Higher Order Inclusion Function Forms and their Applications in Global Optimization

Kotecha Ketan Virendrabhai
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Systems and Control Engineering
Indian Institute of Technology, Bombay

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ABSTRACT A problem of current interest in interval analysis [24] is the construction of inclusion function forms for multidimensional functions, having the property of higher order convergence. Higher order inclusion function forms have applications, for example, in the solution of equations, quadrature, and global optimization problems, where faster convergence could possibly be obtained with their aid. Recently, Lin and Rokne [19] introduced the so-called Taylor-Bernstein (TB) form as a higher order inclusion function form. However, Lin and Rokne's TB form is seized with overwhelming computational difficulties when the domain width is small. In this work, we present the following. (a) First, we propose an improvement of Lin and Rokne's TB form that makes it more practical for small domain widths. We then show that with the improved TB form, higher order convergence can be quite easily obtained through several low to medium dimensional examples. (b) Then, we propose an unconstrained global optimization algorithm that incorporates the improved TB form as an inclusion function form for the objective function. The cut-off test and termination condition are also suitably modified in the proposed algorithm. The performance of the proposed algorithm is then numerically tested and compared with those of the basic Moore-Skelboe algorithm [29] and the Moore-Skelboe algorithm with the Taylor model as an inclusion function. Test results on six benchmark examples of low to medium dimensions show the superior performance of the proposed algorithm. (c) Next, we propose a further improved TB form called the 'combined' TB form, that is more effective than either of the above two TB forms in application problems where the domain shrinks from large to small widths. The combined TB form inherits the higher order convergence property. Through numerical tests involving the six benchmark examples, the combined TB form is shown to compute the range enclosures most efficiently over entire range of domain widths. (d) Lastly, we propose an improved algorithm for unconstrained global optimization in the framework of the Moore-Skelboe algorithm. A novel and powerful feature of the proposed algorithm is that a variety of inclusion function forms for the objective function are incorporated into it - the combined TB form, the Taylor model, and the simple natural inclusion form. Several improvements are also made in the Bernstein step of the combined TB form, such as selection of a more efficient direction for subdivision, and use of cut-off test and monotonicity property to discard those boxes where the global minimizer cannot lie. Further, the incorporation of several inclusion function forms allows the cut-off test and termination condition in the MS algorithm to be made even more effective than in our earlier proposed optimization algorithm. The performance of the improved optimization algorithm is numerically tested and compared on eleven benchmark examples with those of the MS algorithm, the MS algorithm with the Taylor model as inclusion function, and our earlier proposed optimization algorithm. The results of these tests indicate that the proposed algorithm is usually considerably superior to the rest, in terms of the various performance metrics chosen for comparison.

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Introduction

1.1 Motivation

Let $\Re$ be the set of reals, $\mathbf{X} \subseteq \Re^l$ be a right parallelepiped parallel to the axes (also called as a box), and let $\bar{f}(\mathbf{X})$ denote the set of all values of an arbitrary function $f : \mathbf{X} \to \Re$ on $\mathbf{X}$. Let $I(\mathbf{X})$ be the set of all boxes contained in $\mathbf{X}$. Let the width of an interval $\mathbf{X}$ be defined as $w(\mathbf{X}) := \max \mathbf{X} - \min \mathbf{X}$ if $\mathbf{X} \in I(\Re)$, and as $w(\mathbf{X}) := \max \{w(\mathbf{X}_1), \ldots, w(\mathbf{X}_l)\}$, if $\mathbf{X} \in I(\Re^l)$.

Definition 1.1 *[30] A function $F : I(\mathbf{X}) \to I(\Re)$ is said to be an inclusion function for f, if*

$$\bar{f}(\mathbf{Y}) \subseteq F(\mathbf{Y}) \text{ for all } \mathbf{Y} \in I(\mathbf{X})$$

Definition 1.2 *[30] An inclusion function F for f is said to have convergence order α, if*

$$w(F(\mathbf{Y})) - w(\overline{f}(\mathbf{Y})) \leq L w(\mathbf{Y})^{\alpha} \text{ for all } \mathbf{Y} \in I(\mathbf{X}),$$

where L and α are some positive constants.

An important problem in interval analysis of Moore [24] is the construction of inclusion functions for multidimensional functions having the property of higher order convergence, i.e., having a convergence order α that is greater than quadratic. Such inclusion functions have applications, for example, in the solution of equations, quadrature and global optimization problems.

In particular, we consider the global optimization problem of determining arbitrarily good lower bounds for the minimum of $\bar{f}(\mathbf{X})$. Many algorithms based on interval analysis (IA) are available to solve the global optimization problem, see for example, [13], [17], [31] and the references cited therein. IA methods are usually based on branch and bound techniques, that is, they start from the initial box $\mathbf{X}$, subdivide $\mathbf{X}$ and store the subboxes in a list, discarding subboxes which are guaranteed not to contain a global minimizer until the desired accuracy in terms of the width of the intervals in the list is achieved. A basic branch and bound algorithm of IA is the so-called Moore-Skelboe (MS) algorithm [31]. Although the MS algorithm is reliable, it is somewhat slow to converge in 'difficult' problems, when inclusion functions of first and sometimes even second orders are used. Faster convergence could possibly be obtained with higher order inclusion functions.

