pi(S_1)}.$$

But,

$$\frac{\bar{f}(\Pi) - f(S)}{\bar{f}(\hat{\Pi}) - f(S)} \leq \frac{\bar{f}(\Pi) - \bar{f}(\Pi")}{\bar{f}(\hat{\Pi}) - \bar{f}(\Pi")},$$

since $\bar{f}(\Pi) \geq \bar{f}(\hat{\Pi})$ and $f(S) \leq \bar{f}(\Pi")$. It follows that

$$\frac{\bar{f}(\Pi) - f(S)}{\bar{f}(\hat{\Pi}) - f(S)} \leq \frac{p_1 - 1}{p_1}\frac{\beta}{\beta - f^3(\Pi(S_1))}.$$

Thus,

$$(cost\ of\ \Pi) \leq (cost\ of\ \hat{\Pi})(\frac{\beta}{\beta - f^3(\Pi(S_1)})(\frac{p_1 - 1}{p_1}).$$

Now, $f^3(\cdot)$ is a polymatroid rank function on subsets of $\Pi(S_1)$ with the property that

$$f^3(\Pi(S_1)) = f^3(\Pi(S_1) - \{N_{1j}\}), j = 1, \cdots, p_1.$$

As we have seen before (page 561), for such a polymatroid rank function

$$\frac{\sum_{j=1}^{p_1} f^3(N_{1j})}{\sum_{j=1}^{p_1} f^3(N_{1j}) - f^3(\Pi(S_1)))} \leq \mid \Pi(S_1) \mid = p_1.$$

Hence, cost of $\Pi \leq$ (cost of $\hat{\Pi})(p_1 - 1)$.

□

The following exercises ([Narayanan+Roy+Patkar96]) give instances of application of the above ideas. In these cases the bounds come out to be more attractive.

Exercise 13.12 *Let $\mathcal{G}$ be a self loop free graph and let $w(\cdot)$ be a positive weight function on $E(\mathcal{G})$.*

i. *Let $(wI)(X), X \subseteq V(\mathcal{G})$ denote the sum of the weights of the edges incident on X.*
   Let $(wE)(X), X \subseteq V(\mathcal{G})$, denote the sum of the weights of the edges with both end points in X. Then

$$\sum_{v \in X} (wI)(v) - (wE)(X) \equiv (wI)(X)$$

ii. *(We use the notation of Algorithm 13.2). If $f(\cdot) \equiv (wI)(\cdot)$ then $\frac{\beta}{\beta - f^3(\Pi(S_1))} = 2$, since for any self loop free graph $\frac{\sum (wI)(v)}{\sum (wI)(v) - (wI)(V(\mathcal{G}))} = 2$.*

Exercise 13.13 *Let $B \equiv (V_L, V_R, E)$ be a bipartite graph and let $w_R(\cdot)$ be a positive weight function on V_R. Let $w_R\Gamma_L(X), X \subseteq V_L$, denote the sum of the weights of the vertices in V_R which are adjacent to vertices in X. Let $w_R E_L(X), X \subseteq V_L$ denote the sum of the weights of the vertices in V_R which are adjacent* **only** *to vertices in X. Prove:*

i. *$(w_R\Gamma_L)^*(X) = \sum_{v \in X} w_R\Gamma_L(v) - w_R E_L(X)$, where the comodular dual is taken with respect to the vector α with $\alpha(v) \equiv w_R\Gamma_L(v), v \in V_L$.*

ii. *If no vertex in V_R is adjacent to only one vertex in V_L then*

$$\frac{\sum w_R\Gamma_L(v)}{\sum w_R\Gamma_L(v) - w_R\Gamma_L(V_L)} \leq 2.$$

   Hence, if $f(\cdot) \equiv w_R\Gamma_L(\cdot)$, then $\frac{\beta}{\beta - f^3(\Pi(S_1))} \leq 2$.

iii. *In any bipartite graph* $w_R\Gamma_L^*(V_L) - w_R\Gamma_L^*(V_L - v_1) = 0$.

iv. *If* $f(\cdot) \equiv (w_R\Gamma_L)^*(\cdot)$ *then* $\frac{\beta}{\beta - f^3(\Pi(S_1))} \leq q$, *where* q *is the maximum degree of a vertex in* V_R.

13.5 The PLP of Duals and Truncations

In this section we deal with the PLP of functions derived from a submodular function by the natural operations of dualization and Dilworth truncation. The PLP of the different types of duals is essentially the same but seems to bear no apparent relation to the PLP of the original function. The PLP of the truncation $(f - \sigma)_t(\cdot)$, however, is very simply related to that of the original function $f(\cdot)$. We also define a new dual operation to truncation. This is related to copartitions analogous to the way in which Dilworth truncation and partitions are related. One is thereby led to the definition of the Principal Lattice of Copartitions (PLC).

13.5.1 The PLP of Duals

We remind the reader of the two types of duals defined in Section 10.3. If $f(\cdot)$ is a submodular function on subsets of S, then $f^d(X) \equiv f(S) - f(S - X)$ (the supermodular function $f^d(\cdot)$ is the **contramodular** dual of $f(\cdot)$) and $f^*(X) \equiv \sum_{e \in X} \alpha(e) - [f(S) - f(S - X)]$ (the submodular function $f^*(\cdot)$ is the **comodular** dual of $f(\cdot)$ with respect to the weight function $\alpha(\cdot)$). Let $f^c(X) \equiv f(S - X)$ $\forall X \subseteq S$.

The PLP of the submodular function $f^d(\cdot)$ is the collection of all partitions that maximize $\overline{(f(S) + \lambda - f^c)}(\cdot)$, i.e., minimize $\overline{(f^c - f(S) - \lambda)}(\cdot)$. Now $f^*(\cdot) = \alpha(\cdot) - f^d(\cdot)$. We know that adding a weight function to a submodular function $h(\cdot)$ does not alter the partitions which minimize $\overline{(h - \lambda)}(\cdot)$. Hence, a partition minimizes $\overline{(f^* - \lambda)}(\cdot)$ iff it minimizes $\overline{(-f^d - \lambda)}(\cdot)$, i.e., iff it minimizes $\overline{(f^c - f(S) - \lambda)}(\cdot)$. We therefore need only study the PLP of any one of these functions. The following example suggests that there may not be any simple relationship between the PLP of $f(\cdot)$ and $f^*(\cdot)$.

Example 13.5.1 *Consider the graph* $\mathcal{G}$ *of Figure 13.1. It is shown in page 617 that* $E(\mathcal{G})$ *is* Π *- molecular relative to the rank function* $r(\cdot)$. *Let* $E \equiv E(\mathcal{G})$. *Let* $r^*(\cdot)$ *be the comodular dual of* $r(\cdot)$ *taken with respect to the* $|\cdot|$ *function. We find that* $r^*(E) = 6$. *The value of* λ *for which* $\overline{(r^* - \lambda)}(\Pi_0) = (r^* - \lambda)(\Pi_E)$ *is* $7/12$. *Consider the partition* Π *which has* $\{12, 13\}$ *as a block and others as singletons. We find that* $\overline{(r^* - \lambda)}(\Pi) < \overline{(r^* - \lambda)}(\Pi_0)$. *So* E *is not* Π *- molecular relative to* $r^*(\cdot)$.

Some partitions are however common to the PLPs of $f(\cdot)$ and $f^c(\cdot)$ as shown below.

Figure 13.1: The graph $\mathcal{G}$: PLP of Dual not Simply Related

Lemma 13.5.1 *Let $f(\cdot)$ be a submodular function on subsets of S. Let Π be a partition of S s.t. the blocks of Π are separators of $(f - f(\emptyset))(\cdot)$. Then*

i. Π *minimizes* $\overline{(f - f(\emptyset))}(\cdot)$,

ii. the blocks of Π *are separators of* $(f^c - f^c(\emptyset))(\cdot)$ *and hence minimize* $\overline{(f^c - f^c(\emptyset))}(\cdot)$.

Proof :
i. The function $g(\cdot) \equiv (f - f(\emptyset))(\cdot)$ is submodular and takes value zero on the null set. Hence, for any partition Π' of S, (by repeated use of the submodular inequality), we must have $\bar{g}(\Pi') \geq g(S)$. But if the blocks of Π are separators of $g(\cdot)$. we have $\bar{g}(\Pi) = g(S)$. This proves the required result.

ii. Suppose K is a separator of $(f - f(\emptyset))(\cdot)$. Then

$$f^c(K) - f^c(\emptyset) = f(S-K) - f(S),$$
$$f^c(S-K) - f^c(\emptyset) = f(K) - f(S)$$

Thus

$$\begin{aligned}(f^c - f^c(\emptyset))(K) + (f^c - f^c(\emptyset))(S-K) &= f(S-K) + f(K) - 2f(S) \\ &= f(S-K) - f(\emptyset) + f(K) - f(\emptyset) \\ &\quad +2f(\emptyset) - 2f(S)\end{aligned}$$

$$\begin{aligned} &= f(S) - f(\emptyset) + 2f(\emptyset) - 2f(S) \\ &= f(\emptyset) - f(S) \\ &= f^c(S) - f^c(\emptyset), \end{aligned}$$

i.e., K is a separator of $(f^c - f^c(\emptyset))(\cdot)$. The result now follows from the previous part of the present lemma.

□

13.5.2 The PLP of the Truncation

The PLP of $(f - \sigma)_t(\cdot)$ can be obtained immediately from that of $f(\cdot)$, as we show below.
We need the following theorem whose statement and proof are line by line translations of Theorem 11.4.4.

Theorem 13.5.1 *Let $f(\cdot)$ be a submodular function on subsets of S. Let $p(\cdot)$ denote $((f - \sigma)_t - \lambda)(\cdot)$ and let $h(\cdot)$ denote $(f - (\sigma + \lambda))(\cdot)$.*

i. When $\lambda \geq 0$

- *the minimum values of $\overline{p}(\cdot)$ and $\overline{h}(\cdot)$ over partitions in $\mathcal{P}_S$ are equal and if Π minimizes $\overline{p}(\cdot)$ then there exists a finer partition Π' that minimizes $\overline{h}(\cdot)$;*
- *any partition that minimizes $\overline{h}(\cdot)$ also minimizes $\overline{p}(\cdot)$.*

ii. When $\lambda > 0$, Π minimizes $\overline{p}(\cdot)$ iff it minimizes $\overline{h}(\cdot)$.

iii. When $\lambda \geq 0$, there is a unique minimal partition that minimizes both $\overline{p}(\cdot)$ and $\overline{h}(\cdot)$ and when $\lambda = 0$ its blocks are the elementary separators of $(f - \sigma)_t(\cdot)$.

Proof of Theorem 13.5.1:
i. By the definition of truncation,

$$\overline{(f - \sigma)_t}(\Pi) \leq \overline{(f - \sigma)}(\Pi), \forall \Pi \in \mathcal{P}_S.$$

Hence $\overline{p}(\Pi) \leq \overline{h}(\Pi) \quad \forall \Pi \in \mathcal{P}_S$ and $min_{\Pi \in \mathcal{P}_S} \overline{p}(\Pi) \leq min_{\Pi \in \mathcal{P}_S} \overline{h}(\Pi)$. Next, for any partition Π of S, when $\lambda \geq 0$ we have,

$$\overline{p}(\Pi) \equiv \overline{(f - \sigma)_t}(\Pi) - \lambda|\Pi| = \overline{(f - \sigma)}(\Pi_1) - \lambda|\Pi|,$$

for some $\Pi_1 \leq \Pi$. Hence,

$$\overline{p}(\Pi) \geq \overline{(f - \sigma)}(\Pi_1) - \lambda|\Pi_1| = \overline{h}(\Pi_1).$$

We conclude that $min_{\Pi \in \mathcal{P}_S} \overline{p}(\Pi) = min_{\Pi \in \mathcal{P}_S} \overline{h}(\Pi)$ and that if Π minimizes $\overline{p}(\cdot)$ then there exists a finer partition Π' that minimizes both $\overline{h}(\cdot)$ and $\overline{p}(\cdot)$. Let m denote this minimum value. Suppose Π minimizes $\overline{h}(\cdot)$. We then have,

$$m = \overline{(f - \sigma)}(\Pi) - \lambda|\Pi| \geq \overline{(f - \sigma)_t}(\Pi) - \lambda|\Pi| \geq m.$$

Thus Π minimizes $\overline{p}(\cdot)$.

ii. $(\lambda > 0)$. We need to show that if Π minimizes $\overline{p}(\cdot)$ then it also minimizes $\overline{h}(\cdot)$. We claim that in this case

$$\overline{(f-\sigma)_t}(\Pi) = \overline{(f-\sigma)}(\Pi),$$

from which it would follow that the minimum value m of both $\overline{p}(\cdot)$ and $\overline{h}(\cdot)$ equals $\overline{h}(\Pi)$. Suppose otherwise. Then, we must have

$$m = \overline{(f-\sigma)}(\Pi) - \lambda|\Pi| = \overline{(f-\sigma)_t}(\Pi_1) - \lambda|\Pi|,$$

for some $\Pi_1 < \Pi$. Hence,

$$m > \overline{(f-\sigma)_t}(\Pi_1) - \lambda|\Pi_1| \geq m,$$

which is a contradiction. Thus we must have

$$\overline{(f-\sigma)_t}(\Pi) = \overline{(f-\sigma)}(\Pi),$$

and that Π minimizes $\overline{(f-\sigma)}(\cdot)$.

iii. Since $h(\cdot)$ is clearly submodular we must have the minimal minimizing partition to be unique since the minimizing partitions of $\overline{h}(\cdot)$ are precisely the minimizing partitions in the principal lattice of partitions of $(f-\sigma)(\cdot)$ and property PLP1 can be used. From the previous parts of the present theorem it follows that this partition is also the unique minimal partition that minimizes $\overline{p}(\cdot)$. Consider the situation when $\lambda = 0$. Let Π minimize $\overline{p}(\cdot)$. Now if M is any union of blocks of Π, we have, by the submodularity of $(f-\sigma)_t$ and the fact that it takes value zero on $\emptyset$, $(f-\sigma)_t(M) \leq \sum (f-\sigma)_t(N_i)$, where N_i are the blocks of Π contained in M. Thus if $\Pi' \geq \Pi$ then Π' (in particular Π_S) also minimizes $\overline{p}(\cdot)$. It follows that the blocks of Π_{min}, the minimal minimizing partition of $\overline{p}(\cdot)$ must be separators of $(f-\sigma)_t$. On the other hand if Π has its blocks as separators of $(f-\sigma)_t$, by the definition of separators, we must have $\overline{(f-\sigma)_t}(\Pi) = \overline{(f-\sigma)_t}(\Pi_S)$. This completes the proof.

□

The structure of the PLP of $\overline{(f-\sigma)_t}(\cdot)$ is an immediate corollary (see Figure 13.2).

Corollary 13.5.1 *Let $f(\cdot)$ be a submodular function on subsets of S.*

i. The minimum partitions of $\overline{(f-\sigma)}(\cdot)$ and $\overline{(f-\sigma)_t}(\cdot)$ are identical.

ii. The blocks of this partition are the elementary separators of $(f-\sigma)_t(\cdot)$. Hence the maximum partition that minimizes $\overline{(f-\sigma)_t}(\cdot)$ is Π_S.

iii. For $\lambda > 0$, a partition of S minimizes $\overline{(f-\sigma)_t - \lambda}(\cdot)$ iff it minimizes $\overline{(f-\sigma-\lambda)}(\cdot)$.

We thus see that the relation between the PLP of $(f-\sigma)_t(\cdot)$ and that of $f(\cdot)$ is analogous to the relation between the PP of (f, g) and that of $((\lambda f) * g, g)$.

The following result is a consequence of Corollary 13.5.1. It is analogous to Corollary 11.4.1 for convolution.

Figure 13.2: Comparison of PLP of $f(\cdot), (f-\sigma)_t(\cdot), f/\mathbf{S}_{\mathbf{1}_{fus\cdot\Pi_{\lambda_1}}}(\cdot)$

Corollary 13.5.2 *Let $f(\cdot)$be a submodular function on subsets of S. Let $\lambda \geq 0$. Then*

$$((f-\sigma)_t - \lambda)_t(\cdot) = (f - (\sigma + \lambda))_t(\cdot)$$

Exercise 13.14

Let $\mathcal{G}$ be a graph with $E \equiv E(\mathcal{G})$. Let $V(X) \equiv$ set of end points of edges in X, $X \subseteq E(\mathcal{G})$ and $r(X) \equiv$ rank of the subgraph on X. Show that

i. the minimum partition of $\overline{(\mid V \mid -1)(\cdot)}$ is also the minimum partition of $\bar{r}(\cdot)$, that the blocks of this partition are the 2-connected components of $\mathcal{G}$, and that Π_E is the maximum minimizing partition of $\bar{r}(\cdot)$,

ii. the blocks of the maximum minimizing partition of $\overline{(\mid V \mid -1)(\cdot)}$ are the 1-connected components of $\mathcal{G}$.

13.5.3 The Cotruncation Operation and the Principal Lattice of Copartitions

A collection $\{M_1, \cdots, M_k\}$ of subsets of $X \subseteq S$ is called a **copartition** of X relative to S iff $\{S - M_1, \cdots, S - M_k\}$ is a partition of X. 'Optimum' copartitions carry information about the structure of a submodular function very much the same way as optimum partitions do. Analogous to the Dilworth truncation operation we can define the Dilworth cotruncation operation and we have quite naturally a principal lattice of copartitions (PLC).

We use the following notation. A typical copartition would be denoted by Θ. The collection of all copartitions of X relative to S would be denoted by Δ_{XS} (Δ_X for short). If $\Theta' \equiv \{M_1, \cdots, M_k\}$ is a copartition of $X \subseteq S$ relative to S, then $\Pi(\Theta')$ denotes the partition $\Pi' \equiv \{S - M_1, \cdots, S - M_k\}$. Conversely $\Theta(\Pi')$ denotes Θ'.

If $f(\cdot)$ is a real valued function on subsets of S the **copartition associate** of $f(\cdot)$, denoted by $\underline{f}(\cdot)$, is defined as follows:

$$\underline{f}(\Theta) \equiv \sum_{M_i \in \Theta} f(M_i),$$

where Θ is a copartition of $X \subseteq S$ relative to S.

The **upper Dilworth cotruncation of** $f(\cdot)$ (**lower Dilworth cotruncation of** $f(\cdot)$) denoted by $f^{ct}(\cdot)$ (denoted by $f_{ct}(\cdot)$) is defined as follows:

$$f^{ct}(X) \equiv max_{\Theta \in \Delta_{XS}}(\underline{f}(\Theta))$$

$$(f_{ct}(X) \equiv min_{\Theta \in \Delta_{XS}}(\underline{f}(\Theta)))$$

The following lemma shows that $f_{ct}(\cdot)$ is submodular if $f(\cdot)$ is submodular and $f^{ct}(\cdot)$ is supermodular if $f(\cdot)$ is supermodular. We remind the reader that $f^c(X) \equiv f(S - X), X \subseteq S$.

Lemma 13.5.2 *Let $f(\cdot)$ be a real valued function on subsets of S. Then*

i. $f_{ct}(\cdot) = ((f^c)_t)(\cdot)$

ii. $f^{ct}(\cdot) = ((f^c)^t)(\cdot)$.

Proof :

i. We have

$$\begin{aligned} f_{ct}(X) &= min_{\Theta \in \Delta_{XS}}(\underline{f}(\Theta)) \\ &= min_{\Pi \in \mathcal{P}_X}(\overline{f^c}(\Pi)) \\ &= (f^c)_t(X) \end{aligned}$$

ii. The proof is similar to the above.

□

Corollary 13.5.3 *If $f(\cdot)$ is a submodular (supermodular) function on subsets of S then $f_{ct}(\cdot)$ is submodular ($f^{ct}(\cdot)$ is supermodular).*

Proof : We only prove the submodular case. If $f(\cdot)$ is submodular then $f^c(\cdot)$ is submodular. By Theorem 13.2.4 it follows that $(f^c)_t(\cdot)$ must be submodular. The result follows from Lemma 13.5.2.

□

Since we have $\underline{f}(\Theta) = \bar{f}^c(\Pi(\Theta))$ and $(f-\lambda)^c(\cdot) = (f^c-\lambda)(\cdot)$ it follows that

Θ minimizes $\underline{(f-\lambda)}(\cdot)$ over Δ_{SS} iff $\Pi(\Theta)$ minimizes $\overline{(f^c-\lambda)}(\cdot)$ over $\mathcal{P}_S.\cdots(*PLC)$

The above fact $(*PLC)$ is the truncation analogue of Theorem 11.4.5. We are now led naturally to the definition of the principal lattice of copartitions (PLC) of a submodular function $f(\cdot)$ on subsets of S, as the collection of all copartitions which minimize $\underline{(f-\lambda)}(\cdot)$ over Δ_{SS} for some value of λ. Let us define $\Theta_1 \leq \Theta_2$ iff $\Pi(\Theta_1) \leq \Pi(\Theta_2)$. Then the (PLC) of $f(\cdot)$ has properties identical to the PLP of $f(\cdot)$. These can be derived routinely by using the fact $(*PLC)$. The PLC of $(f-\sigma)_{ct}(\cdot)$ is related to the PLC of $f(\cdot)$ exactly the way the PLP of $(f^c-\sigma)_t(\cdot)$ is related to that of $f^c(\cdot)$.

13.6 *The Principal Lattice of Partitions associated with Special Fusions

We saw in the case of the principal partition of $(f(\cdot), g(\cdot))$ ($f(\cdot)$ submodular, $g(\cdot)$ a positive weight function) that if T_1, T_2 are sets in the principal partition with $T_1 \subseteq T_2$ then the principal partition of $(f \diamond (\mathbf{S}-\mathbf{T}_1)/(\mathbf{T}_2-\mathbf{T}_1)(\cdot), g/(\mathbf{T}_2-\mathbf{T}_1)(\cdot))$ would mimic that of $(f(\cdot), g(\cdot))$. Indeed a subset $A \subseteq T_2 - T_1$ belongs to the former iff $(A \cup T_1)$ belongs to the latter and has the same critical value. We show below that a similar situation prevails in the case of the principal lattice of partitions.

We use the following notation:
$f_{\oplus\Pi}(\cdot)$ denotes $\bigoplus_i f/\mathbf{N}_i(\cdot)$, $N_i \in \Pi$, $f_{fus\cdot\Pi}(\cdot)$ is as defined earlier (Section 10.3). (Let Π be a partition ($\equiv S_1, \cdots, S_k$) of S. Then the **fusion of** $f(\cdot)$ **relative to** Π, denoted by $f_{fus.\Pi}(\cdot)$, is defined on subsets of Π by

$$f_{fus.\Pi}(X) \equiv f(\bigcup_{T \in X} T), X \subseteq \Pi).$$

If $\Pi \equiv \{N_1, \cdots, N_k\}$, a partition of Π may be visualized as

$$\{\{N_1, \cdots, N_r\}, \cdots, \{N_{k-t}, \cdots, N_k\}\}.$$

If $\Pi' \geq \Pi$ then $\Pi'_{fus\cdot\Pi}$ denotes the partition of Π with N_{fus} as one of its blocks iff the members of N_{fus} are the set of blocks of Π contained in a single block of Π'. In other words, if in the partition Π' we treat the blocks of Π as singletons, we get the partition $\Pi'_{fus\cdot\Pi}$ of Π. Let Π be a partition of S and let Π_{fus}, a partition of Π. Then $(\Pi_{fus})_{exp.\Pi}$ denotes the partition Π" of S with N a block of Π" iff N is the union of all blocks of Π which are members of a single block of Π_{fus}. In other words, if in the partition Π_{fus} of Π we expand the elements in each block into the corresponding block of Π and take their union, we get the blocks of $(\Pi_{fus})_{exp.\Pi}$. Thus $((\Pi_{fus})_{exp.\Pi})_{fus\cdot\Pi} = \Pi_{fus}$. As usual $\mathcal{L}_\lambda$ denotes the collection of partitions that minimize $\overline{(f-\lambda)}(\cdot)$.

Let partition Π_1 of S minimize $\overline{(f-\theta)}(\cdot)$ and let partition Π_2 of S minimize $\overline{(f-\beta)}(\cdot)$ with $\beta \geq \theta$. We know that $\Pi_1 \geq \Pi_2$.
We now describe the PLP of

i. $f_1(\cdot) \equiv f_{\oplus\Pi_1}(\cdot)$ when $f(\emptyset) = 0$

ii. $f_2(\cdot) \equiv f/\mathbf{S}_1(\cdot)$, where S_1 is a block in a partition Π in the PLP of $f(\cdot)$

iii. $f_3(\cdot) \equiv f_{fus\cdot\Pi_2}(\cdot)$

iv. $f_4(\cdot) \equiv (f_{\oplus\Pi_1})_{fus.\Pi_2}(\cdot)$ when $f(\emptyset) = 0$.

i. Let $\mathcal{L}^1_\lambda$ denote the collection of partitions that minimize $\overline{(f_1-\lambda)}(\cdot)$. Since $f(\emptyset) = 0$, we must have $\theta \geq 0$.
If $\theta = 0$, the blocks of Π_1 are separators of $f(\cdot)$. Hence, $f(\cdot) = f_1(\cdot)$.
If $\theta > 0$, the only minimizing partition coarser than Π_1 for $\overline{(f_1-\theta)}(\cdot)$ would be Π_1 itself (by the definition of $f_1(\cdot)$). Hence, the minimizing partitions for $\overline{(f_1-\theta)}(\cdot)$ (by Theorem 13.3.5) must be finer than Π_1. On partitions finer than Π_1 both $\overline{f}(\cdot)$ and $\overline{f_1}(\cdot)$ coincide. We know that $\overline{(f-\theta)}(\cdot)$ is a minimum at Π_1. We conclude therefore, that $\overline{(f_1-\theta)}(\cdot)$ reaches a minimum at those partitions of the PLP of $f(\cdot)$ which correspond to critical value θ and which are finer than Π_1. Thus, $\mathcal{L}^1_\theta = \mathcal{L}_\theta \cap \{\Pi : \Pi \leq \Pi_1\}$. By a similar argument we conclude that $\mathcal{L}^1_\lambda = \mathcal{L}_\lambda \cap \{\Pi : \Pi \leq \Pi_1\}$

whenever $\lambda > \theta$. But by PLP2, Π^{λ}, the unique maximum partition in $\mathcal{L}_{\lambda}$, is finer than Π_1. Hence, $\mathcal{L}^1_{\lambda} = \mathcal{L}_{\lambda}, \lambda > \theta$. For $\lambda = 0$, $\overline{(f_1 - \lambda)}(\Pi_1) = \overline{(f_1 - \lambda)}(\Pi) \ \forall\, \Pi \geq \Pi_1$ since $f_1(\cdot)$ has the blocks of Π_1 as separators.
For $\lambda = \theta$ we have already seen that Π_1 minimizes $\overline{(f_1 - \lambda)}$. It is clear therefore, that if $\Pi_S > \Pi_1$, the next critical value after θ is 0 and $\mathcal{L}^1_0 = \{\Pi : \Pi \geq \Pi_1\}$. To summarize

i. if $\theta > 0$ for $f_{\oplus\Pi_1}(\cdot)$ $(f(\emptyset) = 0)$

- $(\lambda > \theta)$ $\quad \mathcal{L}^1_{\lambda} = \mathcal{L}_{\lambda}$
- $(\lambda = \theta)$ $\quad \mathcal{L}^1_{\lambda} = \mathcal{L}_{\lambda} \cap \{\Pi : \Pi \leq \Pi_1\}$
- $(0 < \lambda < \theta)$ $\quad \mathcal{L}^1_{\lambda} = \{\Pi_1\}$
- $(\lambda = 0)$ $\quad \mathcal{L}^1_{\lambda} = \{\Pi : \Pi \geq \Pi_1\}$.

ii. if $\theta = 0, f_{\oplus\Pi_1}(\cdot) = f(\cdot)$

ii. Let S_1 be a block of a partition Π in the PLP of $f(\cdot)$ and let $f_2(\cdot) \equiv f/\mathbf{S}_1(\cdot)$. Let Π minimize $\overline{(f - \theta)}(\cdot)$. We know that for $\lambda > \theta$ minimizing partitions of $\overline{(f - \lambda)}(\cdot)$ are finer than Π while for $\lambda < \theta$ minimizing partitions of $\overline{(f - \lambda)}(\cdot)$ are coarser than Π. Use of Lemma 13.2.2 shows us that the partitions of S_1 which minimize $\overline{(f_2 - \lambda)}(\cdot)$ are precisely the partitions of S_1 which are contained in a minimizing partition of $\overline{(f - \lambda)}(\cdot)$. To summarize,
collection of minimizing partitions of $\overline{(f_2 - \lambda)}(\cdot)$
$\equiv$ partitions of S_1 contained in partitions of $\mathcal{L}_{\lambda}$.

iii. We observe, if Π_2 is a partition of S, that

$$\bar{f}_3(\Pi_{fus\cdot\Pi_2}) = \bar{f}_{fus\cdot\Pi_2}(\Pi_{fus\cdot\Pi_2}) = \bar{f}(\Pi),\ \textit{whenever}\ \Pi \geq \Pi_2.$$

If Π_2 is a partition in the PLP of $f(\cdot)$ minimizing $\overline{(f - \beta)}(\cdot)$, then for $\lambda < \beta$, any minimizing partition Π of $\overline{(f - \lambda)}(\cdot)$ is coarser than Π_2. Hence,

- $(\lambda < \beta)$ Π is a minimizing partition of $\overline{(f - \lambda)}$ iff $(\Pi_{fus\cdot\Pi_2})$ is a minimizing partition for $\overline{(f_3 - \lambda)}(\cdot)$.
- $(\lambda = \beta)$ the minimizing partitions of $\overline{(f_3 - \lambda)}(\cdot)$ are of the form $\Pi_{fus\cdot\Pi_2}$ where Π is a minimizing partition for $\overline{(f - \lambda)}(\cdot)$ coarser than Π_2.
- $(\lambda > \beta)$ $(\Pi_2)_{\cdot fus\cdot\Pi_2}$ is the only minimizing partition.

iv. Let $\mathcal{L}^4_{\lambda} \equiv$ collection of partitions which minimize $\overline{(f_4 - \lambda)}(\cdot)$.

$$f_4(\cdot) \equiv (f_{\oplus\Pi_1})_{fus\cdot\Pi_2}(\cdot).$$

By using the ideas of the previous sections of this discussion we conclude that

$$\begin{aligned}
\mathcal{L}^4_\lambda &= \{(\Pi_2)_{fus\cdot\Pi_2}\}, \lambda > \beta \\
&= \{\Pi_{fus\cdot\Pi_2} : \Pi \in \mathcal{L}_\lambda, \Pi \geq \Pi_2\}, \lambda = \beta \\
&= \{\Pi_{fus\cdot\Pi_2} : \Pi \in \mathcal{L}_\lambda\}, \theta < \lambda < \beta \\
&= \{\Pi_{fus\cdot\Pi_2} : \Pi \in \mathcal{L}_\lambda, \Pi \leq \Pi_1\}, \lambda = \theta \\
&= \{(\Pi_1)_{fus\cdot\Pi_2}\}, 0 < \lambda < \theta \\
&= \{\Pi_{fus\cdot\Pi_2}, \Pi \geq \Pi_1\}, \lambda = 0.
\end{aligned}$$

Exercise 13.15 *(Compare second part of Lemma 11.4.5). Let $f(\cdot)$ be a submodular function on subsets of S and let $S_1 \subseteq S$ be s.t.*

$$(f - \lambda)(S_1) = (f - \lambda)_t(S_1).$$

Then there is a partition Π in $\mathcal{L}_\lambda$ s.t. S_1 is contained in a block of Π. In particular Π^λ has a block that contains S_1.

We now restate Theorem 13.3.2 (Uniqueness Theorem) in the language of this section.

Theorem 13.6.1 *Let $f(\cdot)$ be a submodular function on subsets of S. Let $\Pi_0 < \Pi_1 < \cdots < \Pi_t = \Pi_S$ be a strictly increasing sequence of partitions of S and let $\lambda_1, \cdots, \lambda_t$ be a strictly decreasing sequence of real numbers satisfying the following condition for each Π_i, Π_{i+1} and each block T of Π_{i+1} :*
$\Pi_i(T)$, the partition of T in Π_i is $\Pi-$molecular relative to $(f/\mathbf{T})_{fus\cdot\Pi_i(T)}(\cdot)$, corresponding to critical value λ_{i+1}.
Then

i. *$\Pi_0, \cdots, \Pi_t$ is the principal sequence of partitions of $f(\cdot)$ and $\lambda_1, \cdots, \lambda_t$ is its decreasing sequence of critical values.*

ii. *if $\Pi_{i+1} \equiv \{T^1, \cdots, T^t\}$, $\Pi_i \equiv \biguplus_j \Pi_i(T^j)$ then a partition Π is in $\mathcal{L}_{\lambda_{i+1}}$ iff*

  (a) *$\Pi \equiv \Pi(T^1) \uplus \cdots \uplus \Pi(T^t)$, where $\Pi(T^j)$ is a partition of $T^j, j = 1, \cdots, t$.*
  (b) *$(\Pi(T^j))_{fus\cdot\Pi_i(T^j)}$ lies in the PLP of $(f/\mathbf{T}^j)_{fus\cdot\Pi_i(T^j)}$.*

13.7 Building Submodular Functions with desired PLP

The following exercises describe ways of building submodular functions with desired PLP. We begin with $\Pi-$atomic and $\Pi-$molecular structures.

Exercise 13.16 *Let $f(\cdot)$ be submodular on subsets of S. Show that*

i. *there exists a partition strictly coarser than* Π_0 *minimizing* $(\overline{f-\lambda})(\cdot)$ *iff there exists* $N \subseteq S, \mid N \mid > 1$ *s.t.*

$$f(N) - \lambda \leq \sum_{e \in N} f(e) - \lambda \mid N \mid . \qquad (*)$$

ii. S *is* $\Pi-$*atomic relative to* $f(\cdot)$ *iff there exists no proper non-singleton subset of* S *satisfying the above condition* $(*)$ *for*

$$\lambda = \frac{\sum_{e \in S} f(e) - f(S)}{\mid S \mid -1}$$

i.e., for each such proper subset of S,

$$\frac{\sum_{e \in N} f(e) - f(N)}{\mid N \mid -1} < \frac{\sum_{e \in S} f(e) - f(S)}{\mid S \mid -1}.$$

Exercise 13.17 *Let us say that a graph* $\mathcal{G}$ *is* $\Pi-$*atomic* ($\Pi-$*molecular) iff* $E \equiv E(\mathcal{G})$ *is* $\Pi-$*atomic* ($\Pi-$*molecular) relative to the rank function* $r(\cdot)$ *of* $\mathcal{G}$. *Show that a complete graph on* $n \geq 2$ *nodes is* $\Pi-$*atomic.*

Exercise 13.18 *Show that*

i. *a circuit graph is* $\Pi-$*atomic,*

ii. *a tree graph is* $\Pi-$*molecular,*

iii. *every* $\Pi-$*atomic graph is connected.*

iv. *if a simple (no parallel edges) graph* $\mathcal{G}$ *is* $\Pi-$*atomic then the graph* $\mathcal{G}'$, *obtained by adding a node* n' *and joining it to each of the nodes of* $\mathcal{G}$ *by an edge, is* $\Pi-$*atomic.*

Exercise 13.19 *Let* $\mathcal{G}$ *be a graph of nonzero rank such that* $E \equiv E(\mathcal{G})$ *is* **atomic** *relative to* $(r(\cdot), w(\cdot))$, *i.e.,*

$$\frac{w(X)}{r(X)} < \frac{w(E)}{r(E)}, \emptyset \subset X \subset E,$$

where $w(\cdot)$ *is a positive weight function on the edges. Show that*

i. E *is* $\Pi-$*atomic relative to* $(wI)(\cdot)$, *where* $(wI)(Y) \equiv$ *weight of edges incident on the vertex set* Y.

ii. *the critical value of the PP of* $(r(\cdot), w(\cdot))$ *is the same as the critical value of the PLP of* $(wI)(\cdot)$.

Exercise 13.20 i. *Let* $\mathcal{G}$ *be a graph atomic with respect to* $(r(\cdot), w(\cdot))$. *Let bipartite graph* $B \equiv (V_L, V_R, E_B)$ *be associated with* $\mathcal{G}$ *with* $V_L \equiv V(\mathcal{G})$, $V_R \equiv E(\mathcal{G})$ *and* $e \in E_B$ *iff its right end point (as an edge of* $\mathcal{G}$*) is incident on its left endpoint (as* *a vertex of $\mathcal{G}$). Show that V_L is $\Pi-$atomic relative to $(w\Gamma_L)(\cdot)$, $((w\Gamma_L)(X) \equiv$ sum of weights of vertices adjacent to vertices in $X, X \subseteq V_L$).*

ii. *Let $B \equiv (V_L, V_R, E)$ have every vertex of V_R adjacent to every vertex in V_L. Let $(w\Gamma_L)(X) \equiv$ number of vertices adjacent to vertices in $X, X \subseteq V_L$. Show that*

(a) V_L is $\Pi-$molecular relative to $(w\Gamma_L)(\cdot)$,

(b) every partition of V_L is a minimizing partition for $\overline{((w\Gamma_L)- \mid V_R \mid)}(\cdot)$.

iii. *Let us say that a bipartite graph (V_L, V_R, E) is $\Pi-$atomic ($\Pi-$molecular), if V_L is $\Pi-$atomic ($\Pi-$molecular) relative to $(w\Gamma_L)(\cdot)$.
Let B_1, B_2 be two bipartite graphs (V_L, V_R^1, E_1), (V_L, V_R^2, E_2). Let $B_3 \equiv (V_L, V_R^1, \uplus V_R^2, E_1 \uplus E_2)$. Show that B_3 is $\Pi-$atomic if B_1 is $\Pi-$atomic and B_2 is $\Pi-$molecular.*

Building Submodular functions with given principal sequence of partitions

We now present a simple scheme for constructing a submodular function with prescribed principal sequence and critical values. It is assumed that we know how to build a submodular function on a given set which makes the set Π-molecular with a given critical value. Essentially the same scheme would work even if we wish to build with prescribed PLP and critical values.

ALGORITHM 13.3 Algorithm Build Submod for PLP

INPUT *Sequence of partitions $\Pi_1, \cdots, \Pi_t$ to be the principal sequence of partitions on S and a sequence $\lambda_1, \cdots, \lambda_t$ to be the decreasing sequence of critical values.*

OUTPUT *The function $f(\cdot)$ on subsets of S with given principal sequence of partitions and sequence of critical values.*

STEP 1 *For $j = 0, \cdots, t-1$ repeat the following:*
Let $\Pi_j \equiv \{S_1, \cdots, S_k\}$ and let $\Pi(S_i)$ be the partition of $S_i, i = 1, \cdots, k$, in Π_{j-1} so that

$$\Pi_{j-1} = \biguplus \Pi(S_i)$$

For $i = 1, \cdots, k$ repeat the following:
On $\Pi(S_i)$ (note that elements of $\Pi(S_i)$ are blocks of Π_{j-1})
build a submodular function $f_{(j-1)i}(\cdot)$
s.t. $\Pi(S_i)$ is $\Pi-$molecular
with respect to $f_{(j-1)i}(\cdot)$ with critical value λ_j.
Let $f_{j-1}(\cdot) \equiv \bigoplus_i f_{(j-1)i}(\cdot)$

STEP 2 *Build $\hat{f}_1(\cdot)$ on S as follows:*
Let $\Pi_1 \equiv \{T_1, \cdots, T_r\}$

$$\hat{f}_1(X) \equiv \sum_{X \cap T_i \neq \emptyset} f_0(X \cap T_i) + f_1(\bigcup_{X \cap T_i \neq \emptyset} \{T_i\}) \qquad \forall\, X \subseteq S$$

STEP 3 *Do for* $j = 2, \cdots, t-1$
Build $\hat{f}_j(\cdot)$ on S as follows:
Let $\Pi_j \equiv \{S_1, \cdots, S_k\}$

$$\hat{f}_j(X) \equiv \sum_{X \cap S_i \neq \emptyset} \hat{f}_{j-1}(X \cap S_i) + f_j(\bigcup_{X \cap S_i \neq \emptyset} \{S_i\}) \quad \forall\, X \subseteq S$$

Output $\hat{f}(\cdot) \equiv \hat{f}_{t-1}(\cdot)$ as the submodular function with the given principal sequence and critical values.

Justification
Let $\Pi_j \equiv \{S_1, \cdots, S_k\}$, $\Pi_{j+1} \equiv \{T_1, \cdots, T_r\}$.
We observe that (by STEP 3)
$\hat{f}_{fus \cdot \Pi_j}(X) = f_j(X) + \omega_{j-1}(X) + k_i, X \subseteq (T_i)_{fus.\Pi_j}$, where $\omega_{j-1}(\cdot)$ is the weight function defined by $\omega_{j-1}(\{S_{i_1}, \cdots, S_{i_m}\} \equiv \sum_{n=1}^{m} \hat{f}_{j-1}(S_{i_n})$ and k_i is an appropriate constant (sum of the values of $f_{j+1}(\cdot), \cdots, f_{t-1}(\cdot)$ on an element that 'touches' T_i). If T is a block of Π_{j+1} and $\Pi_j(T)$ is its partition in Π_j, then it is clear that $(\hat{f}/\mathbf{T})_{fus \cdot \Pi_j(T)}(\cdot)$ differs from $f_j/(\mathbf{\Pi}_j(T))(\cdot)$ only by the sum of a weight function and a constant. Hence, both these functions have the same PLP and critical value. The validity of the algorithm now follows from Theorem 13.6.1.

□

Exercise 13.21 *Show that Algorithm 13.3 yields a polymatroid rank function if $f_j(\cdot)$ are polymatroid rank functions.*

Exercise 13.22 *Let* $S \equiv \{1, \cdots, 10\}$ *and let* $\Pi_o, \Pi_1 \equiv \{\{1,2,3\}, \{4,5\}, \{6,7\}, \{8,9,10\}\}$ $\Pi_2 \equiv \{\{1,2,3,4,5\}, \{6,7,8,9,10\}\}$, $\Pi_3 \equiv \Pi_S$ *be the principal sequence. Let the critical values be* $5, 4, 2$. *Build a submodular function on subsets of S with the given principal sequence and critical values.*

13.8 Notes

Although the operation of truncation was introduced earlier in the literature [Dilworth44], it has not received the kind of attention that convolution operation has received. It is a curious fact that the first application to electrical networks in the case of both convolution and truncation was for the solution of versions of the hybrid rank [Kishi+Kajitani68], [Ohtsuki+Ishizaki+Watanabe68],[Narayanan90]. Once the first application was found the analogy between the two operations became too strong to be missed and this led to the notion of the PLP [Narayanan91]. This chapter and the next have been written according to the plan of this latter reference. The operation of truncation is an essential tool if we wish to study functions which might be defined to be submodular over restricted classes of subsets [Frank+Tardos88]. This area has not been touched upon in the present book.

13.9 Solutions of Exercises

E 13.1:
i. This is immediate.

ii. (lower truncation) If Π, Π_1, Π_2 minimize $\overline{\lambda f + \beta g}(\cdot)$, $\overline{\lambda f}(\cdot)$, $\overline{\beta g}(\cdot)$, over partitions of S we must have

$$\begin{aligned}\overline{\lambda f + \beta g}(\Pi) &= \overline{\lambda f}(\Pi) + \overline{\beta g}(\Pi)\\ &\geq \overline{\lambda f}(\Pi_1) + \overline{\beta g}(\Pi_2).\end{aligned}$$

The result follows. The statement about weight function follows when we observe that
$\bar{g}(\Pi) = g(S))\ \ \forall \Pi \in \mathcal{P}_S$.

E 13.2: Let $X \subseteq Y \subseteq S$. Let Π, Π' minimize $\bar{f}(\cdot)$ over $\mathcal{P}_X$, $\bar{f}(\cdot)$ over $\mathcal{P}_Y$ respectively such that each block of Π is contained in a block of Π' (Theorem 13.2.3). Then, $f_t(X) \equiv \bar{f}(\Pi)$ and $f_t(Y) \equiv \bar{f}(\Pi')$. Let M be a block of Π' and let $N_1, \cdots, N_k$ be the blocks of Π contained in it. We must have

$$\sum_{i=1}^{k} f(N_i) \leq f(\bigcup N_i) \leq f(M)$$

(Using Lemma 13.2.2 and the fact that $f(\cdot)$ is increasing).
Hence, $\bar{f}(\Pi') \geq \bar{f}(\Pi)$ as required.

E 13.3:
i. If $\{X_1, X_2, \cdots, X_k\}$ is a partition of $X \subseteq S$, then we have, by the submodularity of $g(\cdot)$ and by the fact that $g(\emptyset) = 0$,

$$\sum_{i=1}^{k} g(X_i) \geq g(X).$$

Hence Π_X minimizes $\bar{g}(\cdot)$ over $\mathcal{P}_X$.
Hence, $g_t(X) = g(X)$.
ii. We will show that $g_t(\cdot) \leq f_t(\cdot)$.
Let Π minimize $\bar{f}(\cdot)$ over $\mathcal{P}_X$. Then $f_t(X) = \bar{f}(\Pi)$. Now $\bar{g}(\Pi) \leq \bar{f}(\Pi)$ since $g(\cdot) \leq f(\cdot)$. But $g_t(X) \leq \bar{g}(\Pi)$. Hence, $g_t(X) \leq f_t(X)$.

E 13.4: We have,

$$(pf - q)_t(e) = pk - q = 1\ \forall e \in S.$$

Further, $(pf-q)(\cdot)$ is an increasing integral submodular function. Hence, so must $(pf-q)_t(\cdot)$ be (Exercise 13.2). By definition $(pf-q)_t(\emptyset)=0$ and

$$(pf-q)_t(X\cup e)-(pf-q)_t(X)\leq (pf-q)_t(e)-(pf-q)_t(\emptyset)=1 \quad \forall e\in S-X.$$

We conclude that $(pf-q)_t(\cdot)$ is a matroid rank function.

E 13.5:
i. We have the rank function $r_k(\cdot)$ defined as follows:

$$r_k(X_k)\equiv (r'-(k-1))_t(X_k),$$

where X_k is a collection of k-rank flats of $\mathcal{M}$. We need to show that $r_k(\cdot)$ is the rank function of the matroid $\mathcal{M}_k$.

We first show that the function $(r'-(k-1))(\cdot)$ is integral, submodular and increasing.
Let $X_k, Y_k\subseteq S_k$, the collection of all k-rank flats of $\mathcal{M}$. Let $X, Y\subseteq S$ be the union of the k-rank flats of $\mathcal{M}$ in X_k, Y_k respectively. Let $r_{\mathcal{M}}(\cdot)$ be the rank function of the matroid $\mathcal{M}$. We then have

$$r'(X_k)+r'(Y_k)\equiv r_{\mathcal{M}}(X)+r_{\mathcal{M}}(Y)\geq r_{\mathcal{M}}(X\cup Y)+r_{\mathcal{M}}(X\cap Y).$$

It is clear that union of all the k-rank flats of $\mathcal{M}$ which are members of $X_k\cup Y_k$ is simply $X\cup Y$. On the other hand, the union of the k-rank flats which are members of $X_k\cap Y_k$ is contained in, but may not be equal to, $X\cap Y$. Hence, $r_{\mathcal{M}}(X\cap Y)\geq r'(X_k\cap Y_k)$.
Thus,

$$r'(X_k)+r'(Y_k)\geq r'(X_k\cup Y_k)+r'(X_k\cap Y_k).$$

This is clearly also true of $(r'-(k-1))(\cdot)$. This function is integral and increasing since $r_{\mathcal{M}}(\cdot)$ is integral and increasing.
Further, if X_k is a k-rank flat, $(r'-(k-1))_t(X_k)=1$. Now we can use Exercise 13.4 and infer that $(r'-(k-1))_t(\cdot)$ is a matroid rank function.

ii. Let $\mathcal{A}_{\mathcal{M}}$ be a flat of $\mathcal{M}$ with rank $p>k$. Let A_k be the collection of all k rank flats contained in $\mathcal{A}_{\mathcal{M}}$. We will show that $r_k(A_k)=p-(k-1)$ and that A_k is a flat in $\mathcal{M}_k$.

We need some preliminary notation and a lemma.
Let two k rank flats of a matroid $\mathcal{M}$ on S be said to be **adjacent** iff their intersection has rank $(k-1)$. We say a collection of k rank flats is **connected** iff between any two distinct k rank flats x_1, x_s in the collection we can find a sequence $x_1, x_2, \cdots, x_{t-1}, x_s$ whose successive terms are adjacent. We then have the following lemma.

Lemma 13.9.1 *Let A be a p-rank flat in $\mathcal{M}$ with $p\geq k$. Then the collection A_k of all k rank flats contained in A is connected.*

Proof : If $p=k$ the result is trivially true. Let $p>k$.
Let $x_1, x_s\in A_k$. Let b_1, b_s be maximally independent sets contained in x_1, x_s

respectively, such that $b_1 \cap b_s$ is a maximum. Suppose $\mid b_1 \cap b_s \mid < k-1$. Let $e \in b_s - b_1$. Then $b_1 \cup e$ has rank $k+1$ or k. In either case there exists an element $e' \in b_1 - b_s$ s.t. $b_2 \equiv b_1 \cup e - e'$ has rank k. The closure x_2 of b_2 has rank k and is a member of A_k. Further,
$r(x_2 \cap x_1) = r(b_2 \cap b_1) = k-1$ and $\mid b_2 \cap b_s \mid > \mid b_1 \cap b_s \mid$. Repeating this process we can build a sequence $x_1, x_2, \cdots, x_s$ of members of A_k whose successive terms are adjacent.

□

We first show that

$$(r' - (k-1))_t(A_k) = r'(A_k) - (k-1) = p - k + 1.$$

Let A be any p-rank flat in $\mathcal{M}$ and let $\Pi \equiv \{{X_k}^1, \cdots, {X_k}^s\}$ be any partition of A_k, the collection of k rank flats contained in A. We know that A_k is connected. Hence, without loss of generality we may assume there are pairs of adjacent elements (x_{11}, x_{21}), (x_{22}, x_{31}), (x_{32}, x_{41}), $\cdots$, $(x_{(s-1)2}, x_{s1})$ s.t.

$$x_{i1} \in {X_k}^i, x_{i2} \in \bigcup_{j=1}^{i} {X_k}^j \qquad \forall\ i.$$

Now $r'({X_k}^i) \equiv r_{\mathcal{M}}(X^i_{\mathcal{M}})$, where $X^i_{\mathcal{M}}$ is the union of all the k-rank flats which are members of ${X_k}^i$. Hence,

$$\begin{aligned}
r'({X_k}^1) + r'({X_k}^2) &= r_{\mathcal{M}}(X^1_{\mathcal{M}}) + r_{\mathcal{M}}(X^2_{\mathcal{M}}) \\
&\geq r_{\mathcal{M}}(X^1_{\mathcal{M}} \cup X^2_{\mathcal{M}}) + r_{\mathcal{M}}(X^1_{\mathcal{M}} \cap X^2_{\mathcal{M}}) \\
&\geq r_{\mathcal{M}}(X^1_{\mathcal{M}} \cup X^2_{\mathcal{M}}) + (k-1) \\
(since \qquad r_{\mathcal{M}}(X^1_{\mathcal{M}} \cap X^2_{\mathcal{M}}) &\geq r_{\mathcal{M}}(x_{11} \cap x_{21}) \geq (k-1)) \\
&\geq r'({X_k}^1 \cup {X_k}^2) + (k-1).
\end{aligned}$$

More generally,

$$r'({X_k}^1 \cup \cdots \cup {X_k}^j) + r'({X_k}^{j+1}) \geq r'({X_k}^1 \cup \cdots \cup {X_k}^{j+1}) + (k-1) \qquad j = 1, \cdots s-1$$

Hence,

$$\bar{r'}(\Pi) \geq r'(A_k) + (s-1)(k-1)$$

$$i.e., \qquad \bar{r'}(\Pi) - (k-1)s \geq r'(A_k) - (k-1),$$

$$i.e., \qquad (r' - (k-1))_t(A_k) = r'(A_k) - (k-1) = p - k + 1 \qquad \text{as required.}$$

Next we need to show that A_k is a flat of $\mathcal{M}_k$. Let $x \in S_k - A_k$. We will show that

$$(r' - (k-1))_t(A_k \cup x) > r'(A_k) - (k-1).$$

If Π is a partition of A_k that minimizes $\overline{(r' - (k-1))}(\cdot)$ over partitions of A_k, there exists a partition Π' of $A_k \cup x$ that minimizes $\overline{(r' - (k-1))}(\cdot)$ over partitions of

$A_k \cup x$ and is such that each block of Π is contained in a block of Π' (Theorem 13.2.3). We know that the single block partition $\{A_k\}$ minimizes $\overline{(r' - (k-1))}(\cdot)$ over partitions of A_k.
So either $\{A_k \cup x\}$ or $\{A_k, \{x\}\}$ minimizes $\overline{(r' - (k-1))}(\cdot)$ over partitions of $A_k \cup x$. Now x (as a subset of S) contains elements of $S - A$ and A is a flat of $\mathcal{M}$. So

$$\begin{aligned} r'(A_k \cup x) - (k-1) &= r_{\mathcal{M}}(A \cup x) - (k-1) \\ &> r_{\mathcal{M}}(A) - (k-1) \\ &> r'(A_k) - (k-1). \end{aligned}$$

Further,

$$r'(A_k) + r'(x) - 2(k-1) = (r'(A_k) - (k-1)) + 1.$$

We thus see that

$$(r' - (k-1))_t(A_k \cup x) > (r' - (k-1))_t(A_k) \qquad \text{as required.}$$

E 13.6: Let $g(\emptyset) = \lambda_0$. Then $(g - \lambda_0)(\cdot)$ is a weight function which takes the same value on all partitions of S. Hence, λ_0 is the only critical value of the PLP of $g(\cdot)$ and Π_0, Π_S, the principal sequence of partitions. The PLP contains all partitions of S.