The first paper in the literature concerning construction of inclusion functions with higher order convergence is that of Herzberger [14], who shows that higher order convergence can be obtained for a certain class of intervals. However, his requirement on the function is unrealistically strong. Cornelius and Lohner [6] propose the interpolation and remainder forms for multidimensional functions that enable any convergence order to be obtained in theory. However, in practice, convergence order of at most 4 or 5 is recommended even for unidimensional functions, see [6] and [30, pg. 9]. The same holds for the improved version of these forms for unidimensional functions, as proposed by Neumaier in [28, sec. 2.4]. Alefeld and Lohner [1] propose centered forms with higher order convergence for unidimensional functions. However, because of the strong condition on the functional representation, these higher order centered forms have limited practical value [1, pg. 8]. Berz *et al.* [3], [21] propose the so-called Taylor models for multidimensional functions. Although the accuracy of the so-called remainder interval part of the Taylor model increases in a higher order convergent fashion, the Taylor model itself is known to exhibit only quadratic convergence, see also Kearfott and Arazyan [18]. Lin and Rokne [19] combine the Bernstein form with the Taylor form to obtain the so-called Taylor-Bernstein (TB) form as an inclusion function form of f. However, as the domain width becomes smaller, the required degree of the Bernstein polynomials becomes very large. As a consequence, the Bernstein step becomes very computationally intensive when the domain intervals shrink in widths. Therefore, as an inclusion form for obtaining higher order convergence, the practical utility of the Lin and Rokne's TB form is severely restricted.

It is seen from the foregoing that there is a lack of higher order inclusion function forms that are practically effective, even for low to medium (i.e., even up to say, six) dimensional problems.

1.2 Objectives

Motivated by the above concerns, we have set mainly the following two objectives for the present work:

1. To develop higher order inclusion function form for multidimensional functions that are practically effective, and
2. To develop unconstrained global optimization algorithm with the developed higher order inclusion function form, for efficient determination of arbitrarily good lower bounds on the minimum of $\bar{f}(\mathbf{X})$.

In each case, the practical effectiveness of the proposed tool is to be numerically tested and compared with existing techniques on several 'difficult' problems of different dimensions.

1.3 Contributions

The main contributions of this work can be summarized as the following.

1. An improved TB form is proposed as a higher order inclusion function form for multi-dimensional functions that are sufficiently differentiable[1]. The improved TB form uses Bernstein polynomials [9] for bounding the range of the polynomial obtained from the Taylor form [30] of the given function f. The improved TB form has some important differences from Lin and Rokne's TB form, in the practical way it is constructed.

   The higher order convergence behavior of the improved TB form is numerically tested and compared with that of Lin and Rokne's TB form and also with that of the Taylor model. For the numerical testing, six benchmark examples with dimensions varying from 1 to 6 are considered. In all examples, unlike with the Taylor model and Lin and Rokne's TB form, higher order convergence of orders up to 9 are exhibited with the improved TB form. Moreover, such high orders of convergence are rather *easily* obtained for up to $5-$dim problems with the improved TB form.

2. An algorithm for unconstrained global optimization is proposed in the framework of the MS algorithm. The proposed optimization algorithm uses the improved TB form as an inclusion function form for the objective function. The improved TB form also allows the cut-off test and termination condition of the MS algorithm to be made more effective, and these modifications are incorporated into the proposed algorithm.

[1] The function must be at least $(m+1)$ times differentiable for achieving $(m+1)-$ th convergence order.

The performance of the proposed optimization algorithm is numerically tested and compared with those of the MS algorithm and the MS algorithm with the Taylor model as inclusion function. For the numerical testing, six benchmark examples of varying dimensions are considered. The results of these tests indicate that the proposed algorithm with the improved TB form as inclusion function is considerably more effective for the low to medium dimension problems considered.

3. A combined TB form is proposed as a higher order inclusion function form for multidimensional functions that are sufficiently differentiable. In application problems where the domain shrinks from large to small widths, the combined TB form is more effective than either Lin and Rokne's TB form or the improved TB form. The combined TB form switches between the Lin and Rokne's and improved TB forms depending on the domain width - i.e., it reduces to Lin and Rokne's TB form for large domain widths, and to the improved TB form for small domain widths.

   The higher order convergence behavior of the combined TB form is numerically tested and compared with that of Lin and Rokne's TB form, the improved TB form, and also with that of the Taylor model. For the numerical testing, six benchmark examples with dimensions varying from 1 to 6 are considered. In all examples, while the Lin and Rokne's TB form as well as the improved TB form fail to compute the range enclosures for some domain widths, the combined TB form succeeds in computing the same for any domain width and for any problem dimension. It is also found in these examples that the maximum list length needed by the combined TB form is *nil*, for any domain width and for any problem dimension; moreover, for large to intermediate domain widths, the combined TB form is faster than the improved TB form by as much as $2-4$ orders of magnitude.

4. An improved algorithm for unconstrained global optimization is proposed in the framework of the MS algorithm. A novel and powerful feature of the proposed algorithm is that a variety of inclusion function forms for the objective function are incorporated into it - the combined TB form, the Taylor model, and the simple natural inclusion form. Several improvements are also made in the Bernstein step of the combined TB form, such as selection of a more efficient direction for subdivision, and use of cut-off test and monotonicity property to discard those boxes where the global minimizer cannot lie. Further, the incorporation of several inclusion function forms allows the cut-off test and termination condition in the MS algorithm to be made even more effective than in our earlier proposed optimization algorithm.

The performance of the improved optimization algorithm is numerically tested and compared with those of the MS algorithm, the MS algorithm with the Taylor model as inclusion function, and our earlier proposed optimization algorithm. For the numerical testing, eleven benchmark examples with varying dimensions are considered. The results of these tests indicate that the proposed algorithm is superior to the rest, in terms of the various performance metrics chosen for comparison.

1.4 Organization of the thesis

The rest of the thesis is organized as follows.

In Chapter 2, an improved TB form having the higher order convergence property is proposed. In section 2.2, the essentials of the Bernstein form, Taylor form, and TB form of Lin and Rokne are given. In section 2.3, an improved TB form that is more effective than that of Lin and Rokne's TB form is proposed. In section 2.4, the higher order convergence behavior of the improved TB form is numerically tested and compared with that of the latter and also with that of the Taylor model. In section 2.5, the obtained test results are discussed, and in 2.6, the conclusions of the chapter are given.

In Chapter 3, an algorithm is proposed for global optimization in the framework of the MS algorithm. In section 3.1, the global optimization problem is introduced. In section 3.2, an outline of the MS algorithm is given. In section 3.3, an algorithm is proposed for global optimization with the improved TB form as an inclusion function form. In section 3.4, the performance of the proposed algorithm is numerically tested and compared with those of the MS algorithm and the MS algorithm with the Taylor model as an inclusion function form. The obtained test results are discussed in section 3.5, while the conclusions of the chapter are given in section 3.6.

In Chapter 4, a combined TB form having the higher order convergence property is proposed. In section 4.1, the need for a further improved TB form is presented. In section 4.2, a combined TB form is presented that combines the TB form in Chapter 2 and that of Lin and Rokne's. In section 4.3, the performance of the combined form is numerically tested and compared with those of the constituent forms, as well as with those of the Taylor model and the simple natural inclusion form. The obtained test results are discussed in section 4.4, while the conclusions of the chapter are given in section 4.5.

In Chapter 5, an improved algorithm is proposed for global optimization in the framework of the MS algorithm. In section 5.2, several improvements concerning the Bernstein step are introduced. In section 5.3, the proposed algorithm for global optimization is given. In section 5.4, the performance of the proposed algorithm is numerically tested and compared with those of the MS algorithm, the MS algorithm with the Taylor model as an inclusion function form, and the earlier proposed optimization algorithm. The obtained test results are discussed in section 5.5, while the conclusions of the chapter are given in section 5.6.

In Chapter 6, the conclusions of the work are given. In Appendix A, an outline of the Bernstein approach for the univariate case is given.

2

An improved Taylor-Bernstein form for higher order convergence

2.1 Introduction

As seen in Chapter 1, in the interval analysis literature there is a lack of higher order inclusion function forms that are practically effective, even for low to medium dimensional problems.

In this chapter, we attempt to fill in this gap by proposing an inclusion function form having the higher order convergence property for multidimensional functions. The proposed inclusion function form uses Bernstein polynomials for bounding the range of the polynomial obtained from the Taylor form of the function f. The Bernstein form [9] is combined with the Taylor form [30] to obtain the resulting so-called Taylor-Bernstein (TB) form as an inclusion function form of f. The proposed TB form has some important practical differences in the way it is constructed from the TB form of Lin and Rokne [19]. Specifically, the range of polynomial part of Taylor expansion is computed in the former using Bernstein subdivision, and a vertex condition check is done on every subdivision.

Further, we also numerically test and compare the higher order convergence behavior of the improved TB form with those of Lin and Rokne's TB form and the Taylor model. For the testing, we consider six benchmark examples with dimensions varying from one to six, and examine convergence orders up to nine.

The rest of this chapter is organized as follows. In section 2.2, we outline the essentials of the Bernstein form, Taylor form, and TB form of Lin and Rokne. In section 2.3, we propose an improved TB form that is more effective in practice. In section 2.4, we numerically test and compare the higher order convergence behavior of the improved TB form with that of the TB form of Lin and Rokne, and also with that of the Taylor model. In section 2.5, we summarize the obtained test results. In section 2.6, we draw the conclusions of the chapter.

2.2 Bernstein, Taylor, and TB forms

2.2.1 The Bernstein form

The Bernstein form has established itself as an important tool for finding bounds on the range of multivariate polynomials, see, for instance, [11], [33] and the references cited therein. An introduction to the Bernstein form is given in the book [30]. The salient features of the Bernstein form approach are:

1. The computation of the bounds conveys the information about the sharpness of these bounds.

2. The approach avoids functional evaluations which might be costly if the degree of the polynomial is high.

3. When bisecting a box and applying the Bernstein form to one of the two subboxes to get an enclosure for the range over this subbox, we obtain without any extra cost an enclosure for the range over the other subbox.

4. For sufficiently small boxes the Bernstein form gives the exact range.

In this subsection, we follow the notation in [11]. Let l be the number of variables and $\mathbf{x} = (x_1, \ldots, x_l) \in \Re^l$. A multi-index I is an ordered l-tuple of non-negative integers $I = (i_1, \ldots, i_l)$. For two given multi-indices I, N we write $I \leq N$ if $0 \leq i_k \leq n_k$, $k = 1, \ldots, l$. With $I = (i_1, \ldots, i_{r-1}, i_r, i_{r+1}, \ldots, i_l)$ we associate index $I_{r,k}$ given by $I_{r,k} = (i_1, \ldots, i_{r-1}, i_r + k, i_{r+1}, \ldots, i_l)$ where $0 \leq i_r + k \leq n_r$. Also, we write $\binom{N}{I}$ for $\binom{n_1}{i_1} \ldots \binom{n_l}{i_l}$.

We can expand a given multivariate polynomial into Bernstein polynomials to obtain bounds for its range over an l-dimensional box $\mathbf{X}$. Without loss of generality, consider the unit box $\mathbf{U} = [0, 1]^l$ since any nonempty box $\mathbf{X}$ of $\Re^l$ can be mapped affinely onto this box.