E 13.7:

i. $\overline{\beta f}(\Pi) = \beta(\bar{f}(\Pi))$.

ii. $\overline{((f+g) - \lambda)}(\Pi) = \overline{(f - \lambda)}(\Pi) + g(S)$

iii. Observe that $\overline{(f - \lambda)}(\Pi) = \overline{(f + k - (\lambda + k))}(\Pi)$.

E 13.8: Let Π minimize $\overline{(f - \lambda)}(\cdot)$ over partitions of S.
Now $\overline{(f - \lambda)}(\Pi) = \bar{f}(\Pi) - \lambda \mid \Pi \mid$.
Hence, $\bar{f}(\Pi') - \lambda \mid \Pi' \mid \geq \bar{f}(\Pi) - \lambda \mid \Pi \mid$, the equality holding only if Π' belongs to the PLP of $f(\cdot)$. Since $\mid \Pi' \mid = \mid \Pi \mid$, the required result follows.

E 13.9: We only prove the $\lambda-$gain rate case. Let Π_1, Π_2 have the $\lambda-$gain property. Let N be a block of Π_2. We have by Theorem 13.2.1,

$$\overline{(f - \lambda)}(\Pi_N) + \overline{(f - \lambda)}(\Pi_1) \geq \overline{(f - \lambda)}(\Pi_N \vee \Pi_1) + \overline{(f - \lambda)}(\Pi_N \wedge \Pi_1).$$

By the $\lambda-$gain rate property of Π_1,

$$\overline{(f - \lambda)}(\Pi_1) \leq \overline{(f - \lambda)}(\Pi_N \vee \Pi_1).$$

Hence, $\overline{(f - \lambda)}(\Pi_N) \geq \overline{(f - \lambda)}(\Pi_N \wedge \Pi_1)$.
Let $\{N_1, \cdots, N_k\}$ be the partition of N within $\Pi_N \wedge \Pi_1$. We can replace N by $\{N_1, \cdots, N_k\}$ in Π_2 and repeat the procedure for each block of Π_2. At the end we get

$$\overline{(f - \lambda)}(\Pi_1 \wedge \Pi_2) \leq \overline{(f - \lambda)}(\Pi_2)$$

Interchanging Π_1, Π_2 we get

$$(\overline{f-\lambda})(\Pi_1 \wedge \Pi_2) \leq (\overline{f-\lambda})(\Pi_1).$$

E 13.10: Using the result in Exercise 13.9 it follows that since Π_1, Π_2 have the $\lambda-$gain rate property

$$(\overline{f-\lambda})(\Pi_1) \geq (\overline{f-\lambda})(\Pi_1 \wedge \Pi_2).$$

Hence, $\Pi_1 \wedge \Pi_2$ minimizes $(\overline{f-\lambda})$. Similarly, since, Π_1, Π_2 have the $\lambda-$cost rate property

$$(\overline{f-\lambda})(\Pi_1) \geq (\overline{f-\lambda})(\Pi_1 \vee \Pi_2).$$

The result follows.

E 13.11: We will show that Π_1 satisfies the strict λ_2 cost rate property. The result then follows from Theorem 13.3.5.
Let $\Pi_3 < \Pi_1$. We have

$$\bar{f}(\Pi_3) - \lambda_1 \mid \Pi_3 \mid \geq \bar{f}(\Pi_1) - \lambda_1 \mid \Pi_1 \mid .$$

Hence,

$$\bar{f}(\Pi_3) - \lambda_2 \mid \Pi_3 \mid -(\lambda_1 - \lambda_2) \mid \Pi_3 \mid \geq \bar{f}(\Pi_1) - \lambda_2 \mid \Pi_1 \mid -(\lambda_1 - \lambda_2)|\Pi_1|.$$

But, $\mid \Pi_3 \mid > \mid \Pi_1 \mid$ and $\lambda_1 > \lambda_2$.
Hence, $\bar{f}(\Pi_3) - \lambda_2 \mid \Pi_3 \mid > \bar{f}(\Pi_1) - \lambda_2 \mid \Pi_1 \mid$, which is the strict λ_2 cost rate condition for Π_1.

E 13.12:
i. $\sum_{v\in X}(wI)(v)$ counts weight of edges with both end points in X twice while weights of remaining edges incident on vertices in X are counted only once. Hence, $\sum_{v\in X}(wI)(v) - (wE)(X)$ counts weight of each edge incident on vertices in X once which is also what $(wI)(X)$ does.

ii. $f'(\cdot) \equiv f/\mathbf{S}_1(\cdot)$ is the weighted incidence function of the graph $\mathcal{G} \cdot S_1$, $f^2(\cdot) = (\mathbf{f'}_{fus\cdot\Pi(S_1)})(\cdot)$ is the weighted incidence function of the graph $\mathcal{G}'$, obtained from $\mathcal{G} \cdot S_1$, by fusing the vertices in each block of $\Pi(S_1)$ into single vertices. In the process some edges with both points in the same block of $\Pi(S_1)$ would become self loops. The weight $(wI)(\Pi(S_1)) - (wI)(\Pi(S_1) - \{N\})$ at the fused vertex $\{N\}$ is the weight of self loops incident at it. Let $\mathcal{G}$" be the graph obtained from $\mathcal{G}'$ by deleting self loops. Hence, $f^3(\cdot)$ is the weighted incidence function of the graph $\mathcal{G}$" on $\Pi(S_1)$. Since this graph has no self loops, we must have
$\sum_{v\in V(\mathcal{G}")}(wI)(v) = 2*$(sum of weights of edges in $E(\mathcal{G}")$),
while $f^3(\Pi(S_1)) =$ (sum of weights of edges in $E(\mathcal{G}")$). The required result follows.

E 13.13: Similar ideas may be found in [Narayanan+Roy+Patkar96].

i. follows from the fact that

$$(w_R E_L)(X) = w_R\Gamma_L(V_L) - w_R\Gamma_L(V_L - X)$$

ii. The first part is obvious. We now show that $f^3(\cdot)$ is the weighted left adjacency function of a bipartite graph that satisfies this condition. $f(\cdot)$ is the weighted left adjacency function of B. Next $f'(\cdot) \equiv f/\mathbf{S}_1(\cdot)$ is the weighted left adjacency function of the bipartite graph B_1, that is the subgraph of B on $S_1 \cup \Gamma(S_1)$. $f'_{fus\cdot\Pi(S_1)}(\cdot)$ is the weighted left adjacency function of the bipartite graph B' obtained from B_1 by fusing the vertices in each block of $\Pi(S_1))$ and replacing parallel edges by a single edge. There may now be vertices in V_R which are adjacent only to single left vertices of B'. The weight $(w_R\Gamma)(\Pi(S_1)) - (w_R\Gamma)(\Pi(S_1) - \{N\})$ at the fused vertex $\{N\}$ is the sum of the weights of the vertices in V_R which are adjacent only to the left (fused) vertex $\{N\}$.
Let B" be obtained from B' by deleting such vertices from V_R. Hence, $f^3(\cdot)$ is the weighted left adjacency function of B".
Now, $\sum_{v\in\Pi(S_1)} f^3(v) \geq 2*$(sum of weights of right side vertices in B"),
since no right vertex in B" is adjacent to only one left vertex, while
$f^3(\Pi(S_1)) =$ sum of weights of right vertices in B".
The result follows.

iii.

$$w_R\Gamma^*_L(V_L) = \sum_{v\in V_L} w_R\Gamma_L(v) - (w_RE_L)(V_L)$$

$$w_R\Gamma^*_L(V_L - v_1) = \left(\sum_{v\in(V_L-v_1)} w_R\Gamma_L(v) - (w_RE_L)(V_L - v_1)\right)$$

$$w_R\Gamma^*_L(V_L) - w_R\Gamma^*_L(V_L - v_1) = w_R\Gamma_L(v_1) - w_R\Gamma_L(v_1) = 0.$$

iv. Let us call

$$f(X) \equiv \left(\sum_{v\in X} w_R\Gamma_L(v) - (w_RE_L)(X)\right)$$

the dual weighted left adjacency function of the bipartite graph.
Then as in the case of $w_R\Gamma_L(\cdot)$, $f'(\cdot) \equiv f/\mathbf{S_1}(\cdot)$ would turn out to be the dual weighted left adjacency function of B_1 but $f^2(\cdot) = f'_{fus\cdot\Pi(S_1)}(\cdot)$ is not associated naturally with a bipartite graph. However, $f^3(\cdot)$ is the dual weighted left adjacency function of the bipartite graph B' obtained from B_1 by fusing the vertices in each block N_i of $\Pi(S_1)$ into a single vertex (replacing parallel edges by a single edge). The bipartite graphs B' and B" have the same dual weighted left adjacency function. Thus in this case,

$$\beta = \sum_{v\in\Pi(S_1)} f^3(v) = \sum q_i w_R(e_i) - \sum_{v_j\in\Pi(S_1)} (w_RE_L)(v_j)$$

where the first summation is over all right vertices e_i of B' and q_i is the degree of e_i in B' and

$$f^3(\Pi(S_1)) = \sum q_i w_R(e_i) - \sum w_R(e_i)$$

Thus,

$$\frac{\beta}{\beta - f^3(\Pi(S_1))} = \frac{\sum q_i w_R(e_i) - \sum(w_R E_L)(v_j)}{\sum w_R(e_i) - \sum(w_R E_L)(v_j)}$$

The numerator and denominator omit all right vertices which are adjacent only to single left vertices of B'. Thus, the ratio

$$= \frac{\sum q"_i w_R(e_i)}{\sum w_R(e_i)},$$

where $q"_i$ is the degree of right vertex e_i in $B"$ (obtained by omitting the above mentioned right vertices from B'). Now $q"_i \leq q \equiv$ the maximum degree of a right vertex in B. The result follows.

E 13.14:

i. This follows immediately from Theorem 13.5.1. But we give below a direct proof.

If Π_2 is a partition of $E(\mathcal{G})$ with its blocks, the edge sets of 2-connected components of $\mathcal{G}$, then

$$\bar{r}(\Pi_2) = r(\mathcal{G}) = \bar{r}(\Pi_E).$$

Since $r(\emptyset) = 0$ and $r(\cdot)$ is submodular, $\bar{r}(\Pi_2) \geq r(\mathcal{G})$.
Thus, Π_2 minimizes $\bar{r}(\cdot)$ and Π_E is the maximum minimizing partition of $\bar{r}(\cdot)$.
Now if a graph $\mathcal{G}'$ is 2-connected, $r(X) + r(E(\mathcal{G}') - X) \geq r(\mathcal{G}') + 1$. So if N is any block of Π_2 and $\{N_1, \cdots, N_k\}$ is a partition of N we must have

$$r(N) < r(N_1) + r(N - N_1) \leq r(N_1) + \cdots + r(N_k).$$

Hence, $\bar{r}(\Pi_2) < \bar{r}(\Pi')$ whenever $\Pi' < \Pi_2$.
Hence, Π_2 is the minimum minimizing partition of $\bar{r}(\cdot)$.
Next $r(X) =\mid V \mid$ - number of components of the subgraph on X with vertex set $V(\mathcal{G})$. Hence, $\bar{r}(\Pi) \leq \overline{(\mid V \mid -1)}(\Pi)\ \forall \Pi \in \mathcal{P}_E$. So $\overline{(\mid V \mid -1)}(\Pi) \geq \bar{r}(\Pi) \geq \bar{r}(\Pi_2) = \overline{(\mid V \mid -1)}(\Pi_2)$.
Further, $\overline{(\mid V \mid -1)}(\Pi') \geq \bar{r}(\Pi') > \bar{r}(\Pi_2)$ if $\Pi' < \Pi_2$.
Hence, Π_2 is the minimum minimizing partition of $\overline{(\mid V \mid -1)}$.

ii. Let Π_1 be the maximum minimizing partition of $\overline{(\mid V \mid -1)}(\cdot)$. Then $\overline{(\mid V \mid -1)}(\Pi_1) = \overline{(\mid V \mid -1)}(\Pi_2) = \bar{r}(\Pi_2) = \bar{r}(\Pi_1)$, since the blocks of Π_2 are separators of $\mathcal{G}$. It follows that the subgraph on any block M of Π_1 must be connected (only then would $r(M) = (\mid V \mid -1)(M)$) On the other hand if $E_1, \cdots, E_k$ are the edge sets of the components of $\mathcal{G}$ and $\Pi_1{}' \equiv \{E_1, \cdots, E_k\}$, then $\Pi_1{}' \geq \Pi_1$ and

$$\overline{(\mid V \mid -1)}(\Pi_1) = \bar{r}(\Pi_1) = \bar{r}(\Pi_2) = \bar{r}(\Pi_1{}') = \overline{(\mid V \mid -1)}(\Pi_1{}').$$

We conclude therefore, that $\Pi_1{}' = \Pi_1$.

E 13.15: We observe that there can be no partition finer than Π_{S_1} whose $(\overline{f - \lambda})(\cdot)$ value is lower than that of Π_{S_1}. The required result then follows from Theorem 13.3.5.

E 13.16:

i. Let $\Pi > \Pi_0$ minimize $(\overline{f - \lambda})(\cdot)$ and let N be a non-singleton block of Π.

By Lemma 13.2.2 we must have $(\overline{f-\lambda})(\Pi(N)) \geq (f-\lambda)(N)$, for any partition $\Pi(N)$ of N. The required condition now follows taking $\Pi(N)$ to be the partition of N into singletons. On the other hand, if any non-singleton set T satisfies the given condition $(\overline{f-\lambda})(\Pi_T) \leq (\overline{f-\lambda})(\Pi_0)$. So there must exist a partition strictly coarser than Π_0 that minimizes $(\overline{f-\lambda})(\cdot)$.

ii. If S is $\Pi-$ atomic relative to $f(\cdot)$, the PLP of $f(\cdot)$ has only Π_0, Π_S as members. The only value of λ for which $(\overline{f-\lambda})(\Pi_0) = (\overline{f-\lambda})(\Pi_S)$ is

$$\lambda = \frac{\sum_{e\in S} f(e) - f(S)}{\mid S \mid -1}.$$

Hence, for this value of $\lambda, (\overline{f-\lambda})(\cdot)$ reaches a minimum at Π_0, Π_S and, since S is $\Pi-$atomic, at no other partition. This is equivalent to the non-existence of a proper non-singleton subset satisfying the condition $(*)$ for the above value of λ.

E 13.17: We use the ideas in Exercise 13.16. We need to show that if $N \subset S$ and $\mid N \mid > 1$

$$\frac{\mid N \mid -r(N)}{\mid N \mid -1} < \frac{\mid E \mid -(n-1)}{\mid E \mid -1}.$$

We have, $\mid E \mid = \frac{n(n-1)}{2}$. If a violation is to occur for a given value of $r(N), \mid N \mid$ must be as large as possible. We may, therefore, take N to be the set of edges of the subgraphs on some k nodes, for $2 < k < n$. This would be the complete graph on k nodes. The above inequality is equivalent in this case to

$$\frac{k(k-1) - 2(k-1)}{n(n-1) - 2(n-1)} < \frac{k(k-1) - 2}{n(n-1) - 2}$$

which is easily seen to be true.

E 13.18:

i. For each proper non-singleton subset N we need to show that

$$\begin{aligned} \frac{\mid N \mid -r(N)}{\mid N \mid -1} &< \frac{\mid E \mid -r(E)}{\mid E \mid -1} \\ &< \frac{n-(n-1)}{n-1} = \frac{1}{n-1}, \end{aligned}$$

where n is the number of nodes in the graph.
Now the subgraph on N has to be a forest. So $r(N) = \mid N \mid$. Hence, LHS is zero and the inequality is strict.

ii. In this case it is easily seen that for each partition of $E(\mathcal{G})$.

$$\bar{r}(\Pi) = \mid E(\mathcal{G}) \mid .$$

Thus, $E(\mathcal{G})$ is $\Pi-$molecular with critical value 0.

iii. Suppose $\mathcal{G}$ on $E \equiv E(\mathcal{G})$ is disconnected and has components $N_1, \cdots, N_k$. Since $r(E) = \sum r(N_i)$, it is easily verified that

$$\frac{\mid N_i \mid -r(N_i)}{\mid N_i \mid -1} \geq \frac{\mid E \mid -r(E)}{\mid E \mid -1}$$

for at least one of the N_i.
But this contradicts the fact that $\mathcal{G}$ is $\Pi-$atomic.

iv. We may assume by the previous part that $\mathcal{G}$ is connected. Further, it may be directly verified that the result is true if $r(E) = 1$. The critical value $\lambda_{\mathcal{G}}$, for $r(\cdot) \equiv$ rank function of $\mathcal{G}$, is $\frac{|E|-r(E)}{|E|-1}$. Let $E' \equiv E(\mathcal{G}')$, $r' \equiv$ rank function of $\mathcal{G}'$. When $r(E) \geq 2$, we will show that

$$\frac{\mid E' \mid -r'(E')}{\mid E' \mid -1} > \frac{\mid E \mid -r(E)}{\mid E \mid -1}.$$

$$Now\ LHS \quad = \frac{\mid E \mid}{\mid E \mid +r(E)}$$

We would be done if we could show that

$$\frac{\mid E \mid}{\mid E \mid +r(E)} > \frac{\mid E \mid -r(E)}{\mid E \mid -1}$$

$$i.e., \mid E \mid < (r(E))^2,\ equivalently,\ \frac{n}{2} < (n-1).$$

The last is clearly true for $n > 2$, i.e., for $r(E) \geq 2$.

Next let $N' \neq E'$ contain $k \neq 0$ edges incident on the node n' and let N be the subset obtained by deleting such edges from N'. We will show that

$$\frac{\mid E' \mid -r'(E')}{\mid E' \mid -1} > \frac{\mid N' \mid -r(N')}{\mid N' \mid -1}.$$

We have

$$\frac{\mid N' \mid -r(N')}{\mid N' \mid -1} \leq \frac{\mid N \mid -r(N) + k - 1}{\mid N \mid -1 + k}, \cdots (!)$$

We consider two cases

i. $(E \supset N)$. In this case

$$RHS\ of\ (!) < \frac{\mid E \mid -r(E) + n - 1}{\mid E \mid -1 + n},$$

ii. $(E = N)$. In this case $k < n$. Once again

$$RHS\ of\ (!) < \frac{\mid E \mid -r(E) + n - 1}{\mid E \mid -1 + n}.$$

Since

$$\frac{|E'| - r'(E')}{|E'| - 1} = \frac{\mid E \mid -r(E) + n - 1}{\mid E \mid -1 + n},$$

the result is proved.

E 13.19: We first show that

$$\frac{w(X)}{r(X)} < \frac{w(E)}{r(E)},\ \emptyset \subset X \subset E, \qquad (*)$$

is equivalent to

$$\frac{w(E(Y))}{\mid Y \mid -1} < \frac{w(E)}{\mid V \mid -1}, Y \subset V = V(\mathcal{G}), \mid Y \mid > 1, \qquad (**)$$

where $E(Y)$ is the set of edges with both endpoints in Y. We note that the graph $\mathcal{G}$ must be connected since it is atomic (if $X_1, \cdots, X_k$ are the components we must have

$$\frac{w(X_i)}{r(X_i)} \geq \frac{w(E)}{r(E)},\quad \mathit{for\ some}\ i).$$

Hence, $r(E) = \mid V \mid -1$. Next $\frac{w(E(Y))}{\mid Y \mid -1} \leq \frac{w(E(Y))}{r(E(Y))}$. So if $(*)$ holds so will $(**)$ hold. Next let $\frac{w(X)}{r(X)} \geq \frac{w(E)}{r(E)}$ for some $X, \emptyset \subset X \subset E$. Then if V_1 is the vertex set of the subgraph $\mathcal{G}'$ on X

$$\frac{w(X)}{r(X)} \leq \frac{w(E(V_1))}{r(E(V_1))}$$

since $X \subseteq E(V_1)$ and $r(X) = r(E(V_1))$. If $V_{11}, \cdots, V_{1k}$ are the vertex sets of the components of $\mathcal{G}'$ we must have

$$\frac{w(E(V_{1i}))}{\mid V_{1i} \mid -1} = \frac{w(E(V_{1i}))}{r(E(V_{1i}))} \geq \frac{w(E(V_1))}{r(E(V_1))}\ \mathit{for\ some}\ i.$$

$\mathcal{G}$ has no self loops since it is atomic and has non-zero rank. Hence, $w(E(Y)) > 0$ only if $\mid Y \mid > 1$. We thus see that for some i, $\mid V_{1i} \mid > 1$ and

$$\frac{w(E(V_{1i}))}{\mid V_{1i} \mid -1} \geq \frac{w(E)}{\mid V \mid -1},$$

thus violating $(**)$. Thus $(**)$ implies $(*)$.
Therefore, $(*)$ and $(**)$ are equivalent. Now,

$$w(E(Y)) = \sum_{v \in Y} (wI)(v) - (wI)(Y).$$

Hence, $(**)$ is equivalent to

$$\frac{\sum_{v \in Y} (wI)(v) - (wI)(Y)}{\mid Y \mid -1} < \frac{\sum_{v \in V} (wI)(v) - (wI)(V)}{\mid V \mid -1}, Y \subset V, \mid Y \mid > 1.$$

By Exercise 13.16 this implies that V is $\Pi-$atomic relative to $(wI)(\cdot)$. Further, the critical value of the PLP equals

$$\frac{\sum_{v \in V} (wI)(v) - (wI)(V)}{\mid V \mid -1},$$

which equals $\frac{w(E)}{r(E)}$. The latter is the critical value of the PP of $(r(\cdot), w(\cdot))$.

E 13.20:

i. This is simply a restatement of Exercise 13.19.

ii. Let $\mid V_R \mid = m$ and let $\Pi \equiv \{N_1, \cdots, N_k\}$ be a partition of V_L. We then have

$$\overline{(w\Gamma_L)}(\Pi) = \sum (w\Gamma_L)(N_i) = k \mid V_R \mid .$$

Hence, $\overline{((w\Gamma_L) - \mid V_R \mid)}(\Pi) = 0$.

iii. Let $(w\Gamma_L^1)(\cdot), (w\Gamma_L^2)(\cdot), (w\Gamma_L^3)(\cdot)$ be the respective left weighted adjacency functions of B_1, B_2, B_3. Clearly $(w\Gamma_L^3)(\cdot) = ((w\Gamma_L^1) + (w\Gamma_L^2))(\cdot)$.
Let λ_1, λ_2 be the critical values for $(w\Gamma_L^1)(\cdot), (w\Gamma_L^2)(\cdot)$. Then,

$$\overline{((w\Gamma_L^3) - (\lambda_1 + \lambda_2))}(\Pi) = \overline{((w\Gamma_L^1) - \lambda_1)}(\Pi) + \overline{((w\Gamma_L^2) - \lambda_2)}(\Pi).$$

Thus, a partition minimizes $\overline{((w\Gamma_L^3) - (\lambda_1 + \lambda_2)}(\cdot)$ if it minimizes both $\overline{((w\Gamma_L^1) - \lambda_1)}(\cdot)$ and $\overline{((w\Gamma_L^2) - \lambda_2)}(\cdot)$. Also it is clear that both $\overline{((w\Gamma_L^1) - \lambda_1)}(\cdot)$ and $\overline{((w\Gamma_L^2) - \lambda_2)}(\cdot)$ reach their minimum at Π_0, Π_{V_L} and never simultaneously at any other partition.. Hence, Π_0, Π_{V_L} are the only minimizing partitions for $\overline{((w\Gamma_L^3) - (\lambda_1 + \lambda_2))}(\cdot)$.

E 13.21: Consider STEP 3 of Algorithm Build Submod for PLP.

$$\hat{f}_j(X) \equiv \sum_{X \cap S_i \neq \emptyset} \hat{f}_{j-1}(X \cap S_i) + f_j(\bigcup_{X \cap S_i \neq \emptyset} \{S_i\}).$$

Define $f'_j(X) \equiv f_j(\bigcup_{X \cap S_i \neq \emptyset} \{S_i\})$.
It is easily verified that $f'_j(\cdot)$ is a polymatroid rank function if $f_j(\cdot)$ is one. By induction we may assume $\hat{f}_{j-1}(\cdot)$ is a polymatroid rank function ($\hat{f}_o(\cdot)$ may be taken to be the polymatroid rank function $f_o(\cdot)$).
Now $\hat{f}_j(\cdot)$ is the sum of two polymatroid rank functions and therefore, is also one.

E 13.22: For all the bipartite graphs described below $(w\Gamma_L)(X)$, where X is a subset of left vertices, denotes the number of right vertices adjacent to X. We first build the function $f_o(\cdot), f_1(\cdot), f_2(\cdot)$. We will take these to be left adjacency functions. Let $B_{01} \equiv (V_{L01}, V_{R01}, E_{01})$. Let

$$V_{L01} \equiv \{1, 2, 3\}, \qquad V_{R01} \equiv \{1_{01}, 2_{01}, \cdots, 5_{01}\}$$

Connect each vertex of V_{L01} to each vertex of V_{R01}. Use of second part of Exercise 13.20 reveals that $\{1, 2, 3\}$ is $\Pi-$molecular relative to $(w\Gamma_{L01})(\cdot)$ with a critical value $= \mid V_{R01} \mid = 5$.
We can similarly build B_{02}, B_{03}, B_{04} with $V_{LO2} \equiv \{4, 5\}, V_{RO2} \equiv \{1_{02}, \cdots, 5_{02}\}$ etc.

$$f_0(\cdot) \equiv (w\Gamma_{L0})(\cdot) \equiv (w\Gamma_{L01})(\cdot) \oplus (w\Gamma_{L02})(\cdot) \oplus (w\Gamma_{L03})(\cdot) \oplus (w\Gamma_{L04})(\cdot).$$

Similarly,

$$B_{11} \equiv (V_{L11}, V_{R11}, E_{11})$$

$$V_{L11} \equiv \{{}^a\{1,2,3\}, {}^a\{4,5\}\}, V_{R11} \equiv \{1_{11}, 2_{11}, 3_{11}, 4_{11}\}.$$

Connect each vertex of V_{L11} to each vertex of V_{R11}. This gives $\Pi-$molecularity to V_{L11} with critical value 4.
We similarly build B_{12}. $f_1(\cdot) \equiv (w\Gamma_{L1})(\cdot) \equiv (w\Gamma_{L11})(\cdot) \oplus (w\Gamma_{L12})(\cdot)$.

$$B_{21} \equiv (V_{L21}, V_{R21}, E_{21})$$

$$V_{L21} \equiv \left\{{}^b\{1,2,3,4,5\}, {}^b\{6,7,8,9,10\}\right\}, V_{R21} \equiv \{1_{21}, 2_{21}, 3_{21}\}.$$

Each vertex of V_{L21} is connected to each vertex of V_{R21}. V_{L21} is $\Pi-$molecular relative to $(w\Gamma_{L21})(\cdot)$ with critical value 3.
We build the overall bipartite graph $\hat{B}$ as follows.
Firstly, $\hat{B}_o \equiv B_{01} \oplus B_{02} \oplus B_{03} \oplus B_{04}$.
$\hat{B}_1$ is built as follows. Split ${}^a\{1,2,3\}$ into three copies of itself a_1, a_2, a_3 (If ${}^a\{1,2,3\}$ is connected to v_R by e then a_1, a_2, a_3 would be connected to v_R by e^1, e^2, e^3). Fuse a_1 with 1 of V_{L01}, a_2 with 2 and a_3 with 3. Similarly, ${}^a\{4,5\}$ is split into copies a_4, a_5 which are fused respectively with $4, 5$ of V_{L02}. Let $V_{L12} \equiv \{{}^a\{6,7\}, {}^a\{8,9,10\}\}$; $V_{R12} \equiv \{1_{12}, 2_{12}, 3_{12}, 4_{12}\}$. ${}^a\{6,7\}$ is split into copies a_6, a_7 which are attached to vertices $6, 7$ of B_{03}. Vertex ${}^a\{8,9,10\}$ is split into copies a_8, a_9, a_{10} which are attached to vertices $8, 9, 10$ of B_{04}.
The result is $\hat{B}_1$. Let the left vertex set of $\hat{B}_1$ be called $\{1, \cdots, 10\}$.
$\hat{B} \equiv \hat{B}_2$ is now built by making copies b_1, b_2, b_3, b_4, b_5 of ${}^b\{1,2,3,4,5\}$ and fusing these with vertices $1, 2, \cdots, 5$, respectively of $\hat{B}_1$ and making copies $b_6, b_7, b_8, b_9, b_{10}$ of ${}^b\{6,7,8,9,10\}$ and fusing these with vertices $6, 7, 8, 9, 10$ respectively of $\hat{B}_1$. The result is $\hat{B}$. The left adjacency function of this submodular function has the desired principal sequence and critical values.
We can reduce the number of edges in the above graph by making V_{Rij} into singletons of appropriate weight and defining $(w\Gamma_{Lij})(X) \equiv$ sum of the weights of right side vertices adjacent to X.

13.10 Solutions of Problems

P 13.1: i. This is a direct consequence of Theorem 13.2.2.

ii. We need the following lemma.

Lemma 13.10.1 *Let $f(\cdot), g(\cdot)$ be submodular functions on subsets of S. Let $h_\lambda(\cdot) \equiv (f + \lambda g)(\cdot)$. Let $N \subseteq S$. Let Π be any partition of S. Then*

$$\begin{aligned}\bar{h}_{\lambda_2}(\Pi) + \bar{h}_{\lambda_1}(\Pi_N) \quad &\geq \quad \bar{h}_{\lambda_2}(\Pi \vee \Pi_N) + \bar{h}_{\lambda_1}(\Pi \wedge \Pi_N) \\ &\quad +(\lambda_2 - \lambda_1)(\bar{g}(\Pi \wedge \Pi_N) - \bar{g}(\Pi_N)).\end{aligned}$$

Proof : We have, by the definition of $\bar{h}_{\lambda_i}(\cdot)$,

$$\begin{aligned}\bar{h}_{\lambda_2}(\Pi \vee \Pi_N) + \bar{h}_{\lambda_1}(\Pi \wedge \Pi_N) \quad &= \quad \bar{h}_{\lambda_2}(\Pi \vee \Pi_N) + \bar{h}_{\lambda_2}(\Pi \wedge \Pi_N) \\ &\quad -(\lambda_2 - \lambda_1)(\bar{g}(\Pi \wedge \Pi_N))\end{aligned}$$

By Theorem 13.2.1, the RHS is

$$\begin{aligned}
&\leq \bar{h}_{\lambda_2}(\Pi) + \bar{h}_{\lambda_2}(\Pi_N) - (\lambda_2 - \lambda_1)(\bar{g}(\Pi \wedge \Pi_N)) \\
&\leq \bar{h}_{\lambda_2}(\Pi) + \bar{h}_{\lambda_1}(\Pi_N) - (\lambda_2 - \lambda_1)(\bar{g}(\Pi \wedge \Pi_N) - \bar{g}(\Pi_N))
\end{aligned}$$

The required result now follows immediately.

□

Let N be a block of Π_1. By the definition of Π_2 we have

$$\bar{h}_{\lambda_2}(\Pi_2 \vee \Pi_N) \geq \bar{h}_{\lambda_2}(\Pi_2).$$

Hence, by Lemma 13.10.1 we have

$$\bar{h}_{\lambda_1}(\Pi_2 \wedge \Pi_N) \leq \bar{h}_{\lambda_1}(\Pi_N) - (\lambda_2 - \lambda_1)(\bar{g}(\Pi_2 \wedge \Pi_N) - \bar{g}(\Pi_N)).$$

Now, since $g(\cdot)$ is submodular and $g(\emptyset) = 0$, $\bar{g}(\Pi_2 \wedge \Pi_N) \geq \bar{g}(\Pi_N)$. Since $\lambda_2 > \lambda_1$, it follows that $\bar{h}_{\lambda_1}(\Pi_2 \wedge \Pi_N) < \bar{h}_{\lambda_1}(\Pi_N)$ unless $\bar{g}(\Pi_2 \wedge \Pi_N) = \bar{g}(\Pi_N)$. The former eventuality would allow us to construct a partition with lower $\bar{h}_{\lambda_1}(\cdot)$ value than Π_1 has, which would be a contradiction. Hence,

$$\begin{aligned}
\bar{g}(\Pi_2 \wedge \Pi_N) &= \bar{g}(\Pi_N) \\
\textit{and} \qquad \bar{h}_{\lambda_1}(\Pi_2 \wedge \Pi_N) &= \bar{h}_{\lambda_1}(\Pi_N).
\end{aligned}$$

By repeating this argument for each block of Π_1 we find that

$$\bar{h}_{\lambda_1}(\Pi_2 \wedge \Pi_1) = \bar{h}_{\lambda_1}(\Pi_1).$$

On the other hand, starting with

$$\begin{aligned}
\bar{h}_{\lambda_1}(\Pi_2 \wedge \Pi_N) &\geq \bar{h}_{\lambda_1}(\Pi_N) \\
\text{we get} \qquad \bar{g}(\Pi_2 \wedge \Pi_N) &= \bar{g}(\Pi_N) \\
\text{and } \bar{h}_{\lambda_2}(\Pi_2 \vee \Pi_N) &\leq \bar{h}_{\lambda_2}(\Pi) - (\lambda_2 - \lambda_1)(\bar{g}(\Pi \wedge \Pi_N) - \bar{g}(\Pi_N)) \\
&\leq \bar{h}_{\lambda_2}(\Pi_2).
\end{aligned}$$

Hence, $\Pi_2 \vee \Pi_N$ minimizes $\bar{h}_{\lambda_2}(\cdot)$. Repeating this argument for each block of Π_1, we find that $\Pi_2 \vee \Pi_1$ minimizes $\bar{h}_{\lambda_2}(\cdot)$.

iii. In this case, using the inequality of Lemma 13.10.1, we would get, unless $\Pi_2 \wedge \Pi_N = \Pi_N$ that

$$\bar{h}_{\lambda_1}(\Pi_2 \wedge \Pi_N) < \bar{h}_{\lambda_1}(\Pi_N).$$

This would allow us to build a partition of lower $\bar{h}_{\lambda_1}(\cdot)$ value than Π_1. To avoid this contradiction we must have $\Pi_2 \wedge \Pi_N = \Pi_N$. Thus N is contained in a block of Π_2. Hence, $\Pi_1 \leq \Pi_2$.

Chapter 14

Algorithms for the PLP of a Submodular Function

14.1 Introduction

In this chapter we first present algorithms for the PLP of a general submodular function and later specialize these to important instances of functions based on bipartite graphs. The general algorithms of this chapter are parallel to the algorithms for principal partition in Section 11.6.

The main algorithms here are

i. Algorithm $Min(\bar{f}, S)$ which finds the minimum partition minimizing $\bar{f}(\cdot)$ (the analogous algorithm in the case of PP is $Convolve(f_1, f_2)$),

ii. Algorithm P-sequence of partitions which is a direct translation of algorithm P-sequence for PP,

iii. Algorithm DTL$(f - \lambda)$ which is analogous to Algorithm $\mathcal{B}_{\lambda_{f,g}}$.

At the heart of the algorithms in the case of both PP and PLP is the problem of minimizing a submodular function. Although no practical solution is known to this problem in general, for the instances which are of interest to us in this book, efficient algorithms are indeed available.

We next specialize the general PLP algorithms to the important cases of weighted adjacency and weighted exclusivity functions associated with a bipartite graph. In both these cases the minimization of the basic submodular function reduces to appropriate flow problems which can be solved extremely efficiently. After this we consider some useful techniques for improving the efficiency of our algorithms in those cases where the maximum value of the (integral) submodular function is less than the size of the underlying set. Due to technical reasons we postpone consideration of algorithms for the PLP of a matroid rank function, which exploit the matroid character of the function, to the next chapter.

Finally we consider the relation between PP and PLP associated with a submodular function. Although, in a strict sense, it can be argued that the two entities are unrelated, we show that the PLP of certain functions associated with the bipartite graph are strongly related to the PP of certain other functions across the bipartite graph. Using this relation we show how to build fast algorithms for the PP of the rank function of a graph.

The reader might like to review the notation used in Section 13.6 (e.g. $f_{fus \cdot \Pi}(\cdot)$, $\Pi'_{exp.\Pi}$ etc.).

14.2 Minimizing the Partition Associate of a Submodular function

We begin by considering the problem of minimizing the partition associate $\bar{f}(\cdot)$ of a submodular function $f(\cdot)$ on subsets of S. We know that subtracting a weight function $w(\cdot)$ from $f(\cdot)$ does not alter the minimizing partitions (since $\bar{w}(\Pi)$ remains unaltered over all partitions of S). If $w(e) \equiv f(e)\ \forall e \in S$, the submodular function $(f-w)(\cdot)$ takes zero value on singletons. We will call such a function **zero singleton submodular (z.s.s)**. The minimization of the partition associate of such functions reduces to the detection of 'fusion sets' which we define below:

Definition 14.2.1 *Let $f(\cdot)$ be a z.s.s. function on the subsets of S. A set $T \subseteq S$ is a* **fusion set** *of $f(\cdot)$ iff*

i. $f(T) < 0$.

ii. $f(T) \leq f(R), R \subseteq T$

iii. All subsets of T on which $f(\cdot)$ takes negative value have a common element.

A fusion set is **strong** *if $f(T) < f(R), R \subset T$.*

The next theorem shows that fusion sets can be 'fused' in the process of finding minimizing partitions for $\bar{f}(\cdot)$.

Theorem 14.2.1 *Let $f(\cdot)$ be a z.s.s. function on subsets of S.*

i. Let N be a fusion set of $f(\cdot)$. Then there exists a partition Π of S such that $\bar{f}(\cdot)$ reaches a minimum on it and N is contained in one of the blocks of Π, i.e. $\Pi \geq \Pi_N$.

ii. If N is a strong fusion set of $f(\cdot)$ then every partition Π on which $\bar{f}(\cdot)$ reaches a minimum contains N in one of its blocks.

Proof :
i. Let $\bar{f}(\cdot)$ reach a minimum on partition Π' of S. Since N is a fusion set, whenever $\Pi(N)$ is a partition of $N, \bar{f}(\Pi(N)) \geq f(N)$ since all but atmost one block N' of $\Pi(N)$ would have nonnegative $f(\cdot)$ value and $f(N') \geq f(N)$.
Consider the partition $\Pi' \wedge \Pi_N$. Now N is a disjoint union of some of the blocks of $\Pi' \wedge \Pi_N$. By the above argument the sum of the values of $f(\cdot)$ on these blocks is greater or equal to $f(N)$. Further both Π_N and $\Pi' \wedge \Pi_N$ have $S - N$ partitioned into singletons. We conclude that

$$\bar{f}(\Pi_N) \leq \bar{f}(\Pi' \wedge \Pi_N).$$

Now by Theorem 13.2.1 we have

$$\bar{f}(\Pi') + \bar{f}(\Pi_N) \geq \bar{f}(\Pi' \vee \Pi_N) + \bar{f}(\Pi' \wedge \Pi_N)$$

Hence, $\bar{f}(\Pi') \geq \bar{f}(\Pi' \vee \Pi_N)$.
Clearly $\Pi' \vee \Pi_N$ minimizes $\bar{f}(\cdot)$ and $\Pi' \vee \Pi_N \geq \Pi_N$.

ii. If N is a strong fusion set, unless $\Pi' \wedge \Pi_N$ is equal to Π_N, we must have $\bar{f}(\Pi_N) < \bar{f}(\Pi' \wedge \Pi_N)$ and therefore, $\bar{f}(\Pi') > \bar{f}(\Pi' \vee \Pi_N)$, a contradiction. We conclude that $\Pi_N = \Pi' \wedge \Pi_N$, i.e., N is contained in a block of Π'.

□

The following corollary is an immediate consequence. (See Section 13.6 for definition of $\bar{f}_{fus \cdot \Pi}(\cdot)$ and $\Pi_{exp \cdot \Pi_N}$).

Corollary 14.2.1 *Let $f(\cdot)$ be a z.s.s. function defined on subsets of S. Let N be a fusion set of $f(\cdot)$. Let $\bar{f}_{fus \cdot \Pi_N}(\cdot)$ reach a minimum at Π. Then $\bar{f}(\cdot)$ reaches a minimum at $\Pi_{exp \cdot \Pi_N}$.*

We will use the technique, of repeatedly fusing fusion sets (converting the resulting function into z.s.s. functions) until there are no more fusion sets, to minimize the partition associate of the given submodular function. By definition, fusion sets are nonsingletons. So fusing them will reduce the size of the problem.

14.2.1 Find (Strong) Fusion Set

This pair of subroutines, described below, depends on finding the minimum of a submodular function $f(\cdot)$ over supersets of a given subset T of S. If we have a polynomial algorithm for minimizing an arbitrary submodular function then it can be adapted to this problem as follows:
Consider the submodular function $f_{fus \cdot \Pi_T}(\cdot)$ defined over subsets of Π_T. Let $f_1(\cdot) \equiv f_{fus \cdot \Pi_T}(\cdot) + w_T(\cdot)$, where w_T is the weight function on Π_T which takes the value (lower bound on minimum value of $f(\cdot) - 1 - f(T)$) on $\{T\}$ and zero on all other singletons. Clearly $f_1(\cdot)$ will reach a minimum only over subsets of Π_T which have $\{T\}$ as a member. If $f_1(\cdot)$ reaches a minimum on K, a subset of blocks of Π_T, then $f(\cdot)$ reaches a minimum over subsets containing T, on the subset K' of S which is the union of blocks in K.

The Subroutines Find Fusion Set and Find Strong Fusion Set differ very little from each other. They are therefore presented below in a combined form with the statements of the latter given within brackets only where they differ from the corresponding statements of the former subroutine.

Subroutine Find Fusion Set (f, S)
(**Subroutine Find Strong Fusion Set**(f, S))

INPUT A z.s.s. function $f(\cdot)$ on subsets of S.

OUTPUT A fusion set of $f(\cdot)$
(A strong fusion set of $f(\cdot)$)
or a declaration that none exist.

Initialize $T \leftarrow \emptyset$ or $T \leftarrow$[a subset of S known not to contain a fusion set].

STEP 1 **If**
$S - T = \emptyset$, **then** S contains no fusion set.
(**then** S contains no strong fusion set).
Declare this and STOP.
Else
let $e \in S - T$. Minimize $f(\cdot)$ on subsets of $T \cup e$ that contain e.
Let the minimum be reached on T_e.
(Let T_e be the unique minimal set on which $f(\cdot)$ reaches a minimum).

STEP 2 If $f(T_e) \geq 0$, then $T \cup e$ contains no fusion set.
(If $f(T_e) \geq 0,$ then $T \cup e$ contains no strong fusion set).
$T \leftarrow T \cup e$. GOTO STEP 1.

STEP 3 If $f(T_e) < 0$, then T_e is a fusion set .
(If $f(T_e) < 0,$ then T_e is a strong fusion set).

STOP

Observe that the two subroutines Find Fusion Set and Find Strong Fusion Set differ from each other only in STEP 1 where for the latter case we have to find a minimal minimizing set.

Justification
If T contains no fusion set of $f(\cdot)$ and $e \notin T$, then every set K, contained in $T \cup e$ and such that $f(K) < 0$, necessarily has e as a member. So if $f(T_e) < 0$, since $f(\cdot)$ reaches a minimum, among subsets of $T \cup e$ containing e, on T_e, it follows that T_e is a fusion set. Suppose in addition T_e is a minimal such set. Then every proper subset of T_e will have a non-negative $f(\cdot)$ value and T_e would, therefore, be a strong fusion set.

□

Complexity
Subroutine Find (Strong) Fusion Set clearly requires $O(\mid S \mid)$ submodular function minimizations.

14.2.2 Min($\bar{f}, S$)

We next present an algorithm for minimizing the partition associate $\bar{f}(\cdot)$ of a submodular function $f(\cdot)$. The algorithm outputs the unique minimal partition which minimizes $\bar{f}(\cdot)$.

ALGORITHM 14.1 Algorithm $Min(\bar{f}, S)$

INPUT *A z.s.s. function* $f(\cdot)$ *over subsets of* S.

OUTPUT *A minimal partition that minimizes* $\bar{f}(\cdot)$ *over partitions of* S.

Initialize $\Pi \leftarrow \Pi_0$, $f_{Temp}(\cdot) \leftarrow f(\cdot)$, $T \leftarrow \emptyset$.
COMMENT: T *is the set at which Subroutine Find Strong Fusion set is initialized.*

STEP 1 *Find Strong Fusion Set* (f_{Temp}, Π) *initializing at* T.
If none exist, declare that $\bar{f}(\cdot)$ *reaches a minimum at* Π *and* Π *is the minimal such partition and STOP.*

STEP 2 *Let* N *be a strong fusion set of* $f_{Temp}(\cdot)$. *(*N *is a set of blocks of* Π*).*
Let $N' \equiv \bigcup_{N_i \in N} N_i$.

$$\Pi \leftarrow \Pi \vee \Pi_{N'},$$

$$f_{Temp}(\cdot) \leftarrow (f_{fus \cdot \Pi} - w)(\cdot),$$

where $w(\cdot)$ *is a weight function on* Π *with*
$w(N_i) = f_{fus \cdot \Pi}(N_i) \ \ \forall N_i \in \Pi$,

$$T \leftarrow (T - N) \cup \{N\},$$

GOTO STEP 1.

STOP

Justification
This algorithm is justified directly by Theorem 14.2.1 and Corollary 14.2.1. We need to explain the setting of T to $T - N \cup \{N\}$ in STEP 2. For simplicity, suppose N is the first strong fusion set so far detected. T contains no fusion set but $T \cup e$ contains N. $f_{Temp}(\cdot) = (f_{fus \cdot \Pi_N} - w)(\cdot)$ where $w(\cdot)$ is a weight function on Π_N s.t. $w(e) = f(e) = 0$, $e \notin N$ and $w(\{N\}) = f(N)$. Consider the set $(T - N) \cup \{N\}$ in Π_N. If this contains a fusion set K of $f_{Temp}(\cdot)$, then N must be a member of that fusion set, since $T - N$ contains no fusion set. Further K would have a negative $f_{Temp}(\cdot)$ value. But by the definition of N, $f(N) \leq f((K - \{N\}) \cup N) = f_{fus \cdot \Pi_N}(K)$. Hence

$$f_{Temp}(K) = f_{fus \cdot \Pi_N}(K) - w(K) = f_{fus \cdot \Pi_N}(K) - w(N) = f_{fus \cdot \Pi_N}(K) - f(N) \geq 0.$$

This contradiction shows that K cannot be a fusion set of $f_{Temp}(\cdot)$.

□

Maximal min $(\bar{f}, S)$

The problem of determining the unique maximal partition that minimizes $\overline{(f-\lambda)}(\cdot)$ needs a slight modification of the notion of a fusion set

Definition 14.2.2 *Let $f(\cdot)$ be a z.s.s. function on the subsets of S. A set $T \subseteq S$ is a* **quasi-fusion set** *of $f(\cdot)$ iff*

i. T is a nonsingleton set,

ii. $f(T) \leq 0$,

iii. $f(T) \leq f(R), R \subseteq T$,

iv. all subsets of T on which $f(\cdot)$ takes negative value have a common element.

As in the case of fusion sets (Theorem 14.2.1), and using the same ideas, we can prove that if N is a quasi-fusion set then there exists a partition Π of S such that $\bar{f}(\cdot)$ reaches a minimum on it and N is contained in one of its blocks (i.e. $\Pi \geq \Pi_N$).

Let **Algorithm Maximal min**$(\bar{f}, S)$ denote the algorithm that, given a z.s.s. function $f(\cdot)$ over subsets of S, finds a maximal partition that minimizes $\bar{f}(\cdot)$ over partitions of S. This algorithm can be obtained from Algorithm Min$(\bar{f}, S)$ essentially by replacing 'strong fusion set' by 'quasi- fusion set'. In particular STEP 1 of the new algorithm would read:
'Find Quasi Fusion Set(f_{Temp}, Π). If none exist $\bar{f}(\cdot)$ reaches a minimum at Π and Π is the maximal such partition. STOP'.
However in STEP 2, we should update T to $T - N$ instead of to $(T - N) \cup \{N\}$.
The subroutine Find Quasi Fusion Set(f, S) is given below.

Subroutine Find Quasi Fusion Set(f, S)
INPUT A z.s.s. function $f(\cdot)$ on subsets of S.

OUTPUT A quasi-fusion set of $f(\cdot)$
or a declaration that none exist.

Initialize $T \leftarrow \emptyset$ or $T \leftarrow$ (a subset of S known not to contain a quasi-fusion set).

STEP 1 **If**
$S - T = \emptyset$, **then** S contains no quasi-fusion set.
Declare this and STOP.
Else
let $e \in S - T$. Minimize $f(\cdot)$ on nonsingleton subsets of $T \cup e$ that contain e. Let the minimum be reached on T_e.

STEP 2 If $f(T_e) > 0$, then $T \cup e$ contains no quasi-fusion set.
$T \leftarrow T \cup e$. GOTO STEP 1.

STEP 3 If $f(T_e) \leq 0$, then T_e is a quasi-fusion set.

STOP

Complexity of Algorithm $Min(\bar{f}, S)$
We show later in Section 14.5 that the complexity of Algorithm $Min(\bar{f}, S)$ is the same as that of Subroutine Find Strong Fusion Set. The essential reason for this is that after the detection of a strong fusion set and the construction of the 'fused' function, one does not have to start again from the null set. By similar arguments one can show that the complexity of Algorithm Maximal $min(\bar{f}, S)$ is also the same.

14.3 Construction of the P-sequence of Partitions

Our next algorithm constructs the P-sequence of partitions of a submodular function $f(\cdot)$. As mentioned before this is analogous to Algorithm 11.1 for the principal partition.

Informally, Algorithm 14.2 proceeds as follows. We start with the partition interval (Π_0, Π_S). If for every partition Π between the end partitions, the value of $\overline{(f-\lambda)}(\Pi)$, does not exceed its value at the end partitions, then we are done - the principal sequence is (Π_0, Π_S) and the critical value is $\frac{\overline{f}(\Pi_0)-\overline{f}(\Pi_S)}{|\Pi_0|-|\Pi_S|}$. Otherwise we find the minimal partition, say Π_1, that minimizes the above expression. Now we work with the intervals $(\Pi_0, \Pi_1), (\Pi_1, \Pi_S)$ and look for minimizing partitions within the interval in question. In each case we use a value of λ for which $\overline{(f-\lambda)}(\cdot)$, reaches the same value at both ends of the interval. When we are unable to subdivide the intervals any further we get a sequence of partitions and a sequence of values which, the Uniqueness Theorem (Theorem 13.3.2) assures us, are respectively the principal sequence and the sequence of critical values of $f(\cdot)$.

This algorithm makes use of the subroutine $Subdivide_f(\Pi_1, \Pi_2)$, which subroutine is analogous to $Subdivide_{f,g}(A, B)$ in the case of the principal partition.

$Subdivide_f(\Pi_1, \Pi_2)$

INPUT A submodular function $f(\cdot)$ on subsets of S and a pair of partitions Π_1, Π_2 s.t. $\Pi_1 \leq \Pi_2 \leq \Pi_S$.

OUTPUT The unique minimal minimizing partition Π_{min} for $\overline{(f-\lambda)}(\Pi)$ over the interval $\Pi_1 \leq \Pi \leq \Pi_2$, where $\lambda = \frac{\bar{f}(\Pi_1)-\bar{f}(\Pi_2)}{|\Pi_1|-|\Pi_2|}$.

STEP 1 $\lambda \leftarrow \frac{\bar{f}(\Pi_1)-\bar{f}(\Pi_2)}{|\Pi_1|-|\Pi_2|}$.
Let $f'(\cdot) \equiv f_{fus \cdot \Pi_1}(\cdot)$.
Let $(\Pi_2)_{fus \cdot \Pi_1}$ have $N_1, \cdots, N_k$ as blocks.
Let $f'_j(\cdot) \equiv f'/\mathbf{N}_j(\cdot)$, $j = 1, \cdots, k$.
For $j = 1, \cdots, k$, do
$Min(\bar{f}'_j, N_j)$,
Let Π^j be the unique minimal minimizing partition output.
Let $\Pi \equiv \biguplus_{j=1}^{k} \Pi^j$. Output $\Pi_{exp \cdot \Pi_1}$ as the partition Π_{min}.

STOP

ALGORITHM 14.2 Algorithm P-sequence of Partitions

INPUT *A submodular function $f(\cdot)$ on subsets of S.*

OUTPUT *The principal sequence of partitions of $f(\cdot)$.*

Initialize *Current Partition Sequence* $\leftarrow (\Pi_0, \Pi_S)$
$\lambda_1 \leftarrow \frac{\bar{f}(\Pi_0)-\bar{f}(\Pi_S)}{|\Pi_0|-|\Pi_S|}$
Current Lambda Sequence $\leftarrow (\lambda_1), j \leftarrow 0, \Pi_0$ *is unmarked.*

STEP 1 *Let Current Partition Sequence be* $(\Pi_1^j, \cdots, \Pi_{r_j}^j)$ *and let Current Lambda Sequence be* $(\lambda_1^j, \cdots, \lambda_{r_j-1}^j)$.
If $\Pi_m^j,\ 1 \leq m \leq r_j - 1$, *is unmarked*
then $Subdivide_f(\Pi_m^j, \Pi_{m+1}^j)$.
Else *GOTO STEP 3.*

STEP 2 *Let* $(\Pi_1^j, \cdots, \Pi_{r_j}^j) = (\Pi_1, \cdots, \Pi_k)$
and let $(\lambda_1^j, \cdots, \lambda_{r_j-1}^j) = (\lambda_1, \cdots, \lambda_{k-1})$
Let Π *be the min partition output by* $Subdivide_f(\Pi_m^j, \Pi_{m+1}^j)$
If $\Pi = \Pi_m^j$ **then**

$$\begin{array}{rcl} j \leftarrow j+1 & , & r_j \leftarrow k \\ (\Pi_1^j, \cdots, \Pi_{r_j}^j) & \leftarrow & (\Pi_1, \cdots, \Pi_k) \\ (\lambda_1^j, \cdots, \lambda_{r_j-1}^j) & \leftarrow & (\lambda_1, \cdots, \lambda_{k-1}) \\ Mark\ \Pi_m^j & \& & GOTO\ STEP\ 1 \end{array}$$

Else *(* $\Pi \neq \Pi_m^j$*)*

$$\begin{array}{rcl} j \leftarrow j+1 & , & r_j \leftarrow k+1 \\ \Pi_i^j \leftarrow \Pi_i & , & i \leq m \\ \Pi_{m+1}^j & \leftarrow & \Pi \\ \Pi_{i+1}^j \leftarrow \Pi_i & , & m < i < r_j \\ \lambda_i^j \leftarrow \lambda_i & , & i < m \end{array}$$

$$\begin{array}{rcl} \lambda_m^j \leftarrow \lambda' & = & \left(\dfrac{\bar{f}(\Pi_m^j) - \bar{f}(\Pi)}{\mid \Pi_m^j \mid - \mid \Pi \mid} \right) \\ \lambda_{m+1}^j \leftarrow \lambda" & = & \left(\dfrac{\bar{f}(\Pi) - \bar{f}(\Pi_{m+2}^j)}{\mid \Pi \mid - \mid \Pi_{m+2}^j \mid} \right) \\ \lambda_{i+1}^j \leftarrow \lambda_i & , & m < i < r_j - 1 \end{array}$$

The Current Partition Sequence

$$(\Pi_1^j, \cdots, \Pi_m^j, \Pi_{m+1}^j, \Pi_{m+2}^j, \cdots, \Pi_{r_j}^j) \leftarrow (\Pi_1, \cdots, \Pi_m, \Pi, \Pi_{m+1}, \cdots, \Pi_k)$$

The Current Lambda Sequence

$$(\lambda_1^j, \cdots, \lambda_m^j, \lambda_{m+1}^j, \cdots, \lambda_{r_j-1}^j) \leftarrow (\lambda_1, \cdots, \lambda', \lambda", \cdots, \lambda_{k-1})$$

GOTO STEP 1.

STEP 3 *Output Current Partition Sequence as the principal sequence of partitions and Current Lambda Sequence as the Critical Value Sequence.*

STOP

Justification of Algorithm P-sequence of partitions is directly by use of Uniqueness Theorem (PLP) (Theorem 13.3.2).

□

Complexity
Clearly Algorithm P-sequence of partitions calls Algorithm $Min(\overline{f-\lambda}, \Pi_{\lambda})$ atmost $\mid S \mid$ times and thus requires $O(\mid S \mid^2)$ submodular function minimizations.

Exercise 14.1
Show that Subdivide (Π_1, Π_2) *requires* $O(\mid \Pi_1 \mid \alpha(\mid \Pi_1 \mid - \mid \Pi_2 \mid +1))$ *elementary steps, where submodular function minimization over a set of size* k *is* $O(\alpha(k))$ *elementary steps. Assume that* $\alpha(k)$ *is superlinear.*

Speeding up Algorithm Principal Sequence of Partitions through Balanced Bisection

The following technique used in [Imai83] in connection with the construction of principal partition of the rank function of a graph can also be used to speed up the construction of the principal sequence of partitions [Patkar+Narayanan92b]. We will call this the method of balanced bisection.

Suppose we can find beforehand a set of numbers within which all the critical values are known to lie. Let k be the size of the set.
We can sort the values of this set as $\lambda_1 < \cdots < \lambda_k$. Our aim is to pick a λ for which $\mid \Pi_{\lambda} \mid$ and $\mid \Pi^{\lambda} \mid$ lie on either side of $\frac{|S|}{2}$ (or are equal to $\frac{|S|}{2}$). We do this by binary search among $\lambda_1, \cdots, \lambda_k$ and it takes $\log k$ calls to *Subdivide*, using λ_j in place of λ of the subroutine, i.e., $O(n\alpha(n)\log k)$ elementary steps, where $\alpha(n)$ is the number steps needed to solve a submodular minimization problem over a set of size n (see Exercise 14.1). Suppose we have found $\lambda^1, \Pi^{\lambda^1}$ and Π_{λ^1}. Now we search in the interval $[\lambda_1, \lambda^1], [\lambda^1, \lambda_k]$ for λ^2, λ^3 s.t. $\mid \Pi_{\lambda^2} \mid, \mid \Pi^{\lambda^2} \mid$ are on either sides of $\frac{|S|}{4}$ and $\mid \Pi_{\lambda^3} \mid, \mid \Pi^{\lambda^3} \mid$ are on either side of $\frac{3|S|}{4}$. Using Exercise 14.1 the complexity at this level is $O((n\alpha(\frac{n}{2}) + \frac{n}{2}\alpha(\frac{n}{2}))\log k)$ where $n = \mid S \mid$.

Repeating this procedure we see that the overall complexity is

$$O((n\alpha(n) + 2n\alpha(\frac{n}{2}) + 4n\alpha(\frac{n}{4}) + \cdots)\log k).$$

If for some constant $p, p\alpha(n) > n^2\ \forall n$, then the above reduces to $O(n\alpha(n)(\log n)(\log k))$, where $n = \mid S \mid$ and k = number of possible values (known before hand) that the critical values can take.

Exercise 14.2
Let $f(\cdot)$ be an integral submodular function on subsets of S.
Let $b \leq f(X) \leq a\ \forall X \subseteq S$. Show that every critical value has the form $\frac{p}{q}$ where p, q are integers with $-n(a-b) \leq p \leq n(a-b)$ and $0 < q \leq n = \mid S \mid$. Hence, show that all the critical values are contained in a set with $n(2n(a-b)+1)$ elements.

14.4 Construction of the DTL

By building the P-sequence of partitions, we would have determined the unique minimal and maximal partitions which minimize $(\overline{f-\lambda})(\cdot)$ where $f(\cdot)$ is submodular and λ an arbitrary real number. We now consider the problem of determining all such partitions.
Let Π_1, Π_2 be the minimal and maximal partitions minimizing $(\overline{f-\lambda})(\cdot)$. It is clear that Π is in the DTL of $(f-\lambda)(\cdot)$ iff $\Pi_{fus \cdot \Pi_1}$ is in the DTL of $(f_{fus \cdot \Pi_1} - \lambda)(\cdot)$. The DTL of the latter function does not change by subtracting a weight function (on Π_1). We may assume the weight function to be such that the subtraction results in a z.s.s. function.
Let $f_0(\cdot) \equiv (f_{fus \cdot \Pi_1} - \lambda)(\cdot) - w(\cdot)$ where the weight function is defined by $w(e) \equiv f_{fus \cdot \Pi_1}(e) - \lambda\ \forall e \in \Pi_1$. The function $f_0(\cdot)$ is z.s.s. on Π_1 and $\bar{f}_0(\cdot)$ takes minimum value zero on $(\Pi_1)_{fus.Pi_1}$. Hence, $f_0(\cdot)$ cannot take negative value on any subset of Π_1. For, if it does so on $N \subseteq \Pi_1$, then the partition of Π_1 which has N as a block and the others as singletons would have a negative $\bar{f}_0(\cdot)$ value. Further, arguing similarly, a subset of Π_1 can appear as a block of a minimizing partition of Π_1 iff $f_0(\cdot)$ takes zero value on it. Thus, the problem of determining the DTL of the z.s.s. function $f_0(\cdot)$ reduces to that of determining the subsets of Π_1 on which $f_0(\cdot)$ takes zero value. Each such subset must be contained in a block of $(\Pi_2)_{fus \cdot \Pi_1}$. This block is also a zero subset. We, therefore, consider the problem of determining the **zero sets** of a **z.s.s. function of type (000)** (defined below).

Definition 14.4.1 *A submodular function $f_1(\cdot)$ defined on subsets of T is of* **type (000)** *iff (a) $f_1(\cdot)$ is z.s.s. (b) $f_1(T) = 0$, (c) $min_{R \subseteq T} f_1(R) = 0$.*
The sets on which a type (000) function takes zero value are called **zero sets.**

Definition 14.4.2 *Let $f_1(\cdot)$ be a submodular function of type (000) on subsets of T. The* **zero bipartite graph** *$B(f_1) \equiv (V_L, V_R, E)$ of $f_1(\cdot)$ is defined as follows. $V_L \equiv T, V_R$ is the family of all nonsingleton zero subsets $C_{v_i v_j}$ of T which are minimal with respect to the property of containing $\{v_i, v_j\} \subseteq T, v_i \neq v_j$. A vertex $v \in V_L$ would be adjacent to $C_{v_i v_j} \in V_R$ iff $v \in C_{v_i v_j}$.*

We can now characterize the zero sets of a type (000) function.

Theorem 14.4.1 *Let $B(f_1) \equiv (V_L, V_R, E)$ be the zero bipartite graph of a type (000) function $f_1(\cdot)$ on subsets of T. A set $R \subseteq T$ is a zero set of $f_1(\cdot)$ iff it is the union of the right vertices (each of which is a subset of T) of a connected subgraph of $B(f_1)$.*

Proof : If We will prove this by induction on the number n of right vertices of the connected subgraph of $B(f_1)$. The result is obviously true for $n = 1$. Let it be true for $n < k$. Let $e_1, \cdots, e_k$ be the right vertices of the connected subgraph B_k of $B(f_1)$ and let $\bigcup_{i=1}^{k} e_i = R$. Let B_{k-1} denote a connected subgraph of B_k on $(k-1)$ of the right vertices and the left vertices which are adjacent to them. We lose no generality in assuming that the right vertices are $e_1, \cdots, e_{k-1}$. By the induction assumption $R' \equiv \bigcup_{i=1}^{k-1} E_i$ is a zero set of $f_1(\cdot)$.
By the submodularity of $f_1(\cdot)$ we have, $f_1(R') + f_1(e_k) \geq f_1(R) + f_1(R' \cap e_k)$.
Since B_k is connected we have $R' \cap e_k \neq \emptyset$ (equivalently, a left vertex is adjacent to right vertices in R' as well as to the right vertex e_k). Further, $f_1(\cdot)$ is of type (000) and e_k is a zero set. Hence, $f_1(e_k) \leq f_1(R' \cap e_k)$ and therefore, $f_1(R') \geq f_1(R)$. Thus, R is a zero set of $f_1(\cdot)$.
Only if Suppose R is a zero set of $f_1(\cdot)$. Let $R = \{v_1, \cdots, v_t\}$. Let e_{ij} be a right vertex of $B(f_1)$ such that $e_{ij} \subseteq R$ and $v_i, v_j \in e_{ij}$. Consider the subgraph of $B(f_1)$ on the set of left vertices R and set of right vertices $\{e_{ij} : i \neq j, i, j \in \{1, \cdots, t\}\}$. Clearly this subgraph is connected and $\bigcup e_{ij} = R$.

□

The algorithm for constructing $B(f_1)$ is based on the following simple lemma whose routine proof is omitted.

Lemma 14.4.1 *Let $f_1(\cdot)$ be a submodular function of type (000) defined on subsets of T. Then there is a unique minimal zero set $C_{v_i v_j}$ containing a given distinct pair v_i, v_j of elements of T. If $f_1(\cdot)$ is a type (000) function on subsets of T, to determine $C_{v_i v_j}$ we have to find the minimal set that minimizes $f_1(\cdot)$ over subsets of T that are supersets of $\{v_i, v_j\}$. The zero bipartite graph needs $O(\mid T \mid^2)$ such minimizations.*

ALGORITHM 14.3 Algorithm DTL $(f - \lambda)$
INPUT *A submodular function $(f - \lambda)(\cdot)$ on subsets of S.*

OUTPUT *All the zero bipartite graphs needed to determine DTL $(f - \lambda)$.*

STEP 1 *Determine Π_1, Π_2, the minimum and maximum partitions minimizing $\overline{(f - \lambda)}(\cdot)$*

STEP 2 *Let $T_1, \cdots, T_k$ be the blocks of $(\Pi_2)_{fus \cdot \Pi_1}$. Build type (000) functions $f_1(\cdot), \cdots f_k(\cdot)$ on these sets:*

$$f_j(\cdot) \equiv f_0/\mathbf{T}_j(\cdot), f_0(\cdot) \equiv (f - \lambda)_{fus \cdot \Pi_1}(\cdot) - w(\cdot),$$

where $w(e) \equiv (f_{fus \cdot \Pi_1}(e) - \lambda)\ \forall e \in \Pi_1$.

STEP 3 *Build zero bipartite graphs for each $f_j(\cdot)$. Output these as the zero bipartite graphs needed to determine $DTL\ (f-\lambda)$.*

STOP

Thus, the complexity of **computing the DTL of** $(f-\lambda)(\cdot)$ is $O(\sum \mid T_i \mid^2)$ submodular function minimizations. Hence, (see Exercise 14.4) the **complexity of computing the DTL** of **all the** $(f-\lambda)(\cdot), \lambda$ a critical value, is $O(\mid S \mid^2)$ submodular function minimizations.

Remark: Instead of building the zero bipartite graph, the DTL$(f-\lambda)$ can also be stored in the form of $|S|$ Hasse Diagrams (see page 554).

Exercise 14.3
Show that $C_{v_i v_j} = \{v_k : T - v_k$ contains no zero set which contains $\{v_i, v_j\}\}$.

Exercise 14.4
If Π_1, Π_2 are minimal and maximal partitions of S minimizing $(\overline{f-\lambda})(\cdot)$ and $(\Pi_2)_{fus \cdot \Pi_1}$ has blocks $N_1, \cdots, N_k$, we saw that the zero bipartite graphs corresponding to the critical value λ can be constructed by performing $\sum O(\mid N_i \mid^2)$ submodular function minimizations.
Show that constructing the zero bipartite graphs for all the critical values requires $O(\mid S \mid^2)$ submodular function minimizations.

14.5 Complexity of construction of the PLP

Let $f(\cdot)$ be a submodular function on subsets of S. We will assume that we have an algorithm for minimizing, over supersets of a given subset, an arbitrary submodular function. Further we would assume that the algorithm can return the minimal set on which the minimum occurs.
Let this algorithm be $O(\alpha(n))$. Then Subroutine Find Strong Fusion Set can be seen to have a worst case complexity of $O(n\alpha(n))$. The algorithm $Min(\bar{f}, S)$ is better analyzed directly in terms of the number of submodular function minimizations. Suppose a subset T of size p has no fusion set, but $T \cup e$ has a strong fusion set N. Then the computational labour thus far is $\sum_{j=1}^{p+1} \alpha(j)$. After this step N would be treated as a single element and we would look for strong fusion subsets of the new set. We would now initialize T, in Subroutine Find Strong Fusion Set for the new problem, to $\{N\} \cup \{T - N\}$ instead of $\emptyset$. Thus, the complexity of $Min(\bar{f}, S)$ is also $O(n\alpha(n))$.
Next Algorithm P-sequence of partitions invokes Algorithm $Min(\bar{f}, S)$ at most n times. Therefore, it has complexity $O(n^2\alpha(n))$. (If it is possible to use the method of balanced bisection the complexity would be $O(n\alpha(n)(\log n)(\log k))$ where k is a set of numbers within which the critical values are known to lie). The procedure for constructing all the zero bipartite graphs is $O(n^2\alpha(n))$. Hence, the **overall complexity for constructing the PLP of** $f(\cdot)$ is $O(n^2\alpha(n))$.

Space requirement for the construction of PLP
The P-sequence of partitions of $f(\cdot)$ requires $O(\mid S \mid^2)$ space since atmost $\mid S \mid$

partitions of S have to be stored. If Π_1, Π_2 are minimum and maximum partitions minimizing $(\overline{f-\lambda})(\cdot)$ for a critical value λ and $(\Pi_2)_{fus \cdot \Pi_1} \equiv \{N_1, \cdots, N_k\}$ then DTL of $(f-\lambda)(\cdot)$ requires $\sum(O(\mid N_i \mid^2)$ zero sets of size atmost $\mid S \mid)$. Using the argument of the solution of Exercise 14.4 we conclude that the DTL of all $(f-\lambda)(\cdot)$ requires $O(\mid S \mid^2)$ zero sets of size atmost $\mid S \mid$ i.e., $O(\mid S \mid^3)$ space. Thus, **the overall space requirement for PLP of** $f(\cdot)$ is $O(\mid S \mid^3)$.

14.6 Construction of the PLP of the dual

In practical situations, it is often the case that a submodular function $f(\cdot)$ is easy to handle algorithmically, while the PLP of $f^c(\cdot)$ $(f^c(X) \equiv f(S-X))$ may be of importance. We briefly sketch the essential parts of the construction of the latter PLP using subroutines which are in terms of $f(\cdot)$.

Let us consider the problem of determining the fusion set of $f^c(\cdot)$. Suppose $T \subseteq S$ has no fusion set. Let $e \in T$. We wish to check if $T \cup e$ contains a fusion set. Let $h(\cdot) \equiv (f^c - \lambda) - w(\cdot)$ where $w(\cdot)$ is the weight function which agrees with $(f^c - \lambda)(\cdot)$ on singletons. (Clearly $h(\cdot)$ is a z.s.s. function). We need to minimize $h(X), X \subseteq T \cup e$ and check if this minimum value is negative. Now

$$\begin{aligned} min_{X \subseteq T \cup e} h(X) &= min_{X \subseteq T \cup e}(f(S-X) - \lambda - w(X)) \\ &= min_{X \subseteq T \cup e}(f(S-X) + w'(X) - \lambda) \\ &\quad where\, w'(X) = -w(X) \\ &= min_{Y \supseteq (S-(T\cup e))}(f(Y) + w'(S-Y) - \lambda) \end{aligned}$$

$$\begin{aligned} Let\; w"(e_i) &= \infty, e_i \in S-(T\cup e) \\ &= w'(e_i), e_i \in T \cup e. \end{aligned}$$

Then the above problem reduces to computing the convolution $(f * w")(\cdot)$. For finding a strong fusion set we need to find a maximal Y s.t.

$$(f * w")(S) = f(Y) + w"(S-Y).$$

Thus, Subroutine Find Strong Fusion set for $h(\cdot)$ and Algorithm $Min(f^c, S)$ require computation of $\mid S \mid$ such convolutions of $f(\cdot)$ with appropriate weight functions.

14.7 PLP Algorithms for $(w_R\Gamma)(\cdot)$ and $-(w_R E_L)(\cdot)$

We now specialize the PLP algorithms thus far described to two important situations viz. weighted adjacency function and the weighted exclusivity function associated with a bipartite graph. The reader would notice that the specializations go through very routinely. However we spell them out in detail because

- the situations are very commonly encountered and these algorithms compete very well with tailor made algorithms for problems relevant to instances (e.g. principal partition for the rank function of a graph [Narayanan90], [Patkar+Narayanan92b], the rigidity matroid of a graph (ibid.)) of these functions
- it becomes clearer that the PLP algorithms are particularly natural for bipartite graphs.

Let $B \equiv (V_L, V_R, E)$, let $w_R(\cdot)$ be a positive weight function on V_R, let $\Gamma(X)$, $X \subseteq V_L$, denote the set of vertices adjacent to those in X and let $E_L(X), X \subseteq V_L$, denote the set of vertices adjacent only to those in X. We denote $w_R(\Gamma(\cdot))$, $w_R(E_L(\cdot))$, respectively by $(w_R\Gamma)(\cdot)$ and $w_R E_L(\cdot)$. The reader may find it convenient to review Subsections 3.6.10 and 11.6.3.

14.7.1 PLP of $(w_R\Gamma)(\cdot)$

Detection of Strong Fusion set for $(w_R\Gamma)(\cdot)$ through flow maximization

Let us consider the detection of a strong fusion set for this case. First convert $((w_R\Gamma) - \lambda)(\cdot)$ to a z.s.s. function $h(\cdot)$. Let $g(\cdot)$ be a weight function with

$$g(v) \equiv ((w_R\Gamma) - \lambda)(v)\ \forall v \in V_L$$

Let $h(\cdot) \equiv ((w_R\Gamma) - \lambda)(\cdot) - g(\cdot)$. Suppose we know that the left vertex subset T contains no fusion set (T could be $\emptyset$ for instance) of the z.s.s. function $h(\cdot)$. Let $e \notin T$. We minimize $(w_R\Gamma)(X) - g(X)$, $X \subseteq T \cup e$, $e \in X$, and test whether the minimum value is less than λ. The minimal minimizing set is a strong fusion set if the minimum value is less than λ. Otherwise $T \cup e$ has no fusion set.

The problem of minimizing $(w_R\Gamma)(X) - g(X)$ is equivalent to minimizing $(w_R\Gamma)(X) + g(V_L - X)$ which has already been shown to be a maximum flow problem in Subsection 11.6.3 and also, with a slightly different notation, in Subsection 3.6.10 (see Figure 11.3).

The minimum cut (which has the form $(s \uplus X \uplus \Gamma(X), t \uplus (T \cup e - X) \uplus (V_R - \Gamma(X))$ in this flow graph would have capacity = minimum value of $(w_R\Gamma)(X) + g((T \cup e) - X)$, $X \subseteq T \cup e$ ($T \cup e$ is the left vertex set of the current subgraph B_{curr} of B and the flow graph is (using the notation of Subsection 3.6.10) $F(B_{curr}, \mathbf{w}_L / T \cup e, \mathbf{w}_R / \Gamma(T \cup e))$, where $w_L(\cdot) \equiv g(\cdot)$).

We modify the network flow formulation (as in Exercise 11.33) since we need to find X' that minimizes $(w_R\Gamma)(X) + w_L(T \cup e - X)$ under the condition that $e \in X$. We ensure this by making the capacity of the edge (s, e) large so that the cut $(s, (T \cup e) \uplus V_R \uplus t)$ has a capacity larger than the capacity of the cut $(s \uplus e \uplus \Gamma(e), T \uplus (V_R - \Gamma(e)) \uplus t)$. It turns out that we need only increase the capacity of e from $(w_R\Gamma)(e) - \lambda$ to $(w_R\Gamma)(e) + \delta, \delta > 0$ (see Exercise 14.6) . Figure 14.1 indicates the flow graph with the modification. The original capacity of the edge (s, e) is $w_L(e) \equiv (w_R\Gamma)(e) - \lambda$.

Figure 14.1: Flow graph modification for fusion set detection

There is a minor point that has to be clarified. Since $w_L(v) \equiv (w_R\Gamma)(v) - \lambda$, this value can turn out to be negative. (Strictly speaking this needs the introduction of a new source vertex $\hat{s}$ and an edge $(\hat{s}, s)$ of capacity ∞). It is easily verified that such an element can always be put in the t-part of the cut (see Exercise 14.5). Hence, if there is a fusion set containing v there must also be one not containing it. Therefore, in general, if K has no fusion set, $e \notin K$ and $g(e)$ $(= w_L(e))$ negative, then neither will $K \cup e$ contain a fusion set.
Thus, informally the **Subroutine Find Strong Fusion Set** reduces to the following:
Let T be a subset of V_L known not to contain a fusion set of $w_R\Gamma(\cdot) - \lambda$ and let $e \notin T$. If $g(e)$ is negative we declare that $T \cup e$ contains no fusion set. Let B_{curr} be the subgraph of B on $(T \cup e) \uplus \Gamma(T \cup e)$, let $\mathbf{g}_{curr}$ on $T \cup e$ agree with $\mathbf{g}$ on T and let $g_{curr}(e) = g(e) + \lambda + \delta, \delta > 0$. Let $\mathbf{w}_{curr}$ denote $\mathbf{w}_R/\Gamma(T \cup e)$. Build the flow graph $F(B_{curr}, \mathbf{g}_{curr}, \mathbf{w}_{curr})$. (For the edge from source to left vertex v, the edge capacity is $g_{curr}(v)$, all bipartite graph edges directed from left to right with edge capacity ∞, right vertex v_R to t with edge capacity $w_R(v_R)$.)
Maximize flow and find a min cut of the form

$$(s \uplus X \uplus \Gamma(X), t \uplus ((T \cup e) - X) \uplus (V_R - \Gamma(X)))$$

such that X is minimal, i.e., find the nearest source side min cut (see Theorem 3.6.2).
(Now since we raised the capacity of (s, e) to $g(e) + (\lambda + \delta)$ it would turn out that $X \neq \emptyset$. Further $e \in X$ or there is no fusion set.)

If $(w_R\Gamma)(X) - g(X) < \lambda$, then we declare that X is a strong fusion set. Otherwise $T \cup e$ contains no fusion set.

We thus see that **detection of a strong fusion set requires**
$O(\mid V_L \mid)$ flow maximizations in a flow graph that has
$O(\mid V_L \mid + \mid V_R \mid)$ vertices
$O(\mid E \mid + \mid V_L \mid + \mid V_R \mid)$ edges and
longest path from source to sink of length $O(min(\mid V_L \mid, \mid V_R \mid))$.
Observe that if the problem were to determine a quasi-fusion set the algorithm is essentially the same as above. The difference is

i. a min cut of the above form has to be found s.t. X is **maximal** (i.e., the nearest sink side cut has to be found).

ii. if $((w_R\Gamma)(X) - g(X)) \leq \lambda$ and X is not a singleton, we declare that X is a **quasi-fusion set**.

Algorithm $Min(\bar{f}, S)$ for $(w_R\Gamma)(\cdot)$

Next we consider the specialization of Algorithm $Min(\bar{f}, S)$ to this case. We will begin with a crude version where the Subroutine Find Strong Fusion Set is initialized to null set after detection of a strong fusion set.
Let B_Π denote the bipartite graph obtained from B by fusing the blocks of the partition Π of V_L into single vertices. Let $\Gamma_\Pi(\cdot)$ be the left adjacency function of this graph. It can be verified that

$$(w_R\Gamma_\Pi)(\cdot) = (w_R\Gamma)_{fus \cdot \Pi}(\cdot)$$

Let N be a strong fusion set that has just been detected. We now fuse N to a single element. We use Subroutine Find Strong Fusion Set, on the z.s.s. function obtained from the weighted left adjacency function of the bipartite graph B_{Π_N} by subtracting λ and a suitable weight function. This procedure is repeated until in the last bipartite graph the function (weighted left adjacency function $-\lambda$) has no fusion set. The left vertex set of this bipartite graph can be associated with a partition of V_L (each vertex of the last bipartite graph is obtained by fusing a block of the partition of V_L). This partition is the minimum partition that minimizes $\overline{((w_R\Gamma) - \lambda)}(\cdot)$.

According to (the efficient version of) Algorithm $Min(\bar{f}, S)$ we do not start from scratch when the bipartite graph B_{Π_N} is formed. Suppose, before the fusion set is detected we have the flow graph corresponding to the subgraph of B on $(T \cup e) \uplus \Gamma(T \cup e)$. After N is detected and fused into the vertex v_N we form the new flow graph in which all capacities of edges which are not leading into N remain as before. The capacity of the edge (s, v_N) is taken to be $(w_R\Gamma)(N) - \lambda$. We know that for the new weighted left adjacency function, $(T - N) \cup \{N\}$ does not contain a fusion set. The new flow graph corresponds to the subgraph of B_{Π_N} on

$$((T - N) \cup \{N\}) \uplus \Gamma((T - N) \cup \{N\}).$$

We now run Subroutine Find Strong Fusion set on the function $w_R\Gamma_\Pi(\cdot) - \lambda$ associated with B_{Π_N}, initializing it at $(T - N) \cup \{N\}$. The actual complexity would depend upon the algorithm used for flow maximization (see Subsection 3.6.10).

It is therefore clear that, **in order to complete Algorithm** $Min(\bar{f}, V_L)$,
we have to perform $O(\mid V_L \mid)$ flow maximizations in a flow graph that has
$O(\mid V_L \mid + \mid V_R \mid)$ vertices and
$O(\mid V_L \mid + \mid V_R \mid + \mid E \mid)$ edges and
length $O(min(\mid V_L \mid, \mid V_R \mid))$ of longest path from source to sink,

$Subdivide_f(\Pi_1, \Pi_2)$ **for** $(w_R\Gamma)(\cdot)$
Let $B \equiv (V_L, V_R, E)$ be the bipartite graph under discussion. Then Π_1, Π_2 are partitions of V_L. In this case $f(\cdot) \equiv (w_R\Gamma)(\cdot)$. In STEP 1 of $Subdivide_f(\Pi_1, \Pi_2)$, $f'(\cdot) \equiv f_{fus\cdot\Pi_1}(\cdot)$. Let B_{Π_1} denote the bipartite graph obtained from B by fusing the blocks of Π_1 into single vertices. Let $\Gamma_{\Pi_1}(\cdot)$ denote the left adjacency function of B_{Π_1}. It is clear that $f'(\cdot) = (w_R\Gamma_{\Pi_1})(\cdot)$. As in STEP 2, let $(\Pi_2)_{fus\cdot\Pi_1}$ have $N_1, \cdots, N_k$ as blocks and let $f'_j(\cdot) \equiv f'/\mathbf{N}_j(\cdot)$, $j = 1, \cdots, k$. Let B_{N_j} denote the subgraph of B_{Π_1} on $(N_j \cup \Gamma_{\Pi_1}(N_j))$. Let $\Gamma_{N_j}(\cdot)$ denote the left adjacency function of B_{N_j}. Then $f'_j(\cdot) = (w_R\Gamma_{N_j})(\cdot)$. Now we can do Algorithm $Min(\bar{f}'_j, N_j)$ using a suitable flow graph on B_{N_j} as described above.

Thus, it is clear that **Subroutine** $Subdivide_f(\Pi_1, \Pi_2)$ **requires**
$O(\sum_{j=1}^{k} \mid N_j \mid)$ flow maximizations on a flow graph of size
$O(\mid V_L \mid + \mid V_R \mid))$ vertices,
$O(\mid V_L \mid + \mid V_R \mid + \mid E \mid)$ edges and
longest path from source to sink of length $O(min(\mid V_L \mid, \mid V_R \mid))$.

Thus, **Algorithm P-sequence of partitions requires** atmost $O(\mid V_L \mid^2)$ flow maximizations on a flow graph of the above size.

Exercise 14.5
In Figure 14.1, suppose $w_L(v) \equiv (w_R\Gamma)(v) - \lambda$ is negative. Show that we can always find a min cut such that v belongs to the t side of the cut.

Exercise 14.6
In page 604 it is remarked that increasing the capacity of (s, e) from $((w_R\Gamma)(e) - \lambda)$ to $((w_R\Gamma)(e) + \delta)$, $\delta > 0$, is adequate to ensure that the min cut will contain at least one element of V_L on the source side. Prove this statement.

DTL of $((w_R\Gamma) - \lambda)(\cdot)$

Let $B \equiv (V_L, V_R, E)$ be the bipartite graph under discussion. We follow the notation of Algorithm 14.3. Let Π_1, Π_2 be the minimal and maximal partitions of V_L minimizing $\overline{((w_R\Gamma) - \lambda)}(\cdot)$. We denote $(w_R\Gamma)(\cdot)$ by $f(\cdot)$, $(f_{fus\cdot\Pi_1} - \lambda - w)(\cdot)$ by $f_0(\cdot)$ (where $w(\cdot)$ is the weight function defined by $w(e) \equiv f_{fus\cdot\Pi_1}(e) - \lambda\, \forall e \in \Pi_1$), the blocks of $(\Pi_2)_{fus\cdot\Pi_1}$ by $N_1, \cdots, N_k$ and $f_0/\mathbf{N}_j(\cdot)$ by $f_j(\cdot)$.

Let us consider the problem of determining the zero bipartite graph of the type (000) function $f_j(\cdot)$, **equivalently**, that of determining all $C_{v_iv_k}$ (the minimal zero

set of $f_j(\cdot)$ containing $\{v_i, v_k\}$). This requires the building of B_{Π_1} from the bipartite graph B by fusing each block Π_1 into a single vertex. (See the discussion on page 607.) Let B_{N_j} be the subgraph of B_{Π_1} on $N_j \cup \Gamma(N_j)$. The network associated with B_{N_j} is built as described earlier (edge (s, v) has capacity $((w_R\Gamma)(v) - \lambda), v \in N_j$, edges of B_{N_j} are from left to right and have capacity ∞, edge (v_R, t) has capacity $w_R(v_R), v_R \in \Gamma(N_j)$). We maximize flow and find a min cut corresponding to X such that X is minimal under the condition that $\{v_i, v_k\} \subseteq X$. As in the case of fusion set detection this can be done by increasing the capacity of $(s, v_i), (s, v_k)$ to $(w_R\Gamma_{N_j})(v_i) + \delta$ and $(w_R\Gamma_{N_j})(v_k) + \delta, \delta > 0$. One flow maximization thus yields one set $C_{v_i v_j}$. Thus to determine the zero bipartite graph for B_{N_j} corresponding to λ requires $| N_j |^2$ flow maximizations. By arguing as in the case of a general submodular function we conclude that **computing the DTL of all the** $\overline{(f - \lambda)}(\cdot)$ **for all critical values requires**
$O(| V_L |^2)$ flow maximizations in a flow graph that has
$O(| V_L | + | V_R |)$ vertices,
$O(| V_L | + | V_R | + | E |)$ edges and
longest path from source to sink of length $O(min(| V_L |, | V_R |))$.
Thus, **the construction of the PLP of** $(w_R\Gamma)(\cdot)$ **requires** $O(| V_L |^2)$ flow maximizations in the above flow graph.
We remind the reader that Sleator's algorithm [Sleator80] for this problem would have complexity (as given in Subsection 3.6.10)
$O(| V_L |^2 \ (min(| V_L |, | V_R |))(| E | \log | E |))$.

14.7.2 PLP of $(-w_R E_L)(\cdot)$

We use a flow formulation for this problem also. The function $E_L(\cdot)$ (of the bipartite graph $B \equiv (V_L, V_R, E)$) is inconvenient to work with directly for such a formulation. So we use the fact that

$$E_L(X) \equiv \Gamma(V_L) - \Gamma(V_L - X).$$

Hence, $(w_R E_L)(X) = (w_R\Gamma)(V_L) - (w_R\Gamma)(V_L - X)$.
Let $f(X) \equiv (w_R\Gamma)(V_L - X),\ X \subseteq V_L$. Then,
minimizing $\overline{(-w_R E_L - \lambda + (w_R\Gamma)(V_L))}(\cdot)$ is equivalent to minimizing $\overline{(f - \lambda)}(\cdot)$.
We concentrate on the construction of PLP of the function $f(X) \equiv (w_R\Gamma)(V_L - X)$.

Find Strong Fusion Set for $-w_R E_L(\cdot) - \lambda + (w_R\Gamma)(V_L)$

We first convert $(f - \lambda)(\cdot)$ to a z.s.s. function. Let $q'(\cdot)$ be a weight function with $q'(v) \equiv (f - \lambda)(v) \, \forall v \in V_L$. Denote $-q'(v)$ by $q(v)$. Let $p(\cdot) \equiv (f - \lambda)(\cdot) + q(\cdot)$. Suppose we know that the left vertex subset T contains no fusion set of the z.s.s. function $p(\cdot)$. Let $e \notin T$. We minimize $f(X) + q(X),\ X \subseteq T \cup e,\ e \in X$, and test whether the minimum value is less than λ. The minimal minimizing set is a strong fusion set if the minimum value is less than λ. Otherwise $T \cup e$ has no fusion set.

Figure 14.2: Flow graph for minimization of $f(X) + q(X), X \subseteq Z$

Let us consider the problem of minimizing

$$f(X) + q(X) \equiv (w_R\Gamma)(V_L - X) - \sum_{v \in X} (w_R\Gamma)(V_L - v) + \lambda \mid X \mid, X \subseteq Z \subseteq V_L.$$

This can be posed as the flow problem of Figure 14.2 as we showed in Subsection 11.6.3 (and by the result in Exercise 11.33). The flow graph can be seen to be $F(B, \mathbf{w}_Z, \mathbf{w}_R)$ (Z is the set currently being tested for containing a strong fusion set), where

$$\begin{aligned} w_Z(v) &\equiv \infty, \quad v \in V_L - Z \\ w_Z(v) &\equiv \lambda - (w_R\Gamma)(V_L - v), v \in Z \end{aligned}$$

Maximizing flow yields a min cut of the form

$$(s \uplus Y \uplus \Gamma(Y), t \uplus (V_L - Y) \uplus (V_R - \Gamma(Y)),$$

where $Y \equiv (V_L - X)$, $V_L \supseteq Y \supseteq (V_L - Z)$. Next we need $X \neq \emptyset$ i.e., $Y \subset V_L$. When Z grows from T to $T \cup e$, the capacity of (s, e) falls from ∞ to $(\lambda - (w_R\Gamma)(V_L - e))$. Instead if we make the capacity of (s, e) equal to zero then it is easily verified that the capacity of the cut $(s \uplus V_L \uplus V_R, t)$ is not less than the capacity of the cut

$$(s \uplus (V_L - e) \uplus \Gamma(V_L - e), t \uplus e \uplus (V_R - \Gamma(V_L - e)).$$

Now if X is a fusion set, then the capacity of the cut, corresponding to $V_L - X$, is less than that corresponding to $V_L - e$ in the original flow graph. This would hold true in the modified flow graph also since $e \in X$, if X is a fusion set. Hence in the modified flow graph, if X is a fusion set, the cut corresponding to $Y \equiv V_L - X$ has a lower capacity than the cut $(s \uplus V_L \uplus V_R, t)$. We can find the maximal Y by finding the nearest sink side cut (Theorem 3.6.2). We now check if $f(X) + q(X) < \lambda$ where $X \equiv V_L - Y$. If so, X is a strong fusion set. Otherwise $T \cup e$ contains no fusion set.

One final remark needs to be made for the case where $q(a)$ is negative for some $a \in V_L$. If such an element is in Y then $(w_R\Gamma)(Y - a) + \sum_{v \in (V_L - Y) \cup a} q(v)$ cannot have higher value than $(w_R\Gamma)(Y) + \sum_{v \in (V_L - Y)} q(v)$. So we lose no generality in assuming that $a \in Y$ in the first place when $q(a)$ is negative. In particular it follows that if $q(e)$ is negative $T \cup e$ does not contain a fusion set.

Thus the **Subroutine Find Strong Fusion Set** reduces to the following.
Suppose we know that T contains no fusion set. Let $e \notin T$. If $(\lambda - (w_R\Gamma)(V_L - e))$ is negative $T \cup e$ contains no fusion set. Otherwise build the flow graph $F(B, \mathbf{w'}_T, \mathbf{w}_R)$ corresponding to $T \subseteq V_L$: where

$$\begin{aligned} w'_T(v) &\equiv \infty, \quad v \in V_L - T - e \\ w'_T(v) &\equiv \lambda - (w_R\Gamma)(V_L - v), v \in T \\ w'_T(e) &\equiv 0. \end{aligned}$$

Maximize flow and find the nearest sink side min cut. By theorem 3.6.2 this has the form

$$(s \uplus Y \uplus \Gamma(Y), t \uplus (V_L - Y) \uplus (V_R - \Gamma(Y)).$$

Take $X \equiv V_L - Y$. Check if $f(X) + q(X) < \lambda$.
If Yes declare X to be a strong fusion set.
Otherwise $T \cup e$ contains no fusion set.

We thus see that **detection of a strong fusion set in this case requires**
$O(\mid V_L \mid)$ flow maximizations in a flow graph that has
$O(\mid V_L \mid + \mid V_R \mid)$ vertices,
$O(\mid E \mid + \mid V_L \mid + \mid V_R \mid)$ edges and
longest path from source to sink of $O(min(\mid V_L \mid, \mid V_R \mid))$.

If the problem were to determine a quasi-fusion set the algorithm is essentially the same as above. The difference is

i. Y must be made **minimal**.

ii. if $f(X) + q(X) \leq \lambda$ and X is a nonsingleton set we declare that X is a **quasi-fusion set**.

Exercise 14.7
In order to ensure that the minimization of $(w_R\Gamma)(Y) + q(V_L - Y)$ takes place over $Y \supseteq V_L - Z$, we put capacity of $(s, v) = \infty$ whenever $v \in V_L - Z$. Show that it is adequate to make this capacity λ instead of ∞ provided $\lambda \geq (w_R\Gamma)(V_L)$.

Exercise 14.8
In the flow graph of Figure 14.2 show that it is unnecessary to consider the case where $(\lambda - (w_R\Gamma)(V_L - v))$ is negative.

Algorithm $Min(\bar{f}, S)$ **for** $(-w_RE_L)(\cdot)$

Let B_Π denote the bipartite graph obtained from B by fusing the blocks of the partition Π of V_L into single vertices. Let $w_RE_\Pi(\cdot)$ be the weighted left exclusivity function of this graph. It can be verified that

$$(w_RE_\Pi)(\cdot) = (w_RE_L)_{fus\cdot\Pi}(\cdot)$$

Let N be a strong fusion set that has just been detected. We fuse N to a single element and use Subroutine Find Strong Fusion Set on the function (-weighted left exclusivity function of the bipartite graph $B_{\Pi_N} - \lambda$). This procedure is repeated until in the last bipartite graph this function has no fusion set. The left vertex set of this bipartite graph can be associated with a partition of V_L (each vertex of the last bipartite graph is obtained by fusing a block of the partition of V_L). This partition is the minimum partition that minimizes $\overline{(-(w_RE_L) - \lambda)}(\cdot)$.

A few remarks on the initialization of Find Strong Fusion Set. Suppose T has no fusion set. But $T \cup e$ has the strong fusion set N. This is fused to the vertex v_N and we work thenceforward with the bipartite graph B_{Π_N}. The flow graph (for PLP of $-(w_RE_L)(\cdot)$) corresponding to B_{Π_N} would have capacities of all edges which are not leading into N unchanged. The capacity of the edge (s, v_N) is taken to be $\lambda - (w_R\Gamma)(V_L - N)$. We know that for the new function derived from the current weighted left exclusivity function, $(T - N) \cup \{N\}$ does not contain a fusion set. Subroutine Find Strong Fusion Set is now run initializing it at $(T - N) \cup \{N\}$. It is therefore clear that, **in order to complete Algorithm** $Min(\bar{f}, V_L)$,
we have to perform $O(|V_L|)$ flow maximizations in a flow graph that has
$O(\mid V_L \mid + \mid V_R \mid)$ vertices,
$O(\mid V_L \mid + \mid V_R \mid + \mid E \mid)$ edges,
longest path from source to sink of length $O(min(\mid V_L \mid, \mid V_R \mid))$.

$Subdivide_f(\Pi_1, \Pi_2)$ **for** $-(w_RE_L)(\cdot)$
Let $B \equiv (V_L, V_R, E)$ be the bipartite graph under discussion. Then Π_1, Π_2 are partitions of V_L. In this case $f(\cdot) = -(w_RE_L)(\cdot)$. In STEP 1, $f'(\cdot) \equiv f_{fus\cdot\Pi_1}(\cdot)$. Let B_{Π_1} denote the bipartite graph obtained from B by fusing the blocks of Π_1 into single vertices. Let $E_{\Pi_1}(\cdot)$ denote the left exclusivity function of B_{Π_1}. It is clear that $f'(\cdot) = -(w_RE_{\Pi_1})(\cdot)$. As in STEP 2, let $(\Pi_2)_{fus\cdot\Pi_1}$ have $N_1, \cdots, N_k$ as blocks and let $f'_j(\cdot) \equiv f'/\mathbf{N}_j(\cdot)$, $j = 1, \cdots, k$. Let B_{N_j} denote the subgraph of B_{Π_1} on $(N_j \cup \Gamma_{\Pi_1}(N_j))$. Let $E_{N_j}(\cdot)$ denote the left exclusivity function of B_{N_j}. Then $f'_j = -(w_RE_{N_j})(\cdot)$. Now we can do Algorithm $Min(\bar{f}'_j, N_j)$ using a suitable flow graph on B_{N_j} as described above.
Thus, it is clear that **Subroutine** $Subdivide_f(\Pi_1, \Pi_2)$ **requires**
$O(\sum_{j-1}^{k} \mid N_j \mid)$ flow maximizations on a flow graph of size

$O(\mid V_L \mid + \mid V_R \mid)$ vertices,
$O(\mid V_L \mid + \mid V_R \mid + \mid E \mid)$ edges,
and longest path from source to sink of length $O(min(\mid V_L \mid, \mid V_R \mid))$.
Thus, **Algorithm P-sequence of partitions requires** atmost $\mid V_L \mid^2$ flow maximizations on a flow graph of the above size.

DTL of $(-(w_R E_L)(\cdot) - \hat{\lambda})$

Let $B \equiv (V_L, V_R, E)$ be the bipartite graph under discussion. Let Π_1, Π_2 be the minimal and maximal partitions of V_L minimizing $\overline{h - \hat{\lambda}}(\cdot)$ where $h(\cdot) = -(w_R E_L)(\cdot)$. Let $(\Pi_2)_{fus \cdot \Pi_1}$ have blocks $N_1, \cdots, N_k$. Let $B_{\Pi_1}, B_{N_j}, E_{\Pi_1}, E_{N_j}$, etc. have the same meanings as above. It is clear that $h_{fus \cdot \Pi_1}(\cdot) = -(w_R E_{\Pi_1})(\cdot)$ and $h_{fus \cdot \Pi_1} / \mathbf{N}_j(\cdot) = -(w_R E_{N_j})(\cdot)$.
Let $f'_j(X) \equiv (w_R \Gamma_{N_j})(N_j - X)$, $X \subseteq N_j$ where $\Gamma_{N_j}(\cdot)$ is the left adjacency function of B_{N_j}. The minimizing partitions of N_j for $\overline{(-(w_R E_{N_j}) - \hat{\lambda})}(\cdot)$ are identical to those for $\overline{(f'_j - (\hat{\lambda} + (w_R \Gamma_{N_j})(N_j))}(\cdot)$. Let $\lambda \equiv \hat{\lambda} + (w_R \Gamma_{N_j})(N_j)$ and let $f_j = (f'_j - \lambda)(\cdot) - w(\cdot)$, where $w(\cdot)$ is a weight function on N_j with $w(e) = (f'_j - \lambda)(e) \quad \forall e \in N_j$.
The function $f_j(\cdot)$ is a type (000) function. Our problem is to find minimal zero sets containing given $\{v_i, v_k\} \subseteq N_j$. This problem is the same as the flow problem considered on page 609 with the added condition that $\{v_i, v_k\} \subseteq X$. (Note that in this case $Z = N_j =$ left vertex set of B_{N_j}. The flow graph is as in Figure 14.2.) This can be handled by putting the capacities of edges going into v_i and v_k equal to zero, while maximizing flow. We look for a nearest sink side min cut, i.e., a min cut of the form $(s \uplus Y \uplus \Gamma(Y), t \uplus (N_j - Y) \uplus (\Gamma(N_j) - \Gamma(Y)))$ such that Y is maximal under the condition that $\{v_i, v_k\} \cap Y = \emptyset$ (see Remark below). Then $C_{v_i v_k} = N_j - Y$. Thus, to determine all the $C_{v_i v_k}$ corresponding to λ and N_j requires $O(\mid N_j \mid^2)$ flow maximizations on the flow graph associated with B_{N_j} as given in page 609. To determine all the zero bipartite graphs corresponding to λ requires $O(\sum \mid N_j \mid^2)$ flow maximizations.
To determine all such graphs for all the λ requires
$O(\mid V_L \mid^2)$ flow maximizations on a flow graph that has
$O(\mid V_L \mid + \mid V_R \mid)$ vertices,
$O(\mid V_L \mid + \mid V_R \mid + \mid E \mid)$ edges
and length of longest path from source to sink equal to $O(min(\mid V_L \mid, \mid V_R \mid))$.

Remark: Since $f_j(\cdot)$ is a type (000) function, we can show the following:

- When $\hat{\lambda} = 0$, $f_j(\cdot)$ is a modular function and therefore all subsets of N_j are zero sets.
- When $\hat{\lambda} > 0$, if the capacities of $(s, v_i), (s, v_k)$ are made equal to zero, the capacity of the cut separating t from the rest is greater than the capacity either
  - of the cut separating s from the rest, or

– of the cut $(s \uplus Z \uplus \Gamma(Z), t \uplus (N_j - Z) \uplus (\Gamma(N_j) - \Gamma(Z)))$, where $Z = V_L - \{v_i, v_k\}$.

Thus in any case the nearest sink side min-cut would yield a subset Y s.t. $(v_i, v_k) \cap Y = \emptyset$, and, as mentioned above, we can take $C_{v_i v_k} = N_j - Y$.

As in the case of $(w_R\Gamma)(\cdot)$ the **construction of the PLP in this case also has complexity** (using Sleator's flow algorithm [Sleator80])

$$O(\mid V_L \mid^2 (min(\mid V_L \mid, \mid V_R \mid))(\mid E \mid \log \mid E \mid)).$$

14.8 Structural Changes in Minimizing Partitions

In this section we study the structural changes in maximal and minimal minimizing partitions of $\bar{f}(\cdot)$ ($f(\cdot)$ submodular) as the set grows and how to exploit these changes for improving the efficiency of the PLP algorithms. The key result which we need is Theorem 13.2.3 which assures us that the blocks of both the minimal minimizing partition and the maximal minimizing partition grow as the set grows. We have already seen that the blocks of minimal minimizing partitions of $\overline{(f-\sigma)}(\cdot)$ are elementary separators of $(f-\sigma)_t(\cdot)$ (Theorem 13.5.1). The maximal minimizing partition of $\overline{(f-\sigma)}(\cdot)$ would of course have this property with respect to $(f - \sigma_{next})_t(\cdot)$ where σ_{next} is the next lower critical value after σ. But it also has an interesting and useful property when the set grows without changing its $(f-\sigma)_t(\cdot)$ value, which is the main result of this section. This result, given below, is a routine generalization of ideas present in [Patkar+Narayanan91],[Patkar92]. We denote the maximal (minimal) partition that minimizes $\bar{f}(\cdot)$ over partitions of $X \subseteq S$, by $\Pi_{max}(X)(\Pi_{min}(X))$.

Theorem 14.8.1 *Let $f(\cdot)$ be an increasing submodular function over subsets of S. Let $X \subseteq S$ and let $e \in S - X$. Then $f_t(X) = f_t(X \cup e)$ iff there is a block N of $\Pi_{max}(X)$ s.t. $f(N) = f(N \cup e)$ and $N \cup e$ is a block of $\Pi_{max}(X \cup e)$.*

Proof : Since $f(\cdot)$ is an increasing submodular function so must $f_t(\cdot)$ be (Exercise 13.2). Further by Theorem 13.2.3 the blocks of $\Pi_{max}(X)$ are each contained in some block of $\Pi_{max}(X \cup e)$.
If Clearly the blocks other than $N, N \cup e$ are identical in $\Pi_{max}(X)$ and $\Pi_{max}(X \cup e)$. Further $f(N) = f(N \cup e)$. Hence, $\bar{f}(\Pi_{max}(X)) = \bar{f}(\Pi_{max}(X \cup e))$ and $f_t(X) = f_t(X \cup e)$.

Only If Let $f_t(X) = f_t(X \cup e)$. Let M be the block of $\Pi_{max}(X \cup e)$ that has e as a member. Since each block of $\Pi_{max}(X)$ is contained in a block of $\Pi_{max}(X \cup e)$ it follows that

$$M = N_1 \cup \cdots \cup N_k \cup e,$$

where $N_1, \cdots, N_k$ are blocks of $\Pi_{max}(X)$. The remaining blocks of $\Pi_{max}(X)$, if any, would be blocks also in $\Pi(X \cup e)$. We have $\bar{f}(\Pi_{max}(X \cup e)) = \bar{f}(\Pi_{max}(X))$.