Let $p(\mathbf{x})$ be a multivariate polynomial in l variables with real coefficients. Denote by $N = (n_1, \ldots, n_l)$ the tuple of maximum degrees so that n_k is the maximum degree of x_k in $p(\mathbf{x})$ for $k = 1, \ldots, l$. Denote by $S = \{I : I \leq N\}$ the set containing all the tuples from $\Re^l$ which are 'smaller than or equal' to the tuple N of maximum degrees. Then, we can write an arbitrary l-variate polynomial p in the form

$$p(\mathbf{x}) = \sum_{I \in S} a_I \mathbf{x}^I, \quad \mathbf{x} \in \Re^l \tag{2.1}$$

where for $\mathbf{x} = (x_1, \ldots, x_l) \in \Re^l$ we set $\mathbf{x}^I = x_1^{i_1} x_2^{i_2} \ldots x_l^{i_l}$, where $a_I \in \Re$ represents the corresponding coefficient to each $x^I \in \Re^I$. We refer to N as the degree of p. The I^{th} Bernstein polynomial of degree N is defined as

$$B_I^N(\mathbf{x}) = B_{i_1}^{n_1}(x_1) \ldots B_{i_l}^{n_l}(x_l) \quad \mathbf{x} \in \Re^l$$

where, for $i_j = 0, \ldots, n_j, j = 1, \ldots, l$

$$B_{i_j}^{n_j}(x_j) = \binom{n_j}{i_j} x_j^{i_j} (1 - x_j)^{n_j - i_j}$$

The Bernstein coefficients $b_I(\mathbf{U})$ of p over the unit box $\mathbf{U}$ are given by

$$b_I(\mathbf{U}) = \sum_{J \le I} \frac{\binom{I}{J}}{\binom{N}{J}} a_J\,, \quad I \in S$$

Thus, the Bernstein form of a multivariate polynomial p is defined by

$$p(\mathbf{x}) = \sum_{I \in S} b_I(\mathbf{U}) B_I^N(\mathbf{x})$$

The Bernstein coefficients are collected in an array $B(\mathbf{U}) = (b_I(\mathbf{U}))_{I \in S}$, called a *patch*. Based on the above, we can have an algorithm for finding a patch of Bernstein coefficients.

Algorithm Patch : $B(\mathbf{U})$ =Patch($\mathbf{X}, a_I$)

Inputs: A box $\mathbf{X}$, a polynomial p as in (2.1) of degree N in l-variables with coefficients a_I.

Outputs: A patch $B(\mathbf{U})$ of Bernstein coefficients of p on $\mathbf{U}$.

BEGIN Algorithm

1. Transform the polynomial p (with coefficients a_I) on $\mathbf{X}$ to a polynomial on $\mathbf{U}$. Denote the coefficients of the latter as a'_I .

2. Find the Bernstein coefficients of p on $\mathbf{U}$ as

$$b_I(\mathbf{U}) = \sum_{J \le I} \frac{\binom{I}{J}}{\binom{N}{J}} a'_J\,, \quad I \in S$$

3. Return the patch $B(\mathbf{U}) = (b_I(\mathbf{U}))_{I \in S}$.

END Algorithm

The following result describes the range enclosure property of the Bernstein coefficients.

Lemma 2.1 *[5] : Let p be a polynomial of degree N, and let $\bar{p}(\mathbf{X})$ denote the range of p on the given domain $\mathbf{X}$. Then, the following property holds for a patch $B(\mathbf{U})$ of Bernstein coefficients :*

$$\bar{p}(\mathbf{X}) \subseteq [\min B(\mathbf{U}), \max B(\mathbf{U})]$$

We can find an enclosure of the range of the multivariate polynomial p on $\mathbf{X}$ by transforming the polynomial into Bernstein form. Then, by Lemma 2.1, the coefficients of the expansion in the Bernstein form provide lower and upper bounds for the range.

The obtained range enclosure can be further improved either by degree elevation of the Bernstein polynomial or by subdivision. The subdivision strategy is generally more efficient than the degree elevation strategy [9] and is therefore preferred.

Let $\mathbf{D}$ be any subbox of $\mathbf{U}$ generated by bisection, and suppose the patch $B(\mathbf{D})$ has been already computed. Further suppose $\mathbf{D}$ is bisected along the r-th component direction $(1 \leq r \leq l)$ to produce two further subboxes $\mathbf{D}_A$ and $\mathbf{D}_B$ given by

$$\begin{aligned}\mathbf{D}_A &= \left[\underline{d}_1, \bar{d}_1\right] \times \ldots \times \left[\underline{d}_r, m(d_r)\right] \times \ldots \times \left[\underline{d}_l, \bar{d}_l\right] \\ \mathbf{D}_B &= \left[\underline{d}_1, \bar{d}_1\right] \times \ldots \times \left[m(d_r), \bar{d}_r\right] \times \ldots \times \left[\underline{d}_l, \bar{d}_l\right]\end{aligned}$$

Then, the patches $B(\mathbf{D}_A)$ and $B(\mathbf{D}_B)$ can be obtained from $B(\mathbf{D})$ by executing the following algorithm.

Algorithm Subdivision : $[B(\mathbf{D}_A), B(\mathbf{D}_B), \mathbf{D}_A, \mathbf{D}_B] = \mathrm{SD}(\mathbf{D}, B(\mathbf{D}), r)$

Inputs: The box $\mathbf{D} \subseteq \mathbf{U}$, its patch $B(\mathbf{D})$, and a component direction r $(1 \leq r \leq l)$ in which $\mathbf{D}$ is to be bisected.

Outputs: Subboxes $\mathbf{D}_A$ and $\mathbf{D}_B$, with respective patches $B(\mathbf{D}_A)$ and $B(\mathbf{D}_B)$

BEGIN Algorithm

1. Bisect $\mathbf{D}$ along the r-th component direction to produce the two subboxes $\mathbf{D}_A$ and $\mathbf{D}_B$.

2. Compute patch $B(\mathbf{D}_A)$ as follows.

   (a) Set : $B^{(0)}(\mathbf{D}) \leftarrow B(\mathbf{D})$

   (b) FOR $k = 1, \ldots, n_r$ DO

   $$b_I^{(k)}(\mathbf{D}) = \begin{cases} b_I^{(k-1)}(\mathbf{D}) & : i_r < k \\ \frac{1}{2}\left\{b_{I_{r,-1}}^{(k-1)}(\mathbf{D}) + b_I^{(k-1)}(\mathbf{D})\right\} & : i_r \geq k \end{cases}$$

   To obtain the new coefficients, we apply formula given above for $i_j = 0, \ldots, n_j$, $j = 1, \ldots, r-1, r+1, \ldots, l$.