Hence,

$$f(M) = f(N_1) + \cdots + f(N_k).$$

But

$$f(M) \geq f(M - e) = f(\bigcup_{i=1}^{k} N_i).$$

Hence, the partition Π of X which has $\bigcup_{i=1}^{k} N_i$ as a block and the others as in $\Pi_{max}(X)$ also satisfies $f_t(X) = \bar{f}(\Pi)$. If $k > 1, \Pi \not\leq \Pi_{max}(X)$, which is a contradiction. We conclude that $k = 1$ and, therefore, $M = N \cup e$ where N is a block of $\Pi_{max}(X)$ and further $f(M) = f(N)$ as required.

□

Theorem 14.8.1 is algorithmically useful in the case of an integral submodular function $f(\cdot)$ on subsets of S whose lower truncation $f_t(\cdot)$ is a polymatroid rank function with $f_t(S) <\mid S \mid$. (The reader may like to review Section 14.2 before proceeding further). For instance to build the maximal minimizing partition of $\bar{f}(\cdot)$ over subsets of S:
We start with $X = \{e_1\} \subseteq S$ s.t. $f(e_1) \neq 0$. Suppose we have built $\Pi_{max}(X)$ and $e' \notin X$ we check if for any $N \in \Pi_{max}(X)$, $f(N \cup e') = f(N)$. If so, we set e' aside. If however $f(N \cup e) \neq f(N)$ for each N we grow X to $X \cup e$. Let $\Pi \equiv \Pi_{max}(X) \cup \{\{e\}\}$. Then Π is a partition of $X \cup e$. Let $f_1(\cdot) = f/\mathbf{X} \cup \mathbf{e}(\cdot)$. Let $f_2(\cdot) \equiv (f_1)_{fus \cdot \Pi}(\cdot)$. Let $f_3(\cdot) \equiv f_2(\cdot) - w(\cdot)$ where $w(\cdot)$ is a weight function on Π so that $f_3(\cdot)$ becomes z.s.s. Observe that $\Pi - \{\{e\}\}$ contains no fusion set of $f_3(\cdot)$. We now find the maximal set T_e minimizing $f_3(\cdot)$ over subsets of Π containing e. Let $M = \bigcup_{N_i \in T_e} N_i$. Then $\Pi_{max}(X \cup e)$ has M as a block and the remaining blocks from $\Pi_{max}(X)$. (Here we have used Theorem 13.2.3).

Observe that to build $\Pi_{max}(X \cup e)$ from $\Pi_{max}(X)$ takes only one submodular function minimization (with the additional condition that we find the maximal such set). We repeat this process until we reach a set Y s.t. for each $e' \notin Y$, there exists a block N of $\Pi_{max}(Y)$ s.t. $f(N \cup e') = f(N)$. By Theorem 14.8.1 and by the submodularity of $f(\cdot)$, this happens when $f_t(Y) = f_t(S)$. It follows that to reach this stage we need perform no more than $f_t(S)$ minimizations. In addition of course we have to compute $f(\cdot)$ on $O(\mid S \mid f_t(S))$ sets (no Π_{max} that we encounter can have more than $f_t(S)$ blocks.) Direct application of the algorithms of Section 14.2 would involve $\mid S \mid$ submodular function minimizations.

Example 14.8.1 *[Patkar92],[Patkar+Narayanan92b] Consider the function $\mid V \mid (\cdot)$ acting on edge subsets of graph $\mathcal{G}$. The function $(k \mid V \mid -(2k-1))_t$, where k is a positive integer, is easily seen to be a matroid rank function if $\mathcal{G}$ has no selfloops. So if $f(\cdot) \equiv (k \mid V \mid -(2k-1))_t(\cdot)$, then $\Pi_{max}(S)$ of $\bar{f}(\cdot)$ can be computed using $(k \mid V \mid -(2k-1))_t(S)$ submodular function minimizations (in this case these are max flow computations). We remark that when $k = 2$ we get the function $(2 \mid V \mid -3)_t(\cdot)$ which is the 'rigidity matroid' rank function (currently an active area of research [Asimow+Roth78],[Asimow+Roth79]).*

The algorithm for computing $\Pi_{max}(S)$ also yields, if $f_t(\cdot)$ is a matroid rank function, a basis for the matroid. For, our mode of construction of set Y ensures that there is a sequence of elements $e_1, e_2, \cdots, e_k$ s.t. $Y = \{e_1, \cdots, e_k\}$ and $f_t(e_1) < f_t(e_1, e_2) < \cdots < f_t(Y) = f_t(S)$. Algorithms based on these ideas are competitive with the best algorithms for rigidity matroid computations [Patkar+Narayanan92b], [Patkar+Narayanan92c]. They further have the advantage of being very general and being easy to implement.

Minimizing partitions of $\overline{f}(\cdot)$ where $f_t(\cdot)$ is a matroid rank function

The special case where $f_t(\cdot)$ is a matroid rank function deserves study. The following theorem (a routine generalization of the ideas found in [Patkar92], [Patkar+Narayanan92b]) captures the essential ideas. We note that if $\Pi_{min}(X)$ is known, to find $\Pi_{min}(X \cup e)$, we have to find the minimal minimizing set containing e, for a suitable z.s.s. function on $\Pi_{min}(X) \cup \{\{e\}\}$, just once.

Theorem 14.8.2 *Let X be independent in the matroid $\mathcal{M}$ whose rank function is $f_t(\cdot)$. Then*

i. *$\Pi_{min}(X)$ has only singleton blocks.*

ii. *if $f_t(X \cup e) = f_t(X), e \notin X$, the f-circuit of e with respect to X is the block containing e in the partition $\Pi_{min}(X \cup e)$.*

Proof :
i. If X is independent in M then $f_t(X) =\mid X \mid$. For each $e_i \in X$ we have, $f(e_i) = f_t(e_i) = 1$. Hence, if $\Pi(X)$ is the partition of X with singleton blocks, then $\bar{f}(\Pi(X)) =\mid X \mid= f_t(X)$. Hence, $\Pi_{min}(X) = \Pi(X)$.

ii. Let N be the block of $\Pi_{min}(X \cup e)$ containing e, the remaining blocks being singletons. We have $f_t(N) = f(N)$. Now $N - e \subseteq X$ and hence $N - e$ is independent in $\mathcal{M}$, i.e., $f_t(N - e) =\mid N - e \mid$. Thus,

$$f_t(X) =\mid X \mid= f_t(X \cup e) =\mid X - N \mid +f(N).$$

Therefore,

$$f(N) = f_t(N) =\mid N - e \mid .$$

Hence, the f-circuit N' of e with respect to X is contained in N. Let M be the block containing e in $\Pi_{min}(N')$, the remaining blocks being singletons. Thus

$$\mid N' - 1 \mid= f_t(N') = \bar{f}(\Pi_{min}(N')) =\mid N' - M \mid +f(M).$$

We conclude that $f(M) =\mid M - 1 \mid$. But then the partition $\Pi(X \cup e)$ of $X \cup e$ which has M as a block and the remaining as singletons clearly satisfies

$$\bar{f}(\Pi(X \cup e)) = f(M)+ \mid X - M \mid=\mid X \mid= f_t(X \cup e).$$

Hence, $\Pi(X \cup e) \geq \Pi_{min}(X \cup e)$. We conclude that $N' \supseteq M \supseteq N$. Hence, $N = N'$.

□

The following is an immediate corollary.

Corollary 14.8.1 *Let $f(\cdot)$ be a submodular function on subsets of S s.t. $f_t(\cdot)$ is a matroid rank function. Then a nonsingleton subset $C \subseteq S$ is a circuit of the matroid iff*

i. $f(e) = 1 \, \forall e \in C$.

ii. $\Pi_{min}(C) = \{C\}$

iii. $\Pi_{min}(C - e)$ *has only singleton blocks for each* $e \in C$.

Further if (i) and (ii) are satisfied (iii) need be satisfied only for any one $e \in C$. Then it would be satisfied for all $e \in C$.

Exercise 14.9
Let $B \equiv (V_L, V_R, E)$ be a bipartite graph. Show that

i. no more than $\mid V_R \mid$ flow maximizations are required to compute a partition that minimizes $(\mid \Gamma_L \mid -1)(\cdot)$ and

ii. no more than $(q \mid V_R \mid -p)$ flow maximization are required in the case of $(\mid \Gamma_L \mid -p/q)(\cdot)$.

Exercise 14.10 *Let $f_t(\cdot)$ be a matroid rank function on subsets of S and let X be a basis of the matroid. If N_1, N_2 are two intersecting f-circuits with respect to X, then $N_1 \cup N_2$ is contained in a block of $\Pi_{min}(S)$ of $\bar{f}(\cdot)$.*

Exercise 14.11
Let $f(\cdot)$ be a submodular function on subsets of S. Let $X \subseteq S, e \notin X$ and let $\{e\}$ not be a separator of $(f - \lambda)_t/(\mathbf{X} \cup \mathbf{e})(\cdot)$. Show that

$$(f - \lambda)_t(X \cup e) - (f - \lambda)_t(X) \geq f(X \cup e) - f(X)$$

Exercise 14.12
Let $f(\cdot)$be a polymatroid rank function on subsets of S s.t. $f(e) = k + 1 \; \forall e \in S$, where k is a positive integer. Show that

i. $(f - k)_t(\cdot)$ *is a matroid rank function*

ii. the circuits of this matroid are the strong fusion sets of $(f - k)(\cdot)$

iii. a nonvoid set T is independent in the matroid iff its minimal minimizing partition for $(f - k)(\cdot)$ is the singleton block partition Π_0.

Exercise 14.13
(This exercise needs familiarity with Chapter 12). Let $\mathcal{M}_1, \cdots, \mathcal{M}_k$ be self loop free matroids on S with rank functions $r_1(\cdot), r_k(\cdot)$ respectively. Show that

i. a set K is independent in $\mathcal{M}_1 \vee \cdots \vee \mathcal{M}_k$ iff $\sum_{i=1}^{k} r_i(X) \geq \mid X \mid \ \forall X \subseteq K$.

ii. a set K is independent in the matroid $\mathcal{M} \equiv (\mathcal{M}_1, \cdots, \mathcal{M}_k)_t$ whose rank function is $(r_1 + \cdots + r_k - (k-1))_t(\cdot)$, iff $\sum_{i=1}^{k} r_i(X) \geq \mid X \mid +(k-1) \ \forall X \subseteq K$. (Thus K is independent in $\mathcal{M}_1 \vee \cdots \vee \mathcal{M}_k$, if it is independent in $(\mathcal{M}_1, \cdots, \mathcal{M}_k)_t$).

iii. A circuit of $(\mathcal{M}_1, \mathcal{M}_2)_t$ is a minimal nonvoid subset of S s.t. the restriction of $\mathcal{M}_1, \mathcal{M}_2$ on it have disjoint bases.

Remark: The reader might wonder why the important problem of constructing the PLP of a matroid rank function has been avoided so far. The reason is that the fusion operation is not natural for matroids - when we fuse subsets of the underlying set the matroid rank function loses its matroid character. In order to use the PLP algorithms so far described we need a method by which algorithms involving polymatroid rank functions can be redone in terms of the underlying 'fused matroid rank function'. Such an algorithm is presented in the next chapter (Section 15.3).

14.9 Relation between PP and PLP

In general the principal lattice partitions of a submodular function is unrelated to its principal partition with respect to any simply constructed weight function (see examples below). Curiously, however, the problem of construction of the principal partition of the rank function of a graph relative to a positive weight function $w(\cdot)$ can be posed as that of construction of the principal lattice of partitions of the weighted incidence function $w(I)(\cdot)$, where $w(I)(X) \equiv$ weight of edges incident on vertices in X or of the weighted edge function $w(E)(X)(\equiv$ weight of edges with both end points within X). The fact is algorithmically useful since instead of working with edge sets we can work with vertex partitions. The fastest algorithm presently available for principal partition of the rank function of a graph uses this idea ([Patkar+Narayanan92c]). In this section we present a general result which relates the PLP of a certain natural submodular function on the left vertex subsets of a bipartite graph to the principal partition of a related submodular function on the right vertex subsets. We begin with a couple of examples to show that the PLP of a submodular function is unrelated to its PP with respect to a natural weight function.

Example 14.9.1 *[Narayanan91] Consider the submodular function $\sigma(X) = r(X) - \frac{1}{2} \mid X \mid -\frac{1}{2}$ on the edge subsets of the graph $\mathcal{G}$ in Figure 14.3. We see that $\sigma(\emptyset) = -\frac{1}{2}, \sigma(S) = 0, \sigma(e_i) = 0 \, \forall e_i \in S$. We may further verify that for $X \subseteq S$ we always have $\mid X \mid < 2r(X)$. Hence, the minimum value of $\sigma(\cdot)$ on nonvoid sets is zero. Each partition in the PLP of this function will have blocks on which $\sigma(\cdot)$ takes zero value. The right vertex set of the zero bipartite graph of $\sigma(\cdot)$ is as follows. (These are minimal zero sets of $\sigma(\cdot)$ with respect to the property of containing a*

Figure 14.3: The Graph $\mathcal{G}$:PP and PLP Unrelated

given pair of elements). $\{\{1,2,3\},\{6,7,8\},\{3,4,5\},\{2,6,11\},\{8,9,10\},\{1,2,3,4,5\},$ $\{6,7,8,9,10\},\{1,2,3,6,11\},\{6,7,8,2,11\},\{1,2,3,6,7,8,11\},\{1,2,3,4,5,6,7,8,9,$ $10,11\},E(\mathcal{G})\}$ *An examination of this family reveals that* $\{11\},\{12,13\}$ *are invariant under the automorphisms of the zero bipartite graph of* $\sigma(\cdot)$ *and are distinguishable from each other. Thus, the PLP of* $\sigma(\cdot)$ **discriminates** *between these sets. Now the PLP of* $\sigma(\cdot)$ *is the same as that of* $r(\cdot)$*. So the PLP of* $r(\cdot)$ *discriminates between* 11 *and* 12*. However, the PP of* $r(\cdot)$ *with respect to* $|\cdot|$ *has only* $E(\mathcal{G})$ *and* $\emptyset$ *as members (i.e.,* $E(\mathcal{G})$ *is atomic relative to* $(r(\cdot),|\cdot|)$*. Thus, the PP of* $r(\cdot)$ *cannot discriminate between* 11 *and* 12*.*

Example 14.9.2 *Consider the submodular function* $\rho(\cdot)$ *defined on the subsets of* $\{1,2,3\}$ *as follows:*
$\rho(\emptyset)=-4,\rho(\{1\})=-3,\rho(\{2\})=-4,\ \rho(\{3\})=7,\rho(\{1,2\})=-5,\rho(\{2,3\})=5,$ $\rho(\{1,3\})=5,\rho(\{1,2,3\})=0$*. This submodular function reaches its minimum at* $\{1,2\}$*. The PP of* $(\rho(\cdot),|\cdot|)$ *discriminates between* $\{1,2\}$ *and* $\{3\}$ *(Consider sets that maximize* $|X|-\frac{1}{2}(\rho(X))$*). But it is easily seen by direct inspection that* $\bar{\rho}(\cdot)$ *reaches its minimum only on* $\{\{1,2,3\}\}$ *and* $\{\{1\},\{2\},\{3\}\}$*. So the PLP of* $\rho(\cdot)$ *is unable to discriminate between* $\{1,2\}$ *and* $\{3\}$*.*

The PLP-PP relation across a bipartite graph

Let $B\equiv(V_L,V_R,E)$ be a bipartite graph. Let $w(\cdot)$ be a positive weight function on V_R. As usual, let $E_L(X),X\subseteq V_L$ be the subset of vertices of V_R which are adjacent only to vertices in X and let $w(E_L)(X)$ denote the sum of the weights of vertices adjacent only to those in X. let $\Gamma_R(X), X \subseteq V_R$ be the subset of vertices of V_L which are adjacent to vertices in X. We define $p_R(Y)$, **the partition of** V_L **associated with** Y**,** $Y \subseteq V_R$, as follows: Let the components of the subgraph of B on $Y \uplus \Gamma_R(Y)$ meet V_L in sets $X_1, \cdots, X_k$. Let $\{e_1, \cdots, e_r\} = V_L - \bigcup_{i=1}^k X_i$. Then,

$$p_R(Y) \equiv \{X_1, \cdots, X_k, \{e_1\}, \cdots, \{e_r\}\}.$$

Henceforth we denote $p_R(\cdot)$ by $p(\cdot)$.
Let Π be a partition of V_L. We define $E_L(\Pi)$ to be $\bigcup_{X\in\Pi} E_L(X)$.

Exercise 14.14 *Let $B \equiv (V_L, V_R, E)$. Let Π_1, Π_2 be partitions of V_L and let $Y \subseteq V_R$. Prove,*

i. $p(E_L(\Pi_1)) \leq \Pi_1$.

ii. If $E_L(\Pi_1) = E_L(\Pi_2)$ *show that* $E_L(\Pi_1 \vee \Pi_2) \supseteq E_L(\Pi_1 \wedge \Pi_2) = E_L(\Pi_1)$

iii. Construct an example where $E_L(\Pi_1) = E_L(\Pi_2)$*, but* $E_L(\Pi_1 \vee \Pi_2) \supset E_L(\Pi_1)$.

We now have the following lemma.

Lemma 14.9.1 *Let $B \equiv (V_L, V_R, E)$ be a bipartite graph. Then*

i. $E_L(p(E_L(\Pi))) = E_L(\Pi), \Pi \in \mathcal{P}_{V_L}$

ii. $p(E_L(p(Y))) = p(Y), Y \subseteq V_R$

Proof : By definition of $p(\cdot)$ and $E_L(\cdot)$ it is clear that

$$p(E_L(\Pi)) \leq \Pi, \qquad (*)$$

$$E_L(p(Y)) \supseteq Y. \qquad (**)$$

Further, if $Y_1 \subseteq Y_2$ and $\Pi_1 \leq \Pi_2$ we have

$$p(Y_1) \leq p(Y_2) \qquad (*3)$$

$$E_L(\Pi_1) \subseteq E_L(\Pi_2) \qquad (*4)$$

Using $(**)$ we have $E_L(p(E_L(\Pi))) \supseteq E_L(\Pi)$. Using $(*)$ and $(*4)$ we have $E_L(p(E_L(\Pi))) \subseteq E_L(\Pi)$. This proves the first part of the lemma.
Next, using $(*)$ we have $p(E_L(p(Y))) \leq p(Y)$. Using $(**)$ and $(*3)$ $p(E_L(p(Y))) \geq p(Y)$. This proves the second part of the lemma.

□

We denote $p(E_L(\Pi))$ by $\bar{\Pi}$ and $E_L(p(Y))$ by $\bar{Y}$. We refer to $\bar{\Pi}$ as **closure** of Π and $\bar{Y}$ as **closure** of Y. If $\Pi = \bar{\Pi}(Y = \bar{Y})$ we say that it is **closed**. The above lemma states that $E_L(\Pi), p(Y)$ are closed and that every closed partition (closed set) can be obtained as the image of some $Y \subseteq V_R$ under $p(\cdot)$ (as the image of some partition $\Pi \in \mathcal{P}_{V_L}$ under $E_L(\cdot)$).

It is easily seen (Exercise 10.1) that if $w(\cdot)$ is a positive weight function on V_R, then $w(E_L)(\cdot)$ is a supermodular function on subsets of V_L. By our present notation $w(E_L(\Pi)) \equiv \overline{w(E_L)}(\Pi)$. The following theorem which is the main result of this section, shows that constructing the PLP of $-w(E_L)(\cdot)$ is equivalent to the construction of the PP of $(\mid \Gamma_R \mid -1)_t(\cdot)$ with respect to $w(\cdot)$.

Theorem 14.9.1 *[Narayanan+Kamath94] Let $B \equiv (V_L, V_R, E)$ be a bipartite graph. Let $w(\cdot)$ be a positive weight function on V_R. Then*

i. $\Pi \in \mathcal{P}_{V_L}$ *minimizes* $\overline{(-w(E_L) - \lambda)}(\cdot)$, $\lambda > 0$, *only if it is closed.*

ii. $X \equiv V_R$ *minimizes* $\lambda(\mid \Gamma_R \mid -1)_t(\cdot) - w(\cdot)$, $\lambda > 0$, *only if it is closed.*

iii. *When* $\lambda > 0$, $X \subseteq V_R$ *minimizes* $\lambda(\mid \Gamma_R \mid -1)_t - w(\cdot)$ *iff* $p(X)$ *minimizes* $\overline{(-w(E_L) - \lambda)}(\cdot)$ **equivalently** $\Pi \in \mathcal{P}_{V_L}$ *minimizes* $\overline{(-w(E_L) - \lambda)}(\cdot)$ *iff* $E_L(\Pi)$ *minimizes* $\lambda(\mid \Gamma_R \mid -1)_t(\cdot) - w(\cdot)$.

We need the following lemma for the proof of Theorem 14.9.1.

Lemma 14.9.2 *Let $B \equiv (V_L, V_R, E)$ be a bipartite graph. Let $X \subseteq V_R$. Let $\Pi(X)$ be the partition of X induced by the connected components of the subgraph of B on $X \uplus \Gamma_R(X)$. Then,*

$$
\begin{aligned}
(\mid \Gamma_R \mid -1)_t(X) &= \overline{(\mid \Gamma_R \mid -1)}(\Pi(X)) \\
&= \mid V_L \mid - \mid p(X) \mid .
\end{aligned}
$$

Proof : In order to show that $\Pi(X)$ minimizes $\overline{(\mid \Gamma_R \mid -1)}(\cdot)$ among all partitions of X we need merely show (by Theorem 13.3.5) that on all partitions finer and on all partitions coarser than $\Pi(X)$, $\overline{(\mid \Gamma_R \mid -1)}(\cdot)$ does not take lower value than on $\Pi(X)$.
Let $\Pi' \geq \Pi(X)$. We then have $\overline{\mid \Gamma_R \mid}(\Pi') = \overline{\mid \Gamma_R \mid}(\Pi(X))$ (by the definition of $\Pi(X)$), while $\mid \Pi' \mid \leq \mid \Pi(X) \mid$.
Hence, $\overline{(\mid \Gamma_R \mid -1)}(\Pi(X)) \leq \overline{(\mid \Gamma_R \mid -1)}(\Pi')$.
Next let $\Pi" \leq \Pi(X)$. If $\Pi" \neq \Pi(X)$, there exists a block N of $\Pi(X)$ which is partitioned into blocks $N_1, \cdots, N_k$ with $k \geq 2$ in $\Pi"$. Now the subgraph of B on $N \uplus \Gamma_R(N)$ is connected. Hence, without loss of generality we may assume that $\Gamma_R(N_1), \Gamma_R(N_2)$ have a common element and in general $\Gamma_R(N_1 \uplus \cdots \uplus N_j), \Gamma_R(N_{j+1})$, $j < k$, have a common element. Hence

$$\overline{(\mid \Gamma_R \mid -1)}(\{N_1, \cdots, N_k\}) \geq (\mid \Gamma_R \mid -1)(N).$$

This proves that

$$\overline{(\mid \Gamma_R \mid -1)}(\Pi") \geq \overline{(\mid \Gamma_R \mid -1)}(\Pi(X)).$$

We conclude that $\Pi(X)$ minimizes $\overline{(\mid \Gamma_R \mid -1)}(\cdot)$ and hence,

$$(\mid \Gamma_R \mid -1)_t(X) = \overline{(\mid \Gamma_R \mid -1)}(\Pi(X)).$$

Let $\Pi(X) \equiv \{M_1, \cdots, M_n\}$.
By definition, $p(X) \equiv \{\Gamma_R(M_1), \cdots, \Gamma_R(M_n), \{e_1\}, \cdots, \{e_r\}\}$, where $\{e_1, \cdots, e_r\} = V_L - \Gamma_R(X)$. Hence,

$$\begin{aligned}\overline{(\mid \Gamma_R \mid -1)}(\Pi(X)) &= \sum \mid \Gamma_R(M_i) \mid -n = \mid \Gamma_R(X) \mid -n \\ &= \mid V_L \mid -n - r = \mid V_L - \mid p(X) \mid\end{aligned}$$

□

Proof of Theorem 14.9.1:

i. We have $\bar{\Pi} \leq \Pi$ and by Lemma 14.9.1 $E_L(\bar{\Pi}) = E_L(\Pi)$. If $\bar{\Pi} \neq \Pi$ then $\mid \bar{\Pi} \mid > \mid \Pi \mid$. The result follows.

ii. We have $\bar{Y} \supseteq Y$ and, by the above mentioned lemma, $p(\bar{Y}) = p(Y)$. If $\bar{Y} \neq Y$, then $w(\bar{Y}) > w(Y)$. Using Lemma 14.9.2 we get

$$\begin{aligned}(\mid \Gamma_R \mid -1)_t(\bar{Y}) &= (\mid \Gamma_R \mid -1)_t(Y) \\ &= \mid V_L \mid - \mid p(\bar{Y}) \mid = \mid V_L \mid - \mid p(Y) \mid .\end{aligned}$$

Hence,

$$\lambda(\mid \Gamma_R \mid -1)_t(\bar{Y}) - w(\bar{Y}) < \lambda(\mid \Gamma_R \mid -1)_t(Y) - w(Y).$$

The result follows.

iii. Let $\lambda > 0$ and let $X \subseteq V_R$ minimize $\lambda(\mid \Gamma_R \mid -1)_t(\cdot) - w(\cdot)$. Then X is closed, i.e., $E_L(p(X)) = X$. Now

$$\lambda(\mid \Gamma_R \mid -1)_t(X) - w(X) = \lambda \mid V_L \mid -\lambda \mid p(X) \mid -w(X).$$

Thus, X minimizes $-\lambda \mid p(\cdot) \mid -w(\cdot)$. Since $X = E_L(p(X))$, we see that

$$\overline{(-w(E_L) - \lambda)}(p(X)) = -w(X) - \lambda \mid p(X) \mid .$$

We thus see that the minimum value of $\overline{(-w(E_L) - \lambda)}(\cdot)$ over $\mathcal{P}_{V_L}$ is not greater than the minimum value of $(-\lambda \mid p(\cdot) \mid -w(\cdot))$ over subsets of V_R. On the other hand let Π minimize $\overline{(-w(E_L) - \lambda)}(\cdot)$. We know that Π is closed, i.e., $p(E_L(\Pi)) = \Pi$. We have

$$\begin{aligned}\overline{(-w(E_L) - \lambda)}(\Pi) &= -w(E_L(\Pi)) - \lambda \mid \Pi \mid \\ &= -w(E_L(\Pi)) - \lambda \mid p(E_L(\Pi)) \mid .\end{aligned}$$

Hence the minimum value of $\overline{(-w(E_L) - \lambda)}(\cdot)$ over $\mathcal{P}_{V_L}$ is not less than the minimum value of $(-\lambda \mid p(\cdot) \mid -w(\cdot))$ over subsets of V_R.
We conclude that the minimum values of the above two functions are equal and also that if Π minimizes $\overline{(-w(E_L) - \lambda)}(\cdot)$, then $E_L(\Pi)$ minimizes $(-\lambda \mid p(\cdot) \mid -w(\cdot) + \lambda \mid V_L \mid)$ and if $X \subseteq V_R$ minimizes the latter function then $p(X)$ minimizes $\overline{(-w(E_L) - \lambda)}(\cdot)$. Since the minimizing partition of $\overline{(-w(E_L) - \lambda)}(\cdot)$ (minimizing set of $(-\lambda \mid p(\cdot) \mid -w(\cdot))$) is closed, the result follows using the fact that

$$\lambda(\mid \Gamma_R \mid -1)_t(X) - w(X) = \lambda \mid V_L \mid -w(X) - \lambda \mid p(X) \mid$$

□

Exercise 14.15 *Prove the following generalization of Theorem 14.9.1: Let $w(\cdot)$ be any increasing set function:*

i. $\Pi \in \mathcal{P}_{V_L}$ *minimizes* $\overline{(-w(E_L) - \lambda)}(\cdot), \lambda > 0$, *only if it is closed.*

ii. $X \subseteq V_R$ *minimizes* $\lambda(\mid \Gamma_R \mid -1)_t(\cdot) - w(\cdot), \lambda > 0$ *only if* $\bar{X}$ *minimizes the same function.*

iii. *When* $\lambda > 0, X \subseteq V_R$ *minimizes* $\lambda(\mid \Gamma_R \mid -1)_t(\cdot) - w(\cdot)$ *only if* $p(X)$ *minimizes* $\overline{(-w(E_L) - \lambda)}(\cdot)$.

iv. *When* $\lambda > 0, \Pi \in \mathcal{P}_{V_L}$ *minimizes* $\overline{(-w(E_L) - \lambda)}(\cdot)$ *iff* $E_L(\Pi)$ *minimizes* $\lambda(\mid \Gamma_R \mid -1)_t(\cdot) - w(\cdot)$.

v. *If* $w(\cdot)$ *is supermodular,* $w(E_L)(\cdot)$ *acting on subsets of* V_R*, is a supermodular function. Hence, the minimizing partition of* $\overline{(-w(E_L) - \lambda)}(\cdot), \lambda > 0$ *are precisely those in the PLP of the submodular function* $-w(E_L)(\cdot)$.

14.10 Fast Algorithms for Principal Partition of the rank function of a graph

As we mentioned before, it is an interesting fact that the fastest algorithm currently available for the principal partition of the rank function of a graph is by finding the PLP of the weighted interior edge function on vertex subsets. The basic idea is already outlined in the previous sections. We do the specialization of this idea explicitly to the rank function of a graph below. An advantage of this method is simplicity of implementation.

Let $B \equiv (V_L, V_R, E_B)$,where $V_L \equiv V(\mathcal{G}), V_R \equiv E(\mathcal{G})$ with $v \in V_L$ adjacent in B to $e \in V_R$ iff edge e is incident on vertex v in graph $\mathcal{G}$. Observe that $(\mid \Gamma_R \mid -1)_t(X) = r(X)$, $X \subseteq V_R$ $((\mid \Gamma_R \mid -1)_t(X) = (\mid V \mid -1)_t(X)$, where $V(X) \equiv$ end point set of X). Thus, as discussed in the previous section (or even directly), minimizing $\lambda r(X) - w(X)$ is equivalent to minimizing $\overline{(-w(E_L) - \lambda)}(\cdot)$ over partitions of $V_L \equiv V(\mathcal{G})$. Here $w(E_L)(Y), Y \subseteq V(\mathcal{G})$ is the weight of edges with both end points within Y. Thus, constructing the PP of $(r(\cdot), w(\cdot))$ is equivalent to constructing the PLP of $-w(E_L)(\cdot)(= -w(E)(\cdot))$.

Exercise 14.16 *Do the conversion of the problem of finding the PP of the rank function of a graph to a PLP problem directly.*

Algorithm P-sequence of partitions of $w_R E_L(\cdot)$ yields also the P-sequence of $(r(\cdot), w_R(\cdot))$. Here $V_L \equiv V(\mathcal{G}) = V$ and $V_R \equiv E(\mathcal{G}), w_R(\cdot) \equiv w(\cdot)$. So the complexity of this algorithm (using Sleator's flow algorithm [Sleator80], as shown in Subsection 14.7.2) is $O(\mid V \mid^3 \mid E(\mathcal{G}) \mid \log \mid E(\mathcal{G}) \mid)$. The DTL of $((w_R E)_L(\cdot) - \lambda)$ yields $\mathcal{B}_{\lambda r, w_R}$. The algorithm given in the above section does this in $O(\mid V \mid^3 \mid E(\mathcal{G}) \mid \log \mid E(\mathcal{G}) \mid)$. Using the argument of Exercise 14.4, computing the DTL of all the λs is also $O(\mid V \mid^3 \mid E(\mathcal{G}) \mid \log \mid E(\mathcal{G}) \mid)$. Thus the overall complexity of the construction of PP of $(r(\cdot), w_R(\cdot))$ is $O(\mid V \mid^3 \mid E(\mathcal{G}) \mid \log \mid E(\mathcal{G}) \mid)$.

Now let us consider the unweighted case where $w_R(\cdot) = \mid \cdot \mid$. Here we use the balanced bisection method given in page 599. Clearly the set of possible critical values in this case can be confined to $\{p/q,\ p, q \text{ integers s.t. } 0 < p < \mid E(\mathcal{G}) \mid, 0 < q < |V|\}$. The cardinality of this set is $\mid V \mid\mid E(\mathcal{G}) \mid$.
The Algorithm $Min(\bar{f}, S)$ in this case has a complexity $O(\mid E(\mathcal{G}) \mid\mid V \mid^2 \log \mid E(\mathcal{G}) \mid)$. Using the balanced bisection method would mean that Algorithm P-sequence of partitions has a complexity $O(\mid E(\mathcal{G}) \mid\mid V \mid^2 \log^2 \mid E(\mathcal{G}) \mid)$.
Our present version of DTL would give a complexity $O(\mid E(\mathcal{G}) \mid\mid V \mid^3 \log \mid E(\mathcal{G}) \mid)$ for DTL's of all critical values. A more sophisticated version ([Patkar+Narayanan92b], [Patkar92]) has a complexity of $O(\mid V \mid^2 \mid E(\mathcal{G}) \mid \log \mid E(\mathcal{G}) \mid))$.
Thus by the method of transforming the PP problem into a PLP problem, the complexity for the **principal partition of** $(r(\cdot), \mid \cdot \mid)$, is as follows:

$$\begin{array}{rcl} P-sequence & - & O(\mid E(\mathcal{G}) \mid\mid V \mid^2 \log^2 \mid E(\mathcal{G}) \mid) \\ Complete\ Principal\ partition & - & O(\mid E(\mathcal{G}) \mid\mid V \mid^2 \log^2 \mid E(\mathcal{G}) \mid) \end{array}$$

The complexity by direct edge set based methods, earlier available, in [Imai83] for this (unweighted) case of PP of the rank function of a graph was as under:

$$\begin{array}{rcl} P-sequence & - & O(\mid E(\mathcal{G}) \mid^3 \log \mid E(\mathcal{G}) \mid) \\ overall\ PP & - & O(\mid E(\mathcal{G}) \mid^3 \log \mid E(\mathcal{G}) \mid) \end{array}$$

For fixed k the unweighted problem is better solved by 'remembering' the flow before fusion takes place. By such methods when k is an integer X_k, the minimum set minimizing $kr(X) + |E(\mathcal{G}) - X|$ can be computed in $O(min(k(k+1) \mid V \mid^2, \mid E(\mathcal{G}) \mid^2))$ [Patkar+Narayanan91].

Exercise 14.17 *Show that*

i. *the packing number of a graph $\mathcal{G}$ (maximum number of disjoint spanning forests contained in $\mathcal{G}$) can be determined in*

$$O(\mid E(\mathcal{G}) \mid\mid V \mid \log \mid V \mid \log(\mid E(\mathcal{G}) \mid / \mid V \mid)) \ time$$

and

ii. *the covering number of graph $\mathcal{G}$ (minimum number of disjoint subforests needed to cover $E(\mathcal{G})$) can be determined in $O(\mid E(\mathcal{G}) \mid\mid V \mid \log^2 \mid V \mid)$ time.*

14.11 Solutions of Exercises

E 14.1: Clearly $(\Pi_2)_{fus\cdot\Pi_1}$ can have a block of maximum size atmost $(\mid \Pi_1 \mid - \mid \Pi_2 \mid +1)$. To find the min partition within this block takes $O(k_1\alpha(k_1))$ elementary steps where $k_1 = (\mid \Pi_1 \mid - \mid \Pi_2 \mid +1)$. If $k_2, \cdots, k_r$ are the sizes of the remaining blocks of $(\Pi_2)_{fus\cdot\Pi_1}$, the overall complexity of Subdivide (Π_1, Π_2) is $O(k_1\alpha(k_1) + \cdots k_r\alpha(k_r))$.
Since $\alpha(k)$ is superlinear

$$k_2\alpha(k_2) + \cdots + k_r\alpha(k_r) \quad \leq \quad (k_2 + \cdots + k_r)\alpha(k_1)$$

Thus, the overall complexity in $O((\mid \Pi_1 \mid \alpha(\mid \Pi_1 \mid - \mid \Pi_2 \mid +1))$.

E 14.2: For every critical value λ, the function $(f - \lambda)(\cdot)$ reaches a minimum on atleast two partitions Π_1, Π_2 of S with $\Pi_1 \leq \Pi_2$ so that

$$\lambda = \frac{\bar{f}(\Pi_1) - \bar{f}(\Pi_2)}{\mid \Pi_1 \mid - \mid \Pi_2 \mid}.$$

The numerator clearly lies between $-n(a-b)$ and $n(a-b)$ and the denominator between 0 and n. The possible values $\frac{p}{q}$ can take is therefore in $\{\frac{p}{q} : -n(a-b) \leq p \leq n(a-b),\ 0 < q \leq n\}$. This set clearly has cardinality $n(2n(a-b)+1)$.

E 14.4: Suppose $\Pi_1, \cdots, \Pi_n = \Pi_S$ is the principal sequence of partitions. If $n = 1$ the result is obviously true. Suppose the result is true for $n = (r-1)$.
If $\Pi_1, \cdots, \Pi_{r-1}, \Pi_r = \Pi_S$ is the principal sequence, then by induction assumption to build the zero bipartite graphs of all the partition pairs (Π_j, Π_{j+1}) from Π_0 to Π_{r-1} we require $O(\sum_{i=1}^{k} \mid N_i \mid^2)$ submodular function minimizations, where $N_1, \cdots, N_k$ are the the blocks of Π_{r-1}. To build the zero bipartite graph corresponding to (Π_{r-1}, Π_r) we require $O(k^2)$ minimizations. Assume for simplicity that the sets N_i are nonsingletons. Now for any positive number β,

$$\beta(\sum_{i=1}^{k} \mid N_i \mid^2) + \beta k^2 < \beta \mid S \mid^2, k \geq 2.$$

Hence, we require $O(\mid S \mid^2)$ minimizations to build all the zero bipartite graphs for $n = r$. (If we wish to argue allowing N_i to be singletons, we need to show that

$$m_1^2 + \cdots + m_j^2 + k^2 < (m_1 + \cdots + m_j + k - j)^2,$$

where the $m_i > 1, i = 1, \cdots, j$ and $k \geq j$. This is routine.)

E 14.5: (For reasons of brevity we do not add a new source vertex $\hat{s}$ and an edge $(\hat{s}, s)$ of capacity ∞.)
Let $g(v)(= w_L(v))$ be negative. Suppose there exists a min cut of the form

$$(s \uplus X \uplus \Gamma(X), (V_L - X) \uplus (V_R - \Gamma(X)) \uplus t).$$

If $v \in X$ we shift it from the s-side to the t-side of the cut. The new cut is

$$(s \uplus (X - v) \uplus \Gamma(X - v), (V_L - (X - v)) \uplus (V_R - \Gamma(X - v)) \uplus t).$$

We thus see that the cut capacity is reduced by $w_R(\Gamma(v)-\Gamma(X-v))$. This reduction cannot be negative and the result follows.

E 14.6: We assume that $(w_R\Gamma)(e) - \lambda > 0$, as otherwise it can be seen that $T \cup e$ contains no fusion set. We will show that after the edge capacity of (s, e) is increased to $((w_R\Gamma)(e) + \delta)$ the capacity of the cut $(s, T \cup e \uplus V_R \uplus t)$ is greater than that of the cut $(s \uplus e \uplus \Gamma(e), T \uplus (V_R - \Gamma(e)) \uplus t)$. From this it would follow that the min cut has the form $(s \uplus X \uplus \Gamma(X), ((T \cup e) - X) \uplus (V_R - \Gamma(X)) \uplus t)$ where $X \neq \emptyset$. More careful analysis would show that δ can be taken to be zero, for the cut corresponding to a fusion set would have capacity lower than that due to $\{e\}$.

The capacity of the cut $(s, (T \cup e) \uplus V_R \uplus t)$
$= \sum_{v \in T \cup e}((w_R\Gamma)(v) - \lambda) + (\lambda + \delta)$ - (capacity of the edges directed to s)
$= \sum_{v \in T}((w_R\Gamma)(v) - \lambda) + (w_R(\Gamma(e)) + \delta$ - (capacity of edges to s).
Capacity of the cut $(s \uplus e \uplus \Gamma(e), T \uplus (V_R - \Gamma(e)) \uplus t)$
$= \sum_{v \in T}((w_R\Gamma)(v) - \lambda) + (w_R(\Gamma(e))$ -(capacity of edges directed to the s).
This proves the required result.

E 14.7: Suppose capacity of $(s, v) = \lambda$ and $v \notin Y$. Compare the capacity of the cut

$$(s \uplus Y \uplus \Gamma(Y), t \uplus (V_L - Y) \uplus (V_R - \Gamma(Y)))$$

with that of

$$(s \uplus (Y \cup v) \uplus \Gamma(Y \cup v), t \uplus (V_L - (Y \cup v)) \uplus (V_R - \Gamma(Y \cup v))).$$

The former cut has a capacity greater than that of the latter by $\lambda - w_R(\Gamma(v) - \Gamma(Y))$ which is a positive number. Hence, we may assume that the set Y corresponding to the min cut has v belonging to it, i.e., $Y \supseteq V_L - Z$.

E 14.8: We will show that the critical value of $f(\cdot)$ cannot be lower than $(w_R\Gamma)(V_L - v)$. Observe that if λ is a critical value of $f(\cdot)$ then $\lambda - (w_R\Gamma)(V_L)$ is a critical value of $\underline{-w_R E_L}(\cdot)$. Now $-(w_R E_L)(\cdot)$ takes zero value on the null set. Hence, $\overline{-w_R E_L}(\cdot)$ reaches its minimum at Π_{V_L}. Hence, $-(w_R E_L)(\cdot)$ cannot have a critical value lower than zero. Equivalently, $f(\cdot)$ cannot have a critical value lower than $0 + (w_R\Gamma)(V_L)$. Hence, we do not need to minimize $\overline{(f - \lambda)}(\cdot)$ for $\lambda < (w_R\Gamma)(V_L)$ and therefore, we may take $\lambda \geq (w_R\Gamma)(V_L - e)$ since $w_R(\cdot) \geq 0$.

E 14.11: Consider $\Pi_{max}(X), \Pi_{max}(X \cup e)$ for $(f - \lambda)(\cdot)$. Since $\{e\}$ is not a separator of $(f(\cdot) - \lambda)/\mathbf{X} \cup \mathbf{e})(\cdot)$ it is clear that $\Pi_{max}(X \cup e)$ cannot have e in a singleton block. Let M be the block of $\Pi_{max}(X \cup e)$ containing e. Then $M = N_1 \uplus \cdots \uplus N_k \cup e$ where the N_i are blocks of $\Pi_{max}(X)$. The remaining blocks of $\Pi_{max}(X)$ and $\Pi_{max}(X \cup e)$ are identical. So

$$\begin{aligned}
(f - \lambda)_t(X \cup e) - (f - \lambda)_t(X) &= (f - \lambda)(M) - \sum (f - \lambda)N_i \\
&\geq (f - \lambda)(M) - (f - \lambda)(\cup N_i) \\
&\geq f(X \cup e) - f(X) \\
&\quad (\textit{by submodularity of} f(\cdot)).
\end{aligned}$$

E 14.12:

i. $(f-k)_t(\cdot)$ is increasing, integral, submodular with zero value on null set and 1 on singletons.
ii. We know that circuits of $(f-k)_t(\cdot)$ are minimal sets s.t.

$$(f-k)_t(X) <\mid X \mid \cdots (*).$$

Let X' be a strong fusion set. Then,

$$(f-k)(X') < \sum_{e \in X'} (f-k)(e) = \mid X' \mid$$

Hence,

$$(f-k)_t(X') \leq (f-k)(X') <\mid X' \mid$$

Hence X contains a minimal set satisfying $(*)$.
Next let Y be s.t. $(f-k)_t(Y) <\mid Y \mid$. Let Π be a partition of Y that minimizes $\overline{(f-k)}(\cdot)$. Then $\overline{(f-k)}(\Pi) <\mid Y \mid = \overline{(f-k)}(\Pi_0)$, where Π_0 is the singleton block partition of Y. Thus Π has a nonsingleton block say Z. Now Z contains a minimal subset X whose $(f-k)(\cdot)$ value is less than its size $(= \sum_{e \in X}(f-k)(e))$. Clearly this set is a strong fusion set. So Y contains a strong fusion set.
iii. We have, $\overline{(f-k)}(\Pi_0) = \mid T \mid$, where Π_0 is the partition of T with singleton blocks. Hence, Π_0 is the minimal minimizing partition for $\overline{(f-k)}(\cdot)$ iff $(f-k)_t(T) = \mid T \mid$, i.e., iff T is independent in the matroid.

E 14.13:
i. Follows from Nash-Williams' rank formula.

ii. We use the results in Exercise 14.12. Hence, K is independent $(\mathcal{M}_1, \cdots, \mathcal{M}_2)_t$ iff it contains no strong fusion set, i.e., iff there exists no $X \subseteq K$ s.t.

$$(\sum r_i - (k-1))(X) < \sum_{e \in X}((\sum r_i - (k-1)))(e) = \mid X \mid,$$

i.e., s.t. $\sum r_i(X) <\mid X \mid + (k-1)$.
Equivalently, K is independent iff

$$\sum r_i(X) \geq \mid X \mid + (k-1) \ \forall X \subseteq K.$$

iii. We use Exercise 14.12 Circuits of $(\mathcal{M}_1, \mathcal{M}_2)_t$ are minimal nonvoid sets that satisfy,
$(r_1+r_2)(X) <\mid X \mid +1$, equivalently, $(r_1+r_2)(X) \leq \mid X \mid$. Thus, X is a circuit of $(\mathcal{M}_1, \mathcal{M}_2)_t$ iff

$$0 = \mid X \mid - (r_1+r_2)(X) > \mid Y \mid - (r_1+r_2)(Y) \ \forall Y \subset X, Y \neq \emptyset$$

i.e., iff

$$\begin{aligned} ((r_1+r_2) * \mid \cdot \mid)(X) &= (r_1+r_2)(X) \\ &< (r_1+r_2)(Y) + \mid X - Y \mid, \forall Y \subset X, Y \neq \emptyset \end{aligned}$$

i.e., iff X is a minimal nonvoid set which is covered by disjoint bases of restrictions of $\mathcal{M}_1, \mathcal{M}_2$ on it (i.e., any pair of elements of X is mutually approachable relative to $\mathcal{M}_1 \cdot X, \mathcal{M}_2 \cdot X$).

E 14.14:
i. & **ii**. Let $\Pi_1 \equiv \{N_1, \cdots, N_k\}$, $\Pi_2 \equiv \{M_1, \cdots, M_r\}$. Suppose $E_L(\Pi_1) = E_L(\Pi_2)$. Let $N_{Ri} \equiv E_L(N_i)$, $i = 1, \cdots, k$ and let $M_{Ri} \equiv E_L(M_i)$, $i = 1, \cdots, r$.
Let B_1 be the direct sum of the subgraphs of B on $N_{Ri} \uplus \Gamma_R(N_{Ri})$. Let B_2 be similarly obtained from the subgraphs of B on $M_{Ri} \uplus \Gamma_R(M_{Ri})$. Observe that $\Gamma_R(N_{Ri}) \subseteq N_i$ and $\Gamma_R(M_{Ri}) \subseteq M_i$. We have $\bigcup N_{Ri} = \bigcup M_{Ri}$. Hence the edges incident on these sets must be identical. It is therefore clear that B_1 and B_2 are identical. All vertices of V_L may not appear in B_1. Let such vertices be added as isolated vertices to B_1 and let the resulting bipartite graph be B_{1L}. The components of B_{1L} meet V_L in the partition

$$\Pi \equiv p(\cup N_{Ri}) = p(E_L(\Pi_1)).$$

Now the intersection of each component of B_{1L} with V_L, by its construction, must be contained in one of the N_i as well as one of the M_i. Hence, $\Pi \leq \Pi_1$ and $\Pi \leq \Pi_2$. (This proves the first part of the exercise).
Therefore, $\Pi \leq \Pi_1 \wedge \Pi_2 \leq \Pi_1$ and hence, $E_L(\Pi) \subseteq E_L(\Pi_1 \wedge \Pi_2) \subseteq E_L(\Pi_1)$. But $E_L(\Pi) = E_L(\Pi_1)$. We conclude that $E_L(\Pi_1 \wedge \Pi_2) = E_L(\Pi_1)$.
From the definition of the function $E_L(\cdot)$ it follows that $E_L(\Pi_1 \vee \Pi_2) \supseteq E_L(\Pi_1)$.
iii. We give an example where $E_L(\Pi_1 \vee \Pi_2) \neq E_L(\Pi_1)$. Let $\Pi_1 \equiv \{N_1, N_2\}$, $\Pi_2 \equiv \{M_1, M_2, M_3\}$, where $M_1 \subseteq N_1, M_3 \subseteq N_2, M_2 \cap N_1 \neq \emptyset$ and $M_2 \cap N_2 \neq \emptyset$. Let vertex $v \in V_R$ be adjacent to $a \in M_1$ as well as to $b \in M_2 \cap N_2$, but to no other vertices. It is clear that $v \in E_L(\Pi_1 \vee \Pi_2) = E_L(N_1 \cup N_2)$ but $v \notin E_L(\Pi_1)$ and $v \notin E_L(\Pi_2)$.

E 14.15:
i. Here the proof is identical to part (i) of Theorem 14.9.1.
ii. We have $w(\bar{Y}) \geq w(Y)$ and $p(\bar{Y}) = p(Y)$. We would then have

$$\lambda(\mid \Gamma_R \mid -1)_t(\bar{Y}) - w(\bar{Y}) \leq \lambda(\mid \Gamma_R \mid -1)_t(Y) - w(Y),$$

from which the result follows.
iii.& **iv**. Proof is similar to part (iii) of Theorem 14.9.1.
v. See Problem 10.3.

E 14.16: Observe first that the minimum of $\lambda r(X) - w(X)$ will occur only on closed sets (under the formation of circuits). (If X is not closed then the closure $\bar{X}$ has the same rank and greater weight than X). With each subset $X \subseteq E(\mathcal{G})$ we can associate the partition $p(X)$ of $V(\mathcal{G})$, whose blocks are the vertex sets of components of the subgraph $\mathcal{G}'$ of $\mathcal{G}$ on X with vertex set $V(\mathcal{G})$.
Now $r(\bar{X}) = V(\mathcal{G})$ - number of components of $\mathcal{G}'$.
$= \mid V_L \mid - \mid p(\bar{X}) \mid$.
Hence,

$$\lambda r(\bar{X}) - w(\bar{X}) = \lambda(\mid V_L \mid - \mid p(\bar{X}) \mid) - \overline{w(E)}(p(\bar{X})),$$

where we have used the fact that $w(\bar{X})$ is the weight of edges with both endpoints within the same block of $p(\bar{X})$, i.e., $w(\bar{X}) = \overline{w(E)}(p(\bar{X}))$. On the other hand, $\lambda(\mid V_L \mid - \mid \Pi \mid) - \overline{w(E)}(\Pi)$, for any partition Π of $V(\mathcal{G})$, is not less than $\lambda r(Z) - w(Z)$, where Z is the set of edges each of which has both end points within the same block of Π. Hence, minimizing $\lambda r(X) - w(X)$ is equivalent to minimizing $\overline{w(E)}(\Pi) - \lambda \mid \Pi \mid$.

E 14.17: We assume the graph is connected. Otherwise we test each component in turn.(Better bounds for these problems is available. See for instance [Patkar92],[Patkar+Narayanan91],[Patkar+Narayanan92c]).

i. We need to find the least integer k in $\left[0, \frac{|E|}{|V|}\right]$ for which $\mid E(V) \mid -k(\mid V \mid -1)$ reaches its maximum on $E(\mathcal{G})$. Let $B \equiv (V_L, V_R, E_B) \equiv (V(\mathcal{G}), E(\mathcal{G}), E_B)$ be the bipartite graph associated with the graph $\mathcal{G}$ in the usual way. Build the flow graph $F_B^k(B, \mathbf{w}_L, \mathbf{w}_R)$, where $w_L(v) = k \; \forall v \in V_L$, $w_R(e) = 1 \; \forall e \in V_R$. If $(s, V_L \uplus V_R \uplus t)$ is a min cut in this flow graph, then maximum would be reached on $E(\mathcal{G})$. One such maximization takes $O(\mid E \mid\mid V \mid \log \mid V \mid)$ time. Finding the least integer k needs the above maximization to be done $\log(\frac{|E|}{|V|})$ times.

ii. In this case we have to find the maximum k in $[0, \mid E \mid]$ for which the maximum value of $\mid E(V) \mid -k(\mid V \mid -1)$ is not negative. (The covering number would be k, if for this maximum value of k the above expression has the maximum value zero, otherwise the covering number would be $(max\ value\ of\ k+1)$). The flow graph is as above. Flow maximization has to be done $\log \mid E \mid = \log \mid V \mid$ times. So complexity is $O(\mid E \mid\mid V \mid \log^2 \mid V \mid)$ time.

Chapter 15

The Hybrid Rank Problem

15.1 Introduction

The hybrid rank problem in its various manifestations provides a strong link between the theories of submodular functions and of electrical networks. In this chapter we present four different non-equivalent formulations of this problem. Each of these formulations is related to a practically important network problem and each of the first three motivates an important area of research in submodular function theory. The first of these formulations - 'the topological degree of freedom problem' is the most basic and is due to G.Kron (see [Kron39],[Kron63] [Amari62] for related material). It involves the convolution of the rank and nullity functions of a graph and was solved using Kishi-Kajitani's 'principal partition' [Kishi+Kajitani68] by Ohtsuki et al.[Ohtsuki+Ishizaki+Watanabe68] in 1967 (see also [Kishi+Kajitani69] and [Ohtsuki+Ishizaki+Watanabe70]). The other formulations are generalizations of the first. One of these involves minimizing the partition associate of a submodular function while another involves solving the membership problem for a polymatroid given a matroid expansion. The last of the formulations is a vector space generalization which resembles convolution but leads to an optimization problem for which a solution is presently unknown.

15.2 The Hybrid Rank Problem - First Formulation

The Problem (first formulation): Given a graph $\mathcal{G}$, partition $E(\mathcal{G})$ into X and $E(\mathcal{G}) - X$ such that $r(\mathcal{G} \cdot X) + \nu(\mathcal{G} \times (E(\mathcal{G}) - X))$ is minimized.

This problem has already been mentioned in Subsection 6.5.6. When the network is linear, if we write nodal equations for $\mathcal{N}_{AL}$ and loop equations for $\mathcal{N}_{BK}$, the total number of equations would be $r(\mathcal{G} \cdot A) + \nu(\mathcal{G} \times B)$. So one could ask

for the partition A, B for which the above expression reaches a minimum value. Although the notation used here is different, this is essentially the reason G.Kron posed the 'topological degree of freedom' problem.
In order to be historically accurate we present five problems from the literature and give their combined solution, in brief first, and fill in details subsequently.

i. Forest of minimum size hybrid representation (Kishi and Kajitani [Kishi+Kajitani69]). Let a forest t be represented by a pair of sets (A_t, B_t) where $A_t \subseteq t, t \cap B_t = \emptyset$ such that $(A_{t_1}, B_{t_1}) = (A_{t_2}, B_{t_2})$ iff $t_1 = t_2$. Observe that there can be several pairs representing the same forest, for instance $(t, \phi), (\phi, E(G) - t)$ both represent t. We call $|A_t \bigcup B_t|$ the **size of the representation** (A_t, B_t). Find a forest which has the representation of minimum size.

ii. Maximum distance between two trees (Kishi and Kajitani [Kishi+Kajitani69]) Find two forests in a given graph which have the maximum distance between them (distance between two forests t_1 and t_2 is $|t_1 - t_2|$) i.e., the size of their union is the largest possible.

iii. The topological degree of freedom of an electrical network (Ohtsuki et al [Ohtsuki+Ishizaki+Watanabe70]) Select a minimum sized set of branch voltages and branch currents from which, by using Kirchchoff's voltage equations and Kirchhoff's current equations, we can find either the voltage or the current associated with each branch. The minimum size is called the **topological degree of freedom of the network**, equivalently, the **hybrid rank of the graph.**

iv. The Shannon switching game [Edmonds65b]
$\mathcal{G}$ is a graph with one of its edges say e_M 'marked'. There are two players - a 'cut' player and a 'short' player. The cut player, during his turn, removes (opens) an edge leaving the end points in place. The short player, during his turn, fuses the end points of an edge and removes it. Neither player is allowed to touch e_M. The aim of the cut player is to destroy all the paths between the end points of e_M (equivalently, destroy all circuits containing e_M). The aim of the short player is to fuse the end points of e_M (equivalently, destroy all cutsets containing e_M). The problem is to analyse this game to characterize situations where the cut or short player, playing second, can always win and to determine the winning strategy.

v. The maximum rank of a cobase submatrix (Iri [Iri68],[Iri69])
For a rectangular $(m \times n)$ matrix with linearly independent rows, let us call an $m \times (n - m)$ submatrix a **cobase** submatrix iff the remaining set of columns are from an identity matrix. The **term rank** of a matrix is the maximum number of nonzero entries in the matrix which belong to distinct rows and distinct columns. Find

- a cobase matrix of maximum rank, and
- a cobase matrix of minimum term rank

among all matrices row equivalent to the given matrix.

For the above five problems the solution involves essentially the same strategy: Find a set X (or a minimal set X_{min} or a maximal set X_{max}) such that

$$(2r * |\cdot|)(E(G)) = 2r(X) + |E(G) - X|.$$

Select a forest t which has maximal intersection with X. The representation $(t \cap X, (E(G)-t) \bigcap (E(G)-X))$ has the least size among all representations of all forests.

The maximum distance turns out to be the same as the above minimum size of representation. In [Kishi+Kajitani69] Kishi and Kajitani gave an algorithm for building a pair of maximally distant forests which is essentially the well known algorithm for building a base of the union of two matroids (see [Edmonds65a] for the case where the matroids are identical - essentially the same algorithm works for the general case).

Let t_X be a forest of the subgraph on X. Let $L_{\overline{X}}$ be a coforest of the graph on $G \times (E(G) - X)$ (the graph obtained by fusing the end points of edges in X and removing them). Select the branch voltages of t_X and the branch currents of $L_{\overline{X}}$ as the desired set of variables. As is easily seen, the topological degree of freedom is also the same as the minimum size of representation of a forest.

If $e_M \in X_{min}$, the short player can always win. If $e_M \in (E(G) - X_{max})$ the cut player can always win. If $e_M \in X_{max} - X_{min}$, whoever plays first can always win. The winning strategies involve the construction of appropriate maximally distant forests during every turn.

The solution is similar for the last problem. Let S be the set of all columns and let $r(\cdot)$ be the rank function on the collection of subsets of S. Then the maximum rank of a cobase matrix = the minimum term rank of a cobase matrix =
= $(2r* \mid \cdot \mid)(S) - r(S)$. Select two maximally distant bases (bases $\equiv$ maximally independent columns). Perform row operations so that an identity matrix appears coresponding to one of these. The submatrix corresponding to the complement of this base is the desired cobase matrix which has both maximum rank as well as minimum term rank.

Now we go into details of the solution to the above problems. First we note that the problem of building two maximally distant forests is the same as finding two bases of $\mathcal{M}(\mathcal{G})$ whose union is a base of $\mathcal{M}(\mathcal{G}) \vee \mathcal{M}(\mathcal{G})$. This can be solved by using the Algorithm Matroid Union of Section 12.2. The others are stated as exercises for which we have provided solutions. We need the following lemma, based on the matroid union operation. The reader needs to be familiar with Section 12.2 in order to understand the proof of the lemma.

Lemma 15.2.1 *Let t_1, t_2 $(\bar{t}_1, \bar{t}_2)$ be maximally distant forests (coforests) of graph $\mathcal{G}$. Let $E \equiv E(\mathcal{G})$. Then*

i. there exists a set $A \subseteq E (A^* \subseteq E)$ s.t.

(a) $A \supseteq E - (t_1 \cup t_2)$ $(A^* \supseteq E - (\bar{t}_1 \cup \bar{t}_2))$,

(b) $A \cap t_1, A \cap t_2$ *are disjoint forests of* $\mathcal{G} \cdot A$ *(*$A^* \cap \bar{t}_1, A^* \cap \bar{t}_2$ *are disjoint coforests of* $\mathcal{G} \times A^*$*).*

ii. *A set* $A \subseteq E$ *satisfies the above properties iff it satisfies*

$$(2r * \mid \cdot \mid)(E) = 2r(A) + \mid E - A \mid = \mid t_1 \cup t_2 \mid$$

(A set $A^* \subseteq E$ *satisfies the above properties iff it satisfies*

$$(2\nu * \mid \cdot \mid)(E) = 2\nu(A^*) + \mid E - A^* \mid = \mid \bar{t}_1 \cup \bar{t}_2 \mid,$$

where $\nu(A^*) = \nu(\mathcal{G} \times A^*)$*),*

iii. *there is a unique minimal set* $X_{min}(Y_{min})$ *and a unique maximal set* $X_{max}(Y_{max})$ *which satisfy the above properties of* $A(A^*)$*,*

iv. *An edge e belongs to* X_{min} *(e belongs to* Y_{min}*) iff there exist maximally distant forests* t_1, t_2 *(coforests* $\bar{t}_1, \bar{t}_2$*) s.t.* $e \in (E - (t_1 \cup t_2)), (e \in (E - (\bar{t}_1 \cup \bar{t}_2)))$*,*

v. $X_{max} = E - Y_{min}$ *and* $X_{min} = E - Y_{max}$*.*

Proof : It is easily verified that t_1, t_2 are maximally distant iff $\bar{t}_1, \bar{t}_2$ are maximally distant. We use the Algorithm Matroid Union of Section 12.2 and build a maximally distant pair of forests t_1, t_2. Let R be the set of all vertices of $\mathcal{G}(t_1, t_2)$ (which are edges of $\mathcal{G}$) reachable from $E - (t_1 \cup t_2)$.
Let R^* be the set of all vertices of $\mathcal{G}(\bar{t}_1, \bar{t}_2)$ reachable from $E - (\bar{t}_1 \cup \bar{t}_2)$. Then

i(a) by definition $R \supseteq E - (t_1 \cup t_2)$ and $R^* \supseteq E - (\bar{t}_1 \cup \bar{t}_2)$.

i(b) By Lemma 12.3.3 we have, $t_1 \cap R, t_2 \cap R$ are pairwise disjoint forests of $\mathcal{G} \cdot R$ and $\bar{t}_1 \cap \bar{t}_2 \cap R^*$ are pairwise disjoints coforests of $\mathcal{G} \times R^*$ (note that $(\mathcal{M}(\mathcal{G}))^* \cdot T = (\mathcal{M}(\mathcal{G} \times T))^*$).

ii. By the same lemma, R, R^* minimize respectively $2r(X) - \mid X \mid, X \subseteq E$ and $2\nu(X) - \mid X \mid, X \subseteq E$. Hence,

$$(2r * \mid \cdot \mid)(E) = \min_{X \subseteq E}(2r(X) - \mid X \mid) + \mid E \mid = 2r(R) + \mid E - R \mid .$$

Now, $2r(R) + \mid E - R \mid = 2r(R) - \mid R \mid + \mid E \mid = - \mid E - (t_1 \cup t_2) \mid + \mid E \mid = \mid t_1 \cup t_2 \mid$.

Similarly,

$$(2\nu * \mid \cdot \mid)(E) = 2\nu(R^*) + \mid E - R^* \mid = \mid \bar{t}_1 \cup \bar{t}_2 \mid .$$

If A is any other set satisfying properties (i(a)) and (i(b)), then it is easily seen that

$$2r(A) - \mid A \mid = - \mid E - (t_1 \cup t_2) \mid$$

so that

$$2r(A) + \mid E - A \mid = \mid t_1 \cup t_2 \mid = (2r * \mid \cdot \mid)(E).$$

If A violates (i(a)) it cannot contain R which by Lemma 12.3.3 is the minimal set s.t.

$(2r * \mid \cdot \mid)(E) = 2r(R)+ \mid E - R \mid$. If A violates (i(b)), an easy computation shows that

$$2r(A)+ \mid E - A \mid > \mid t_1 \cup t_2 \mid = 2r(R)+ \mid E - R \mid .$$

The dual has an identical proof.

iii. The function $2r(X)+ \mid E - X \mid$ is submodular and hence has a unique minimal and a unique maximal set minimizing it. (The unique minimal set is in fact R as mentioned above). Minimizing this function has earlier been shown to be equivalent to possessing the first two properties stated in this lemma.
Similarly for the dual.

iv. By Lemma 12.3.3, $E - R$ is the set of coloops of $\mathcal{M}(\mathcal{G}) \vee \mathcal{M}(\mathcal{G})$. Thus every element in R can be put outside some base of $\mathcal{M}(\mathcal{G}) \vee \mathcal{M}(\mathcal{G})$, equivalently, for each $e \in R$ there exist maximally distant forests t_1', t_2' of $\mathcal{G}$ s.t. $e \notin t_1' \cup t_2'$. Similarly for the dual.

v. We have $\nu(X) = \mid X \mid - r(E) + r(E - X)$. So

$$2\nu(X)+ \mid E - X \mid = (2r(E - X)+ \mid X \mid)+ \mid E \mid - 2r(E)$$

Hence, a set minimizes $2\nu(X)+ \mid E - X \mid$ iff its complement minimizes $2r(X)+ \mid E - X \mid$. The result is an immediate consequence.

□

Exercise 15.1 *Show that the minimum size among all representations of forests in $\mathcal{G}$ equals*

$$min_{X \subseteq E}(r(\mathcal{G} \ . \ X) + \nu(\mathcal{G} \times (E - X)),$$

where $E \equiv E(\mathcal{G})$.

Exercise 15.2 *Show that a subset $A \subseteq E(\mathcal{G})$ minimizes $r(\mathcal{G} \ . \ X) + \nu(\mathcal{G} \times (E - X))$ iff it minimizes $2r(X)+ \mid E - X \mid$, where $E \equiv E(\mathcal{G})$.*

Exercise 15.3 *Show that two forests t_1, t_2 of a graph $\mathcal{G}$ are* **maximally distant** *iff*

$$\mid t_2 - t_1 \mid = min_{X \subseteq E} r(\mathcal{G} \ . \ X) + \nu(\mathcal{G} \times (E - X)),$$

where $E \equiv E(\mathcal{G})$.

Exercise 15.4 *Describe the best strategy for the Shannon switching game for each of the three cases listed in the brief solution to the problem.*

Exercise 15.5 *Justify the algorithm given in the brief solution to the 'maximum rank of a cobase matrix problem' and hence prove that the maximum rank of a cobase matrix equals the minimum term rank.*

15.3 The Hybrid Rank Problem - Second Formulation

15.3.1 Introduction

Consider the problem of analyzing a network $\mathcal{N}$ whose devices are naturally partitioned into blocks with the device characteristic of each block available to us in both the conductance and the resistance form, i.e.,

$$\mathbf{v}_b - \mathbf{R}_b \mathbf{i}_b = \mathbf{s}_b \textit{ or } \mathbf{G}_b \mathbf{v}_b - \mathbf{i}_b = \hat{\mathbf{s}}_b$$

($\mathbf{R}_b$ and $\mathbf{G}_b$ are not necessarily diagonal matrices). When we break up the devices into A and B to use the hybrid analysis of Chapter 6 we would not like to split the blocks since the variables in each block are coupled. So we are led to the following **second formulation of the hybrid rank problem:**

Let $\mathcal{G}$ be a graph and let Π be a partition of $E(\mathcal{G})$. Find a partition $\{A, B\}$ of $E(\mathcal{G})$, that minimizes $r(\mathcal{G} \cdot A) + \nu(\mathcal{G} \times B)$ under the condition that A (and therefore B) is a union of blocks of Π.

We know that minimizing $r(\mathcal{G} \cdot A) + \nu(\mathcal{G} \times B)$ over any family of subsets is the same as maximizing $\mid A \mid -2r(\mathcal{G} \cdot A)$ $(= -(r(\mathcal{G} \cdot A) + \nu(\mathcal{G} \times B) + r(\mathcal{G}) - \mid E(\mathcal{G}) \mid))$ over the same family of subsets. This is an instance of the membership problem for a polymatroid, given a matroid expansion, described in the next subsection.

In practice, by using the method of multiport decomposition (see Chapter 8), one can replace each subnetwork by a forest subgraph, which would be adequate to represent the topological relationship between different subnetworks. Below, when we model the second formulation in terms of matroids, we therefore assume that the expansion has an independent set in place of a single element of the polymatroid. We give two ways of solving the above problem. The first solves a more general version in terms of matroids. The second converts this problem into a series of simple flow problems.

15.3.2 Membership Problem with Matroid Expansion

We begin with the **matroid version** of the **second formulation of the hybrid rank problem:**

Let $\mathcal{M}$ be a matroid on $\hat{S}$ and let S be a partition of $\hat{S}$. Let

$$f(\cdot) \equiv r_{fus \cdot S}(\cdot) \ (i.e., f(X) \equiv r(\bigcup_{N_i \in X} N_i)).$$

Let $g(\cdot)$ be a weight function on S defined by $g(e) \equiv f(e) \ \ \forall e \in S$. Find the subset of S which minimizes $f(X) + f^*(S - X)$, where $f^*(\cdot)$ is the comodular dual of $f(\cdot)$ relative to the weight function $g(\cdot)$.

The following exercise shows that the above problem is related to the membership problem for a polymatroid, given a matroid expansion.

Exercise 15.6 *Let $f(\cdot)$ be a polymatroid rank function on subsets of S. Let $f^*(\cdot)$ be the dual of $f(\cdot)$ relative to the weight function $g(\cdot)$ where $g(e) \equiv f(e)\ \forall e \in S$. Let*

$$\textit{the hybrid rank of } f(\cdot) \equiv \min_{X \subseteq S} f(X) + f^*(S - X).$$

Show that

i. *the hybrid rank $f(\cdot) =$ the hybrid rank of $f^*(\cdot)$.*

ii. *a subset K minimizes $f(X) + f^*(S - X)$ iff it maximizes $g(X) - 2f(X)$.*

iii. *the hybrid rank of a contraction, a restriction or a minor of $f(\cdot)$ cannot be more than the hybrid rank of $f(\cdot)$.*

We now state the **membership problem for a polymatroid, given a matroid expansion:**
Let $\mathcal{M}$ be a matroid on $\hat{S}$ and let S be a partition of $\hat{S}$. Let

$$f(\cdot) \equiv r_{fus \cdot S}(\cdot)\ (i.e., f(X) \equiv r(\bigcup_{N_i \in X} N_i)).$$

Find the subset of S which maximizes $g(X) - f(X), X \subseteq S$, where $g(\cdot)$ is a weight function on S.

We give a simple general solution to this problem using the matroid union operation. For the case where the matroid is graphic, a more efficient solution is given later. First we introduce some convenient notation.

Let $f(\cdot)$ be an integral polymatroid rank function on subsets of $S \equiv \{e_1, \cdots, e_n\}$. Let $\hat{S}$ be the set obtained from S by replacing each e_i by the set $\hat{e}_i$, with the condition that when e_i, e_j are distinct we have $\hat{e}_i \cap \hat{e}_j = \emptyset$. Let $H(\cdot)$ be a function from subsets of S to subsets of $\hat{S}$ defined by $H(X) \equiv \bigcup_{e_i \in X} \hat{e}_i, X \subseteq S$. Let $\mathcal{M}$ be a matroid on $\hat{S}$ whose rank function $r(\cdot)$ satisfies the following:

$$f(X) = r(H(X)) \qquad \forall\ X \subseteq S.$$

We say that the $\mathcal{M}$ or $r(\cdot)$ is an **expansion** of $f(\cdot)$ while $f(\cdot)$ is a **fusion** (**aggregation**) of $r(\cdot)$. Henceforth we take $\mid \hat{e} \mid = r(\hat{e}) = f(e)\ \forall\ e \in S$.
Let $g(\cdot)$ be a positive integral weight function on S. If for some e', $g(e') > f(e')$, it is easy to see (Exercise 15.9) that e' must necessarily belong to any set that maximizes $g(X) - f(X)$. We can therefore work with $S - e'$ in place of S, $f \diamond (\mathbf{S} - \mathbf{e}')(\cdot)$ in place of $f(\cdot)$ and $g/(\mathbf{S} - \mathbf{e}')(\cdot)$ in place of $g(\cdot)$. So, without loss of generality, we may assume that $g(e) \leq f(e)\ \forall e \in S$.
Let $\mathcal{M}_\alpha$ be the matroid on $\hat{S}$ with rank function $\alpha(\cdot)$ defined by

$$\alpha(\cdot) = r_{e_1} \oplus \cdots \oplus r_{e_n}(\cdot),$$

where

$$\begin{aligned} r_{e_i}(\hat{X}) &\equiv \mid \hat{X} \mid, \hat{X} \subseteq \hat{e}_i, \mid \hat{X} \mid \leq r(\hat{e}_i) - g(e_i) \\ &\equiv r(\hat{e}_i) - g(e_i),\ \textit{otherwise.} \end{aligned}$$

(i.e., $\mathcal{M}_\alpha$ is a direct sum of uniform matroids on the $\hat{e}_i$).

Exercise 15.7 Derived matroids are expansions of derived polymatroids: *Let $f(\cdot)$ be a polymatroid rank function on subsets of S and let $\mathcal{M}$ on $\hat{S}$ be a matroid expansion of $f(\cdot)$ with rank function $r(\cdot)$. Further let $\mid \hat{e} \mid = r(\hat{e}) \ \forall \ e \in S$. Let $\hat{T} \equiv H(T)$. Then*

i. $f/\mathbf{T}(\cdot)$ *has the expansion* $\mathcal{M} \cdot \hat{T}$

ii. $f \diamond \mathbf{T}(\cdot)$ *has the expansion* $\mathcal{M} \times \hat{T}$

iii. let $f^*(\cdot)$ denote the dual of $f(\cdot)$ with respect to the weight function $g(\cdot)$ defined by $g(e) \equiv f(e) \ \forall \ e \in S$. Then $\mathcal{M}^*$ is an expansion of $f^*(\cdot)$.

The following theorem assumes the notation of page 635 for $g(\cdot), f(\cdot), r(\cdot), \alpha, \mathcal{M}$, $\mathcal{M}_\alpha$ etc.

Theorem 15.3.1 *Let $f(\cdot)$ be an integral polymatroid rank function and let $g(\cdot)$ be a positive weight function with $g(e) \leq f(e) \ \forall e \in S$. Let $r(\cdot), \alpha(\cdot), \mathcal{M}, \mathcal{M}_\alpha$ be as above. Then,*

i. $g(X) - f(X) = \mid H(X) \mid - (r + \alpha)(H(X)), X \subseteq S$.

ii. Let $\hat{R}$ be the set of noncoloops of $\mathcal{M} \vee \mathcal{M}_\alpha$ and let $\hat{R}_1$ be its closure relative to $\alpha(\cdot)$. Then $\hat{R}_1$ maximizes $\mid \hat{X} \mid - (r + \alpha)(\hat{X}), \hat{X} \subseteq \hat{S}$,
$\hat{R}_1 = H(R_1)$ for some $R_1 \subseteq S$, and
$g(R_1) - f(R_1) = \mid \hat{R}_1 \mid - (r + \alpha)(\hat{R}_1)$.

iii.

$$\max_{X \subseteq S} g(X) - f(X) = \max_{\hat{X} \subseteq \hat{S}} \mid \hat{X} \mid - (r + \alpha)(\hat{X}), \hat{X} \subseteq \hat{S}.$$

iv. The vector $\mathbf{g}$ *belongs to the polymatroid P_f iff there exists a base b of $\mathcal{M}$ s.t. $|b \cap \hat{e}| \geq g(\hat{e}) \ \forall \ e \in S$.*

We need the following lemma for proving this theorem.

Lemma 15.3.1 *Let $\mathcal{M}_1, \mathcal{M}_2$ be matroids on $\hat{S}$ and let $r_1(\cdot), r_2(\cdot)$ respectively, be their rank functions. Let b_1, b_2 be maximally distant bases of $\mathcal{M}_1, \mathcal{M}_2$ respectively (i.e., $b_1 \cup b_2$ is a base of $\mathcal{M}_1 \vee \mathcal{M}_2$). Let $\hat{R}$ be the set of noncoloops of $\mathcal{M}_1 \vee \mathcal{M}_2$, and let $\hat{R}_1$ denote its closure in the matroid $\mathcal{M}_1$. Then*

i. $\hat{R}_1$ maximizes $\mid \hat{X} \mid - (r_1 + r_2)(\hat{X}), \hat{X} \subseteq \hat{S}$,

ii. $\hat{R}_1 - \hat{R}$ is a set of coloops of $\mathcal{M}_2 \cdot \hat{R}_1$,

iii. $\hat{R}_1$ contains no coloops of $\mathcal{M}_1 \cdot \hat{R}_1$,

iv. $b_1 \cap \hat{R}_1, b_2 \cap \hat{R}_1$ are disjoint bases of $\mathcal{M}_1 \cdot \hat{R}_1, \mathcal{M}_2 \cdot \hat{R}_2$ respectively.

Proof : By Lemma 12.3.3, $\hat{R}$ maximizes $\mid \hat{X} \mid -(r_1+r_2)(\hat{X}), \hat{X} \subseteq \hat{S}$. If $\hat{R}_1 = \hat{R}$ we are done. Let $\hat{R}_1 \supset \hat{R}$. We must have

$$\mid \hat{R}_1 \mid -(r_1+r_2)(\hat{R}_1) \leq \mid \hat{R} \mid -(r_1+r_2)(\hat{R}).$$

But $r_1(\hat{R}_1) = r_1(\hat{R})$. Hence, $\mid \hat{R}_1 - \hat{R} \mid \leq r_2(\hat{R}_1) - r_2(\hat{R})$. Thus, the inequalities have to be equalities. Hence,
i. $\hat{R}_1$ maximizes $\mid \hat{X} \mid -(r_1+r_2)(\hat{X}), \hat{X} \subseteq \hat{S}$ and
ii. $\hat{R}_1 - \hat{R}$ is a set of coloops of $\mathcal{M}_2 \cdot \hat{R}_1$.
We know that $\hat{R}$ contains no coloops of $\mathcal{M}_1 \cdot \hat{R}$ by Lemma 12.3.3 and $\hat{R}_1 - \hat{R}$ is spanned by $\hat{R}$ in $\mathcal{M}_1$. Hence,
iii. $\hat{R}_1$ contains no coloops of $\mathcal{M}_1 \cdot \hat{R}_1$.
Now by the above mentioned lemma, $b_1 \cap \hat{R}, b_2 \cap \hat{R}$ are disjoint bases of $\mathcal{M}_1 \cdot \hat{R}, \mathcal{M}_2 \cdot \hat{R}$ respectively and $(\hat{S} - \hat{R}) \subseteq b_1 \cup b_2$. Since $\hat{R}_1 - \hat{R}$ is dependent on $b_1 \cap \hat{R}$ in $\mathcal{M}_1 \cdot \hat{R}$, it is clear that $b_1 \cap \hat{R}$ is a base of $\mathcal{M}_1 \cdot \hat{R}_1$. We know that $b_2 \cap \hat{R}$ is a base of $\mathcal{M}_2 \cdot \hat{R}$. Using (ii) above it follows that $b_2 \cap \hat{R}_1$ is a base of $\mathcal{M}_2 \cdot \hat{R}_1$. Thus,
iv. $b_2 \cap \hat{R}_1, b_1 \cap \hat{R}_1$ are disjoint bases of $\mathcal{M}_1 \cdot \hat{R}_1, \mathcal{M}_2 \cdot \hat{R}_1$, respectively.

□

Proof of Theorem 15.3.1:
i. By definition $H(X) \equiv \bigcup_{e_i \in X} \hat{e}_i$ and $\mid \hat{e}_i \mid = f(e_i) = r(\hat{e}_i)$. Hence,

$$\mid H(X) \mid = \mid \bigcup_{e_i \in X} \hat{e}_i \mid = \sum_{e_i \in X} r(\hat{e}_i).$$

We are given that $r(\cdot), f(\cdot)$ satisfy $f(X) = r(H(X)), X \subseteq S$ and that

$$\alpha(H(X)) = \sum_{e_i \in X} r_{e_i}(\hat{e}_i) = \sum_{e_i \in X} (r(\hat{e}_i) - g(e_i)).$$

Hence,

$$\begin{aligned} \mid H(X) \mid -(r+\alpha)(H(X)) &= \sum_{e_i \in X} r(\hat{e}_i) - f(X) - (\sum_{e_i \in X} (r(\hat{e}_i) - g(e_i))) \\ &= g(X) - f(X) \textit{ as required.} \end{aligned}$$

ii. By Lemma 15.3.1, we know that $\hat{R}_1$ maximizes $\mid \hat{X} \mid -(r+\alpha)(\hat{X}), \hat{X} \subseteq \hat{S}$. Next let $\hat{e}_i \cap \hat{R}_1 \neq \emptyset$. We will show that $\hat{e}_i \subseteq \hat{R}_1$. By the definition of $\alpha_{e_i}(\cdot)$ any subset of $\hat{e}_i$ whose size does not exceed $(r(\hat{e}_i) - g(\hat{e}_i))$ is independent in $\mathcal{M}_\alpha$. If $\mid \hat{e}_i \cap \hat{R}_1 \mid \leq r(\hat{e}_i) - g(\hat{e}_i)$, then $\hat{e}_i \cap \hat{R}_1$ would be a set of coloops of $\mathcal{M}_\alpha \cdot \hat{R}_1$. Since, by Lemma 15.3.1, $\hat{R}_1$ contains no coloops of $\mathcal{M}_\alpha \cdot \hat{R}_1$, we conclude that $\mid \hat{e}_i \cap \hat{R}_1 \mid > r(\hat{e}_i) - g(\hat{e}_i) = \alpha_{e_i}(\hat{e}_i)$. Next $\hat{R}_1$ is closed relative to $\alpha(\cdot)$. But all elements in $\hat{e}_i$ are dependent on any subset of $\hat{e}_i$ of cardinality $\alpha(\hat{e}_i)$. Hence, $\hat{R}_1 \supseteq \hat{e}_i$. Thus, $\hat{R}_1 = H(R_1)$ for some R_1. Hence,

$$g(R_1) - f(R_1) = \mid \hat{R}_1 \mid -(r+\alpha)(\hat{R}_1). \qquad (*)$$

iii. (i) above implies that

$$max_{X \subseteq S} g(X) - f(X) \leq max_{\hat{X} \subseteq \hat{S}} \mid \hat{X} \mid -(r+\alpha)(\hat{X})$$

while (ii) implies the reverse inequality. This proves the required result.

iv. The vector $\mathbf{g}$ belongs to P_f iff

$$max_{X \subseteq S} g(X) - f(X) = max_{\hat{X} \subseteq \hat{S}} \mid \hat{X} \mid -(r+\alpha)(\hat{X}) \leq 0.$$

This happens iff there are bases b, b_α of $\mathcal{M}, \mathcal{M}_\alpha$ respectively s.t. $b \cup b_\alpha = \hat{S}$, i.e., iff

$$(b \cap \hat{e}) \cup (b_\alpha \cap \hat{e}) = \hat{e} \ \forall \ e \in S$$

i.e., iff $\mid b \cap \hat{e} \mid + r(\hat{e}) - g(\hat{e}) \geq r(\hat{e})$
i.e., iff $\mid b \cap \hat{e} \mid \geq g(\hat{e})$ *as required.*

□

Suppose we are given a matroid expansion of $f(\cdot)$. It is easy to build an expansion of $kf(\cdot), k$ a positive integer, as follows. Build k disjoint copies of $\hat{S}_1, \cdots, \hat{S}_k$. An element $e_i \in S$ now has $kf(e_i)$ copies in $\bigcup_{i=1}^{k} \hat{S}_i$. Let us call this set $\hat{e}_i(k)$. Build copies $\mathcal{M}_1, \cdots, \mathcal{M}_k$ of the matroid $\mathcal{M}$ on $\hat{S}_1, \cdots, \hat{S}_k$ with rank functions $r_1(\cdot), \cdots, r_k(\cdot)$ respectively. Let $r_+(\cdot) \equiv \bigoplus_{i=1}^{k} r_i(\cdot)$. Define $H_k(X) \equiv \bigcup_{e_i \in X} \hat{e}_i(k)$, $X \subseteq S$. Since $f(X) = r(\bigcup_{e_i \in X} \hat{e}_i)$ we must have

$$kf(X) = r_+(\bigcup_{e_i \in X} \hat{e}_i(k)).$$

We can, therefore, handle the problem of maximizing $g(X) - kf(X), X \subseteq S, k \in Z_+$ which arises in connection with the construction of the principal partition of $(f(\cdot), g(\cdot))$, by using Theorem 15.3.1.

Exercise 15.8 Concerning Theorem 15.3.1 - Good expansion:
Let $\mathcal{M}$ be the expansion of $f(\cdot)$. Show that

i. If C is a circuit of $\mathcal{M} \cdot \hat{e}$ and $a \in C$, then $\mathcal{M} \cdot (\hat{S} - a)$ is an expansion of $f(\cdot)$ with $\hat{e} - a$ in place of $\hat{e}$.

ii. There exists an expansion of $f(\cdot)$ with no more than $\sum_{e \in S} f(e)$ elements.

Exercise 15.9 Concerning Theorem 15.3.1 - Better expansion:
Let $h(e) \equiv f(S) - f(S - e), e \in S$. Prove:

i. If $g(e) > f(e)$ then e belongs to every set that maximizes $g(X) - f(X), X \subseteq S$.

ii. $(f-h)(\cdot)$ is an integral polymatroid rank function and a set maximizes $g(X) - f(X), X \subseteq S$ iff it maximizes $(g-h)(X) - (f-h)(X), X \subseteq S$.

iii. If $\mathcal{M}$ is an expansion for $f(\cdot)$, then the expansion of $(f-h)(\cdot)$ can be built as follows: For each $e \in S$ find a base b_e of $\mathcal{M}$. s.t. $\mid b_e \cap (\hat{S} - \hat{e}) \mid = r(\mathcal{M} \cdot (\hat{S} - \hat{e}))$. Let $h(e) \equiv \mid b_e \cap \hat{e} \mid$. Let $\mathcal{M}_{red} \equiv \mathcal{M} \times (\hat{S} - \bigcup_{e \in S}(b_e \cap \hat{e}))$. Then $\mathcal{M}_{red}$ is an expansion of $(f-h)(\cdot)$.

Exercise 15.10
Let $f(\cdot)$ be an integral polymatroid rank function on subsets of S. Let $f^*(\cdot)$ be the comodular dual of $f(\cdot)$ relative to $g(\cdot)$ where $g(e) \equiv f(e), e \in S$. Let the notation for matroid expansion be as in Theorem 15.3.1. If $\mathcal{M}$ is an expansion of $f(\cdot)$ on $\hat{S}$, we say a pair $(\hat{T}_1, \hat{T}_2), \hat{T}_1, \hat{T}_2 \subseteq \hat{S}$ is a **common independent pair** *of $\mathcal{M}, \hat{\mathcal{M}}$ relative to S iff*

- *T_1, T_2 are independent in $\mathcal{M}, \hat{\mathcal{M}}$ respectively and*
- $\mid \hat{T}_1 \cap \hat{e} \mid = \mid \hat{T}_2 \cap \hat{e} \mid \ \forall\ e \in S$.

The **size** *of $(\hat{T}_1, \hat{T}_2)$ is defined to be $\mid \hat{T}_1 \mid \ (= \mid \hat{T}_2 \mid)$.*
Prove:

i. If $\mathcal{M}$ is an expansion of $f(\cdot)$, $\mathcal{M}^*$ is an expansion of $f^*(\cdot)$. Further if $f(S) = f(S-e) \ \ \forall e \in S$, then $f^*(e) = f(e)$.

ii. (Assuming $\mid \hat{e} \mid = f(e)\ \forall e \in S$)

$$\begin{aligned} min_{X \subseteq S}(f(X) + f^*(S-X)) &= max\ (size\ of\ common\ independent \\ &\quad pair\ of\ \mathcal{M}, \mathcal{M}^*\ relative\ to\ S) \\ &\geq max\ size\ of\ common\ independent\ set \\ &\quad of\ \mathcal{M}, \mathcal{M}^*. \end{aligned}$$

Further, there is an expansion $\mathcal{M}_1$ of $f(\cdot)$ s.t. inequality above becomes an equality.

Complexity of solving the membership problem given a matroid expansion

Before using the present method, the size of the problem has to be reduced as in Exercise 15.8 and Exercise 15.9, i.e.,

i. we may assume that $\mid \hat{e} \mid = f(e)$

ii. if $g(e) > f(e)$ we work with $(S-e)$ in place of S and $f \diamond (\mathbf{S}-\mathbf{e})(\cdot)$ in place of $f(\cdot)$ and

iii. we work with $(f-h)(\cdot)$ and $(g-h)(\cdot)$, in place of $f(\cdot), g(\cdot)$ (where $h(e) = f(S) - f(S-e), e \in S$ as in Exercise 15.9).

Without loss of generality, we may assume that $g(e) \leq f(e) \ \ \forall e \in S$ and $h(\cdot) = 0$. The rank of the matroid expansion of $f(\cdot)$ is $f(S)$ and the size of $\hat{S}$ is $\sum_{e_i \in S} f(e_i)$. Let $m \equiv \max_{e_i \in S} f(e_i)$. Now the complexity of Algorthm Matroid Union is in terms of calls to the independence oracle and some elementary steps. The independence oracle for $\mathcal{M}_\alpha$ in the present case is trivial since $\mathcal{M}_\alpha$ is the direct sum of uniform matroids. So we will speak only of calls to the independence oracle of $\mathcal{M}$.

Let r denote $r(\mathcal{M})$. Let b, b_α be the current bases of $\mathcal{M}, \mathcal{M}_\alpha$. To build $\mathcal{G}(b, b_\alpha)$ takes at most $(\mid \hat{S} \mid -r)r$ calls to the independence oracle of $\mathcal{M}$. This has to be done at most $(\mid \hat{S} \mid -r)$ times. So the **number of calls to the independence oracle** is $O(r(\mid \hat{S} \mid -r)^2)$. The complexity in terms of $\mid S \mid$, noting that $r, \mid \hat{S} \mid$ are not more than $m \mid S \mid$, is $O(m^3 \mid S \mid^3)$. Next we consider the complexity of performing the bfs in $\mathcal{G}(b, b_\alpha)$, which takes us from $\hat{S} - (b \cup b_\alpha)$ to an element in $b \cap b_\alpha$. The number of edges of this graph is $O((\mid \hat{S} \mid -r)r)$. So the complexity of the search is also $O((\mid \hat{S} \mid -r)r)$. Thus, the **number of elementary steps in Algorithm Matroid Union** is $O(r(\mid \hat{S} \mid -r)^2)$ which again reduces to $O(m^3 \mid S \mid^3)$. If m is less than $\mid S \mid$, the present method compares favourably with that of the general pseudopolynomial algorithm of Cunningham [Cunningham84] ($O(m \mid S \mid^{6.5} log(m \mid S \mid))$.)

The membership problem, as it is usually stated, does not assume the availability of a matroid expansion. So the present algorithm cannot strictly be regarded as a solution to that problem.

Exercise 15.11 *Rewrite the algorithm for the solution of the membership problem given a matroid expansion, using only bases of the matroid $\mathcal{M}$.*

15.3.3 Membership Problem with Graphic Matroid Expansion

The special case where we need to maximize $g(X) - \lambda f(X), X \subseteq S$, given a graphic matroid expansion of $f(\cdot)$, is more relevant to electrical network analysis. For this case two alternative flow based procedures are possible. Neither of these methods attempts to build maximally distant trees. The optimum set, or a set contained in it, appears either in the source or the sink side of a min cut in the flow graph. The first of these procedures is edge based and is given below. The second procedure would be evident from the discussions in Section 14.9 on how to convert certain principal partition problems related to one side of a bipartite graph into principal lattice of partition problems (see Theorem 14.9.1). The complexity of the procedure, presented here, is substantially better than the one made possible by Theorem 15.3.1. We, however, follow the notation of the above mentioned theorem.

The flow technique that we describe below is similar to that of Imai [Imai83] in that the flow graphs are identical. The difference lies in the following: We grow sets until a minimal nonvoid set can be found with positive value of $h^t(X)$ (defined below). This we contract and work with an updated $h^t(\cdot)$. We continue this procedure until no sets can be found with a positive value of current $h^t(\cdot)$. At this stage the union of all the contracted sets gives us the desired optimum set. Imai computes the optimal set using the 'fundamental functions' of the concerned polymatroid.

Let $\mathcal{G}$ be a graph with $\hat{S} \equiv E(\mathcal{G})$ and rank function $r(\cdot)$. Let S be a partition of $\hat{S}$ and let $H(X) \equiv \bigcup_{e_i \in X} e_i,\ X \subseteq S$. Let $f(\cdot)$ be defined on subsets of S s.t. $f(X) \equiv r(H(X))$. Henceforth, in this subsection, $H(X)$ would be **denoted** by $\hat{X}$.

Let $g(\cdot)$ be a weight function on S. We further **assume** that

$$\textit{the subgraph of } \mathcal{G} \textit{ on } \hat{e} \textit{ is connected for each } e \in S. \qquad (*)$$

This assumption is necessary for the following procedure to work. (Note that such an assumption would not be required for a procedure based on Theorem 15.3.1).
Let $V(X) \equiv$ set of end points of edges of $\mathcal{G}$ in members of $X, X \subseteq S$. We then have, using Assumption $(*)$, $f(X) = (\mid V \mid -1)_t(X)$. Let $h(X) \equiv$
$\equiv g(X) - \lambda(\mid V \mid -1)(X)$, $X \subseteq S$. We remind the reader that $h^t(X) \equiv$
$\equiv max_{\Pi \in \mathcal{P}_X} \bar{h}(\Pi)$. In order to maximize $g(X) - \lambda f(X), X \subseteq S$, we need to maximize $h^t(X) \equiv g(X) - \lambda(\mid V \mid -1)_t(X)$, $X \subseteq S$.

Let us start with a set X_o s.t. $h^t(X) \leq 0 \;\; \forall X \subseteq X_o$. Let $e \notin X_o$. We find the minimal nonvoid subset X_e that maximizes $h(X)$ among subsets of $X_o \cup e$ by using an algorithm called say $Max(S, h)$.
If $h(X_e) \leq 0$, we conclude that $X_o \cup e$ satisfies $h^t(X) \leq 0 \; \forall X \subseteq X_o \cup e$. For, if Π is any partition of $Y \subseteq X_o \cup e$, then $\bar{h}(\Pi) = \sum_{N_i \in \Pi} h(N_i) \leq 0$.
If $h(X_e) > 0$ we contract $\mathcal{G}$ to $\mathcal{G} \times (\hat{S} - \hat{X}_e)$. The function $V(\cdot)$ is now defined over subsets of $S - X_e$, with $V(X) \equiv$ set of end points of edges of $\mathcal{G} \times (\hat{S} - \hat{X}_e)$ which are present in members of $X, X \subseteq S - X_e$. The function $r(\cdot)$ is now the rank function of $\mathcal{G} \times (\hat{S} - \hat{X}_e)$ and $g(\cdot)$, the restriction of the original weight function to $(S - X_e)$. The functions $h(\cdot)$, $h^t(\cdot)$ are defined as before in terms of $g(\cdot)$ and $V(\cdot)$.
We initialize the algorithm $Max(S - X_e, h)$ at $X_o - X_e$ and repeat the process.
The process terminates when the current $S - K$ has no nonvoid subset at which $h(\cdot)$ takes a positive value. We then declare K to be the minimal set that maximizes

$$g(X) - \lambda(\mid V \mid -1)_t(X), X \subseteq S.$$

The flow formulation

(The discussion that follows needs familiarity with Subsection 3.6.10 and Subsection 11.6.3. It would help to have Figure 11.3 at hand. For the present discussion one may replace, in that figure, X by X_e, the vertex v_1 by e, $w_L(v_1)$ by ∞, v_2 by e_i, $w_L(v_2)$ by $g(e_i)$ and $w_R(v)$ by 1).

We now consider the problem of maximizing $h(X)$ among nonvoid subsets of Z. This is equivalent to minimizing $\lambda(\mid V \mid (X)) + g(Z - X), \emptyset \subset X \subseteq Z$. This is a flow problem (as described in Subsection 3.6.10 and in Subsection 11.6.3). But we have to be careful to confine ourselves only to nonvoid subsets. As in Exercise 11.33 we do this by forcing the newly added element e to be a member of the sets over which optimization is carried out. The flow graph for this problem is built as follows: First build the bipartite graph $B \equiv (Z, V(Z), E_Z)$ with Z as the left vertex set, $\mathcal{G}$ being the current graph, $V(Z)$ in $\mathcal{G}$ as the right vertex set, an edge between $v \in V(Z)$ and $e_i \in Z$ iff in the graph $\mathcal{G}$ there is an edge in $\hat{e}_i$ that is incident on v. Now add a source vertex s and join it to each vertex in Z, a sink vertex t and join it to each vertex in $V(\mathcal{G})$. The capacities are:

- edge (s, e_i) has capacity $g(e_i), e_i \neq e, e_i \in Z$,

- edge (s, e) has capacity ∞,
- (e_i, v) has capacity ∞, $\forall v \in V(Z)$,
- (v, t) has capacity λ, $\forall v \in V(Z)$.

The nearest source side min cut of this flow graph would have the form $(s \uplus X_e \uplus V(X_e), t \uplus (Z - X_e) \uplus (V(Z) - V(X_e))), e \in X_e$. As discussed in Subsection 3.6.10, $g(Z - X_e) + \lambda(|V|(X_e))$ would have the minimum value among all nonvoid subsets of Z and X_e would be a minimal nonvoid such set. Hence, $h(X_e)$ would have the maximum value among all nonvoid subsets of Z and X_e would be a minimal nonvoid such set.

From this flow graph, we can build the flow graph corresponding to $\mathcal{G} \times (\hat{Z} - \hat{X}_e)$ by first building the subgraph B' of B on $X_e \cup \Gamma_L(X_e)$. Next $\Gamma_L(X_e)$ is partitioned into $V_1, \cdots V_k$ corresponding to the connected components $B'_1, \cdots, B'_k$ of B'. $V_1, \cdots, V_k$ are now made into single nodes, X_e and all edges incident on X_e are deleted. Single edges go from V_1 to $t, \cdots, V_k$ to t, each with capacity λ. (It can however be shown easily that the bipartite graph B' is connected, i.e., $k = 1$). All other vertices and edges of the original flow graph and the capacities associated with the latter are left unchanged.

Justification

To justify the above algorithm for maximizing $h^t(X) \equiv g(X) - \lambda(|V) - 1)_t(X)$, we only need to explain the contraction step.
Let $h(X) \leq 0$ $, \emptyset \subset X \subseteq X_o$ and let X_e be the minimal set that maximizes $h(\cdot)$ among nonvoid subsets of $X_o \cup e$. If $h(X_e) > 0$, it follows that $\bar{h}(\cdot)$ reaches its maximum among partitions of X_e at $\{X_e\}$ and $h(X_e) = h^t(X_e)$. Further it is clear since $h(X) \leq 0$ $, \emptyset \subset X \subset X_e$ that $h^t(X) \leq 0$ $\forall X \subset X_e$. Since $h^t(\cdot)$ is a supermodular function, use of the supermodular inequality and the fact that $h^t(X) < h^t(X_e)$ $\forall X \subset X_e$ would reveal that X_e is a subset of any set that maximizes $h^t(\cdot)$ over subsets of S. So we can work with the contraction of $h^t(\cdot)$ over subsets of $S - X_e$, and if Y is the minimal set that maximizes the latter function, $Y \cup X_e$ would be the minimal set that maximizes $h^t(\cdot)$. Now

$$(h^t) \diamond (\mathbf{S} - \mathbf{X}_e)(X) = g \diamond (\mathbf{S} - \mathbf{X}_e)(X) - \lambda r \diamond (\hat{\mathbf{S}} - \hat{\mathbf{X}}_e)(\hat{X}), X \subseteq S.$$

Since $g(\cdot)$ is a weight function, contraction is the same as restriction. The function $r \diamond (\hat{\mathbf{S}} - \hat{\mathbf{X}}_e)(\cdot)$ is the rank function of the graph $\mathcal{G} \times (\hat{S} - \hat{X}_e)$.

Exercise 15.12 Improvement of graphic expansion: *Let $\mathcal{G}$ be a graph on $\hat{S}$ with $r(\cdot)$ as its rank function. Let $r(\cdot)$ be the expansion for $f(\cdot)$ on subsets of S. We follow the notation of Exercise 15.9. Show that*

i. The matroid $\mathcal{M}_{red} \equiv \mathcal{M}(\mathcal{G} \times (\hat{S} - \bigcup_{e \in S}(b_e \cap \hat{e})))$, where $\mathcal{M}(\mathcal{G})$ denotes the matroid whose independent sets are subforests of $\mathcal{G}$, is an expansion of $f(\cdot)$.

ii. *Let us assume without loss of generality that* $\mid \hat{e} \mid = r(e)\ \forall e \in S$ *(see Exercise 15.8). In the graph* $\mathcal{G}_{red} \equiv \mathcal{G} \times (\hat{S} - \bigcup_{e \in S}(b_e \cap \hat{e}))$*, if* $\mathcal{G} \,.\, \hat{e}$ *is connected, then* $\mathcal{G}_{red} \cdot (\hat{e} - b_e \cap \hat{e})$ *would be a tree with no node incident only on edges of* $\hat{e}$*. Further no edge of the tree would be a cutset of* $\mathcal{G}_{red}$.

iii. *Assume, for simplicity, that* $\mathcal{G}_{red} \cdot (\hat{e} - b_e \cap \hat{e})$ *is connected for each* e*. If we replace each* $(\hat{e} - b_e \cap \hat{e})$ *in* $\mathcal{G}_{red}$ *by a tree on the same set of nodes, the rank function of the resulting graph would be an expansion for* $f(\cdot)$.

We note that the set $\hat{S}$ that occurs in the above discussions would be the edge set of a reduced network. In the present case it can be built very easily from the original graph of the network. (For matroids, in Exercise 15.9, we contracted certain interior 'hidden' elements within each $e \in S$). Here each e represents a connected subgraph. For solving the present membership problem we can replace each subgraph by a tree on the 'boundary nodes' of the subgraph (see Exercise 15.12), contracting branches appropriately to eliminate cutsets. So each e would represent no more than $r(\mathcal{G}_{red} \cdot e)$ edges in the new reduced graph. However, $g(e)$ remains the same as before. In most practical network problems large subnetworks have comparatively few boundary nodes. Thus the $\hat{S}$ specified in the above discussions would be much smaller in size than the edge set of the original graph (say $\approx 10\%$). If this is kept in mind, it would be clear from the discussion below that the algorithm we have presented is practical in the sense that it can be **included in the preprocessing stage of a circuit simulator**.

Complexity

Now we discuss the complexity of the above algorithm - first for general λ and later for $\lambda = 2$, in both cases with $g(e) = f(e) = r(\hat{e})\quad \forall e \in S$. We remind the reader that $|\hat{S}| = \sum_{e \in S} f(e)$.

i. General λ: We have to perform $\mid S \mid$ flow maximizations. The number of edges in the flow graph is $O(\mid \hat{S} \mid)$. The complexity of one flow maximization using the Sleator algorithm is $O(\mid S \mid (\mid \hat{S} \mid log \mid \hat{S} \mid))$ elementary steps. (The Sleator algorithm [Sleator80] proceeds in stages. Each stage has complexity $O(\mid \hat{S} \mid \log \mid \hat{S} \mid)$. The number of stages is the length of the longest undirected path from source to sink and is of $O(min(\mid S \mid, \mid V \mid))$. Here we may assume $\mid S \mid \ll \mid V \mid$). So the overall complexity is $O(\mid S \mid^2 \mid \hat{S} \mid \log \mid \hat{S} \mid)$. If $m \equiv \max_{e \in S} f(e)$, this reduces to $O(m \mid S \mid^3 \log(m \mid S \mid))$ while the earlier method based on Theorem 15.3.1 was $O(m^3 \mid S \mid^3)$. When $\mid S \mid \leq \sqrt{\mid \hat{S} \mid}$, the above complexity is $O(\mid \hat{S} \mid^2 \log \mid \hat{S} \mid)$.

ii. $\lambda = 2$: We assume that $g(e) = f(e) = r(\hat{e})$ and that $|S|$ is $O(|V(\mathcal{G})|)$. In the following discussion $V(X)$ denotes the endpoint set in graph $\mathcal{G}$ of edges belonging to members of X, but $V(\mathcal{G})$ denotes the vertex set of the graph $\mathcal{G}$. (Note that $\mathcal{G}$ itself is a reduced graph built from the original graph by replacing each subnetwork by a tree on its boundary nodes and contracting some branches of the tree, if required, to eliminate cutsets).

We note that, at any stage of the algorithm, we have say X_o on the left side of the bipartite graph and $V(X_o)$, the end vertex set associated with X_o, on the right hand side of the bipartite graph. If now we find that $h^t(X) \leq 0 \ \ \forall X \subseteq X_o$, this means in particular that

$$g(X) - 2(\mid V \mid -1)(X) \leq 0 \ \ \forall X, \emptyset \subset X \subseteq X_o.$$

Thus, $g(X) < 2(\mid V \mid (X))$. Now $g(e)$ would be the number of edges in $\hat{e}$ which is one less than the number of edges incident at e in the bipartite graph $(X_o, V(X_o), E_X)$. Hence the total number of edges in this bipartite graph is not more than $2(\mid V \mid (X_o)) + |X_0|$ and the associated flow graph has $O(\mid V \mid (X_o))$ number of edges. When we add a vertex e to X_o either

- $g(X) - 2(\mid V \mid -1)(X) \leq 0 \ \ \forall X, \emptyset \subset X \subseteq X_o \cup e$, or
- $X_o \cup e$ has a subset X_e with $g(X_e) > 2(\mid V \mid -1)(X_e)$.