   (c) Set : $B(\mathbf{D}_A) \leftarrow B^{(n_r)}(\mathbf{D})$

3. Find patch $B(\mathbf{D}_B)$ from intermediate values in above step, as follows

   (a) FOR $k = 0$ to n_r DO

   $$b_{i_1,\ldots,n_r-k,\ldots,i_l}(\mathbf{D}_B) = b_{i_1,\ldots,n_r,\ldots,i_l}^{(k)}(\mathbf{D})$$

(b) Set : $B(\mathbf{D}_B) \leftarrow (b_I(\mathbf{D}_B))_{I\in S}$

4. RETURN $\mathbf{D}_A$, $\mathbf{D}_B$, $B(\mathbf{D}_A)$ and $B(\mathbf{D}_B)$

END Algorithm

The following result gives a condition called the *vertex condition*, which can be used to verify if the enclosure given by the Bernstein coefficients is the range.

Lemma 2.2 *[5] : Let p be a polynomial of degree N. Let $B(\mathbf{U})$ be a patch on $\mathbf{U}$. Then,*

$$\begin{aligned} \bar{p}(\mathbf{U}) &= [\min B(\mathbf{U}), \max B(\mathbf{U})] \\ &\Leftrightarrow \min B(\mathbf{U}) \ \textit{resp.} \ \max B(\mathbf{U}) \ \textit{occurs at some } I \in S_0 \end{aligned}$$

where, S_0 is a special subset of the index set S defined by

$$S_0 = \{0, n_1\} \times \ldots \times \{0, n_l\}$$

The above vertex condition also holds for any subbox $\mathbf{D} \subseteq \mathbf{U}$, see [22]. Combining the tool of Bernstein subdivision and the vertex condition, we can repeatedly improve the bounds till they are exact, i.e., till the vertex condition is satisfied on every subdivision. This leads to the following algorithm for computing the range of p on $\mathbf{X}$.

Algorithm Bounder : $\bar{p}(\mathbf{X}) = \text{Bounder}(\mathbf{X}, a_I)$

Inputs: A box $\mathbf{X}$, a polynomial p as in (2.1) of degree N in l-variables and having coefficients a_I.

Outputs: The range $\bar{p}(\mathbf{X})$.

BEGIN Algorithm

1. (Compute patch $B(\mathbf{U})$) Execute Algorithm Patch

$$B(\mathbf{U}) = \text{Patch}(\mathbf{X}, a_I)$$

2. (Initialize lists) Set $\mathcal{L} \leftarrow \{(\mathbf{U}, B(\mathbf{U}))\}$, $\mathcal{L}^{sol} \leftarrow \{\}$.

3. (Select item for processing) If $\mathcal{L}$ is empty, go to step 7. Otherwise, pick the first item from $\mathcal{L}$, denote it as $(\mathbf{D}, B(\mathbf{D}))$, and delete the item entry from $\mathcal{L}$.

4. (Check vertex condition on patch) If $(\mathbf{D}, B(\mathbf{D}))$ satisfies the vertex condition in Lemma 2.2, that is, if $\min B(\mathbf{D})$ resp. $\max B(\mathbf{D})$ occurs at some $I \in S_0$, enter the item in list $\mathcal{L}^{sol}$ and return to previous step.

5. (Subdivide and find new patches) Execute Algorithm Subdivision

$$[B(\mathbf{D}_A), B(\mathbf{D}_B), \mathbf{D}_A, \mathbf{D}_B] = \mathrm{SD}(\mathbf{D}, B(\mathbf{D}), r)$$

where, r is chosen to vary cyclically[1] from 1 to l.

6. (Add new entries to list) Enter the new items $(\mathbf{D}_A, B(\mathbf{D}_A))$ and $(\mathbf{D}_B, B(\mathbf{D}_B))$ at end of list $\mathcal{L}$, and return to step 3.

7. (Compute the polynomial range) Compute the range $\bar{p}(\mathbf{X})$ as the minimum to maximum over all the second entries of the items present in list $\mathcal{L}^{sol}$.

8. RETURN $\bar{p}(\mathbf{X})$.

END Algorithm

2.2.2 *The Taylor form*

In this subsection, we first introduce some further notation as in [30]. Let

$$\lambda = \{\lambda_1, \ldots, \lambda_l\}, \quad |\lambda| = \lambda_1 + ... + \lambda_l, \quad \lambda! = \lambda_1! \ldots \lambda_l!, \quad D^\lambda f(x) = \frac{\partial^{\lambda_1 + \ldots + \lambda_l} f(x)}{\partial x_1^{\lambda_1} \ldots \partial x_l^{\lambda_l}} \tag{2.2}$$

Let $I(\mathbf{X})$ be the set of all boxes contained in $\mathbf{X}$. Let the width of an interval $\mathbf{X}$ be defined as $w(\mathbf{X}) = \max \mathbf{X} - \min \mathbf{X}$ if $\mathbf{X} \in I(\Re)$, and as $w(\mathbf{X}) = \max\{w(\mathbf{X}_1), \ldots, w(\mathbf{X}_l)\}$, if $\mathbf{X} \in I(\Re^l)$. Let the midpoint of an interval $\mathbf{X}$ be defined as $m(\mathbf{X}) = (\min \mathbf{X} + \max \mathbf{X})/2$ if $\mathbf{X} \in I(\Re)$, and as $m(\mathbf{X}) = \{m(\mathbf{X}_1), \ldots, m(\mathbf{X}_l)\}$, if $\mathbf{X} \in I(\Re^l)$. Let $\bar{f}(\mathbf{X})$ denote the range of f on $\mathbf{X}$. A function $F : I(\mathbf{X}) \to I(\Re)$ is an inclusion function for f, if $\bar{f}(\mathbf{Y}) \subseteq F(\mathbf{Y})$ for all $\mathbf{Y} \in I(\mathbf{X})$. An inclusion function F for f is said to have convergence order α, if $w(F(\mathbf{Y})) - w(\bar{f}(\mathbf{Y})) \leq L w(\mathbf{Y})^\alpha$ for all $\mathbf{Y} \in I(\mathbf{X})$, where L and α are positive constants.