In the former case, by the argument given above, the new bipartite graph has number of edges not more than $2(\mid V \mid (X_o \cup e)) + |X_0|$. In the latter case, we contract X_e, i.e., delete all nodes in X_e and all edges incident on these nodes, merge all vertices of $V(X_e)$ into one with a single edge leading to the sink with capacity 2. We further replace parallel edges from left vertex set to right vertex set by a single edge of the same capacity (∞). Hence, once again, the new bipartite graph has number of edges not more than $2(\mid V \mid (X_o)) + |X_0|$. Thus, in both cases, the new flow graph also has $O(\mid V(\mathcal{G}) \mid)$ edges. It follows that the number of edges in the current flow graph is always $O(\mid V(\mathcal{G}) \mid)$.

Now let us use the naive flow algorithm in which we send units of flow from source to sink. The cut that separates the sink from the rest of the nodes has a capacity $2 \mid V_R \mid$, where V_R is the right vertex set of the present bipartite graph. Suppose we find a min cut in the flow graph associated with the bipartite graph $B \equiv (Z, V(Z), E_Z)$. We get the min cut

$$(s \uplus X_e \uplus V(X_e), t \uplus (Z - X_e) \uplus (V(Z) - V(X_e))).$$

If $g(X_e) > 2(\mid V \mid -1)(X_e)$, we go through a contraction process. Observe that this contraction process would not disturb the flow that is incident on $(Z - X_e) \uplus (V(Z) - V(X_e))$. Whatever flow is lost touches X_e as well as $V(X_e)$. The single node that is left of $V(X_e)$ after merging can now be taken to carry zero flow. (Any arcs from $Z - X_e$ that may be incident on this vertex earlier would have carried zero flow since they are backward arcs with respect to the cut). In the flow graph corresponding to the contracted graph the capacity of the cut that separates sink from the rest of the nodes equals $(2(\mid V(Z) \mid -$ number of nodes that have disappeared during contraction)). For each contraction the loss of flow cannot exceed 2(number of nodes that have disappeared $+1$). It is therefore clear that the total units of flow that have to be sent from source to sink in all the stages of the algorithm cannot exceed (arguing conservatively) $4 \mid V(\mathcal{G}) \mid$. To send one unit of flow from source to sink in a flow graph with integral capacities takes $O(\mid \ edge\ set\ of\ flow\ graph \mid)$.

In our case this is $O(\mid V(\mathcal{G}) \mid)$. Hence the overall complexity of our algorithm for the case $\lambda = 2$ is $O(\mid V(\mathcal{G}) \mid^2)$.

Remark: In Section 14.9, we have shown that maximizing $g(X) - \lambda(|V|-1)_t(X)$, $X \subseteq S$ is equivalent to minimizing $\overline{(-g(E) - \lambda)}(\cdot)$, where $E(Y), Y \subseteq V(\mathcal{G})$ is the set of elements of S all of whose member edges have both endpoints in Y. The minimization problem has been described in detail in Subsection 14.7.2. The complexity of this algorithm for this problem is also the same as that of the method described in the present subsection.

Exercise 15.13 *Why is the assumption, that the subgraph of $\mathcal{G}$ on $\hat{e}$ is connected for each $e \in S$, necessary?*

Exercise 15.14 *Give a procedure for finding the minimal set that minimizes*

$$(f - \sigma)_t(X) + g(S - X) \; \forall X \subseteq S,$$

where $f(\cdot)$ is a submodular function on subsets of S and $g(\cdot)$ is a positive weight function on S. You may assume that a subroutine is available for finding the minimal set that minimizes $f(X) + g(Z - X) \;\; \forall X \subseteq Z \subseteq S, \emptyset \subset X$.

Exercise 15.15 *i. Let $f(\cdot)$ be an integral polymatroid rank function and let k be an integer. Show that the circuits of the matroid $\mathcal{M}$ whose rank function is $(f - k)_t * \mid \cdot \mid$ are precisely the minimal sets s.t. $(f - k)(X) <\mid X \mid$.*

ii. Let $\mathcal{G}$ be a selfloop free graph and let $V(X)$ denote the set of end points of edges in X. Show that the circuits of the matroid, whose rank function is $(k \mid V \mid -1)_t * \mid \cdot \mid$, are precisely the minimal sets of edges s.t. the subgraphs on them have average degree $\geq 2k$.

iii. Let $B \equiv (V_L, V_R, E)$ be a bipartite graph, with all vertices in V_L having the same degree d. Show that the circuits of the matroid, whose rank function is $(k \mid \Gamma_L \mid -1)_t * \mid \cdot \mid$, are precisely the minimal subsets Z of V_L s.t. the subgraph on $Z \uplus \Gamma_L(Z)$ has average right degree $\geq dk$.

15.3.4 PLP of the rank function of a matroid

A good application of the technique of solving the membership problem for a polymatroid using a matroid expansion, is in the construction of the PLP of a matroid rank function. We briefly indicate the portion of the PLP algorithm which needs the above technique. It is recommended that the reader review Section 14.2 and Subsection 15.3.2.

Let $r(\cdot)$ be a matroid rank function on subsets of S. The basic problem is to minimize $\overline{(r - \lambda)}(\cdot)$ over partitions of S. Suppose we know that the minimal minimizing partition is coarser than a partition Π of S. Let $f(\cdot) \equiv (r - \lambda)_{fus \cdot \Pi}(\cdot)$ and let $f'(\cdot)$ be the z.s.s function $f(X) - \sum_{e_i \in X} f(e_i)$. Let it be known that $X_o \subseteq \Pi$ contains no strong fusion set of $f'(\cdot)$. We add $e \in \Pi - X_o$ to X_o and minimize $f'(\cdot)$ over subsets of $X_o \cup e$. But this is exactly the same as maximizing

$g(X)-f(X), X\subseteq X_o\cup e$ where $g(e_i)\equiv f(e_i), e_i\in X$. However λ would in general be a fraction with denominator q. So, in order to make $g(\cdot)$ and $f(\cdot)$ integral, we have to multiply throughout by q. Once this is done, the problem reduces to the membership problem for the polymatroid P_f given a matroid expansion.
Maximizing $qg(X)-qf(X)$ would involve working with the direct sum of q identical copies of the original matroid. Using the algorithm given in the solution to Exercise 15.11, we can show that maximizing $qg(X)-qf(X)$ requires $O(rq^2(|S|-r)^2)$ calls to the independence oracle and $O(rq^2(|S|-r)^2)$ elementary steps. (There are q bases in the different copies. Information about f-circuits can be put in a graph with $q(|S|-r)r$ edges. To build such a graph takes $O(q(|S|-r)r)$ calls to the independence oracle. Doing bfs in the graph is $O(q(|S|-r)r)$. This has to be done atmost $q(|S|-r)$ times. So the total number of elementary steps is $O(rq^2(|S|-r)^2)$ and the number of calls to independence oracle is also $O(rq^2(|S|-r)^2)$).

The complexity of the PLP algorithm for $r(\cdot)$ using this technique in terms of calls to the independence oracle of the matroid (which is weaker than the rank oracle) is as follows (In general q can be as large as $|S|$):

i. To minimize $\overline{(r-\lambda)}(\cdot)$ –

$$O(\mid S\mid^3 (\mid S\mid -r)^2 r).$$

ii. To build the principal sequence of partitions using balanced bisection –

$$O(\mid S\mid^3 (\mid S\mid -r)^2 r\log\mid S\mid).$$

iii. To build the DTLs –

$$O(\mid S\mid^4 (\mid S\mid -r)^2 r).$$

15.4 The Hybrid Rank Problem - Third Formulation

15.4.1 Introduction

In this section we discuss the third formulation of the hybrid rank problem [Narayanan90],[Narayanan91], which arises naturally when we study the method of network analysis by topological transformations. This method has been discussed in Subsection 7.3.3.

Perhaps the most practically useful of topological transformations are the ones that involve node pair fusion and node fission (see Problem 7.9). We remind the reader briefly of the method: We attach 'virtual voltage sources' across some pairs of nodes and introduce 'virtual current sources' between two halves of a split node and solve the entire network in terms of the values of these sources. We next find the values of these sources for which the currents through the virtual voltage sources and the voltages across the virtual current sources become zero. This would give the solution of the original network. The point relevant to the present chapter in this procedure is that when the network $\mathcal{N}$ is solved in terms of the sources, the **equations correspond to a network** $\mathcal{N}_{new}$, in which the virtual voltage and current sources are zero (i.e., in which the pair of nodes associated with a voltage source is shorted and the two halves of the node between which the current source is connected are split). Each virtual source contributes an additional variable. Our aims are twofold: (i) $\mathcal{N}_{new}$ should have a 'good' topology, and (ii) the number of additional variables should be a minimum. The third formulation of the hybrid rank problem arises when we insist that the blocks of a given partition Π_s of the edge set of $\mathcal{N}$ should be separators (unions of 2 connected components) in $\mathcal{N}_{new}$ and the number of additional variables required to reach $\mathcal{N}_{new}$ be minimized. In practice the subnetworks of $\mathcal{N}$ on the blocks of Π_s would be connected. This condition is also mathematically convenient. The first formulation turns out to be the special case where Π_s has singleton blocks (see Exercise 15.17).

We now state the **third formulation of the hybrid rank problem:**
Let $\mathcal{G}$ be a graph and let Π_s be a specified partition of $E(\mathcal{G})$ s.t. $\mathcal{G} \cdot N_i$ is connected for each $N_i \in \Pi_s$. Find a minimum length sequence of node pair fusions and node fissions which, when performed on $\mathcal{G}$, result in a graph $\mathcal{G}_{new}$ in which each circuit intersects only one of the blocks of Π_s (equivalently each cutset intersects only one of the blocks of Π_s).

In the above formulation **one node pair fusion** is the merging of two nodes into a single node. All edges which were incident on the original pair of nodes would now be incident on the merged node. An edge with both the original vertices as end points would now be a self loop. **One node fission** is the splitting of one node into two - the non-selfloop edges which were originally incident on the parent node would each be incident on precisely one of the two child nodes. The self loop edges could now be incident on one of the two child nodes as self loops or become ordinary edges lying between the two child nodes (see Figure 15.1).

We solve this problem through the following stages:

- First we show that a fusion followed by fission can always achieve the effect of a fission followed by fusion – this is routine. So we lose no generality in assuming that all fusions occur first. A sequence of fusions has the same effect as fusing the blocks of an appropriate partition of $V(\mathcal{G})$.

- Next we consider the special case of the above problem where only fissions are allowed. This problem is similar to that of finding the nullity of a graph and is easy to solve.

- We associate with each partition Π of $V(\mathcal{G})$ a number equal to the **sum** of $(| V(\mathcal{G}) | - | \Pi |)$ **and** the minimum number of fissions required for converting the graph $\mathcal{G}_{fus \cdot \Pi}$ to a graph which has N_i as separators.
  We now attempt to find the partition Π_{min} of $V(\mathcal{G})$ which minimizes this number.

Figure 15.1: Node pair fusion and node fission

It turns out that the minimizing partitions mentioned above are precisely the ones that minimize $\overline{(\mid \Gamma \mid -2)}(\cdot)$ where $|\Gamma|(N_i)$ denotes the number of blocks of Π_s which have edges incident on vertices in N_i. Equivalently we need to find Π s.t. $\overline{(\mid \Gamma \mid -2)}(\Pi) = (\mid \Gamma \mid -2)_t(V(\mathcal{G}))$. In Subsection 14.7.1, the solution to this problem has been described in detail.
The **desired sequence of fusions and fissions would be:**
first the fusions corresponding to Π_{min} then the minimum sequence of fissions required to convert $\mathcal{G}_{fus \cdot \Pi_{min}}$ to a graph which has N_i as separators.

15.4.2 Fusions and Fissions

We begin by showing that the effect of a node fission followed by a node pair fusion can always be achieved by an appropriate fusion followed by an appropriate fission. Since the proof is routine but tedious we simplify it by ignoring arrows.
Let node a have edges $e_{a1}, \cdots, e_{ak}$ incident on it. Let the node fission split a into a_1 on which $e_{a1}, \cdots, e_{at}$ are incident and a_2 on which $e_{a(t+1)}, \cdots, e_{ak}$ are incident. Now let a_1 be merged with a vertex d on which $e_{d1}, \cdots, e_{dr}$ are incident. (Some of e_{di} could be the same as some e_{aj}). If d is a_2 there is nothing to prove. So let us assume that d is different from a_2. Suppose d has no common edge with a_2. Clearly this effect can be achieved by first merging d with a and splitting a_2 with

$e_{a(t+1)}, \cdots, e_{ak}$ incident on it from the combined node. If d has $e_1, \cdots, e_p$ common with a_2, then while a is merged with d these edges would become selfloops and when a_2 has been split away they would lie between a_2 and the merged node containing d. The case where a has selfloops which become ordinary edges between a_1, a_2 can be handled similarly. This proves the required result.

Exercise 15.16 *Construct an example to show that fission followed by fusion can be weaker than fusion followed by fission.*

Fusion rank and fission rank of a graph relative to a partition of its edges

Let $\mathcal{G}$ be a graph and Π_s, a partition of $E(\mathcal{G})$. The **fusion rank of $\mathcal{G}$ relative to Π_s** is the minimum length of a sequence of node pair fusions needed to destroy every circuit that intersects more than one block of Π_s. The **fission rank of $\mathcal{G}$ relative to Π_s** is the minimum length of a sequence of node fissions needed to destroy every circuit that intersects more than one block of Π_s. The **hybrid rank of $\mathcal{G}$ relative to Π_s** is the minimum length of a sequence of node pair fusions and node fissions needed to destroy every circuit that intersects more than one block of Π_s.
Since the effect of a fission followed by a fusion can always be achieved by a fusion followed by a fission, we lose no generality in confining ourselves to sequences which have all the node pair fusions first, followed by the node fissions. So, in order to solve the hybrid rank problem, we need to first solve the problem of constructing a minimum length sequence of node fissions which would make the blocks of the partition into separators. This we do now.

Exercise 15.17 *Let $\mathcal{G}$ be a graph and let Π_s be the partition of $E(\mathcal{G})$ into singletons. Show that, relative to Π_s*

i. *fusion rank* $= r(\mathcal{G})$ *- number of coloops*

ii. *fission rank* $= \nu(\mathcal{G})$ *- number of selfloops*

iii. *hybrid rank* $= \min_{X\subseteq E(\mathcal{G})} r(\mathcal{G} \,.\, X) + \nu(\mathcal{G} \times (E - X))$.

We first present an informal algorithm for constructing a minimum length sequence of node fissions to make blocks of Π_s into separators and, using it, derive a result about fission rank.

Construction of minimum length sequence of node fissions

i. For each N_i construct a coforest L_i of $\mathcal{G} \cdot N_i$. Delete $\bigcup_i L_i$ from $\mathcal{G}$. Let $\mathcal{G}_{red} \equiv \mathcal{G} \,.\, (E(\mathcal{G}) - \bigcup L_i)$. Let $N_i' \equiv N_i - L_i, N_i \in \Pi_s$. Let $\Pi_s' \equiv \{N_i', N_i \in \Pi_s\}$.

ii. We will call a node of $\mathcal{G}_{red}$ a 'boundary node' if edges of more than one block of Π_s' are incident on it. For each node common to more than one block of Π_s', do the following:

- Replace node v common to blocks $N'_1, \cdots, N'_k$ by nodes $v_1, \cdots, v_k$ s.t. v_i is incident only on edges of N'_i. Build a tree on $v_1, \cdots, v_k$. We will call such a tree, a 'fission tree'.

At the end of the above 'do loop' we have a graph $\mathcal{G}'$ derived from $\mathcal{G}_{red}$.

iii. Edges in the fission trees contain a coforest of $\mathcal{G}'$. Delete such a coforest and short the remaining fission tree edges. The graph that results is essentially (except for hinged nodes), $\oplus(\mathcal{G}_{red} \cdot N'_i)$. Each $\mathcal{G}_{red} \cdot N'_i$ is a forest subgraph of $\mathcal{G} \ . \ N_i$.

iv. Add back edges of L_i for each N_i. We are now left with the graph $\oplus\mathcal{G} \ . \ N_i$ (except for hinged nodes). We observe that this graph could have been obtained without going through the process of deleting L_i and adding them back, but following the rest of the algorithm regarding construction of fission trees - deleting some of their edges and shorting the remaining.

The sequence of node fissions in the above algorithm corresponds to a particular coforest of $\mathcal{G}'$ contained in the fission trees with each node fission corresponding to one coforest edge. Hence, length of the sequence of node fissions = nullity of $\mathcal{G}'$. Now any coforest of $\mathcal{G}_{red}$ is also a coforest of $\mathcal{G}'$. Hence nullity of $\mathcal{G}'$ equals

$$\textit{nullity of } \mathcal{G}_{red} = \nu(\mathcal{G} \ . \ (E(\mathcal{G}) - \bigcup_i L_i)) = \nu(\mathcal{G}) - \nu(\mathcal{G} \times (\bigcup_i L_i)).$$

We note that $\mathcal{G} \times (\bigcup_i L_i)$ is obtained by shorting forests of $\mathcal{G} \ . \ N_i$ so that coforest edges become selfloops. Hence,

$$\nu(\mathcal{G} \times (\bigcup L_i)) = \mid \bigcup L_i \mid = \sum_i \nu(\mathcal{G} \ . \ N_i).$$

Hence, length of the above sequence of node fissions $= \nu(\mathcal{G}) - \sum \nu(\mathcal{G} \ . \ N_i)$. It is thus clear that the fission rank of $\mathcal{G} \leq (\nu(\mathcal{G}) - \sum \nu(\mathcal{G} \ . \ N_i))$.
On the other hand, the fission rank of $\mathcal{G}$ cannot be less than that of $\mathcal{G}_{red}$. The only circuits of $\mathcal{G}_{red}$ are those that intersect more than one block of Π'_s. Hence,

$$\textit{fission rank of } \mathcal{G}_{red} = \textit{ nullity of } \mathcal{G}_{red} \ = \ \nu(\mathcal{G}) - \sum \nu(\mathcal{G} \ . \ N_i).$$

Hence fission rank of $\mathcal{G} \geq (\nu(\mathcal{G}) - \sum \nu(\mathcal{G} \ . \ N_i))$. It is a straight forward computation to check that

$$\nu(\mathcal{G}) - \sum \nu(\mathcal{G} \ . \ N_i) = \sum r(\mathcal{G} \ . \ N_i) - r(\mathcal{G}).$$

We therefore have the following result.

Theorem 15.4.1 *Let $\mathcal{G}$ be a graph and let Π_s be a partition of $E \equiv E(G)$ (with $\mathcal{G} \ . \ N_i, N_i \in \Pi_s$, not necessarily connected). Then the fission rank of $\mathcal{G}$ relative to Π_s is equal to*

$$\sum_{N_i \in \Pi_s} r(\mathcal{G} \cdot N_i) - r(\mathcal{G}) = \nu(\mathcal{G}) - \sum_{N_i \in \Pi_s} \nu(\mathcal{G} \ . \ N_i)$$

For our discussion of the third formulation of the hybrid rank the fusion rank of $\mathcal{G}$ relative to Π_s is unimportant. Nevertheless, for completeness, the result about fusion rank is presented in the following exercise. We note that fission rank computations do not require any connectedness assumption on the blocks of Π_s whereas those of fusion rank do require such an assumption.

Exercise 15.18 Fusion rank: *Let $\mathcal{G}$ be a graph and let Π_s be a partition of $E(\mathcal{G})$ s.t. $\mathcal{G} \,.\, N_i, N_i \in \Pi_s$ is connected.*

i. Show that the fusion rank of $\mathcal{G}$ relative to Π_s equals $r(\mathcal{G}) - \sum r(\mathcal{G} \times N_i) = \sum \nu(\mathcal{G} \times N_i) - \nu(\mathcal{G})$.

ii. Give a counter example where $\mathcal{G} \,.\, N_i$ are not connected and the above formula for fusion rank becomes invalid.

The fusion - fission number of a partition Π of $V(\mathcal{G})$
Let us now consider the situation where we use both fusions and fissions, with all the fusions occurring first.
Any sequence of node pair fusions would ultimately fuse certain groups of nodes into single nodes. Hence, as far as the effect of these node pair fusions on the graph is concerned, we may identify them with a partition of $V(\mathcal{G})$ each block of which would be reduced to a single node by the fusions. The number of node pair fusions required to convert a set of nodes V to a single node is $(\mid V \mid -1)$. Hence, if Π is a partition of $V(\mathcal{G})$, the number of node pair fusions required to go from $\mathcal{G}$ to $\mathcal{G}_{fus \cdot \Pi}$ ($\equiv$ *the graph obtained from $\mathcal{G}$ by fusing blocks of Π into single vertices*) is $\mid V(\mathcal{G}) \mid - \mid \Pi \mid$. This number we would henceforth call, the **fusion number of** Π. The fission rank of $\mathcal{G}_{fus \cdot \Pi}$ relative to a partition Π_s of $E(\mathcal{G})$ would be called the **fission number of Π relative to** Π_s. The sum of the fusion number and the fission number of Π relative Π_s would be called the **fusion - fission number of Π relative to** Π_s. Our task is to find a partition of $V(\mathcal{G})$ which minimizes this number.

We now define a bipartite graph which relates Π_s to $V(\mathcal{G})$. Let $B_{\mathcal{G}}$ be the bipartite graph associated with $\mathcal{G}$, with $V_L \equiv V(\mathcal{G})$ and $V_R \equiv E(\mathcal{G})$, with $e \in V_R$ adjacent to $v \in V$ iff edge e is incident on v in $\mathcal{G}$. Let $B(\Pi_s)$ be the bipartite graph obtained from $B_{\mathcal{G}}$ by fusing the right vertices in the blocks of Π_s and replacing parallel edges by single edges.
We then have the following result.

Theorem 15.4.2 *Let $\mathcal{G}$ be a connected graph. Let Π_s be a partition of $E(\mathcal{G})$ s.t. $\mathcal{G} \,.\, N_i$ is connected for each $N_i \in \Pi_s$. Let Π be a partition of $V(\mathcal{G})$. Let $|\Gamma_L|(\cdot)$ be the left adjacency function of $B(\Pi_s)$. Then*

i. the fusion - fission number of Π relative to Π_s equals

$$\overline{(\mid \Gamma_L \mid -2)}(\Pi) + \mid V(\mathcal{G}) \mid - \mid \Pi_s \mid +1$$

ii. the hybrid rank of $\mathcal{G}$ relative to Π_s equals

$$min(\overline{(\mid \Gamma_L \mid -2)}(\Pi) + \mid V(\mathcal{G}) \mid - \mid \Pi_s \mid +1),$$

Π *a partition of* $V(\mathcal{G})$.

Proof :
i. The fusion - fission number of Π relative to Π_s

$$\begin{aligned} &= \textit{fusion number of } \Pi + \textit{fission number of } \Pi \textit{ relative to } \Pi_s \\ &= \mid V(\mathcal{G}) \mid - \mid \Pi \mid + \sum_{N_i \in \Pi_s} r(\mathcal{G}_{fus \cdot \Pi} \cdot N_i) - r(\mathcal{G}_{fus \cdot \Pi}) \end{aligned}$$

(by Theorem 15.4.1).
We note that $r(\mathcal{G}_{fus \cdot \Pi}) = (\mid \Pi \mid -1)$.

We now need to compute $r(\mathcal{G}_{fus \cdot \Pi} \cdot N_i)$. We have assumed that in the graph $\mathcal{G}, \mathcal{G}\ .\ N_i$ is connected. Hence, $\mathcal{G}_{fus \cdot \Pi} \cdot N_i$ must also be connected. The number of vertices, that this graph has, equals the number of blocks of Π which meet the vertex set of N_i. Thus the sum of the cardinalities of the vertex sets of all the $\mathcal{G}_{fus \cdot \Pi} \cdot N_i$ can be obtained by summing, over all $V_j \in \Pi$, the number of N_i which are adjacent to V_j. In other words

$$\sum_{N_i \in \Pi_s} \mid V(\mathcal{G}_{fus \cdot \Pi} \cdot N_i) \mid = \sum_{V_i \in \Pi} \mid \Gamma_L \mid (V_i).$$

Further we note that

$$r(\mathcal{G}_{fus \cdot \Pi} \cdot N_i) = \mid V(\mathcal{G}_{fus \cdot \Pi}) \cdot N_i \mid -1.$$

Thus, the fusion - fission number of Π relative to Π_s

$$\begin{aligned} &= \mid V(\mathcal{G}) \mid +1 - 2 \mid \Pi \mid + \sum_{V_i \in \Pi} \mid \Gamma_L \mid (V_i) - \mid \Pi_s \mid \\ &= \overline{(\mid \Gamma_L \mid -2)}(\Pi) + \mid V(\mathcal{G}) \mid - \mid \Pi_s \mid +1 \end{aligned}$$

ii. This is immediate since the hybrid rank of $\mathcal{G}$ relative to Π_s is the minimum of the fusion - fission numbers of partitions of $V(\mathcal{G})$.

□

We thus see that finding a partition which has the least fusion - fission number relative to Π_s is equivalent to finding a partition which minimizes $\overline{(\mid \Gamma_L \mid -2)}(\Pi)$ over partitions of the left vertex set of the bipartite graph $B(\Pi_s)$. Finding all such partitions is the problem of determining the DTL of $\overline{(\mid \Gamma_L \mid -2)}(\cdot)$ and if we have a variable λ in place of 2 we get the PLP problem. Efficient algorithms for these problems have been given in Subsection 14.7.1. However, in this particular case $(\lambda = 2)$ the naive algorithm which sends units of flow from source to sink does very well, provided, after each fusion, we make use of the previous flow.
We assume that the reader is familiar with the algorithm for $(w_R \Gamma_L)(\cdot)$ given in the above mentioned section. We now make a few observations regarding $B(\Pi_s)$ when it arises from an electrical network subdivided into subnetworks and thence arrive at some conclusions about a naive algorithm for minimizing $\overline{(\mid \Gamma_L \mid -2)}(\cdot)$.

i. V_L can be taken to be the set of boundary nodes. Internal nodes (which are touching only one subnetwork) can easily be seen to appear as singletons in any partition minimizing $\overline{(\mid \Gamma_L \mid -2)}(\cdot)$. During the optimization process the internal nodes and the (bipartite graph) edges incident on them can be deleted from $B(\Pi_s)$.

ii. $\mid V_L \mid \gg \mid V_R \mid$ since V_R is the set of subnetworks.

iii. Number of edges in $B(\Pi_s)$ is $O(\mid V_L \mid)$ since, typically, a vertex that is not the ground vertex, would have degree at most 4 or 5 in a large network. In $B(\Pi_s)$ the degree would be even lower than this.

iv. In the flow graph for this problem (Figure 14.1) the capacity of the edge (s,v) is $|\Gamma_L|(v) - 2$. So the capacity of the cut separating the source from the rest of the nodes is $O(\mid V_L \mid)$.

v. To send (or to withdraw) a unit flow from source to sink we have to do $O(\mid E \mid)$ work where E is the set of edges of the current flow graph. In this case $\mid E \mid$ is $O(\mid V_L \mid)$. Each time a fusion set is detected we have to reduce the capacity of the edge between the source and the merged node. This might necessitate (under the above mentioned assumption about degree of a boundary node) withdrawal of units of flow of $O(\mid X_e \mid)$ where X_e is the fusion set. The total withdrawal of units of flow throughout the algorithm is therefore $O(\mid V_L \mid)$. Thus the total units of flow sent and withdrawn is $O(\mid V_L \mid)$.

It follows that, in practice, to find a partition that minimizes $\overline{(\mid \Gamma_L \mid -2)}(\cdot)$ for $B(\Pi_s)$ arising from a network, the complexity of the naive algorithm, which sends units of flow from source to sink but remembers past flow after fusion, is $O(\mid V_L \mid^2)$. Here V_L = set of boundary nodes which, as we have remarked before, could be as low as 10% of the size of the vertex set of the network. Such an algorithm can reasonably be **included in the preprocessing stage of a circuit simulator**.

Exercise 15.19 When every subnetwork contains the ground node: *Let $\mathcal{G}$ be a graph and let Π_s be a partition of $E(\mathcal{G})$. Let v_g be a boundary node that is incident on each block of Π_s. Show that*

i. an optimal sequence of fusions and fissions is simply to fuse all boundary nodes to v_g.

ii. the hybrid rank relative to Π_s = number of boundary nodes – 1.

In network analysis by decomposition this situation arises when each of the subnetworks into which the given network is decomposed has a ground node.

15.4.3 Relation between the Hybrid Rank of a Graph and its Hybrid Rank relative to a Partition

We have already seen that the first hybrid rank formulation is a special case (by taking Π_s to have singleton blocks) of the third formulation. We bring out yet

another relation between the two formulations in this subsection.
We show that if each block of the partition corresponds to a tree graph, the hybrid rank relative to Π_s is the minimum of the hybrid ranks of the graphs that one obtains by replacing each of the above trees by a cospanning tree. (It would then be natural to ask what happens if we generalize the notion of cospanning trees to the case of cospanning independent vector sets. This would give us the fourth formulation of the hybrid rank problem).

We begin with some preliminary definitions and notation. Let $\mathcal{G}$ be graph and let Π_s be a partition of $E(\mathcal{G})$ s.t. $\mathcal{G} \,.\, N_i$ is a tree graph for each $N_i \in \Pi_s$. Let $\mathcal{G}'$ be another graph with $V(\mathcal{G}') = V(\mathcal{G})$ and let Π'_s be a partition of $E(\mathcal{G}')$ s.t. $\mathcal{G}' \cdot N_i{}'$ is a tree graph for each $N_i{}' \in \Pi'_s$. We say $(\mathcal{G}, \Pi_s), (\mathcal{G}', \Pi'_s)$ are **partition equivalent** iff there is a bijection $\tau : \Pi_s \rightarrow \Pi'_s$ such that $\mathcal{G}' \cdot (\tau(N_i))$ cospans $\mathcal{G} \,.\, N_i$ for each $N_i \in \Pi_s$ (i.e., both the tree graphs have the same vertex sets). The hybrid rank of $\mathcal{G}$ relative to Π_s is the same as the hybrid rank of $\mathcal{G}'$ relative to Π'_s by Theorem 15.4.2 since $B(\Pi_s), B(\Pi'_s)$ are isomorphic (identical under the right vertex mapping $\tau(\cdot)$).

Exercise 15.20 Simplifying subnetworks keeping third hybrid rank invariant: *Let $\mathcal{G}$ be a graph and let Π_s be a partition of $E(\mathcal{G})$ s.t. $\mathcal{G} \,.\, N_i$ is connected for each $N_i \in \Pi_s$. Replace N_i by t_i, where $\mathcal{G} \,.\, t_i$ is a tree subgraph of $\mathcal{G} \,.\, N_i$. Let the resulting graph be $\mathcal{G}_t$. Show that*

- *i. the hybrid rank of $\mathcal{G}_t$ relative to Π'_s, where $t_i \in \Pi'_s$ iff $N_i \in \Pi_s$, is equal to the hybrid rank of $\mathcal{G}$ relative to Π_s.*
- *ii. there is no loss of generality in assuming that $\mathcal{G}_t$ has no cutsets within each t_i.*

The following lemma is needed in the proof of the main result.

Lemma 15.4.1 *Let $\mathcal{G}$ be a graph and let Π_s be a partition of $E(\mathcal{G})$ such that $\mathcal{G} \,.\, N_i$ is a complete graph for each $N_i \in \Pi_s$. Then, the hybrid rank of $\mathcal{G}$ relative to Π_s*

$$= \min_{X \subseteq E} 2r(X) - \sum r(X \cap N_i) + \sum r(N_i) - r(\mathcal{G}).$$

Proof : The expression on the right reaches its minimum on the closure of Y (with respect to $r(\cdot)$) if it reaches its minimum on $Y \subseteq E$. So let us assume that Y is closed. If Y is closed there is a partition Π of $V(\mathcal{G})$ s.t. $E(\Pi) = Y$ $(E(\Pi) \equiv \bigcup_{N_i \in \Pi} E(N_i))$. Then $r(Y) = \mid V(\mathcal{G}) \mid - \mid \Pi \mid$. In terms of Π the expression on the right becomes

$$2 \mid V(\mathcal{G}) \mid -2 \mid \Pi \mid - \sum_{K_j \cap V_i \neq \phi} (\mid K_j \cap V_i \mid -1) + \sum r(N_i) - r(\mathcal{G}),$$

where K_j are blocks of Π and V_i, the vertex set of $\mathcal{G} \,.\, N_i$

$$= 2 \mid V(\mathcal{G}) \mid -2 \mid \Pi \mid - \sum \mid V_i \mid + \mid \bar{\Gamma} \mid (\Pi) + \sum r(N_i) - r(\mathcal{G}),$$

where $\Gamma(K_j)$ is the collection of N_i's that K_j meets
$= \overline{(|\Gamma|-2)}(\Pi) - \mid \Pi_s \mid + \mid V(\mathcal{G}) \mid +1$, since $\sum(\mid V_i \mid -r(N_i)) =\mid \Pi_s \mid$ and $|V(\mathcal{G})| = r(\mathcal{G})+1$. Thus, the minimum of the above expression over all partitions of $V(\mathcal{G})$ equals the minimum of the expression on the RHS of the statement of the theorem over all $X \subseteq E$. But by Theorem 15.4.2 the minimum of the above expression over all partitions of $V(\mathcal{G})$ equals the hybrid rank of $\mathcal{G}$ relative Π_s.
This proves the lemma.

□

Theorem 15.4.3 *Let $\mathcal{G}$ be a graph and let Π_s be a partition of $E(\mathcal{G})$ such that $\mathcal{G} \,.\, t_i$ is a tree graph for each $t_i \in \Pi_s$. Then the hybrid rank of $\mathcal{G}$ relative to Π_s = minimum hybrid rank of $\mathcal{G}'$ where $(\mathcal{G}, \Pi_s)$, $(\mathcal{G}', \Pi'_s)$ are partition equivalent to each other.*

Proof : By Theorem 15.4.2, if each of $\mathcal{G} \,.\, t_i$ were replaced by a complete graph on the same set of nodes the hybrid rank of the new graph $\mathcal{G}_{new}$ relative to $\Pi_s(new)$ would be the same as the hybrid rank of $\mathcal{G}$ relative to Π_s (where $\Pi_s(new)$ has the edge set N_i of the complete graph on $V(\mathcal{G} \,.\, t_i)$ as the block in place of t_i), since the bipartite graphs $B(\Pi_s), B(\Pi_s(new))$ are isomorphic. Now consider a graph $\mathcal{G}'$ s.t. $(\mathcal{G}, \Pi_s), (\mathcal{G}', \Pi'_s)$ are equivalent. Let $E' \equiv E(\mathcal{G}')$. Let $A' \subseteq E'$ be s.t. hybrid rank of $\mathcal{G}'$

$$\begin{aligned} &= r(\mathcal{G}' \cdot A') + \nu(\mathcal{G}' \times (E' - A')) \\ &= r(\mathcal{G}' \cdot A') + \mid E' - A' \mid - r(\mathcal{G}') + r(\mathcal{G}' \cdot A') \\ &= 2r(\mathcal{G}' \cdot A') + \mid E' - A' \mid - r(\mathcal{G}'). \end{aligned}$$

Let $r_{new}(\cdot)$ be the rank function of the graph $\mathcal{G}_{new}$ and let A_n be the closure of A' in this graph. Then, since $r_{new}(A_n \cap N_i) \geq \mid A' \cap t_i \mid$, the above RHS
$\geq 2r_{new}(A_n) + \sum(r_{new}(N_i) - r_{new}(A_n \cap N_i)) - r(\mathcal{G}_{new})$
$\geq$ hybrid rank of $\mathcal{G}$ relative to Π_s (using Lemma 15.4.1).

We will now build a graph $\hat{\mathcal{G}}$ and a partition $\hat{\Pi}_s$ for which the above inequality becomes an equality. Suppose A_n minimizes the expression (in X)

$$2r_{new}(X) - \sum r_{new}(X \cap N_i) + \sum r_{new}(N_i) - r(\mathcal{G}_{new}).$$

In each N_i, let t_{Ai} be a forest of $\mathcal{G} \,.\, (A_n \cap N_i)$. Let t_{Ai} be grown to the tree $\hat{t}_i$ of $\mathcal{G} \,.\, N_i$. Let $\hat{A} = \bigcup_i t_{Ai}$. Clearly A_n is the closure of $\hat{A}$ in $\mathcal{G}_{new}$. Now build $\hat{\mathcal{G}}$ by replacing each N_i by $\hat{t}_i$. Let $\hat{\Pi}_s$ have $\hat{t}_i$ as the blocks. Then the hybrid rank of $\hat{\mathcal{G}}$

$$\begin{aligned} &\leq 2r(\hat{\mathcal{G}} \cdot \hat{A}) + \mid \hat{E} - \hat{A} \mid - r(\hat{\mathcal{G}}) \\ &\leq 2r_{new}(A_n) + \sum(r_{new}(N_i) - r_{new}(A_n \cap N_i)) - r(\mathcal{G}_{new}), \\ &\leq \text{the hybrid rank of } \hat{\mathcal{G}}, \text{ relative to } \hat{\Pi}_s. \end{aligned}$$

(since $\mid \hat{A} \cap t_i \mid = r_{new}(A_n \cap N_i)$). It follows that, for $\hat{\mathcal{G}}$, the hybrid rank equals the hybrid rank relative to $\hat{\Pi}_s$. This proves the theorem.

Figure 15.2: Partition Equivalent Graphs

□

Example: In Figure 15.2 consider the graphs $\mathcal{G}$ and $\mathcal{G}'$ with partitions Π_s, Π'_s, respectively, of their edges, with

$$\Pi_s \equiv \{\{e_{11}, e_{12}\}, \{e_{21}\}, \{e_{31}, e_{32}\}, \{e_{41}\}\}$$
$$\Pi'_s \equiv \{\{e'_{11}, e'_{12}\}, \{e'_{21}\}, \{e'_{31}, e'_{32}\}, \{e'_{41}\}\}\,.$$

It can be seen that, if t_i is a block of Π_s, there is a corresponding block t'_i of Π'_s s.t. $\mathcal{G}\ .\ t_i$ and $\mathcal{G}\ .\ t'_i$ cospan. Thus, $(\mathcal{G}, \Pi_s), (\mathcal{G}', \Pi'_s)$ are partition equivalent. The optimum sequence of fusions and fissions is: fuse (a, c), then split b separating 4th block edges from 1st block edges. The hybrid rank of $\mathcal{G}$ is 3 while that of $\mathcal{G}'$ equals 2, which latter is also the hybrid rank of $\mathcal{G}$ relative to Π_s (also of $\mathcal{G}'$ relative to Π'_s). It may be observed that $\mathcal{G}'$ has been built the way $\hat{\mathcal{G}}$ has been in the proof of Theorem 15.4.3.

15.5 The Hybrid Rank Problem - Fourth Formulation

15.5.1 Introduction

The ideas in this section are straight-forward generalizations of the corresponding notions for the third formulation. The method of network analysis by topological transformations motivates both the formulations. It is suggested that the reader review Chapter 7, in particular, Subsection 7.3.3. There we saw that **if** $\mathcal{V}, \mathcal{V}'$ are voltage spaces of graphs $\mathcal{G}, \mathcal{G}'$ respectively, and if $\mathcal{V}_{EP}$ is s.t. for some $\mathcal{V}_P, \mathcal{V}'_P$

$$\mathcal{V} = \mathcal{V}_{EP} \leftrightarrow \mathcal{V}_P$$

$$\mathcal{V}' = \mathcal{V}_{EP} \leftrightarrow \mathcal{V}'_P,$$

then we can write the KCE, KVE of $\mathcal{G}$ in a bordered form with the thickness of the border equal to $\mid P \mid$ and the inner matrix as the coefficient matrix of KCE, KVE of $\mathcal{G}'$. We could choose a good inner matrix structure and for the desired structure try to minimize $\mid P \mid$. In particular we could partition E into $\{E_1, \cdots, E_k\}$ and insist that $\mathcal{V}'$ have E_i as separators. The third formulation arises when we insist that $\mathcal{V}_{EP}$ and $\mathcal{V}'$ be the voltage spaces of graphs. The present (fourth and last!) formulation arises when we relax this assumption.

We now state the **fourth formulation of the hybrid rank problem:**
Given a vector space $\mathcal{V}$ on E and a partition Π_s of E into $\{E_1, \cdots, E_k\}$, find spaces $\mathcal{V}_{EP}, \mathcal{V}_P$ on $E \uplus P, P$ respectively such that

i. $\mathcal{V}_{EP} \leftrightarrow \mathcal{V}_P = \mathcal{V}$,

ii. there exists a vector space $\mathcal{V}'_P$ on P s.t. $\mathcal{V}_{EP} \leftrightarrow \mathcal{V}'_P$ has $E_1, \cdots, E_k$ as separators,

iii. $\mid P \mid$ is a minimum under the above conditions.

We would call the above minimum size of P the **generalized hybrid rank of $\mathcal{V}$ relative to Π_s.**

There is a slightly different way of phrasing the above problem which is illuminating.
If $\mathcal{V}, \mathcal{V}'$ are vector spaces on E the **distance** $d(\mathcal{V}, \mathcal{V}')$ $(\equiv r(\mathcal{V} + \mathcal{V}') - r(\mathcal{V} \cap \mathcal{V}'))$ between them can be equivalently defined to be the minimum size of a set P for which there exist vector spaces $\mathcal{V}_{EP}$ on $E \uplus P, \mathcal{V}_P, \mathcal{V}'_P$ on P, s.t.

$$\mathcal{V}_{EP} \leftrightarrow \mathcal{V}_P = \mathcal{V}$$

$$\mathcal{V}_{EP} \leftrightarrow \mathcal{V}'_P = \mathcal{V}'$$

(see Exercises 7.11,7.14). Thus the generalized hybrid rank of $\mathcal{V}$ relative to Π_s is the **minimum distance from $\mathcal{V}$ to another space $\mathcal{V}'$ on** E which has the blocks of Π_s as separators.

We do not know how to solve the fourth formulation of the hybrid rank problem. However, we present evidence in this section that, we hope, should suggest that the problem is interesting and relevant.

Exercise 15.21 Submodularity of the distance function: *Show that*

i. $d(\mathcal{V}, \mathcal{V}_1) + d(\mathcal{V}, \mathcal{V}_2) \geq d(\mathcal{V}, \mathcal{V}_1 + \mathcal{V}_2) + d(\mathcal{V}, \mathcal{V}_1 \cap \mathcal{V}_2)$

ii. $d(\mathcal{V}, \mathcal{V}_1) = d(\mathcal{V}, \mathcal{V} \cap \mathcal{V}_1) + d(\mathcal{V} \cap \mathcal{V}_1, \mathcal{V}_1)$

iii. $d(\mathcal{V}, \mathcal{V}_1) = d(\mathcal{V}, \mathcal{V} + \mathcal{V}_1) + d(\mathcal{V} + \mathcal{V}_1, \mathcal{V}_1)$

15.5.2 Generalized Fusions and Fissions

Notions of fusion rank and fission rank, relative to a partition, are generalized in this section. We remind the reader that the fusion rank (fission rank) is the minimum number of node pair fusions (node fissions) needed to make the blocks of the partition into separators. We note that when such fusions are performed, if $\mathcal{G} \,.\, E_j$ are connected, the graph that results is $\oplus_j \mathcal{G} \times E_j$, where E_j are the blocks of the partition. When the fissions corresponding to fission rank are performed, the graph that results is $\oplus_j \mathcal{G} \,.\, E_j$ (in both cases hinged nodes might be present between the different graphs on the E_j).

We begin with some preliminary definitions. Let $\mathcal{V}$ be a vector space on E and let $\Pi_s \equiv \{E_1, \cdots, E_k\}$ be a partition of E. Then the **fusion rank of $\mathcal{V}$ relative to** Π_s is $d(\mathcal{V}, \oplus_j(\mathcal{V} \times E_j))$ and the **fission rank of $\mathcal{V}$ relative to** Π_s is $d(\mathcal{V}, \oplus_j(\mathcal{V} \cdot E_j))$. Hence, the fusion rank of $\mathcal{V}$ relative to Π_s is $r(\mathcal{V}) - \sum_j r(\mathcal{V} \times E_j)$ and the fission rank of $\mathcal{V}$ relative to Π_s is $\sum_j r(\mathcal{V} \cdot E_j) - r(\mathcal{V}) = \nu(\mathcal{V}) - \sum_j \nu(\mathcal{V} \cdot E_j)$, where $\nu(\mathcal{V})$ denotes the nullity of $\mathcal{V}$. The reader may verify that these numbers agree with the fusion and fission rank for a graph when $\mathcal{G} \,.\, E_j$ is connected for each $E_j \in \Pi_s$. The problem of constructing $\mathcal{V}_{EP}, \mathcal{V}_P, \mathcal{V}'_P$ in the cases where $\mathcal{V}' \equiv \oplus_j(\mathcal{V} \times E_j)$ or $\mathcal{V}' \equiv \oplus_j(\mathcal{V} \,.\, E_j)$ so that $\mid P \mid$ is minimum is relatively easy to solve (see Subsection 7.3.3 and also Problem 7.9).

Exercise 15.22 Fission and fusion ranks as distances: *Let $\mathcal{V}$ be a vector space on E and let $\Pi_s \equiv \{E_1, \cdots, E_k\}$ be a partition of E. Show that*

i. if $\mathcal{V}_1 \supseteq \mathcal{V}$ and has E_j as separators, then $\mathcal{V}_1 \supseteq \oplus_j \mathcal{V} \,.\, E_j$. Hence $d(\mathcal{V}, \mathcal{V}_1) \geq$ fission rank of $\mathcal{V}$.

ii. if $\mathcal{V}_2 \subseteq \mathcal{V}$ and has E_j as separators, then $\mathcal{V}_2 \subseteq \oplus_j \mathcal{V} \times E_j$. Hence $d(\mathcal{V}, \mathcal{V}_2) \geq$ fusion rank of $\mathcal{V}$.

Exercise 15.23 Duality of fission and fusion: *Let $\mathcal{V}$ be a vector space on E and let Π_s be a partition of E. Show that the fission rank of $\mathcal{V}$ relative to Π_s is equal to the fusion rank of $\mathcal{V}^{\perp}$ relative to Π_s.*

We now prove a simple result which gives the 'range' over which the nearest vector space, which has the blocks of Π_s as separators, can vary. We also show that there is a unique largest and a unique smallest nearest space and finally that the hybrid ranks of $\mathcal{V}$ and $\mathcal{V}^{\perp}$, relative to a partition of the underlying set, are equal.

Theorem 15.5.1 *Let $\mathcal{V}$ be a vector space on E and let $\Pi_s \equiv \{E_1, \cdots, E_k\}$ be a partition of E. Let vector spaces $\mathcal{V}_1, \mathcal{V}_2$ on E have $E_j, j = 1, \cdots, k$ as separators and further be such that*

$$d(\mathcal{V}, \mathcal{V}_1) = d(\mathcal{V}, \mathcal{V}_2) \leq d(\mathcal{V}, \mathcal{V}'),$$

whenever vector space $\mathcal{V}'$ has $E_j, j = 1, \cdots, k$ as separators. Then the following hold.

i. $\oplus_j \mathcal{V} \cdot E_j \supseteq \mathcal{V}_1 \supseteq \oplus_j \mathcal{V} \times E_j$

ii. $d(\mathcal{V}, \mathcal{V}_1) = d(\mathcal{V}, \mathcal{V}_1 + \mathcal{V}_2) = d(\mathcal{V}, \mathcal{V}_1 \cap \mathcal{V}_2)$

iii. *There is a unique maximal space $\mathcal{V}^m$ and a unique minimal space $\mathcal{V}_m$ on E which have the blocks of Π_s as separators and are at the same (minimum) distance from $\mathcal{V}$ as $\mathcal{V}_1$.*

iv. *$\mathcal{V}_1^\perp$ has $E_j, j = 1, \cdots, k$ as separators and $d(\mathcal{V}^\perp, \mathcal{V}_1^\perp) \leq d(\mathcal{V}^\perp, \mathcal{V}')$ whenever $\mathcal{V}'$ has the E_j as separators. Further, $d(\mathcal{V}, \mathcal{V}_1) = d(\mathcal{V}^\perp, \mathcal{V}_1^\perp)$.*

v. *The generalized hybrid ranks of $\mathcal{V}$ and $\mathcal{V}^\perp$ relative to Π_s are equal.*

Proof : We remind the reader that

$$d(\mathcal{V}, \mathcal{V}') \equiv r(\mathcal{V} + \mathcal{V}') - r(\mathcal{V} \cap \mathcal{V}').$$

i. Suppose $\mathcal{V}_1 \cap (\oplus_j \mathcal{V} \times E_j) \subset \oplus_j \mathcal{V} \times E_j$. Consider the space $\mathcal{V}_n = \mathcal{V}_1 + (\oplus_j \mathcal{V} \times E_j)$. Since both $\mathcal{V}_1$ and $\oplus_j \mathcal{V} \times E_j$ have E_j as separators, $\mathcal{V}_n$ also will have E_j as separators. Now

$$\oplus_j (\mathcal{V} \times E_j) \subseteq \mathcal{V}.$$

Hence, $\mathcal{V} + \mathcal{V}_n = \mathcal{V} + \mathcal{V}_1$. But $\mathcal{V} \cap \mathcal{V}_n \supset \mathcal{V} \cap \mathcal{V}_1$. Hence, $d(\mathcal{V}, \mathcal{V}_n) < d(\mathcal{V} \cap \mathcal{V}_1)$, a contradiction. We can similarly prove (using sum in place of intersection)

$$\mathcal{V}_1 \subseteq \oplus_j (\mathcal{V} \ . \ E_j).$$

The result also follows by using duality, i.e., working with $\mathcal{V}^\perp$ in place of $\mathcal{V}$ and using the facts that

$$(\mathcal{V} \ . \ E_j)^\perp = \mathcal{V}^\perp \times E_j, (\mathcal{V} + \mathcal{V}')^\perp = \mathcal{V}^\perp \cap (\mathcal{V}')^\perp$$

and $d(\mathcal{V}, \mathcal{V}') = d(\mathcal{V}^\perp, (\mathcal{V}')^\perp)$.

ii. It can be verified that (see Exercise 15.21), $d(\mathcal{V}, \mathcal{V}') + d(\mathcal{V}, \mathcal{V}") \geq$ $\geq d(\mathcal{V}, (\mathcal{V} + \mathcal{V}")) + d(\mathcal{V}, (\mathcal{V} \cap \mathcal{V}"))$, for any spaces $\mathcal{V}, \mathcal{V}', \mathcal{V}"$ on E. Now if $\mathcal{V}_1, \mathcal{V}_2$ both have E_j as separators, so will $\mathcal{V}_1 + \mathcal{V}_2$ as well as $\mathcal{V}_1 \cap \mathcal{V}_2$ have. If $\mathcal{V}_1, \mathcal{V}_2$ are used in place of $\mathcal{V}', \mathcal{V}"$ in the above inequality, since $d(\mathcal{V}, \mathcal{V}_1)$ is the minimum possible under the condition that $\mathcal{V}_1$ has E_j as separators, it follows that the inequality should be an equality. So $\mathcal{V}_1 + \mathcal{V}_2, \mathcal{V}_1 \cap \mathcal{V}_2$ also are at minimum distance from $\mathcal{V}$ under the condition that they have E_j as separators.

iii. If there are two distinct minimal (maximal) spaces with the specified property, then their intersection (sum) would have the same property and be at the same distance from $\mathcal{V}$. This would contradict minimality (maximality).

iv. Let $\mathcal{V}_3, \mathcal{V}_4$ be vector spaces on E. We have

$$\begin{aligned} d(\mathcal{V}_3^{\perp}, \mathcal{V}_4^{\perp}) &= r(\mathcal{V}_3^{\perp} + \mathcal{V}_4^{\perp}) - r(\mathcal{V}_3^{\perp} \cap \mathcal{V}_4^{\perp}) \\ &= \mid E \mid - r(\mathcal{V}_3 \cap \mathcal{V}_4) - \mid E \mid + r(\mathcal{V}_3 + \mathcal{V}_4) \\ &= d(\mathcal{V}_3, \mathcal{V}_4). \end{aligned}$$

If $\mathcal{V}_1$ has E_j as a separator,

$$\mathcal{V}_1 \cdot E_j = \mathcal{V}_1 \times E_j.$$

So, $(\mathcal{V}_1 \cdot E_j)^{\perp} = (\mathcal{V}_1 \times E_j)^{\perp}$
i.e., $\mathcal{V}_1^{\perp} \times E_j = \mathcal{V}_1^{\perp} \cdot E_j$.
Hence, $\mathcal{V}_1^{\perp}$ also has E_j as separators. Thus, if $\mathcal{V}_1$ is the nearest space to $\mathcal{V}$ with E_j as separators, $\mathcal{V}_1^{\perp}$ is the nearest space to $\mathcal{V}^{\perp}$ with E_j as separators and, further,

$$d(\mathcal{V}, \mathcal{V}_1) = d(\mathcal{V}^{\perp}, \mathcal{V}_1^{\perp}).$$

v. This is immediate from the above.

□

15.5.3 Port Decomposition and Generalized Hybrid Rank

In Sections 15.3 and 15.4, we saw that as far as topological interrelationships between subnetworks is concerned, one could replace the subnetworks by forest subgraphs which do not contain cutsets of the overall graph. We have a similar notion for the present formulation also. We give a few simple results based on multiport decomposition from which all such ideas follow. Finally, in Theorem 15.5.2, we show that, instead of working with $\mathcal{V}_E$, we can work with $\mathcal{V}_P$, the coupler space in a minimal multiport decomposition of $\mathcal{V}_E$, for all hybrid rank related algorithms.

We begin with some preliminary definitions and results. The reader needs to be familiar with ideas such as generalized minors, matched and skewed sums etc. from Chapter 7 and with multiport decomposition, minimal multiport decomposition etc. from Chapter 8. Let $\mathcal{V}_{EP}, \mathcal{V}_{PQ}$ be vector spaces on $E \uplus P, P \uplus Q$ respectively with $E \cap Q = \emptyset$. We say the **ordered pair** $(\mathcal{V}_{EP}, \mathcal{V}_{PQ})$ is **compatible** iff $\mathcal{V}_{EP} \cdot P \supseteq \mathcal{V}_{PQ} \cdot P$ and $\mathcal{V}_{EP} \times P \subseteq \mathcal{V}_{PQ} \times P$ (see Section 8.2). Since contraction and restriction are (orthogonal) duals, it is clear that $(\mathcal{V}_{EP}, \mathcal{V}_{PQ})$ is compatible iff $(\mathcal{V}_{EP}^{\perp}, \mathcal{V}_{PQ}^{\perp})$ is compatible. Let $\mathcal{V}_E$ have the multiport decomposition $((\mathcal{V}_{E_jP_j})_k, \mathcal{V}_P)$. We say that the **multiport decomposition** is **compatible** iff $\mathcal{V}_{E_jP_j} \cdot P_j \supseteq \mathcal{V}_P \cdot P_j$ and $\mathcal{V}_{E_jP_j} \times P_j \subseteq \mathcal{V}_P \times P_j$, for $j = 1, \cdots, k$. One can show that the decomposition is compatible iff $(\oplus_j \mathcal{V}_{E_jP_j}, \mathcal{V}_P)$ is compatible (Exercise 8.11). We now list a few simple lemmas without proof. These lemmas are parts of exercises or problems as indicated and their proofs are given in the corresponding solutions.

Lemma 15.5.1 *Let $\mathcal{V}_{EP}, \mathcal{V}_{PQ}, \mathcal{V}_{QR}$ be vector spaces on $E \uplus P, P \uplus Q, Q \uplus R$ with E, P, Q, R pairwise disjoint. Then*

$$(\mathcal{V}_{EP} \leftrightarrow \mathcal{V}_{PQ}) \leftrightarrow \mathcal{V}_{QR} = \mathcal{V}_{EP} \leftrightarrow (\mathcal{V}_{PQ} \leftrightarrow \mathcal{V}_{QR})$$

(for proof see Exercise 7.6).

Lemma 15.5.2 *Let $(\mathcal{V}_{EP}, \mathcal{V}_{PQ})$ be a compatible ordered pair of vector spaces. Then $(\mathcal{V}_{EP}, \mathcal{V}_{EP} \leftrightarrow \mathcal{V}_{PQ})$ is compatible and $\mathcal{V}_{EP} \leftrightarrow (\mathcal{V}_{EP} \leftrightarrow \mathcal{V}_{PQ}) = \mathcal{V}_{PQ}$.*

(For proof see Exercise 7.7).

Lemma 15.5.3 *Let $\mathcal{V}_{EP} \equiv \oplus_j \mathcal{V}_{E_jP_j}$. Then $\mathcal{V}_{EP} \leftrightarrow \mathcal{V}_P$ has E_j as separators, if $\mathcal{V}_P$ has P_j as separators.*

(For proof see Exercise 8.6).

Lemma 15.5.4 *Let $((\mathcal{V}_{E_jP_j})_k, \mathcal{V}_P)$ be a compatible multiport decomposition of $\mathcal{V}_E$. Let the following set functions be defined on subsets of $S \equiv \{1, \cdots, k\}$:*

$$\rho_E(I) \equiv r(\mathcal{V}_E \cdot (\bigcup_{i \in I} E_i)),$$

$$\omega_E(I) \equiv \sum_{i \in I} r(\mathcal{V}_{E_iP_i} \times E_i),$$

$$\rho_P(I) \equiv r(\mathcal{V}_P \cdot (\bigcup_{i \in I} P_i)),$$

$$\omega_P(I) \equiv \sum_{i \in I} r(\mathcal{V}_{E_iP_i} \times P_i).$$

Then

i. *$\rho_E(\cdot), \rho_P(\cdot)$ are polymatroid rank functions while $\omega_E(\cdot), \omega_P(\cdot)$ are modular functions*

ii. $(\rho_E - \omega_E)(\cdot) = (\rho_P - \omega_P)(\cdot)$

iii. *the fusion rank of $\mathcal{V}_E$ relative to $\{E_1, \cdots, E_k\}$ is equal to the fusion rank of $\mathcal{V}_P$ relative to $\{P_1, \cdots, P_k\}$*

iv. *the fission rank of $\mathcal{V}_E$ relative to $\{E_1, \cdots, E_k\}$ is equal to the fission rank of $\mathcal{V}_P$ relative to $\{P_1, \cdots, P_k\}$.*

Proof : For proof of the first two parts see Problem 8.8.

iii. The fusion rank of $\mathcal{V}_E$ relative to $\{E_1, \cdots, E_k\} = (\rho_E - \omega_E)(S) =$
$= (\rho_P - \omega_P)(S) =$ fusion rank of $\mathcal{V}_P$ relative to $\{P_1, \cdots, P_k\}$.

iv. We observe that if $((\mathcal{V}_{E_jP_j})_k, \mathcal{V}_P)$ is a compatible multiport decomposition of $\mathcal{V}_E$ then $((\mathcal{V}^{\perp}_{E_jP_j})_k, \mathcal{V}^{\perp}_P)$ is a compatible multiport decomposition of $\mathcal{V}^{\perp}_E$ (by Theorem 8.2.1 and the facts that contractions and restrictions are orthogonal duals and that if $\mathcal{V}_1 \subseteq \mathcal{V}_2$ then $\mathcal{V}_1^{\perp} \supseteq \mathcal{V}_2^{\perp}$). We note also that the fission rank of $\mathcal{V}_E$ relative to $\{E_1, \cdots, E_k\}$ equals the fusion rank of $\mathcal{V}^{\perp}_E$ relative to $\{E_1, \cdots, E_k\}$ (Exercise 15.23).

Let $\rho^*_E(\cdot)$ be defined by replacing $\mathcal{V}_E$ by $\mathcal{V}^{\perp}_E$ in the definition of $\rho_E(\cdot)$, $\omega^*_E(\cdot)$ by replacing $\mathcal{V}_{E_iP_i}$ by $\mathcal{V}^{\perp}_{E_iP_i}$ in the definition of $\omega_E(\cdot)$, $\rho^*_P(\cdot)$ by replacing $\mathcal{V}_P$ by $\mathcal{V}^{\perp}_P$

in the definition of $\rho_P(\cdot)$ and $\omega^*_P(\cdot)$ by replacing $\mathcal{V}_{E_iP_i}$ by $\mathcal{V}^{\perp}_{E_iP_i}$ in the definition of $\omega_P(\cdot)$. It is clear that $(\rho^*_E - \omega^*_E)(\cdot) = (\rho^*_P - \omega^*_P)(\cdot)$. Hence the fission rank of $\mathcal{V}_E$ relative to $\{E_1, \cdots, E_k\}$ = the fusion rank of $\mathcal{V}^{\perp}_E$ relative to $\{E_1, \cdots, E_k\}$ = $(\rho^*_E - \omega^*_E)(S) = (\rho^*_P - \omega^*_P)(S)$ = fusion rank of $\mathcal{V}^{\perp}_P$ relative to $(P_1, \cdots, P_k)$ = fission rank of $\mathcal{V}_P$ relative to $(P_1, \cdots, P_k)$.

□

Theorem 15.5.2 *Let $(\mathcal{V}_{E_1P_1}, \cdots, \mathcal{V}_{E_kP_k}; \mathcal{V}_P)$ be a minimal multiport decomposition of vector space $\mathcal{V}_E$ on E and let*

$$\mathcal{V}_{EP} \equiv \oplus_j \mathcal{V}_{E_jP_j}.$$

Then

i. *the fusion rank of $\mathcal{V}_E$, relative to $\Pi_s \equiv \{E_1, \cdots, E_k\}$, equals the rank of $\mathcal{V}_P$*

ii. *the fission rank of $\mathcal{V}_E$ relative to $\Pi_s \equiv \{E_1, \cdots, E_k\}$ equals the nullity of $\mathcal{V}_P$*

iii. *if $\mathcal{V}_{PQ}, \mathcal{V}_Q$ are s.t. $(\mathcal{V}_{PQ} \leftrightarrow \mathcal{V}_Q) = \mathcal{V}_P$ then*

$$(\mathcal{V}_{EP} \leftrightarrow \mathcal{V}_{PQ}) \leftrightarrow \mathcal{V}_Q = \mathcal{V}_E$$

and further if $(\mathcal{V}_{PQ} \leftrightarrow \mathcal{V}'_Q)$ has $P_1, \cdots, P_k$ as separators then $(\mathcal{V}_{EP} \leftrightarrow \mathcal{V}_{PQ}) \leftrightarrow \mathcal{V}'_Q$ has $E_1, \cdots, E_k$ as separators

iv. *if $\mathcal{V}_{EQ}, \mathcal{V}_Q$ are s.t. $(\mathcal{V}_{EQ} \leftrightarrow \mathcal{V}_Q) = \mathcal{V}_E$ then*

$$(\mathcal{V}_{EP} \leftrightarrow \mathcal{V}_{EQ}) \leftrightarrow \mathcal{V}_Q = \mathcal{V}_P$$

and further if $(\mathcal{V}_{EQ} \leftrightarrow \mathcal{V}'_Q)$ has $E_1, \cdots E_k$ as separators then $(\mathcal{V}_{EP} \leftrightarrow \mathcal{V}_{EQ}) \leftrightarrow \mathcal{V}'_Q$ has $P_1, \cdots P_k$ as separators

v. *the generalized hybrid rank of $\mathcal{V}_E$ relative to Π_s equals the generalized hybrid rank of $\mathcal{V}_P$ relative to $\{P_1, \cdots, P_k\}$.*

Proof : We first observe that a minimal decomposition is a compatible decomposition (Theorem 8.4.1), and, therefore (Exercise 8.11), the ordered pair $(\oplus_j \mathcal{V}_{E_jP_j}, \mathcal{V}_P)$ is compatible. Further, by the above mentioned theorem, when the decomposition is minimal,

$$r(\mathcal{V}_P \times P_i) = r(\mathcal{V}^{\perp}_P \times P_i) = 0, \qquad i = 1, \cdots, k.$$

i. Follows from Lemma 15.5.4 when we observe that the fusion rank of $\mathcal{V}_P$ relative to $\{P_1, \cdots, P_k\}$ is its rank since $r(\mathcal{V}_P \times P_i) = 0, \quad i = 1, \cdots, k$.

ii. Follows from the above mentioned lemma when we observe that the fission rank of $\mathcal{V}_P$ relative to $\{P_1, \cdots, P_k\}$ is its nullity $(r(\mathcal{V}^{\perp}_P))$, since $r(\mathcal{V}^{\perp}_P \times P_i) = 0$, $i = 1, \cdots, k$.

iii. We have $(\mathcal{V}_{EP} \leftrightarrow \mathcal{V}_P) = \mathcal{V}_E$. Hence, $\mathcal{V}_{EP} \leftrightarrow (\mathcal{V}_{PQ} \leftrightarrow \mathcal{V}_Q) = \mathcal{V}_E$. Hence by Lemma 15.5.1
$$(\mathcal{V}_{EP} \leftrightarrow \mathcal{V}_{PQ}) \leftrightarrow \mathcal{V}_Q = \mathcal{V}_E.$$
Next, by the same lemma,
$$(\mathcal{V}_{EP} \leftrightarrow \mathcal{V}_{PQ}) \leftrightarrow \mathcal{V}'_Q = \mathcal{V}_{EP} \leftrightarrow (\mathcal{V}_{PQ} \leftrightarrow \mathcal{V}'_Q).$$
By Lemma 15.5.3, since $\mathcal{V}_{EP} \equiv \oplus_j \mathcal{V}_{E_j P_j}$ and $(\mathcal{V}_{PQ} \leftrightarrow \mathcal{V}'_Q)$ has P_j as separators, $\mathcal{V}_{EP} \leftrightarrow (\mathcal{V}_{PQ} \leftrightarrow \mathcal{V}'_Q)$ has E_j as separators.

iv. Since $((\mathcal{V}_{E_j P_j})_k, \mathcal{V}_P)$ is a minimal decomposition, $(\oplus_j \mathcal{V}_{E_j P_j}, \mathcal{V}_P)$ is compatible. Then, by Lemma 15.5.2
$$(\oplus_j \mathcal{V}_{E_j P_j}) \leftrightarrow ((\oplus_j \mathcal{V}_{E_j P_j}) \leftrightarrow \mathcal{V}_P) = \mathcal{V}_P,$$
i.e., $(\oplus_j \mathcal{V}_{E_j P_j}) \leftrightarrow \mathcal{V}_E = \mathcal{V}_P$, i.e., $((\mathcal{V}_{E_j P_j})_k, \mathcal{V}_E)$ is a decomposition of $\mathcal{V}_P$. The result now follows by arguing as in (iii) above. (Note that in (iii) above we did not use minimality of decomposition).

v. This follows directly from (iii) and (iv) above and the definition of generalized hybrid rank (page 657).

□

Remark: In the above theorem it may be noted that the fifth part depends only on **compatibilty** of the decomposition and not on its **minimality**.

If $\mathcal{V}_E$ is the voltage space of a graph $\mathcal{G}$ and $\{E_1, \cdots, E_k\}$, a given partition of E, we can build a minimal multiport decomposition $((\mathcal{V}_{E_j P_j})_k, \mathcal{V}_P)$ such that $\mathcal{V}_P$ is the voltage space of a graph $\mathcal{G}_P$ (see Algorithm (Port minimization1) of Chapter 8). In $\mathcal{G}_P$ each $\mathcal{G}_P \cdot P_j, j = 1, \cdots, k$, would appear as a forest graph containing no cutsets of $\mathcal{G}_P$. By using the theorems listed in this subsection it follows that for computing the generalized hybrid rank we can work with $\mathcal{G}_P$ rather than $\mathcal{G}$. The results of Section 15.4 imply that the same is true in the case of the third formulation as well as for computing the minimum length fusions and fissions sequence.

Exercise 15.24 A polymatroid membership problem: *Give a polynomial algorithm for the following problem:*
Find if there exists, an independent set of columns of a representative matrix of $\mathcal{V}_E$ *that contains precisely* k_j *columns from* $E_j, j = 1, \cdots, k$.

15.5.4 Relation between the Hybrid Rank of a Representative Matrix of a Vector Space and its Generalized Hybrid Rank relative to a Partition

In this subsection we **relate** the generalized hybrid rank of a vector space relative to a partition of the underlying set **to** the hybrid rank of the matroid of a modified representative matrix. Analogous to the case of the third formulation (Subsection 15.4.3), we show that the hybrid rank of a vector space on E, relative to a partition $\Pi_s \equiv \{E_1, \cdots, E_k\}$ of E, is the minimum of the hybrid ranks of the matroids associated with matrices which are obtained by replacing the columns in E_j by a set of independent cospanning columns.

A few preliminary definitions:
Let $\mathcal{M}$ be a matroid on E. The **hybrid rank of** $\mathcal{M}$ has already been defined to be

$$\min_{K \subseteq E} r(\mathcal{M} \cdot K) + \nu(\mathcal{M} \times (E - K)).$$

Let $\mathbf{A}$ be a matrix. The **matroid** $\mathcal{M}(\mathbf{A})$ is the matroid on the set of columns of $\mathbf{A}$ where a subset is independent in $\mathcal{M}(\mathbf{A})$ iff the corresponding set of columns are independent in $\mathbf{A}$. We say $\mathcal{M}(\mathbf{A})$ is **associated** with $\mathbf{A}$. Let $\mathcal{V}_E$ be a vector space on E. Then $\mathbf{A}$ **represents** $\mathcal{V}_E$ if its rows form a basis for $\mathcal{V}_E$. If $\mathbf{A}$ represents $\mathcal{V}_E$, we denote by $\mathbf{A}^*$, a representative matrix of $\mathcal{V}_E^{\perp}$. The **hybrid rank of** $\mathcal{V}_E$ is the hybrid rank of $\mathcal{M}(\mathbf{A})$, which can be seen to be

$$\min_{K \subseteq E} r(\mathcal{V} \,.\, K) + r(\mathcal{V}^{\perp} \cdot (E - K)),$$

$$equivalently, \ \min_{K \subseteq E} r(\mathcal{V} \,.\, K) + \nu(\mathcal{V} \times (E - K)).$$

Let $\Pi_s \equiv \{E_1, \cdots, E_k\}$ be a partition of E. If $((\mathcal{V}_{E_j P_j})_k, \mathcal{V}_P)$ is a minimal multiport decomposition of $\mathcal{V}_E$, we know (by Theorem 8.4.1) that $r(\mathcal{V}_P \cdot P_i) = \mid P_i \mid$ and $r(\mathcal{V}_P^{\perp} \cdot P_i) = \mid P_i \mid, i = 1, \cdots, k$. Further, by Theorem 15.5.2, the generalized hybrid rank of $\mathcal{V}_E$ relative to $\{E_1, \cdots, E_k\}$ equals that of $\mathcal{V}_P$ relative to $\{P_1, \cdots, P_k\}$. More can be said using the above mentioned theorem: if we know how to find the nearest $\mathcal{V}_P'$ (to $\mathcal{V}_P$) which has P_i as separators, we also know how to find the nearest $\mathcal{V}_E'$ (to $\mathcal{V}_E$) which has E_i as separators. So, for all practical purposes, we may pretend that we are working with $\mathcal{V}_P, \{P_1, \cdots, P_k\}$, corresponding to a minimal decomposition of $\mathcal{V}_E$, relative to $\{E_1, \cdots, E_k\}$. Now if $((\mathcal{V}_{E_j P_j})_k, \mathcal{V}_P)$ is a minimal multiport decomposition of $\mathcal{V}_E$, we know by Theorem 8.4.1, that the columns corresponding to P_i are independent in representative matrices of both $\mathcal{V}_P$ as well as $\mathcal{V}_P^{\perp}$. Therefore, whenever convenient, we assume that each set of columns E_i is independent in $\mathbf{A}$ as well as in $\mathbf{A}^*$.

Let $\mathbf{A}$ be a matrix with E as its column set and let $\Pi_s \equiv \{E_1, \cdots, E_k\}$ be a partition of E. Let $\mathcal{C}_j$ be the vector space spanned by the columns of E_j. We say a matrix $\mathbf{A}'$ **is equivalent to A** relative to $\{\mathcal{C}_j, j = 1, \cdots, k\}$, if its columns can be partitioned into $\{E_1', \cdots, E_k'\}$ where the E_j' are bases of $\mathcal{C}_j, j = 1, \cdots, k$.
If $\mathcal{V}'$ is a vector space on E, then $d(\mathcal{V}, \mathcal{V}') = d(\mathcal{V}, \mathcal{V} \cap \mathcal{V}') + d(\mathcal{V} \cap \mathcal{V}', \mathcal{V}')$ (see Exercise 15.21). If $\mathcal{V}'$ is the **nearest** space to $\mathcal{V}$ with blocks of Π_s as separators, it follows that no proper subspace $\mathcal{V}$" of $\mathcal{V}'$ can contain $\mathcal{V} \cap \mathcal{V}'$ as well as have blocks of Π_s as separators (otherwise $d(\mathcal{V}, \mathcal{V}") \leq d(\mathcal{V}, \mathcal{V} \cap \mathcal{V}') + d(\mathcal{V} \cap \mathcal{V}', \mathcal{V}") < d(\mathcal{V}, \mathcal{V}')$). Now $\mathcal{V}' = \oplus_j (\mathcal{V}' \cdot E_j) \supseteq \oplus_j ((\mathcal{V} \cap \mathcal{V}') \cdot E_j) \supseteq \mathcal{V} \cap \mathcal{V}'$. Hence, $\mathcal{V}' = \oplus_j ((\mathcal{V} \cap \mathcal{V}') \cdot E_j)$ and $d(\mathcal{V} \cap \mathcal{V}', \mathcal{V}')$ is the fission rank of $\mathcal{V} \cap \mathcal{V}'$. The problem of finding the nearest $\mathcal{V}'$ with the desired properties is therefore equivalent to finding a subspace $\mathcal{V}_1$ of $\mathcal{V}$ for

which $(d(\mathcal{V}, \mathcal{V}_1)$ + fission rank of $\mathcal{V}_1)$ is a minimum. Let us call this number the **fusion - fission number of** $\mathcal{V}_1$ relative to $(\mathcal{V}, \Pi_s)$. We then have the following simple result whose routine computational proof we omit.

Theorem 15.5.3 *Let $\mathcal{V}$ be a vector space on E. Let $\Pi_s \equiv \{E_1, \cdots, E_k\}$ be a partition of E. Let $\mathcal{V}_1$ be a subspace of $\mathcal{V}$. Then*

i. the fusion - fission number of $\mathcal{V}_1$ relative to $(\mathcal{V}, \Pi_s)$

$$= \sum_j r(\mathcal{V}_1 \cdot E_j) - 2r(\mathcal{V}_1) + r(\mathcal{V})$$

ii. the generalized hybrid rank of $\mathcal{V}$ relative to Π_s

$$= \min_{\mathcal{V}_1 \subseteq \mathcal{V}} (\sum_j r(\mathcal{V}_1 \cdot E_j) - 2r(\mathcal{V}_1) + r(\mathcal{V}))$$

Exercise 15.25 Fission - fusion number: *We know that $d(\mathcal{V}, \mathcal{V}_1) =$ $= d(\mathcal{V}, \mathcal{V} + \mathcal{V}_1) + d(\mathcal{V} + \mathcal{V}_1, \mathcal{V}_1)$. One may associate with a superspace $\mathcal{V}'$ of $\mathcal{V}$ the 'fission - fusion number' $d(\mathcal{V}, \mathcal{V}') + d(\mathcal{V}', \mathcal{V}_1)$, where $d(\mathcal{V}', \mathcal{V}_1)$ is the fusion number of $\mathcal{V}'$ relative to the partition $\Pi_s \equiv \{E_1, \cdots, E_k\}$. Show that*

i. the fission - fusion number of a superspace $\mathcal{V}'$ of $\mathcal{V}$ equals

$$-\sum_j r(\mathcal{V}' \times E_j) + 2r(\mathcal{V}') - r(\mathcal{V})$$

ii. the fission - fusion number of a superspace $\mathcal{V}'$ of $\mathcal{V}$ relative to $(\mathcal{V}, \Pi_s)$ equals the fusion - fission number of the subspace $(\mathcal{V}')^\perp$ of $\mathcal{V}^\perp$ relative to $(\mathcal{V}^\perp, \Pi_s)$.

iii. the minimum of the fission - fusion numbers of superspaces of $\mathcal{V}$ equals the minimum of the fusion - fission numbers of subspaces of $\mathcal{V}$= generalized hybrid rank of $\mathcal{V}$ relative to Π_s.

The reader would notice that the above result is analogous to Theorem 15.4.2. Both the results may be regarded as of 'row' type. ($\mathcal{V}_1$ is a row subspace, the voltage space of $\mathcal{G}_{fus \cdot \Pi}$ is a subspace of the voltage space of $\mathcal{G}$). Our ultimate aim is to prove a result analogous to Theorem 15.4.3 where hybrid rank relative to Π_s is shown to be equal to the minimum of hybrid ranks over 'equivalent' spaces. To do this we need to get a 'column' version of the hybrid rank relative to Π_s. We need a few definitions to proceed further.

Let $\mathbf{A}$ be the representative matrix of a vector space $\mathcal{V}_E$. Then the row space $\mathcal{R}(\mathbf{A}) = \mathcal{V}_E$. The column space $\mathcal{C}(\mathbf{A})$ is the span of the columns of $\mathbf{A}$. We relate subspaces of $\mathcal{R}(\mathbf{A})$ to subspaces of $\mathcal{C}(\mathbf{A})$ as follows. Let $\mathcal{V}_1 \subseteq \mathcal{R}(\mathbf{A})$. Then the **column annihilator with respect to A of** $\mathcal{V}_1$, denoted by $\mathcal{A}_c(\mathcal{V}_1, \mathbf{A})$, is the collection of vectors of the form $\mathbf{Ax}$ s.t. $\lambda^T \mathbf{Ax} = 0$ whenever $\lambda^T \mathbf{A}$ belongs to $\mathcal{V}_1$. Similarly the **row annihilator with respect to A of** $\mathcal{C}_1 \subseteq \mathcal{C}(\mathbf{A})$, denoted by

$\mathcal{A}_r(\mathbf{A},\mathcal{C}_1)$, is the collection of vectors of the form $\lambda^T\mathbf{A}$ s.t. $\lambda^T\mathbf{Ax} = 0$ whenever $\mathbf{Ax}$ belongs to $\mathcal{C}_1$. When it is clear from the context, we would omit reference to the matrix $\mathbf{A}$ and write $\mathcal{A}_c(\mathcal{V}_1), \mathcal{A}_r(\mathcal{C}_1)$ in place of $\mathcal{A}_c(\mathcal{V}_1,\mathbf{A})$, $\mathcal{A}_r(\mathbf{A},\mathcal{C}_1)$ respectively. It is clear that $\mathcal{A}_c(\mathcal{V}_1), \mathcal{A}_r(\mathcal{C}_1)$ are vector spaces whether or not $\mathcal{V}_1, \mathcal{C}_1$ are vector spaces. By routine linear algebra we now prove the following simple result.

Lemma 15.5.5 *Let* $\mathbf{A}$ *be a representative matrix of* $\mathcal{V}_E$. *Let* $E' \subseteq E$. *Let* $\mathcal{V}_1$ *be a subspace of* $\mathcal{V}_E$ *and let* $\mathcal{C}_1$ *be a subspace of* $\mathcal{C}(\mathbf{A})$. *Then*

i. if $\mathcal{C}_1 = \mathcal{A}_c(\mathcal{V}_1)$ *then*

(a) $r(\mathcal{C}_1 \cap \mathcal{C}(\mathcal{A}_1)) + r(\mathcal{V}_1 \cdot E') = r(\mathcal{V}_E \cdot E')$

(b) $r(\mathcal{C}_1) + r(\mathcal{V}_1) = r(\mathcal{V}_E)$.

ii. $\mathcal{C}_1 = \mathcal{A}_c(\mathcal{V}_1)$ *iff* $\mathcal{V}_1 = \mathcal{A}_r(\mathcal{C}_1)$.

Proof :

i(a) Let $\mathcal{V}_E$ have the representative matrix $\mathbf{A} \equiv (\mathbf{A}'\vdots\mathbf{A}")$, where $\mathbf{A}'$ denotes the submatrix of $\mathbf{A}$ composed of all rows of $\mathbf{A}$ and E' as the set of columns. Let $\mathcal{V}_1$ have the representative matrix $(\mathbf{L})(\mathbf{A}'\vdots\mathbf{A}")$. Then $\mathcal{C}_1$ is the collection of all vectors $(\mathbf{A}'\vdots\mathbf{A}")^{\mathbf{x}_1}_{\mathbf{x}_2}$ s.t. $\mathbf{L}(\mathbf{A}'\vdots\mathbf{A}")^{\mathbf{x}_1}_{\mathbf{x}_2} = 0$ and $\mathcal{C}_1 \cap \mathcal{C}(\mathbf{A}')$ is the collection of all vectors $\mathbf{A}'\mathbf{x}_1$ s.t. $\mathbf{L}(\mathbf{A}')\mathbf{x}_1 = 0$. Now the rows of $\mathbf{L}(\mathbf{A}')$ span $\mathcal{V}_1 \cdot E'$. The collection of all vectors orthogonal to this space is the solution space of $(\mathbf{LA}')\mathbf{x}_1 = 0$. Let us call this space $\mathcal{V}_{x_1}$. Clearly

$$r(\mathcal{V}_{x_1}) = \mid E' \mid - r(\mathcal{V}_1 \cdot E').$$

Consider the linear transformation

$$\mathbf{A}' : \mathcal{V}_{x_1} \to \mathcal{C}_1 \cap \mathcal{C}(\mathbf{A}')$$

where $\mathbf{x}_1 \in \mathcal{V}_{x_1}$ is mapped to $\mathbf{A}'\mathbf{x}_1$. This is an onto mapping and its null space is the space of all vectors $\mathbf{x}_1 \in \mathcal{V}_{x_1}$ s.t. $\mathbf{A}'\mathbf{x}_1 = 0$. This space is $(\mathcal{V}_E \cdot E')^\perp \cap \mathcal{V}_{x_1}$. However, $(\mathcal{V}_E \cdot E')^\perp \subseteq \mathcal{V}_{x_1}$. So the null space is $(\mathcal{V}_E \cdot E')^\perp$. Hence,

$$\begin{aligned} r(\mathcal{C}_1 \cap \mathcal{C}(\mathbf{A}')) &= r(\mathcal{V}_{x_1}) - r(\mathcal{V}_E \cdot E')^\perp \\ &= \mid E' \mid - r(\mathcal{V}_1 \cdot E') - \mid E' \mid + r(\mathcal{V}_E \cdot E') \\ &= r(\mathcal{V}_E \cdot E') - r(\mathcal{V}_1 \cdot E'). \end{aligned}$$

i(b) This follows from the above by putting $E' = E$.

ii. By the definition of $\mathcal{A}_c(\cdot), \mathcal{A}_r(\cdot)$, if $\mathcal{C}_1 = \mathcal{A}_c(\mathcal{V}_1)$ it is clear that $\mathcal{V}_1 \subseteq \mathcal{A}_r(\mathcal{C}_1)$. Further $\mathcal{C}_1 \subseteq \mathcal{A}_c(\mathcal{A}_r(\mathcal{C}_1))$. Since $\mathcal{C}_1 = \mathcal{A}_c(\mathcal{V}_1)$, we have $r(\mathcal{C}_1) = r(\mathcal{V}_E) - r(\mathcal{V}_1)$. If $\mathcal{V}_1 \subset \mathcal{A}_r(\mathcal{C}_1)$, since $\mathcal{C}_1 \subseteq \mathcal{A}_c(\mathcal{A}_r(\mathcal{C}_1))$ we must have

$$\begin{aligned} r(\mathcal{C}_1) &\leq r(\mathcal{V}_E) - r(\mathcal{A}_r(\mathcal{C}_1)) \\ &< r(\mathcal{V}_E) - r(\mathcal{V}_1), \end{aligned}$$

a contradiction. So $\mathcal{V}_1 = \mathcal{A}_r(\mathcal{C}_1)$.
The same contradiction results if $\mathcal{C}_1 \subset \mathcal{A}_c(\mathcal{A}_r(\mathcal{C}_1))$. So $\mathcal{C}_1 = \mathcal{A}_c(\mathcal{A}_r(\mathcal{C}_1))$. Now if $\mathcal{V}_1 = \mathcal{A}_r(\mathcal{C}_1)$ we have $\mathcal{A}_c(\mathcal{V}_1) = \mathcal{A}_c(\mathcal{A}_r(\mathcal{C}_1)) = \mathcal{C}_1$.

□

We now give an expression for the hybrid rank of $\mathcal{V}$ relative to Π_s in terms of column subspaces of a representative matrix of $\mathcal{V}$. The result is analogous to Lemma 15.4.1.

Lemma 15.5.6 *Let $\mathcal{V}$ be a vector space on E and let $\Pi_s \equiv \{E_1, \cdots, E_k\}$ be a partition of E. Let $\mathbf{A}$ be a representative matrix of $\mathcal{V}$ and let $\mathcal{C}_j$ denote the space spanned by the columns $E_j, j = 1, \cdots, k$. Then the generalized hybrid rank of $\mathcal{V}$ relative to Π_s*

$$= \min_{\mathcal{C}}(2r(\mathcal{C}) - \sum_j r(\mathcal{C} \cap \mathcal{C}_j) + \sum_j r(\mathcal{C}_j) - r(\mathcal{V})),$$

where $\mathcal{C}$ is a subspace of $\mathcal{C}(\mathcal{A})$.

Proof : By Theorem 15.5.3, the generalized hybrid rank of $\mathcal{V}$ relative to Π_s

$$= \min_{\mathcal{V}' \subseteq \mathcal{V}}(\sum_j r(\mathcal{V}' \cdot E_j) - 2r(\mathcal{V}') + r(\mathcal{V})).$$

Let $\mathcal{C}'$ denote $\mathcal{A}_c(\mathcal{V}')$. We know by Lemma 15.5.5, that every subspace $\mathcal{C}'$ of $\mathcal{C}(\mathcal{A})$ can be written in the form $\mathcal{A}_c(\mathcal{V}')$ for some subspace $\mathcal{V}' \subseteq \mathcal{V}$. By the same lemma

$$\begin{aligned} r(\mathcal{V}' \cdot E_j) &= r(\mathcal{V} \cdot E_j) - r(\mathcal{C}' \cap \mathcal{C}_j) \\ r(\mathcal{V}') &= r(\mathcal{V}) - r(\mathcal{C}'). \end{aligned}$$

So

$$min_{\mathcal{V}' \subseteq \mathcal{V}}(\sum_j r(\mathcal{V}' \cdot E_j) - 2r(\mathcal{V}') + r(\mathcal{V}))$$

$$= \min_{\mathcal{C}' \subseteq \mathcal{C}(\mathcal{A})}(\sum_j (r(\mathcal{V}\ .\ E_j) - r(\mathcal{C}' \cap \mathcal{C}_j)) - 2(r(\mathcal{V}) - r(\mathcal{C}')) + r(\mathcal{V}).$$

The desired result follows when we note that $r(\mathcal{V}\ .\ E_j) = r(\mathcal{C}_j)$.

□

We now present the main result of this section. We have already indicated that, restricting oneself to the case where the blocks of Π_s are independent in a representative matrix of $\mathcal{V}_E$, does not entail any loss in generality.

Theorem 15.5.4 *Let $\mathcal{V}$ be a vector space on E and let $\Pi_s \equiv \{E_1, \cdots, E_k\}$ be a partition of E. Let $\mathbf{A}$ be a representative matrix of $\mathcal{V}$ with E denoting its set of columns. Let $\mathcal{C}_j$ be the span of the columns E_j for $j = 1, \cdots, k$.*
Then, the generalized hybrid rank of $\mathcal{V}$ relative to Π_s = min (hybrid rank of $\mathbf{A}'$), where $\mathbf{A}'$ is equivalent to $\mathbf{A}$ relative to $\{\mathcal{C}_j, j = 1, \cdots, k\}$.

Proof : Let $\mathbf{A}'$ be the matrix with column set E', equivalent to $\mathbf{A}$ relative to $\{\mathcal{C}_j, j = 1, \cdots, k\}$. Let $\mathcal{M}' \equiv \mathcal{M}(\mathbf{A}')$. Let K be the subset of columns of $\mathbf{A}'$ s.t. $r(\mathcal{M}' \cdot K) + \nu(\mathcal{M}' \times (E' - K)) =$ hybrid rank of $\mathbf{A}'$. Let $\mathcal{C}(K)$ be the vector space spanned by the columns of K. Let $\{E_j{}', j = 1, \cdots, k\}$ be the partition of the columns of $\mathbf{A}'$ s.t. $E_j{}'$ is a basis for $\mathcal{C}_j, j = 1, \cdots, k$. Now

$$r(\mathcal{M}' \cdot K) = r(\mathcal{C}(K)),$$

$$\mid E' - K \mid = \sum_j (\mid E_j{}' \mid - \mid K \cap E_j{}' \mid) \geq \sum_j (r(\mathcal{C}_j) - r(\mathcal{C}_j \cap \mathcal{C}(K)))$$

$$\textit{and} \qquad r(\mathcal{M}' \times (E' - K)) = r(\mathcal{V}) - r(\mathcal{V} \ . \ K) = r(\mathcal{V}) - r(\mathcal{C}(K)).$$

Hence

$$\nu(\mathcal{M}' \times (E' - K)) \geq \sum_j (r(\mathcal{C}_j) - r(\mathcal{C}_j \cap \mathcal{C}(K))) - (r(\mathcal{V}) - r(\mathcal{C}(K))).$$

Thus, the hybrid rank of $\mathbf{A}'$

$$\begin{aligned} &\geq \ 2r(\mathcal{C}(K)) - \sum_j r(\mathcal{C}(K) \cap \mathcal{C}_j) + \sum_j r(\mathcal{C}_j) - r(\mathcal{V}) \\ &\geq \ \textit{hybrid rank of } \mathcal{V} \textit{ relative to } \Pi_s. \end{aligned}$$

Next suppose $\hat{\mathcal{C}}$ minimizes the expression

$$2r(\mathcal{C}) - \sum_j r(\mathcal{C} \cap \mathcal{C}_j) + \sum_j r(\mathcal{C}_j) - r(\mathcal{V})$$

over all subspaces of $\mathcal{C}(\mathbf{A})$. Let us choose $E_j{}'$ so that $\hat{\mathcal{C}} \cap \mathcal{C}_j$, has as basis, a subset K_j' of columns of $E_j{}'$ for $j = 1, \cdots, k$. Let $\mathbf{A}'$ be the matrix whose column set is $E' \equiv \uplus_j E_j{}'$. Let $K \equiv \uplus_j K_j'$. Now

$$\mid E' - K \mid = \sum_j (r(\mathcal{C}_j) - r(\hat{\mathcal{C}} \cap \mathcal{C}_j)).$$

Hence,

$$\begin{aligned} r(\mathcal{M}' \cdot K) + \nu(\mathcal{M}' \times (E' - K)) \ &= \ 2r(\mathcal{C}(K)) - \sum_j r(\hat{\mathcal{C}} \cap \mathcal{C}_j) + \sum_j r(\mathcal{C}_j) - r(\mathcal{V}) \\ &\leq \ 2r(\hat{\mathcal{C}}) - \sum_j r(\hat{\mathcal{C}} \cap \mathcal{C}_j) + \sum_j r(\mathcal{C}_j) - r(\mathcal{V}). \end{aligned}$$

Hence hybrid rank of $\mathbf{A}' \leq$ hybrid rank of $\mathcal{V}$ relative to Π_s. Since $\mathbf{A}'$ is equivalent to $\mathbf{A}$ under $\{\mathcal{C}_j, j = 1, \cdots, k\}$, this proves the required result.

□

Exercise 15.26 Hybrid rank of extension cannot be smaller: *Let $\mathcal{V}$ be a vector space on E and let $\mathcal{V}_{EP}$ be an extension of $\mathcal{V}$, i.e., there exists a vector space $\mathcal{V}_P$ on P s.t.*

$$\mathcal{V}_{EP} \leftrightarrow \mathcal{V}_P = \mathcal{V}.$$

Show that the hybrid rank of $\mathcal{V}$ is not greater than the hybrid rank of $\mathcal{V}_{EP}$.

Exercise 15.27 First and fourth formulations of hybrid rank: *Let $\mathcal{V}$ be a vector space on E and Π_s be the partition of E into singleton blocks. Show that the generalized hybrid rank of $\mathcal{V}$ relative to Π_s equals the hybrid rank of $\mathcal{V}$.*

15.5.5 Nesting Property of Optimal Subspaces

We saw in Subsection 15.5.4 that the generalized hybrid rank problem involves the (alternative) minimization of two functions:

i. $\sum_j r(\mathcal{V} \,.\, E_j) - 2r(\mathcal{V}), \mathcal{V} \subseteq \mathcal{V}_E$

ii. $2r(\mathcal{C}) - \sum_j r(\mathcal{C} \cap \mathcal{C}_j), \mathcal{C} \subseteq \mathcal{C}(\mathbf{A})$, where $\mathbf{A}$ is a representative matrix of $\mathcal{V}_E$.

We will show that both these functions are submodular, that if the subspaces are optimal so are their sum and intersection and, finally, if we replace 2 by λ, the spaces satisfy a nesting property like the sets in the principal partition of a submodular function defined over subsets of a given set.

Let $\mathcal{V}_E$ be a vector space on E and let $f(\cdot)$ be a real valued function on subspaces of $\mathcal{V}_E$. We say $f(\cdot)$ is **submodular** if

$$f(\mathcal{V}_1) + f(\mathcal{V}_2) \geq f(\mathcal{V}_1 + \mathcal{V}_2) + f(\mathcal{V}_1 \cap \mathcal{V}_2)$$

and **supermodular** if

$$f(\mathcal{V}_1) + f(\mathcal{V}_2) \leq f(\mathcal{V}_1 + \mathcal{V}_2) + f(\mathcal{V}_1 \cap \mathcal{V}_2).$$

The function is **modular** if it is both submodular and supermodular. We now prove a simple lemma on the properties of some commonly encountered functions.

Lemma 15.5.7 *Let $r(\cdot)$ be the rank function on the collection of subspaces of $\mathcal{V}_E$. Then,*

i. $\lambda r(\cdot)$ *is a modular function,*

ii. $\sum_j r(\mathcal{V} \,.\, E_j)$, *where $\{E_1, \cdots, E_k\}$ is a specified partition of E, is a submodular function on subspaces of $\mathcal{V}_E$,*

iii. $\sum_j r(\mathcal{V} \cap \mathcal{V}^j)$, *where $\mathcal{V}^1, \cdots, \mathcal{V}^k$ are specified subspaces of $\mathcal{V}_E$, is a supermodular function on subspaces of $\mathcal{V}_E$.*

Proof : i. This is well known but we give the proof for completeness. Let $b_\cap$ be a basis of $\mathcal{V}_1 \cap \mathcal{V}_2$. If we grow $b_\cap$ first to a basis b_1 of $\mathcal{V}_1$, next grow $b_\cap$ to a basis b_2 of $\mathcal{V}_2$, then it is easy to see that $b_1 \cup b_2$ is a basis of $\mathcal{V}_1 + \mathcal{V}_2$ and that $b_\cap = b_1 \cap b_2$. Since

$$| b_1 | + | b_2 | = | b_1 \cap b_2 | + | b_1 \cup b_2 |,$$

we have,

$$r(\mathcal{V}_1) + r(\mathcal{V}_2) = r(\mathcal{V}_1 \cap \mathcal{V}_2) + r(\mathcal{V}_1 + \mathcal{V}_2).$$

ii. We have

$$r(\mathcal{V}_1 \cdot E_j) + r(\mathcal{V}_2 \cdot E_j) = r(\mathcal{V}_1 \cdot E_j + \mathcal{V}_2 \cdot E_j) + r((\mathcal{V}_1 \cdot E_j) \cap (\mathcal{V}_2 \cdot E_j)).$$

Now

$$\mathcal{V}_1 \cdot E_j + \mathcal{V}_2 \cdot E_j = (\mathcal{V}_1 + \mathcal{V}_2) \cdot E_j,$$

and

$$(\mathcal{V}_1 \cdot E_j) \cap (\mathcal{V}_2 \cdot E_j) \supseteq (\mathcal{V}_1 \cap \mathcal{V}_2) \cdot E_j.$$

Hence,

$$r(\mathcal{V}_1 \cdot E_j) + r(\mathcal{V}_2 \cdot E_j) \geq r((\mathcal{V}_1 + \mathcal{V}_2) \cdot E_j) + r((\mathcal{V}_1 \cap \mathcal{V}_2) \cdot E_j).$$

Let $r_j(\mathcal{V}) \equiv r(\mathcal{V} \cdot E_j)$. It is then clear that $\sum_{j=1}^{k} r_j(\cdot)$ is a submodular function.

iii. We have

$$r(\mathcal{V}_1 \cap \mathcal{V}^j) + r(\mathcal{V}_2 \cap \mathcal{V}^j) = r((\mathcal{V}_1 \cap \mathcal{V}^j) + (\mathcal{V}_2 \cap \mathcal{V}^j)) + r(\mathcal{V}_1 \cap \mathcal{V}_2 \cap \mathcal{V}^j).$$

Now

$$(\mathcal{V}_1 \cap \mathcal{V}^j) + (\mathcal{V}_2 \cap \mathcal{V}^j) \subseteq (\mathcal{V}_1 + \mathcal{V}_2) \cap \mathcal{V}^j.$$

Hence,

$$r(\mathcal{V}_1 \cap \mathcal{V}^j) + r(\mathcal{V}_2 \cap \mathcal{V}^j) \leq r((\mathcal{V}_1 + \mathcal{V}_2) \cap \mathcal{V}^j) + r((\mathcal{V}_1 \cap \mathcal{V}_2) \cap \mathcal{V}^j).$$

Let $r^j(\mathcal{V}) \equiv r(\mathcal{V} \cap \mathcal{V}^j)$. It is then clear that $\sum_{j=1}^{k} r^j(\cdot)$ is a supermodular function.

□

The following theorem is now immediate.

Theorem 15.5.5 *Let $\mathcal{V}_E$ be a vector space on E and let $\Pi_s \equiv \{E_1, \cdots, E_k\}$ be a partition of E. Let $\mathbf{A}$ be a representative matrix of $\mathcal{V}_E$. Let $\mathcal{C}_1, \cdots, \mathcal{C}_k$ be subspaces of $\mathcal{C}(\mathbf{A})$. Let*

$$f_\lambda(\mathcal{V}) \equiv \sum_j r(\mathcal{V} \ . \ E_j) - \lambda r(\mathcal{V}), \mathcal{V} \subseteq \mathcal{V}_E$$

and

$$g_\lambda(\mathcal{C}) \equiv \lambda r(\mathcal{C}) - \sum_j r(\mathcal{C} \cap \mathcal{C}_j), \mathcal{C} \subseteq \mathcal{C}(\mathbf{A}).$$

Then $f_\lambda(\cdot)$ is submodular over subspaces of $\mathcal{V}_E$ and $g_\lambda(\cdot)$ is submodular over subspaces of $\mathcal{C}(\mathbf{A})$.

Exercise 15.28 *Give an example where*

i. $(\mathcal{V}_1 \cdot E_j) \cap (\mathcal{V}_2 \cdot E_j) \supset (\mathcal{V}_1 \cap \mathcal{V}_2) \cdot E_j$

ii. $(\mathcal{V}_1 \cap \mathcal{V}^j) + (\mathcal{V}_2 \cap \mathcal{V}^j) \subset (\mathcal{V}_1 + \mathcal{V}_2) \cap \mathcal{V}^j$.

Exercise 15.29 A case where $g_\lambda(\cdot)$ reaches its minimum on the zero subspace: *Let $\mathcal{C}_s$ be a vector space and let $\mathcal{C}_1, \cdots, \mathcal{C}_k$ be given subspaces of $\mathcal{C}_s$. Let $g(\mathcal{C}) \equiv 2r(\mathcal{C}) - \sum_{j=1}^{k} r(\mathcal{C} \cap \mathcal{C}_j)$.*
Suppose $\mathcal{C}_1, \cdots, \mathcal{C}_t$ are such that

$$\sum_{j=1}^{t} r(\mathcal{C}_j) = r(\mathcal{C}_s)$$

and

$$r(\bigcup_{j=t+1}^{k} \mathcal{C}_j) = \sum_{j=t+1}^{k} r(\mathcal{C}_j).$$

Show that the function $g(\cdot)$ reaches its minimum, among subspaces of $\mathcal{C}_s$, at $\{\mathbf{0}\}$.

Our next result brings out the analogy with principal partition. We use the notation of Theorem 15.5.5.

Theorem 15.5.6 *Let $h(\cdot)$ be a submodular function over subspaces of a vector space $\mathcal{V}_E$ on E. Let $\mathbf{A}$ be a representative matrix of $\mathcal{V}_E$. Let $\{E_1, \cdots, E_k\}$ be a partition of the column set E of $\mathbf{A}$. Let $\mathcal{C}_j, j = 1, \cdots, k$ be the span of the column set $E_j, j = 1, \cdots, k$. Let*

$$f_\lambda(\mathcal{V}) \equiv \sum_j r(\mathcal{V} \ . \ E_j) - \lambda r(\mathcal{V}), \mathcal{V} \subseteq \mathcal{V}_E$$

and

$$g_\lambda(\mathcal{C}) \equiv \lambda r(\mathcal{C}) - \sum_j r(\mathcal{C} \cap \mathcal{C}_j), \mathcal{C} \subseteq \mathcal{C}(\mathbf{A}).$$

Then,

i. if $\mathcal{V}_1, \mathcal{V}_2$ minimize $h(\cdot)$ over subspaces of $\mathcal{V}_S$, then so do $\mathcal{V}_1 + \mathcal{V}_2$ and $\mathcal{V}_1 \cap \mathcal{V}_2$,

ii. there is a unique minimal and a unique maximal subspace that minimizes $h(\cdot)$,

iii. if $\lambda_1 > \lambda_2$ and $\mathcal{V}_1, \mathcal{V}_2$ minimize $f_{\lambda_1}(\cdot), f_{\lambda_2}(\cdot)$ respectively then $\mathcal{V}_1 \supseteq \mathcal{V}_2$,

iv. if $\lambda_1 > \lambda_2$ and $\mathcal{C}_1, \mathcal{C}_2$ minimize $g_{\lambda_1}(\cdot), g_{\lambda_2}(\cdot)$ respectively then $\mathcal{C}_1 \subseteq \mathcal{C}_2$.

Proof :

i. We have $h(\mathcal{V}_1) + h(\mathcal{V}_2) \geq h(\mathcal{V}_1 + \mathcal{V}_2) + h(\mathcal{V}_1 \cap \mathcal{V}_2)$. So if $\mathcal{V}_1, \mathcal{V}_2$ minimize $h(\cdot)$, the above inequality must reduce to an equality. Hence $\mathcal{V}_1 + \mathcal{V}_2$ and $\mathcal{V}_1 \cap \mathcal{V}_2$ minimize $h(\cdot)$.

ii. This is an immediate consequence of the previous part of this theorem.

iii. By Theorem 15.5.5, we know that $f_{\lambda}(\cdot)$ is submodular. We have

$$\begin{aligned}
f_{\lambda_1}(\mathcal{V}_1) + f_{\lambda_2}(\mathcal{V}_2) &= f_{\lambda_1}(\mathcal{V}_1) + f_{\lambda_1}(\mathcal{V}_2) + (\lambda_1 - \lambda_2) r(\mathcal{V}_2) \\
&\geq f_{\lambda_1}(\mathcal{V}_1 + \mathcal{V}_2) + f_{\lambda_1}(\mathcal{V}_1 \cap \mathcal{V}_2) + (\lambda_1 - \lambda_2) r(\mathcal{V}_2) \\
&\geq f_{\lambda_1}(\mathcal{V}_1 + \mathcal{V}_2) + f_{\lambda_2}(\mathcal{V}_1 \cap \mathcal{V}_2) \\
&\quad + (\lambda_1 - \lambda_2)(r(\mathcal{V}_2) - r(\mathcal{V}_1 \cap \mathcal{V}_2)).
\end{aligned}$$

Since $\lambda_1 > \lambda_2$, unless $r(\mathcal{V}_2) = r(\mathcal{V}_1 \cap \mathcal{V}_2)$, we must have LHS $> f_{\lambda_1}(\mathcal{V}_1 + \mathcal{V}_2) + f_{\lambda_2}(\mathcal{V}_1 \cap \mathcal{V}_2)$, a contradiction, since $\mathcal{V}_1, \mathcal{V}_2$ minimize $f_{\lambda_1}(\cdot), f_{\lambda_2}(\cdot)$ respectively. We conclude that $r(\mathcal{V}_2) = r(\mathcal{V}_1 \cap \mathcal{V}_2)$ and hence $\mathcal{V}_2 \subseteq \mathcal{V}_1$.

iv. The proof is similar to the above and is omitted.

□

15.6 Solutions of Exercises

E 15.1: Let us assume that $\mathcal{G}$ does not have self loops or coloops since these any way do not figure in a minimal representation and do not affect the minimum value of $r(\mathcal{G} \cdot X) + \nu(\mathcal{G} \times (E - X))$ (by including all the self loops in X and coloops in $(E - X)$).

Let (A_t, B_t) be the representation of a forest t of $\mathcal{G}$. Let T be the set of coloops of $\mathcal{G} \cdot (E - B_t)$. There can be only one forest (namely, t) of $\mathcal{G} \cdot (E - B_t)$ that contains A_t (by the definition of a representation). Hence, $t = A_t \cup T$.
Let $L \equiv E - B_t - (A_t \cup T)$. In the graph $\mathcal{G}$, edges of L must be spanned by A_t (i.e., there must be paths between their endpoints containing only edges of A_t) since T is the set of coloops of $\mathcal{G} \cdot (E - B_t)$.
Let $A \equiv A_t \cup L$. Now A_t is a forest of $\mathcal{G} \cdot A$. Hence, T is a forest and B_t, a coforest of $\mathcal{G} \times (E - A)$. Hence,

$$| A_t \cup B_t | = r(\mathcal{G} \cdot A) + \nu(\mathcal{G} \times (E - A)).$$

On the other hand, given any set $A \subseteq E$, $(A_t,\ B_t)$ is a representation of a forest if A_t is a forest of $\mathcal{G} \cdot A$ and B_t, a coforest of $\mathcal{G} \times (E - A)$. For, there is only one forest T of $\mathcal{G} \times (E - A)$ that does not intersect B_t and $A_t \cup T$ is a forest of $\mathcal{G}$. Thus, the problem of finding a minimum representation is equivalent to finding a partition $\{A, E - A\}$ of $\mathcal{G}$ s.t. A minimizes $r(\mathcal{G} \cdot X) + \nu(\mathcal{G} \times (E - X))$.