Let $f : \mathbf{X} \to \Re$ be a function that is $m+1$ times differentiable on $\mathbf{X}$. Then, the Taylor expansion of f of order m is given as

$$f(\mathbf{x}) = \underbrace{f(\mathbf{c}) + \sum_{|\lambda|=1}^{m} \frac{D^\lambda f(c)}{\lambda!} (\mathbf{x} - \mathbf{c})^\lambda}_{p(\mathbf{x})} + \underbrace{\sum_{|\lambda|=m+1} \frac{f^{(\lambda)}(\boldsymbol{\xi})}{\lambda!} (\mathbf{x} - \mathbf{c})^{m+1}}_{r(\mathbf{x})}, \ \mathbf{x} \in \mathbf{X} \tag{2.3}$$

where, $\mathbf{c} = m(\mathbf{X})$ and $\boldsymbol{\xi} \in \mathbf{X}$. We call $p(\mathbf{x})$ the polynomial part and $r(\mathbf{x})$ the remainder part of the Taylor expansion.

[1] That is, r varies starting from 1 through l, and then again from 1 through l, and so on. Besides cyclical, other strategies for subdivision exist, and their efficiency investigated in [10].

Assume an inclusion function of $(m+1)$-th derivative of f exists and is bounded, and furthermore that it has the isotonicity property [30]. Then, the corresponding Taylor form of order m, denoted by F_{Taylor}, can be expressed as [19] :

$$F_{Taylor}(\mathbf{X}) = \bar{p}(\mathbf{X}) + R(\mathbf{X}) \tag{2.4}$$

where $\bar{p}(\mathbf{X})$ is the range of the polynomial part $p(\mathbf{x})$ on $\mathbf{X}$, and $R(\mathbf{X})$ is any inclusion for the range of the remainder part $r(\mathbf{x})$ on $\mathbf{X}$. Lin and Rokne [19] show that the Taylor form has convergence order $m+1$.

Theorem 2.3 *[19] Assume that the Taylor form of order m is as defined above. Then,*

$$\begin{aligned} \overline{f}(\mathbf{X}) &\subseteq F_{Taylor}(\mathbf{X}) \\ w(F_{Taylor}(\mathbf{X})) - w(\overline{f}(\mathbf{X})) &= O(w(\mathbf{X})^{m+1}) \end{aligned} \tag{2.5}$$

2.2.3 The TB form of Lin and Rokne

The Taylor form provides an enclosure for the range of f over $\mathbf{X}$ with convergence order $m+1$. However, it requires the computation of the range of a multivariate polynomial $\bar{p}(\mathbf{X})$. Lin and Rokne [19] proposed an algorithm that uses Bernstein form to find a (generally non-sharp) enclosure of $\bar{p}(\mathbf{X})$, so that the resulting TB form, still possesses the property of $m+1$ convergence order given by (2.5).

We give below the Lin and Rokne algorithm for finding an enclosure of the range of f on $\mathbf{X}$. Note that this algorithm uses the Taylor form of order m and Bernstein polynomials of sufficiently high degree N' given by (2.7) below, and that a generally non-sharp enclosure of the range of the polynomial part p of Taylor expansion is computed and used.

Algorithm LR [19] : $F_{LR}(\mathbf{X}) = \mathrm{LR}(\mathbf{X}, f, m)$

Inputs: The box $\mathbf{X}$, an expression for the function f, and the order m of Taylor form to be used.

Output: An enclosure $F_{LR}(\mathbf{X})$ of the range of f on $\mathbf{X}$.

1. For the given function f, compute the coefficients of p in (2.3) and also the remainder interval $R(\mathbf{X})$. This may be done automatically on a computer equipped with interval arithmetic using Moore's recursive technique for Taylor coefficients computation, see [23], [24].

2. Relate the obtained Taylor coefficients to those of the power form in (2.1), and denote the coefficients in this form as a_I .

3. Compute the l-tuple of indices D given by

$$D = (d_1, \ldots, d_l)\,, \text{ where } d_1, \ldots, d_l \geq [1/w\,(\mathbf{X})]^{m+1} \tag{2.6}$$

and then the l-tuple of indices N' given by

$$N' = \left(n'_1, \ldots, n'_l\right), \quad \text{where } n'_k = \max\{n_k, d_k\}\,, \quad k = 1, \ldots, l \tag{2.7}$$

and construct $S' = \{I : I \leq N'\}$.