E 15.2: We have, $A \subseteq E$ minimizes $r(\mathcal{G} \cdot X) + \nu(\mathcal{G} \times (E - X))$ iff it minimizes $r(\mathcal{G} \cdot X) + \nu(\mathcal{G} \times (E - X)) + r(\mathcal{G})$, i.e., iff it minimizes

$$r(\mathcal{G} \cdot X) + \nu(\mathcal{G} \times (E - X)) + r(\mathcal{G} \cdot X) + r(\mathcal{G} \times (E - X)),$$

i.e., iff it minimizes $2r(\mathcal{G} \cdot X) + | E - X |$.

E 15.3: Let t_1, t_2 be two forests of $\mathcal{G}$ and let $A \subseteq E$. Then $t_2 - t_1$ is a subforest as well as a subcoforest of $\mathcal{G}$. Hence, $(t_2 - t_1) \cap A$ is a subforest of $\mathcal{G} \ . \ A$ and $(t_2 - t_1) \cap (E - A)$ is a subcoforest of $\mathcal{G} \times (E - A)$. Hence,

$$| t_2 - t_1 | \leq r(\mathcal{G} \ . \ A) + \nu(\mathcal{G} \times (E - A)).$$

Thus, it is clear that, if t_1, t_2 are two forests of $\mathcal{G}$ and $A \subseteq E$ s.t. the above inequality becomes an equality, then t_1, t_2 must be maximally distant and A must be minimizing the expression

$$r(\mathcal{G} \ . \ X) + \nu(\mathcal{G} \times (E - X)).$$

Next let t_1, t_2 be maximally distant and let A be as in Lemma 15.2.1 (i(a) & i(b)).
We have, $t_2 - t_1 = ((t_2 - t_1) \cap A) \uplus ((t_2 - t_1) \cap (E - A))$.
Now, $t_2 \cap A = ((t_2 - t_1) \cap A)$ is a forest of $\mathcal{G} \ . \ A$. Hence, $t_2 \cap (E - A)$ is a forest of $\mathcal{G} \times (E - A)$. Similarly $t_1 \cap (E - A)$ is also a forest of $\mathcal{G} \times (E - A)$. Further, $t_1 \cup t_2 \supseteq (E - A)$. Hence, $(t_2 - t_1) \cap (E - A)$ is a coforest of $\mathcal{G} \times (E - A)$.
We thus see that

$$| t_2 - t_1 | = r(\mathcal{G} \ . \ A) + \nu(\mathcal{G} \times (E - A)).$$

E 15.4: (Original solution due to Lehman, Edmonds [Edmonds65b]. Version in terms of principal partition due to Bruno and Weinberg [Bruno+Weinberg71]).
Notation: By $X_{min}(X_{max})$ of $\mathcal{G}'$ we mean the unique minimal (maximal) set that satisfies $2r(X) + | E(\mathcal{G}') - X | = (2r * | \cdot |)(E(\mathcal{G}'))$, where $r(\cdot)$ is the rank function of $\mathcal{G}'$, i.e., $r(T) = r(\mathcal{G}'.T)$. By $Y_{min}(Y_{max})$ of $\mathcal{G}'$ we mean the unique minimal (maximal) set that satisfies

$$2\nu(X) + | E(\mathcal{G}') - X | = (2\nu * | \cdot |)(E(\mathcal{G}')),$$

where $\nu(\cdot)$ is the nullity function of $\mathcal{G}'$, i.e., $\nu(T) = \nu(\mathcal{G}' \times T)$. All the fundamental circuits are according to the graph $\mathcal{G}$. Lemma 15.2.1 is used repeatedly.

Case 1: Let $e_M \in X_{min}$ of $\mathcal{G}$.
Then there exist two maximally distant forests t_1, t_2 of $\mathcal{G}$ such that $e_M \notin t_1 \cup t_2$ and $t_1 \cap X_{min}, t_2 \cap X_{min}$ are disjoint forests of $\mathcal{G} \ . \ X_{min}$.
If the **short player plays first**, he contracts a branch $e' \in L(e_M, t_1)$. Let $e \in L(e', t_2)$. Edges e', e belong to X_{min} of $\mathcal{G}$ and $t_1 - e', t_2 - e$ are maximally distant forests of $\mathcal{G} \times (E - e')$ $((t_1 - e') \cup (t_2 - e) \supseteq E - X_{min}$ and $t_1 - e', t_2 - e$ have the maximum possible intersection, among all forests of $\mathcal{G} \times (E - e')$, with $X_{min} - e')$.
Then $e_M \in X_{min}$ of $\mathcal{G} \times (E - e')$ since $e_M \notin (t_1 \cup t_2 - e - e')$.

Let the **cut player play first.** We consider all the alternative situations. In each case we show that either after the cut player's 1st move, or after the short player has responded, e_M belongs to the X_{min} of a reduced graph. Continuing this procedure we would finally reach a graph in which there are three parallel edges, one of which is e_M. For this graph it is clear that the short player would always win whether he plays first or second. Let e be the edge that is being deleted.

i. If $e \notin t_1 \cup t_2$, then t_1, t_2 continue to be maximally distant forests of $\mathcal{G} \ . \ (E - e)$. Hence, e_M belongs to $E - (t_1 \cup t_2)$ and therefore to X_{min} of $\mathcal{G} \ . \ (E - e)$.

ii. Let e not belong to X_{min} of $\mathcal{G}$. Then $e \in t_1 \cup t_2 - X_{min}$. Let, without loss of generality, $e \in t_2 - X_{min}$. Extend $t_2 - e$ to a new forest t_2' by adding a suitable edge of t_1. This latter edge cannot belong to X_{min} since edges of $t_1 \cap X_{min}$ are spanned by edges of $t_2 \cap X_{min}$. Now t_1, t_2' would be maximally distant forests in the graph $\mathcal{G} \,.\, (E-e)$ ($t_1 \cup t_2' \supseteq E - e - X_{min}$ and t_1, t_2' have the maximum possible intersection, among all forest of $\mathcal{G} \,.\, (E-e)$, with X_{min}) and $e_M \in E - e - (t_1 \cup t_2)$. Thus, e_M belongs to X_{min} of $\mathcal{G} \,.\, (E-e)$.

iii. Let $e \in (t_1 \cup t_2) \cap X_{min}$. Now we pick maximally distant forests t_1, t_2 of $\mathcal{G}$ s.t. $e \notin t_1' \cup t_2'$. By Lemma 15.2.1 this is possible since $e \in X_{min}$ of $\mathcal{G}$. If $e_M \notin t_1' \cup t_2'$, it is clear that $e_M \in X_{min}$ of $\mathcal{G} \,.\, (E-e)$. So let us assume that $e_M \in t_1' \cup t_2'$. Suppose $e_M \in t_2'$. Then, $(t_2' - e_M)$ can be extended to a forest of $\mathcal{G} \,.\, (E-e)$ using an edge $e' \in t_1'$ s.t. $e' \in L(e_M, t_1')$. Now t_1', t_2' intersect X_{min} in forests of $\mathcal{G} \,.\, X_{min}$. Hence, $e' \in X_{min}$. If now we short e' (i.e., if this is the short player's move) $t_2' - e_M$ and $t_1' - e'$ become maximally distant forests of $\mathcal{G}' \equiv \mathcal{G} \,.\, (E-e) \times (E - e - e')$ ($(t_2' - e_M) \cup (t_1' - e') \supseteq E - e - e' - X_{min}$ and further $(t_2' - e_M), (t_1' - e)$ have maximum possible intersection with $X_{min} - e - e'$ in $\mathcal{G}'$). Thus, in $\mathcal{G}'$, e_M lies outside the maximally distant pair of forests $t_1 - e, t_2' - e_M$. Hence, e_M belongs to X_{min} of $\mathcal{G}'$. This completes Case 1.

Case 2: Let $e_M \in Y_{min}$.
We use arguments dual to those used in the previous case to show that the cut player can always win playing first or second. In particular this means that in the argument we replace
rank of $\mathcal{G} \,.\, T$ by nullity of $\mathcal{G} \times T$, forests by coforests, X_{min} by Y_{min}, deletion (contraction) by contraction (deletion), fundamental circuit with respect to a forest by fundamental cutset with respect to a coforest.
The final graph that we reach in this case will have three series edges with e_M one of them.

Case 3: Let $e_M \in X_{max} - X_{min} = Y_{max} - Y_{min}$.
In this case the one who plays first would win. Observe that if t_1, t_2 are maximally distant forests of $\mathcal{G}$ then $\mathcal{G} \,.\, X_{max}$ has $t_1 \cap X_{max}, t_2 \cap X_{max}$ as disjoint forests. Also $(E - (t_1 \cup t_2)) \subseteq X_{min}$. Hence, in this case $e_M \in t_1 \cup t_2$. Let $e_M \in t_1$ and let the short player play first. Let $e \in t_2 \cap X_{max}$ s.t. $e_M \in L(e, t_1)$. Let e be contacted by the short player. It is then clear that $t_2 - e, t_1 - e_M$ are maximally distant forests of $\mathcal{G} \times (E-e)$. Now we can use the arguments of Case 1 to show that the short player must win treating the cut player's next move as the 'first move'.
The situation where the cut player plays first can be handled by arguments dual to the above.

E 15.5: We first show (in the following Lemma) that the term rank of a given cobase matrix cannot be less than the rank of any cobase matrix.

Lemma 15.6.1 *Let* $\mathbf{Q}, \mathbf{Q}'$ *be row equivalent matrices of the form given below:*

$$\begin{array}{cc} & \begin{array}{cccc} T_1 & T_2 & T_3 & T_4 \end{array} \\ \mathbf{Q} = & \left[\begin{array}{cccc} \mathbf{I}_1 & \mathbf{0} & \mathbf{Q}_{13} & \mathbf{Q}_{14} \\ \mathbf{0} & \mathbf{I}_2 & \mathbf{Q}_{23} & \mathbf{Q}_{24} \end{array} \right] \end{array}$$

$$\mathbf{Q}' = \begin{array}{c} \begin{array}{cccc} T_1 & T_2 & T_3 & T_4 \end{array} \\ \left[\begin{array}{cccc} \mathbf{I}_1 & \mathbf{Q}'_{12} & \mathbf{0} & \mathbf{Q}'_{14} \\ \mathbf{0} & \mathbf{Q}'_{22} & \mathbf{I}_2 & \mathbf{Q}'_{24} \end{array} \right] \end{array}.$$

Then the term rank of the matrix $\begin{bmatrix} \mathbf{Q}_{13} & \mathbf{Q}_{14} \\ \mathbf{Q}_{23} & \mathbf{Q}_{24} \end{bmatrix}$ *is not less than the rank of the matrix* $\begin{bmatrix} \mathbf{Q}'_{12} & \mathbf{Q}'_{14} \\ \mathbf{Q}'_{22} & \mathbf{Q}'_{24} \end{bmatrix}$.

Proof : Observe that the second set of rows of $\mathbf{Q}'$ is obtained by linear combination of the second set of rows of $\mathbf{Q}$. Hence, $\mathbf{Q}_{23}$ is a nonsingular matrix. From this fact it can be inferred that the term rank of

$$\begin{bmatrix} \mathbf{0} & \mathbf{Q}_{14} \\ \mathbf{I}_2 & \mathbf{Q}_{24} \end{bmatrix}$$

must be less than or equal to the term rank of

$$\begin{array}{c} \begin{array}{cc} T_3 & T_4 \end{array} \\ \left[\begin{array}{cc} \mathbf{Q}_{13} & \mathbf{Q}_{14} \\ \mathbf{Q}_{23} & \mathbf{Q}_{24} \end{array} \right] \end{array}.$$

But the rank of $\begin{bmatrix} \mathbf{Q}'_{12} & \mathbf{Q}'_{14} \\ \mathbf{Q}'_{22} & \mathbf{Q}'_{24} \end{bmatrix}$ is the same as the rank of $\begin{bmatrix} \mathbf{0} & \mathbf{Q}_{14} \\ \mathbf{I}_2 & \mathbf{Q}_{24} \end{bmatrix}$, which must be less than or equal to its term rank. The result follows.

□

Next we display a cobase matrix where term rank = rank. This cobase matrix is constructed according to the algorithm given in the brief solution earlier. So the following lemma justifies the algorithm and has the consequence that the maximum rank = minimum term rank.

Lemma 15.6.2 *Let* $\mathbf{Q}$ *be the matrix shown below with set of columns* S

$$\begin{array}{c} \begin{array}{ccc} b_1 & b_2 - b_1 & c \end{array} \\ \left[\begin{array}{ccc} \mathbf{Q}_{11} & \mathbf{Q}_{12} & \mathbf{Q}_{13} \end{array} \right] \end{array},$$

where column sets b_1, b_2 *are maximally distant bases of the column set of* $\mathbf{Q}$ *and* $c \equiv S - (b_1 \cup b_2)$. *Let*

$$\mathbf{Q}' = \begin{array}{c} \begin{array}{ccc} b_1 & b_2 - b_1 & c \end{array} \\ \left[\begin{array}{ccc} \mathbf{I} & \mathbf{P} & \mathbf{R} \end{array} \right] \end{array}$$

be row equivalent to $\mathbf{Q}$. *Then the matrix* $\begin{bmatrix} \mathbf{P} & \mathbf{R} \end{bmatrix}$ *has term rank = rank.*

Proof : By Lemma 15.2.1 there exists a set A of columns s.t. $A \supseteq c$ and $A \cap b_1$, $A \cap b_2$ are disjoint and span all of A. Hence, perhaps after rearranging rows, $\mathbf{Q}'$ would have the form shown below (where the columns correspond from left to right respectively to $b_1 \cap A, b_1 - A, b_2 \cap A, b_2 - b_1 - A, S - (b_1 \cup b_2)$)

$$\begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{Q}'_{13} & \mathbf{Q}'_{14} & \mathbf{Q}'_{15} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{Q}'_{24} & \mathbf{0} \end{bmatrix},$$

with $\mathbf{Q}'_{13}$ being a nonsingular matrix. Clearly the term rank of the matrix $\left[\begin{array}{ccc}\mathbf{Q}'_{13} & \mathbf{Q}'_{14} & \mathbf{Q}'_{15}\end{array}\right]$ equals $\mid b_2 \cap A \mid$ while that of $\mathbf{Q}'_{24}$ does not exceed $\mid b_2 - b_1 - A \mid$.

Thus the matrix

$$\mathbf{Q} = \left[\begin{array}{ccc}\mathbf{Q}'_{13} & \mathbf{Q}'_{14} & \mathbf{Q}'_{15} \\ \mathbf{0} & \mathbf{Q}'_{24} & \mathbf{0}\end{array}\right],$$

whose columns correspond from left to right to $b_2 \cap A, b_2 - b_1 - A, c$ respectively, has term rank not exceeding $\mid b_2 - b_1 \mid$. But this matrix has the columns corresponding to $b_2 - b_1$ linearly independent and therefore has rank equal to $\mid b_2 - b_1 \mid$. Since rank of any matrix cannot exceed its term rank, this proves the result.

□

E 15.6:

i. Immediate from the definition.

ii. We have

$$\begin{aligned} f(X) + f^*(S - X) &= f(X) + g(S - X) - f(S) + f(X) \\ &= 2f(X) - g(X) + g(S) - f(S). \end{aligned}$$

The result follows.

iii. We show the result for contraction. Since restriction of $f(\cdot)$ corresponds to contraction of $f^*(\cdot)$, by (i) above the result would be true also for restriction. Since every minor is a restriction followed by a contraction the result would follow for minors.
Let $f_1(\cdot) = f \diamond \mathbf{T}(\cdot)$. We then have, for $X \subseteq T$,

$$\begin{aligned} f_1(X) + f_1^*(T - X) &= f(X \cup (S - T)) - f(S - T) + f^*(T - X) \\ &\leq f(X) + f(S - T) - f(S - T) + f^*(T - X) \\ &\leq f(X) + f^*(S - X) \end{aligned}$$

(where we have used the facts that $f_1^*(T - X) = f^*(T - X)$ and that $f^*(\cdot)$ is an increasing function).
So min of LHS $\leq$ min of RHS.

E 15.7:
We prove only the statement about the dual. We have

$$\begin{aligned} f^*(T) &= g(T) - f(S) + f(S - T) \\ &= \sum_{e \in T} f(e) - r(\hat{S}) + r(\hat{S} - \hat{T}) \\ &= \mid \hat{T} \mid - r(\hat{S}) + r(\hat{S} - \hat{T}) \\ &= r^*(\hat{T}), \end{aligned}$$

where $r^*(\cdot)$ is the rank function of $\mathcal{M}^*$.

E 15.8:

i. Let $r'(\cdot)$ be the rank function of $\mathcal{M} \cdot (\hat{S} - a)$. If $e \notin X$, it is clear that $f(X) =$

$r'(\bigcup_{e_i \in X} \hat{e}_i)$. If $e \in X, r'((\bigcup_{e_i \in X-e} \hat{e}_i) \cup (\hat{e} - a)) = r(\bigcup_{e_i \in X} e_i) = f(X)$ since a is dependent on $\hat{e} - a$ in the matroid $\mathcal{M}$. This proves the result.

ii. Repeating the operation in the previous part, we can destroy all circuits of $\mathcal{M}$ contained in each $\hat{e}$ retaining the property of being an expansion of $f(\cdot)$ for the resulting matroid. So there is no loss of generality in assuming that $\mathcal{M} \cdot \hat{e}$ contains no circuit for each $e \in S$. But then $\mid \hat{e} \mid = r(\mathcal{M} \cdot \hat{e}) = f(e)$. Hence, $\mid \hat{S} \mid = \sum \mid \hat{e} \mid = \sum_{e \in S} f(e)$.

E 15.9:

i. Suppose Y maximizes $g(X) - f(X), X \subseteq S$ and $e \notin Y$. Then

$$g(Y \cup e) - f(Y \cup e) = g(Y) - f(Y) + [g(e) - (f(Y \cup e) - f(Y))]$$

Since $f(\cdot)$ is a polymatroid rank function we have

$$f(Y \cup e) - f(Y) \leq f(e) < g(e).$$

Thus, $g(Y \cup e) - f(Y \cup e) > g(Y) - f(Y)$, a contradiction.

ii. We need only verify that $(f - h)(\cdot)$ is an integral polymatroid rank function (the other part being valid for any weight function $h(\cdot)$). We have $(f - h)(e) = f(e) - h(e) = f(e) - (f(S) - f(S - e))$. So $(f - h)(e) \geq 0$, since $f(\cdot)$ is a polymatroid rank function. Now $h(\cdot)$ is a weight function and, therefore, $(f - h)(\cdot)$ is submodular. Further $f(\emptyset) = (f - h)(\emptyset) = 0$. We conclude that $(f - h)(\cdot)$ is a polymatroid rank function. The integrality is obvious.

iii. Let $r'(\cdot)$ denote the rank function of $\mathcal{M}_{red}$. For each $e \in S$, let e' denote $(\hat{e} - b_e \cap \hat{e})$ and let $X' \equiv \{e', e \in X\}, X \subseteq S$. We need to show that

$$(f - h)(X) = r'(\bigcup_{e' \in X'} e') \quad \forall\, X \subseteq S. \qquad (*)$$

We need the following preliminary lemma:

Lemma 15.6.3 *Let b_{S-X} be a base of $\mathcal{M} \cdot (\bigcup_{e \in S-X} \hat{e})$. Then $(\bigcup_{e \in X} (b_e \cap \hat{e})) \cup b_{S-X}$ is independent.*

Proof : Suppose the set is dependent. Then there is a minimal subset Y of X s.t. $(\bigcup_{e \in Y} (b_e \cap \hat{e})) \cup b_{S-X}$ is dependent. Let $e_1 \in Y$. Let $K \equiv (\bigcup_{e \in Y - e_1} (b_e \cap \hat{e})) \cup b_{S-X}$. Grow K into a base b_2 of $\mathcal{M} \cdot (\hat{S} - \hat{e}_1)$. Now $b_{e_1} \cap \hat{e}_1$ is a minimal intersection of a base of $\mathcal{M}$ with $\hat{e}_1$ and, therefore, is a base of $\mathcal{M} \times \hat{e}_1$. Hence, $b_2 \cup (b_{e_1} \cap \hat{e}_1)$ is a base of $\mathcal{M}$. But this contradicts the fact that a subset of this set, namely $K \cup (b_{e_1} \cap \hat{e}_1)$ is dependent.

□

Now LHS of $(*)$ equals

$$f(X) - \sum_{e \in X} (f(S) - f(S - e)) = f(X) - \sum_{e \in X} \mid b_e \cap \hat{e} \mid,$$

while RHS equals (by the definition of contraction),

$$
\begin{aligned}
r((\bigcup_{e'\in X'} e') \cup (\bigcup_{e\in S} \hat{e}\cap b_e)) \quad - \quad & r(\bigcup_{e\in S} \hat{e}\cap b_e) \\
&= r((\bigcup_{e\in X} \hat{e}) \cup (\bigcup_{e\in (S-X)} \hat{e}\cap b_e)) - r(\bigcup_{e\in S} \hat{e}\cap b_e) \\
&= r(b_X \cup (\bigcup_{e\in (S-X)} \hat{e}\cap b_e)) - r(\bigcup_{e\in S} \hat{e}\cap b_e), \\
&\textit{where } b_X \textit{ is a base of } \mathcal{M}\cdot(\bigcup_{e\in X} \hat{e}) \\
&= r(b_X) + r(\bigcup_{e\in (S-X)} \hat{e}\cap b_e) - r(\bigcup_{e\in S} \hat{e}\cap b_e) \\
&(\textit{by Lemma } 15.6.3) \\
&= f(X) - r(\bigcup_{e\in X} \hat{e}\cap b_e), \textit{ using the same lemma} \\
&= f(X) - \sum_{e\in X} \mid b_e\cap \hat{e} \mid .
\end{aligned}
$$

E 15.10:

i. Routine.

ii. By the polymatroid intersection theorem (Theorem 11.2.3)

$$\min_{X\subseteq S} f(X) + f^*(S-X) = \max_{x\in P_f\cap P_{f*}} \mathbf{x}(S),$$

(the equality being satisfied with integral $\mathbf{x}$ if $f(\cdot)$ is integral). Now $\mathbf{x}$ is an integral independent vector of P_f (P_{f*}), iff there exists an independent set T_1 of $\mathcal{M}$ (independent set T_2 of $\mathcal{M}^*$) s.t.

$$\mid T_1\cap \hat{e} \mid = \mid T_2\cap \hat{e} \mid = x(e)\ \forall e\in S$$

(by the last part of Theorem 15.3.1). This proves the min max equality. Now if T is a common independent set of $\mathcal{M}, \mathcal{M}^*$ the vector $\mathbf{y}$ defined by $y(e) \equiv \mid T\cap \hat{e} \mid$ is a vector in $P_f\cap P_{f*}$. Hence, $\mathbf{y}(S) \leq \max_{x\in P_f\cap P_{f*}} \mathbf{x}(S)$, as required.

Let $\mathcal{M}_1$ be the expansion of $f(\cdot)$ described in page 444. This matroid is obtained as follows: Replace each e by $f(e)$ $(= g(e))$ parallel copies making up the set $\hat{e}$. Let $\hat{f}(\cdot)$ denote the new polymatroid rank function. The rank function of $\mathcal{M}_1$ is given by $r_1(\cdot) \equiv (\hat{f}* \mid \cdot \mid)(\cdot)$. Now the minimal set that minimizes $f(X)+f^*(S-X)$, $X\subseteq S$ can be seen to be the minimal set that maximizes $g(X) - 2f(X)$ $X\subseteq S$. Let $H(Y) \equiv$ set of all elements in $\hat{S}$ parallel to elements in Y, $Y\subseteq S$. Now by symmetry arguments one can show that (Lemma 12.4.1) Y maximizes $g(X) - 2f(X)$, $X\subseteq S$, iff $H(Y)$ maximizes $\mid Z \mid -2r_1(Z)$, $Z\subseteq \hat{S}$ and further if Y is the minimal set which maximizes $g(X) - 2f(X)$, $H(Y)$ is the minimal set maximizing the corresponding expression. By Lemma 12.4.1, it would also follow that

$$\max_{X\subseteq S} g(X) - 2f(X) = \max_{Z\subseteq \hat{S}} \mid Z \mid -2r_1(Z).$$

Let $b_\vee$ be a base of $\mathcal{M}_1 \vee \mathcal{M}_1$. If $b_\vee = b \vee b'$ where b, b' are bases of $\mathcal{M}_1$ we must have $b_\vee - b$ as a common independent set of $\mathcal{M}_1$ and $\mathcal{M}_1^*$. Now if $\hat{Y} \equiv H(Y)$ is the minimal set that maximizes $\mid Z \mid -2r_1(Z),\ Z \subseteq \hat{S}$ then $(b_\vee - b) \cap \hat{Y}$ would be a base of $\mathcal{M}_1 \cdot \hat{Y}$ (Lemma 12.3.3). If we use a 'matroid translation' of Lemma 15.2.1, we can show that $(b_\vee - b) \cap (\hat{S} - \hat{Y})$ would be a base of $\mathcal{M}_1^* \cdot (\hat{S} - \hat{Y})$. Thus

$$\mid b_\vee - b \mid = r_1(\hat{Y}) + {r_1}^*(\hat{S} - \hat{Y}) = f(Y) + f^*(S - Y).$$

This proves the equality we required.

E 15.11: Order the elements of S as $(e_1, \cdots, e_n)$. Start from any base b_0 of $\mathcal{M}$. Let $g_{0i} \equiv |b_0 \cap e_i|, i = 1, \cdots, n$. Let $e_1, \cdots, e_k$ be the 'deficient' elements for which $g_{0i} < g(e_i)$. For each element in $\hat{e}_1 - b_0, \cdots, \hat{e}_k - b_0$, construct f-circuits relative to b_0. Suppose the f-circuits contain elements of $\hat{e}_{11}, \cdots, \hat{e}_{1k_1}$. For each of the elements in $\hat{e}_{11} - b_0, \cdots, \hat{e}_{1k_1} - b_0$ construct f-circuits relative to b_0. Repeat this procedure until you reach elements of a set $\hat{e}_q$ for which $g_{0q} > g(e_q)$ ('saturated' element).
We now have a 'path' (listing only vertices) say $a_1, a_2', a_3, \cdots, a_i', a_{i+1}, \cdots, a_{t-1}, a_t'$, where the unprimed elements are outside b_0, a_i', a_{i+1} belong to the same $\hat{e}_r$ and further $a_{j+1}' \in L(a_j, b_0)$ and $a_p' \notin L(a_j, b_0)$ whenever $p > j + 1$. Now we push unprimed elements of the path into b_0 and drop the primed ones. If $a_1 \in \hat{e}_i$ and $a_t' \in \hat{e}_q$ say, then the resulting base has one more element of $\hat{e}_i$ and one less element of $\hat{e}_q$ than b_0 has. Of all other sets $\hat{e}_k$ the updated base has the same number of elements as before. Repeating this procedure we either find a base which has for each e, number of elements in $\hat{e}$ not less than $g(e)$ or a base b_f from whose deficient elements it is not possible to reach saturated elements. The set of all elements which can be reached from deficient elements gives the minimal set that maximizes $g(X) - f(X), X \subseteq S$.

E 15.12:
i. This follows from the fact that $\mathcal{M}(\mathcal{G} \times T) = \mathcal{M}(\mathcal{G}) \times T$ and Exercise 15.9 (third part).

ii. Since $\mid \hat{e} \mid = r(e)$, the graph $\mathcal{G}\ .\ \hat{e}$, if connected, must be a tree graph. Now in $\mathcal{G}_{red}$, it must be true that $r'(\hat{S}_{red}) = r'(\hat{S}_{red} - (\hat{e} - b_e \cap \hat{e}))$, where $r'(\cdot)$ is the rank function of $\mathcal{G}_{red}, \hat{S}_{red} \equiv \hat{S} - \bigcup_{e\in S}(\hat{e} \cap b_e)$ (by the exercise referred to above). This means that $\hat{e} - (b_e \cap \hat{e})$ cannot contain a cutset of $\mathcal{G}_{red}$. In particular, there can be no node in $\mathcal{G}_{red}$ to which only edges of $\hat{e}$ are incident.

iii. Let $e' \equiv \hat{e} - b_e \cap \hat{e}$ and $e" \equiv$ the tree on the same set of nodes as e'.
Let $\mathcal{G}_{big}$ contain both e' and $e"$ for each e while $\mathcal{G}"_{red}$ contains only $e"$ for each e. It is clear that $\mathcal{G}_{big} \cdot (\bigcup_{e\in S} e') = \mathcal{G}_{red}$ and $\mathcal{G}_{big} \cdot (\bigcup_{e\in S} e") = \mathcal{G}"_{red}$. Let $r_b(\cdot)$ be the rank function of $\mathcal{G}_{big}$ and let $r"(\cdot)$ be the rank function of $\mathcal{G}"_{red}$. Then it is clear that

$$r'(\bigcup_{e\in X} e') = r_b(\bigcup_{e\in X} (e' \cup e")) = r"(\bigcup_{e\in X} e")\ \ \forall X \subseteq S.$$

This proves the required result.

E 15.13: We assume $f(X) = (\mid V \mid -1)_t(X)$ in the procedure. In particular, $f(e) = (\mid V \mid -1)(e)$. This latter is valid only if the subgraph of $\mathcal{G}$ on $\hat{e}$ is connected.

E 15.14: This procedure identical to the one described for finding the minimal set that minimizes $(\mid V \mid -1)_t(X) + g(S - X)\ \forall X \subseteq S$. We note that $\mid V \mid (\cdot)$ is submodular and replacing 1 by σ does not change the problem.

E 15.15:
i. The circuits of $\mathcal{M}$ are minimal sets s.t. $(f - k)_t(X) <\mid X \mid$. Let Y be a circuit of $\mathcal{M}$ and let $\Pi(Y)$ minimize $\overline{(f - k)}(\cdot)$ over subsets of Y. If N is any block of $\Pi(Y)$ clearly $(f - k)_t(N) = (f - k)(N)$. One of the blocks of $\Pi(Y)$, say N_1, must satisfy $(f - k)(N_1) <\mid N_1 \mid$. Thus, $(f - k)_t(N_1) <\mid N_1 \mid$. By the minimality of Y we must have $N_1 = Y$. So $(f - k)(Y) <\mid Y \mid$. So Y contains a minimal set Z s.t. $(f - k)(Z) <\mid Z \mid$. On the other hand, if $(f - k)(Z) <\mid Z \mid$, we must have $(f - k)_t(Z) \leq (f - k)(Z) <\mid Z \mid$. Hence, Z contains a circuit of $\mathcal{M}$.

The remaining parts of the exercise are a direct application of the above result. We remind the reader that average right degree of the left regular bipartite graph is $\frac{d|V_L|}{|V_R|}$, where d is the degree of a left vertex.

E 15.16: Consider a node a with e_1, e_2 incident on it and a node d with e_3, e_4 incident on it. Let e_1, e_2, e_3, e_4 be distinct elements. Now merge (a, d) and split it into p, q with e_1, e_3 incident on p and e_2, e_4 incident on q. Clearly this effect cannot be achieved by a fission followed by a fusion, for, a fission in the beginning can touch only one of the two nodes a, d whereas both the nodes have to be split in order to achieve the effect.

E 15.17:
i. Let B be the set of coloops of $\mathcal{G}$. Let $\mathcal{G}_1 = \mathcal{G} \cdot (E - B)$. Select any forest of $\mathcal{G}_1$ and fuse the endpoints of each edge in this forest. Every edge in $\mathcal{G}_1$ would then become a selfloop. So fusion rank $\leq r(\mathcal{G}_1) \equiv r(\mathcal{G}) - \mid B \mid$.
Next suppose a set of fusions reduces the graph to a set of self loops. Add edges across pairs of nodes in the original graph corresponding to the fusions. Call the new graph $\mathcal{G}_2$. (Note that $r(\mathcal{G}_2) = r(\mathcal{G})$). Let this set be T. Now contraction of T can reduce rank of $\mathcal{G}_2$ by at most $\mid T \mid$ and unless rank of $\mathcal{G}_2$ is reduced by at least $r(\mathcal{G}_1)$ there would be a nonsingleton cutset containing edges of $\mathcal{G}_1$ (since the contraction of an edge in T can reduce the rank of $\mathcal{G}_1$ atmost by 1). Hence, $\mid T \mid \geq r(\mathcal{G}_1)$. So fusion rank $\geq r(\mathcal{G}) - \mid B \mid$.
The result follows.

ii. Let S_l be the set of selfloops of $\mathcal{G}$. Let $\mathcal{G}_3 = \mathcal{G} \times (E - S_l)$. Select any coforest L of $\mathcal{G}_3$ and split one of the ends of each edge of L. Each edge of $\mathcal{G}_3$ would then become a coloop. So fission rank $\leq \nu(\mathcal{G}_3) = \nu(\mathcal{G}) - \mid S_l \mid$.
Next suppose a set of node fissions reduces the graph to a set of coloops. Split nodes in $\mathcal{G}$ and add edges between the split nodes corresponding to the node fissions. Call the new graph $\mathcal{G}_4$. (Note that $\nu(\mathcal{G}_4) = \nu(\mathcal{G})$). Let the additional set of edges be K. Now deletion of K can reduce the nullity of $\mathcal{G}_4$ by at most $\mid K \mid$ and unless nullity

of $\mathcal{G}_4$ is reduced by at least $\nu(\mathcal{G}_3)$ there would be a nonsingleton circuit containing edges of $\mathcal{G}_3$. Hence, $\mid K \mid \geq \nu(\mathcal{G}_3)$. So fission rank $\geq \nu(\mathcal{G}) - \mid S_l \mid$.
The result follows.

iii. Suppose $A \subseteq E(\mathcal{G})$ minimizes

$$r(\mathcal{G} \, . \, X) + \nu(\mathcal{G} \times (E - X)), \; X \subseteq E(\mathcal{G}).$$

Select a forest of $\mathcal{G} \, . \, A$ and a coforest of $\mathcal{G} \times (E - A)$. Fuse the end points of each edge in the forest of $\mathcal{G} \, . \, A$ and cut one end of each coforest edge of $\mathcal{G} \times (E - A)$. This would result in all edges of A becoming self loops and all edges of $(E - A)$ becoming coloops. Thus, the hybrid rank relative to $\Pi_s \leq$ hybrid rank of $\mathcal{G}$. Next any set of node pair fusions can be associated with a set of additional edges T added across existing nodes of $\mathcal{G}$ which when contracted would perform the same task as the fusions. Any sequence of node fissions can be associated with a set of additional edges K, added by splitting nodes (and putting the edge across), which when deleted would perform the same task as the fissions. Let the new graph, after addition of such edges, be called $\mathcal{G}_5$. Now $\mathcal{G} = \mathcal{G}_5 \cdot (E_5 - T) \times (E(\mathcal{G}))$. If the endpoints of the edges of T are fused and one end of each edge in K is split every edge of $\mathcal{G}_5$ becomes a separator. Hence, $\mid T \uplus K \mid \geq$ hybrid rank of $\mathcal{G}_5 \geq$ hybrid rank of $\mathcal{G}$ (since $\mathcal{G}$ is a minor of $\mathcal{G}_5$ and by Exercise 15.6, hybrid rank of a graph $\geq$ hybrid rank of $\mathcal{G}$).
This proves the required result.

E 15.18:
i. For each block N_i, we can partition the boundary nodes into $V_{i1}, \cdots, V_{it}$, where V_{ij} is the set of vertices common between N_i and the j^{th} component of $\mathcal{G} \, . \, (E(\mathcal{G}) - N_i)$. On each set V_{ij} of vertices we build a tree t_{ij}. We call such trees 'fusion trees'. When these trees are added to the graph $\mathcal{G}$ we get a new graph $\mathcal{G}'$. It is clear that $r(\mathcal{G}') = r(\mathcal{G})$. If all the t_{ij} were contracted we would be left with the graph $\oplus_i \mathcal{G} \times N_i$ (with hinged nodes present between the $\mathcal{G} \times N_i$). To see this consider any one of the N_i, say N_1. If each t_{1j} were contracted it is clear that we would be left with $\mathcal{G} \times N_1$ since t_{1j} was built on V_{1j}, which is the set of boundary nodes common between N_1 and one component of $\mathcal{G} \, . \, (E - N_1)$. Contracting t_{1j} has the same effect on N_1 as contracting this component of $\mathcal{G} \, . \, (E - N_1)$. Now each edge of $t_{ij}, i \neq 1$ is spanned by a path in $\mathcal{G} \, . \, (E - N_1)$. Once the t_{1j} are contracted, any further contraction of $t_{ij}, i \neq 1$ would not affect N_1. Let $\mathcal{G}_{red} \equiv \mathcal{G}' \cdot (\bigcup_{i,j} t_{ij})$. It is clear that contraction of a forest of $\mathcal{G}_{red}$ would convert $\mathcal{G}$ to $\oplus(\mathcal{G} \times N_i)$ and node pair fusions less in number than these on $\mathcal{G}'$ would leave at least one edge in $\bigcup t_{ij}$ of rank one. The end points of this edge have paths between them using only the edges of some N_i and only the edges of $E - N_i$, which means there is a circuit intersecting more than one of the N_i. We therefore conclude that the fusion rank of $\mathcal{G}$ relative to $\Pi_s = r(\mathcal{G}_{red})$. Now

$$r(\mathcal{G}_{red}) = r(\mathcal{G}') - r(\mathcal{G}' \times E(\mathcal{G})) = r(\mathcal{G}) - \sum r(\mathcal{G} \times N_i) = \sum \nu(\mathcal{G} \times N_i) - \nu(\mathcal{G}).$$

ii. Let $\mathcal{G}$ be a circuit for which one orientation is $(e_1 a e_2 b e_3 c e_4 d e_1)$. Let $N_1 \equiv \{e_1, e_3\}$ and $N_2 \equiv \{e_2, e_4\}$. Now $\mathcal{G} \times N_1, \mathcal{G} \times N_2$ have rank one each. We have

$r(\mathcal{G}) = 3$. Hence, $r(\mathcal{G}) - \sum r(\mathcal{G} \times N_i) = 1$. But, with one node pair fusion, two circuits result, at least one of which intersects both N_1 and N_2.

E 15.19: Let V_b be the set of boundary nodes. Let Π_b be any partition of V_b. We will show that $(|\Gamma| - 2)(V_b) \leq \overline{(|\Gamma| - 2)}(\Pi_b)$.
LHS $= \mid \Pi_s \mid -2$. Now Π_b has a block V_g containing v_g. Further $(|\Gamma|-2)(V_g) =\mid \Pi_s \mid -2$. If V_i is any other block of Π_b it is clear that $|\Gamma|(V_i) \geq 2$ so that $(|\Gamma|-2)(V_i) \geq 0$ (since each node in V_i is incident on edges of at least two different blocks of Π_s). Hence RHS $\geq\mid \Pi_s \mid -2$. By using the inequality of Theorem 13.2.1 it is clear that there must be a minimizing partition Π of $\overline{(|\Gamma| - 2)}(\cdot)$ that contains V_b in one of its blocks. But fusing all boundary nodes to v_g is sufficient to make every block of Π_s into a separator. Thus, the fusion - fission rank of Π_{V_b} (the partition that has V_b as a block and the others as singletons) relative to Π_s equals $(\mid V_b \mid -1)$. Further the fusion rank of $\Pi_{V_b} =\mid V_b \mid -1$ and the fusion rank of any coarser partition would be greater or equal to this number. We conclude that (i) Π_{V_b} has the least fusion - fission rank relative to Π_s and (ii) the hybrid rank of $\mathcal{G}$ relative to $\Pi_s =$ fusion - fission rank of Π_{V_b} relative to $\Pi_s =\mid V_b \mid -1$.

E 15.20:
i. The result follows from Theorem 15.4.2 since $B(\Pi_s), B(\Pi'_s)$ become identical under the right vertex mapping $N_i \rightarrow t_i$.

ii. If there are cutsets within some t_i, either there would be an internal vertex in $V(\mathcal{G} \,.\, t_i)$ or a vertex v_c at which branches of both t_i and t_j come together but the branches of t_i by themselves form a cutset. In the former case $B(\Pi'_s)$ has left vertices which are adjacent only to one right vertex. It is easily seen that such vertices should be singletons in the minimal partition that minimizes $\overline{(|\Gamma| - 2)}(\cdot)$. In the second case, if the cut vertex v_c is split, $\mathcal{G}_t$ breaks into 2 connected components. The hybrid rank problem can be solved for each 2-connected component separately. (This would yield a minimal length fusion - fission sequence which makes the t_i into separators in the original graph).

E 15.21:
Below we have used the fact

$$r(\mathcal{V}) + r(\mathcal{V}') = r(\mathcal{V} + \mathcal{V}') + r(\mathcal{V} \cap \mathcal{V}').$$

i.

$$\begin{aligned} LHS &= r(\mathcal{V} + \mathcal{V}_1) + r(\mathcal{V} + \mathcal{V}_2) - r(\mathcal{V} \cap \mathcal{V}_1) - r(\mathcal{V} \cap \mathcal{V}_2) \\ &= r(\mathcal{V} + \mathcal{V}_1 + \mathcal{V}_2) + r((\mathcal{V} + \mathcal{V}_1) \cap (\mathcal{V} + \mathcal{V}_2)) \\ &\quad -r(\mathcal{V} \cap \mathcal{V}_1 + \mathcal{V} \cap \mathcal{V}_2) - r(\mathcal{V} \cap \mathcal{V}_1 \cap \mathcal{V}_2) \end{aligned}$$

Now

$$\begin{aligned} (\mathcal{V} + \mathcal{V}_1) \cap (\mathcal{V} + \mathcal{V}_2) &\supseteq \mathcal{V} + (\mathcal{V}_1 \cap \mathcal{V}_2) \\ and \qquad \mathcal{V} \cap \mathcal{V}_1 + \mathcal{V} \cap \mathcal{V}_2 &\subseteq \mathcal{V} \cap (\mathcal{V}_1 + \mathcal{V}_2) \end{aligned}$$

Hence,

$$LHS \;\geq\; r(\mathcal{V} + \mathcal{V}_1 + \mathcal{V}_2) - r(\mathcal{V} \cap (\mathcal{V}_1 + \mathcal{V}_2)) + r(\mathcal{V} + (\mathcal{V}_1 \cap \mathcal{V}_2)) - r(\mathcal{V} \cap \mathcal{V}_1 \cap \mathcal{V}_2),$$

which is the RHS.

ii.

$$\begin{aligned} d(\mathcal{V},\mathcal{V}_1) &= r(\mathcal{V}+\mathcal{V}_1) - r(\mathcal{V}\cap\mathcal{V}_1) \\ &= r(\mathcal{V}) + r(\mathcal{V}_1) - 2r(\mathcal{V}\cap\mathcal{V}_1) \\ &= r(\mathcal{V}+\mathcal{V}\cap\mathcal{V}_1) - r(\mathcal{V}\cap\mathcal{V}_1) + r(\mathcal{V}_1+\mathcal{V}\cap\mathcal{V}_1) - r(\mathcal{V}\cap\mathcal{V}_1) \\ &= d(\mathcal{V},\mathcal{V}\cap\mathcal{V}_1) + d(\mathcal{V}_1,\mathcal{V}\cap\mathcal{V}_1). \end{aligned}$$

iii.

$$\begin{aligned} d(\mathcal{V},\mathcal{V}_1) &= r(\mathcal{V}+\mathcal{V}_1) - r(\mathcal{V}\cap\mathcal{V}_1) \\ &= r(\mathcal{V}+\mathcal{V}_1) - r(\mathcal{V}) + r(\mathcal{V}+\mathcal{V}_1) - r(\mathcal{V}_1) \\ &= d(\mathcal{V},\mathcal{V}+\mathcal{V}_1) + d(\mathcal{V}_1,\mathcal{V}+\mathcal{V}_1). \end{aligned}$$

E 15.22:

i. Since $\mathcal{V}_1 \supseteq \mathcal{V}$, we must have $\mathcal{V}_1 \cdot E_j \supseteq \mathcal{V} \,.\, E_j$. But $\mathcal{V}_1 = \oplus_j \mathcal{V}_1 \cdot E_j$. The result follows.

ii. Since $\mathcal{V}_2 \subseteq \mathcal{V}$ we must have $\mathcal{V}_2^\perp \supseteq \mathcal{V}^\perp$. Hence, $\mathcal{V}_2^\perp \cdot E_j \supseteq \mathcal{V}^\perp \cdot E_j$ and therefore $(\mathcal{V}_2^\perp \cdot E_j)^\perp \subseteq (\mathcal{V}^\perp \cdot E_j)^\perp$ i.e., $\mathcal{V}_2 \times E_j \subseteq \mathcal{V} \times E_j$. Since $\mathcal{V}_2 = \oplus_j \mathcal{V}_2 \times E_j$, the result follows.

E 15.23: We have

$$\sum r(\mathcal{V}\cdot E_j) - r(\mathcal{V}) = |E| - \sum r(\mathcal{V}^\perp \times E_j) - |E| + r(\mathcal{V}^\perp),$$

where E_j are the blocks of Π_s. The result follows.

E 15.24: Lemma 15.5.4 implies that the matroid associated with a representative matrix of $\mathcal{V}_P$ is an expansion for the function $\rho_P(\cdot) - \omega_P(\cdot) = \rho_E(\cdot) - \omega_E(\cdot)$ (using the notation of the same lemma). So we check if it is possible to find an independent set of columns of a representative matrix of $\mathcal{V}_P$ that contains precisely $k_j - \omega_E(j) + \omega_P(j)$ columns from $E_j, j = 1, \cdots, k$. (When $((\mathcal{V}_{E_jP_j})_k; \mathcal{V}_P)$ is a minimal decomposition of $\mathcal{V}_E$, we have $\omega_P(\cdot) = 0$). We can do this by using the algorithm given in the solution to Exercise 15.11.

E 15.25:
i. $d(\mathcal{V},\mathcal{V}') = r(\mathcal{V}') - r(\mathcal{V})$, since $\mathcal{V}' \supseteq \mathcal{V}$
$d(\mathcal{V}',\mathcal{V}_1) = r(\mathcal{V}') - \sum_j r(\mathcal{V}' \times E_j) =$ fission number of $\mathcal{V}'$.
So $d(\mathcal{V},\mathcal{V}') + d(\mathcal{V}',\mathcal{V}_1) = 2r(\mathcal{V}') - \sum_j r(\mathcal{V}' \times E_j) - r(\mathcal{V})$.

ii. Let $\mathcal{V}_2 = (\mathcal{V}')^\perp$. We then have the fusion - fission number of $\mathcal{V}_2$ relative to $(\mathcal{V}^\perp, \Pi_s)$ equal to

$$\sum_j r(\mathcal{V}_2 \cdot E_j) - 2r(\mathcal{V}_2) + r(\mathcal{V}^\perp) \quad = \quad \sum_j r((\mathcal{V}' \times E_j)^\perp)$$

$$
\begin{aligned}
& -2(\mid E \mid -r(\mathcal{V}')) + r(\mathcal{V}^{\perp}) \\
= \quad & \mid E \mid - \sum_j r(\mathcal{V}' \times E_j) + 2r(\mathcal{V}') \\
& -r(\mathcal{V}) + \mid E \mid -2 \mid E \mid \\
= \quad & 2r(\mathcal{V}') - \sum_j r(\mathcal{V}' \times E_j) - r(\mathcal{V}).
\end{aligned}
$$

iii. By Theorem 15.5.2, the generalized hybrid ranks of $\mathcal{V}$ and $\mathcal{V}^{\perp}$ are equal. We know that the minimum of the fusion - fission numbers of all the subspaces of $\mathcal{V}^{\perp}$ equals the generalized hybrid rank of $\mathcal{V}^{\perp}$ (Theorem 15.5.3). By the previous part of the present exercise, the minimum of the fusion - fission numbers of all the subspaces of $\mathcal{V}^{\perp}$ equals the minimum of the fission - fusion numbers of all the superspaces of $\mathcal{V}$. The result follows.

E 15.26: Let $\mathcal{V}'_{EP}$ have each element of $E \uplus P$ as a separator and let $\mathcal{V}_{EPQ}$ be s.t.

$$\mathcal{V}_{EPQ} \leftrightarrow \mathcal{V}_Q = \mathcal{V}_{EP},$$

$$\mathcal{V}_{EPQ} \leftrightarrow \mathcal{V}'_Q = \mathcal{V}'_{EP}$$

for some spaces $\mathcal{V}_Q, \mathcal{V}'_Q$ where E, P, Q are pairwise disjoint. Now $\mathcal{V}'_{EP} \leftrightarrow \mathcal{V}_P$ would still have each element of E as a separator. Further,

$$(\mathcal{V}_{EPQ} \leftrightarrow \mathcal{V}_Q) \leftrightarrow \mathcal{V}_P = (\mathcal{V}_{EPQ} \leftrightarrow \mathcal{V}_P) \leftrightarrow \mathcal{V}_Q,$$

since E, P, Q are pairwise disjoint. Hence, $\mathcal{V}_E = (\mathcal{V}_{EPQ} \leftrightarrow \mathcal{V}_P) \leftrightarrow \mathcal{V}_Q$.
Next, $\mathcal{V}'_{EP} \leftrightarrow \mathcal{V}_P = (\mathcal{V}_{EPQ} \leftrightarrow \mathcal{V}'_Q) \leftrightarrow \mathcal{V}_P = (\mathcal{V}_{EPQ} \leftrightarrow \mathcal{V}_P) \leftrightarrow \mathcal{V}'_Q$.
Hence, $\mathcal{V}'_E, \mathcal{V}_E$ both have the extension $(\mathcal{V}_{EPQ} \leftrightarrow \mathcal{V}_P)$ and $\mathcal{V}'_E$ has each element of E as a separator. Hence,
$d(\mathcal{V}_{EP}, \mathcal{V}'_{EP}) \geq d(\mathcal{V}'_E, \mathcal{V}_E) \geq$ hybrid rank of $\mathcal{V}_E$. In particular we could have chosen $\mathcal{V}'_{EP}$ so that $d(\mathcal{V}_{EP}, \mathcal{V}'_{EP}) =$ hybrid rank of $\mathcal{V}_{EP}$. The result follows.

E 15.27: We know (by Theorem 15.5.4) that the generalized hybrid rank of $\mathcal{V}$ relative to Π_s equals the minimum (hybrid rank of $\mathbf{A}'$), where $\mathbf{A}'$ is equivalent to the representative matrix $\mathbf{A}$ of $\mathcal{V}_E$ relative to $\{\mathcal{C}_j, j = 1, \cdots, k\}$ where $\mathcal{C}_j$ is the span of the column $E_j \in E$. It is easy to see that $\mathcal{M}(\mathbf{A}) = \mathcal{M}(\mathbf{A}')$. So hybrid rank of $\mathbf{A} =$ hybrid rank of $\mathbf{A}'$. This proves the result.

E 15.28:

i. Let $\mathcal{V}_1$ be spanned by the vector (1 1 1) and $\mathcal{V}_2$ by (0 1 1). Let E_2 be the singleton composed of the second column of the matrix

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

Clearly $(\mathcal{V}_1 \cdot E_2 \cap \mathcal{V}_2 \cdot E_2)$ is spanned by (1). But $(\mathcal{V}_1 \cap \mathcal{V}_2) \cdot E_2$ has only the zero vector.

ii. Let $\mathcal{V}_1, \mathcal{V}_2$ be as defined above and let $\mathcal{V}^1$ be the space spanned by (1 2 2). Clearly $(\mathcal{V}_1 \cap \mathcal{V}^1), (\mathcal{V}_2 \cap \mathcal{V}^1)$ are both zero spaces, while $(\mathcal{V}_1 + \mathcal{V}_2) \cap \mathcal{V}^1$ is spanned by (1 2 2).

E 15.29: Let $\mathcal{C}_{min}$ minimize $g(\cdot)$ among subspaces of $\mathcal{C}_s$.

We must have

$$\sum_{j=1}^{t} r(\mathcal{C}_j \cap \mathcal{C}_{min}) \leq r(\mathcal{C}_s \cap \mathcal{C}_{min}) = r(\mathcal{C}_{min})$$

and

$$r(\mathcal{C}_{min}) \geq r((\bigcup_{j=t+1}^{k} \mathcal{C}_j) \cap \mathcal{C}_{min}) \geq \sum_{j=t+1}^{k} r(\mathcal{C}_j \cap \mathcal{C}_{min}).$$

Thus, $2r(\mathcal{C}_{min}) \geq \sum_{j=1}^{k} r(\mathcal{C}_j \cap \mathcal{C}_{min})$. So $g(\mathcal{C}_{min}) \geq 0$ while $g(\{\mathbf{0}\}) = 0$.

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