4. Find a patch $B\,(\mathbf{U})$ of Bernstein coefficients of p on $\mathbf{U}$ by executing Algorithm Patch : $B\,(\mathbf{U})$ =Patch($\mathbf{X}, a_I$) with S' used in place of S in this Algorithm. Then, compute an enclosure for the range of $\bar{p}\,(\mathbf{X})$ as

$$B^* = [\min B\,(\mathbf{U})\,, \max B\,(\mathbf{U})] \tag{2.8}$$

5. Compute an enclosure for the range of f over $\mathbf{X}$ as

$$F_{LR}\,(\mathbf{X}) = B^* + R\,(\mathbf{X}) \tag{2.9}$$

6. RETURN $F_{LR}\,(\mathbf{X})$.

END Algorithm

Lin and Rokne [19] showed that the TB form computed in the above algorithm retains the property of $m+1$ convergence order shown by the Taylor form:

Theorem 2.4 *[19] Let $F_{LR}\,(\mathbf{X})$ be as computed in Algorithm LR. Then,*

$$\begin{aligned} \overline{f}(\mathbf{X}) &\subseteq F_{LR}\,(\mathbf{X}) \\ w(F_{LR}\,(\mathbf{X})) - w(\overline{f}(\mathbf{X})) &= O(w(\mathbf{X})^{m+1}) \end{aligned}$$

2.3 Proposed improved TB form

As seen from (2.6), D becomes large quite quickly as $w\,(\mathbf{X})$ becomes smaller, leading to high degrees $N' \gg N$ of the Bernstein polynomials in (2.7). As a consequence, the Bernstein step of Algorithm LR becomes very computationally intensive as the domain intervals shrink in widths.

We therefore propose an algorithm that uses a different Bernstein step based on Bernstein polynomials of degree N (note that N is the minimum degree of Bernstein polynomials we can possibly use) and is equipped with the tools of subdivision and vertex condition checks.

We further propose to use in step 1 of our algorithm, the Taylor model technique of Berz *et al.* [3], [20] for computing the Taylor coefficients in parallel with the remainder interval. Berz *et al.* have shown that the Taylor model technique is more computationally efficient and gives tighter results than a direct implementation of Moore's recursive techniques.

The algorithm proposed below computes an enclosure for the range of f on $\mathbf{X}$ using the Taylor form of order m and Bernstein polynomials of degree N. We emphasize that the range of polynomial part of Taylor expansion is computed in this algorithm using Bernstein subdivision, and a vertex condition check is done on every subdivision.

Algorithm TB : $F_{TB}(\mathbf{X}) = TB(\mathbf{X}, f, m)$

Inputs: The box $\mathbf{X}$, an expression for the function f, and the order m of Taylor form to be used.

Output: An enclosure $F_{TB}(\mathbf{X})$ of the range of f on $\mathbf{X}$.

1. For the given function f, compute Taylor coefficients of p in (2.3) in parallel with the remainder interval $R(\mathbf{X})$ using the Taylor model technique of Berz *et al.* [3].

2. Relate the obtained Taylor coefficients to those of the power form in (2.1), and denote the coefficients in this form as a_I .

3. Find the range $\bar{p}(\mathbf{X})$ on $\mathbf{X}$ using Algorithm Bounder :
$$\bar{p}(\mathbf{X}) = \text{Bounder}(\mathbf{X}, a_I) \tag{2.10}$$

4. Using $R(\mathbf{X})$ obtained in step 1 and $\bar{p}(\mathbf{X})$ obtained in step 3, compute an enclosure for the range of f over $\mathbf{X}$ as
$$F_{TB}(\mathbf{X}) = \bar{p}(\mathbf{X}) + R(\mathbf{X}) \tag{2.11}$$

5. RETURN $F_{TB}(\mathbf{X})$.

END Algorithm

It is trivial to show that the TB form computed in the proposed algorithm also has the property of $m+1$ convergence order :

Theorem 2.5 *Let $F_{TB}(\mathbf{X})$ be as computed in Algorithm TB. Then,*

$$\begin{aligned} \overline{f}(\mathbf{X}) &\subseteq F_{TB}(\mathbf{X}) \\ w(F_{TB}(\mathbf{X})) - w(\overline{f}(\mathbf{X})) &= O(w(\mathbf{X})^{m+1}) \end{aligned}$$

Proof. >From (2.4) and (2.11), F_{TB} is a Taylor form F_{Taylor}. Now apply Theorem 2.3. ■

2.4 Numerical tests

We numerically investigate the higher order convergence property of the above inclusion function forms on some benchmark examples. The selected examples are of low to medium dimensions. For all our computations, we use a PC/Pentium III 800 MHz 256 MB RAM machine with a FORTRAN 90 compiler, and version 8.1 of the COSY-INFINITY package of Berz *et al.* *[2], [15]*. We also investigate the performance of the Taylor model as an inclusion function form in these examples.

In each example, we compute the intervals

$F_{TM}(\mathbf{X})$ - using Taylor model of Berz *et al.* [20], computed with the COSY-INFINITY package.

$F_{LR}(\mathbf{X})$ - using TB form of Lin and Rokne.

$F_{TB}(\mathbf{X})$ - using the proposed TB form.

$F_{inner}(\mathbf{X})$ - using *inner* estimates of the range computed with the well-known Moore-Skelboe optimization algorithm of interval analysis (see, for instance, [31]).

Let $\mathbf{X} = [a, b]$, $\mathbf{Y} = [c, d]$ be any two intervals. Then, following [6], as a measure of the overestimation we use the Hausdorff metric

$$\mathcal{H}(\mathbf{X}, \mathbf{Y}) = |[a, b], [c, d]| = \max\{|a - c|, |b - d|\}$$

Consider a sequence of nested intervals $\left\{\mathbf{X}^{(i)}\right\}$. We wish to find and compare for each form, the reduction in overestimation with decrease in the domain interval widths. Consider first the form F_{TM}. Let

$$\mathcal{H}_{TM}\left(\mathbf{X}^{(i-1)}\right) := \mathcal{H}\left(\bar{f}(\mathbf{X}^{(i-1)}), F_{TM}\left(\mathbf{X}^{(i-1)}\right)\right) \tag{2.12}$$

As a measure of the reduction in overestimation obtained with form F_{TM} over successive nested intervals $\mathbf{X}^{(i-1)}$ and $\mathbf{X}^{(i)}$, we use the ratio

$$\mathcal{R}_{TM}\left(\mathbf{X}^{(i-1)}, \mathbf{X}^{(i)}\right) := \frac{\mathcal{H}_{TM}\left(\mathbf{X}^{(i-1)}\right)}{\mathcal{H}_{TM}\left(\mathbf{X}^{(i)}\right)} = \frac{\mathcal{H}\left(\bar{f}(\mathbf{X}^{(i-1)}), F_{TM}\left(\mathbf{X}^{(i-1)}\right)\right)}{\mathcal{H}\left(\bar{f}(\mathbf{X}^{(i)}), F_{TM}\left(\mathbf{X}^{(i)}\right)\right)}$$

Define

$$\mathcal{R}^*\left(\mathbf{X}^{(i-1)}, \mathbf{X}^{(i)}\right) := \left(\frac{w\left(\mathbf{X}^{(i-1)}\right)}{w\left(\mathbf{X}^{(i)}\right)}\right)^{m+1}$$

If F_{TM} is an inclusion function form having convergence order $m + 1$, then

$$\mathcal{R}_{TM}\left(\mathbf{X}^{(i-1)}, \mathbf{X}^{(i)}\right) \to \mathcal{R}^*\left(\mathbf{X}^{(i-1)}, \mathbf{X}^{(i)}\right) \tag{2.13}$$

(where the tending is from above) for "small" enough $w\left(\mathbf{X}^{(i-1)}\right), w\left(\mathbf{X}^{(i)}\right)$.

In practice, the exact range $\bar{f}$ is generally difficult to compute, so the overestimation can be generally found relative only to some *inner* estimate F_{inner} of the range. However, we can easily show that if the $(m+1)$- th convergence order property holds relative to F_{inner}, then it implies that the same holds relative to the exact range $\bar{f}$. That is, it is sufficient if we can show the $(m+1)$- th convergence order property with F_{inner} used in place of $\bar{f}$ in above definitions. To avoid introducing more notation, in the sequel we use the quantities given in (2.12) through (2.13), with F_{inner} replacing $\bar{f}$ throughout.

Similarly, we can define $\mathcal{H}_{LR}$, $\mathcal{H}_{TB}, \mathcal{R}_{LR}, \mathcal{R}_{TB}$ for the forms F_{LR} and F_{TB}. For brevity of notation, we drop the arguments $\mathbf{X}^{(i-1)}, \mathbf{X}^{(i)}$ of all $\mathcal{H}$ and $\mathcal{R}$.

Example 2.1 *Gritton's second problem in Chemical Engineering [18]: The* $1-\dim$ *function is*

$$\begin{aligned} f(x) \;\; = \;\; & -371.93625 - 791.2465656 * x + 4044.944143 * x^2 + 978.1375167 * x^3 \\ & -16547.8928 * x^4 + 22140.72827 * x^5 - 9326.549359 * x^6 - 3518.536872 * x^7 \\ & +4782.532296 * x^8 - 1281.47944 * x^9 - 283.4435875 * x^{10} + 202.6270915 * x^{11} \\ & -16.17913459 * x^{12} - 8.88303902 * x^{13} + 1.575580173 * x^{14} + 0.1245990848 * x^{15} \\ & -0.03589148622 * x^{16} - 0.0001951095576 * x^{17} + 0.0002274682229 * x^{18} \end{aligned}$$

The domain is $\mathbf{X}^{(i)} = \left[-1 + 2^{-i}\left[-1, 1\right]\right]$.

Example 2.2 *Jennrich and Sampson function [26, problem 6]. The* $2-\dim$ *function is*

$$f(x) = \sum_{i=1}^{10} f_i(x)^2, \qquad f_i(x) = 2 + 2i - (\exp(ix_1) + \exp(ix_2))$$

The domain is $\mathbf{X}^{(i)} = \left[-1 + 2^{-i}\left[-1, 1\right]\right]^2$.

Example 2.3 *Levy function [32, Problem L8, pp. 204] The 3*$-\dim$ *function is*

$$\begin{aligned} f(x) &= \sum_{i=1}^{2}(y_i - 1)^2(1 + 10\sin^2(\pi y_{i+1})) + \sin^2(\pi y_1) + (y_3 - 1)^2, \\ y_i &= 1 + \frac{(x_i - 1)}{4} \; i = 1...3 \end{aligned}$$

The domain is $\mathbf{X}^{(i)} = \left[-12 + 2^{-i}\left[-1, 1\right]\right]^3$.

Example 2.4 *Trigonometric function [26, problem 26]. The* $4-\dim$ *function is*

$$f(x) = \sum_{i=1}^{4} f_i(x)^2, f_i(x) = 4 - \sum_{j=1}^{4}\cos x_j + i(1 - \cos x_i) - \sin x_i$$

The domain is $\mathbf{X}^{(i)} = \left[1.75 + 2^{-i}\left[-1, 1\right]\right]^4$.

Example 2.5 *Griewank function [32, Problem Griew5, pp. 205] The* $5-\dim$ *function is*

$$f(x)=\sum_{i=1}^{5}\frac{x_i^2}{400}-\prod_{i=1}^{5}\cos\left(\frac{x_i}{\sqrt{i}}\right)+1$$

The domain is $\mathbf{X}^{(i)}=\left[-600+2^{-i}\left[-1,1\right]\right]^5$.

Example 2.6 *Trigonometric function [26, problem 26]. The* $6-\dim$ *function is*

$$f(x)=\sum_{i=1}^{6}f_i(x)^2, f_i(x)=6-\sum_{j=1}^{6}\cos x_j+i(1-\cos x_i)-\sin x_i$$

The domain is $\mathbf{X}^{(i)}=\left[1.75+2^{-i}\left[-1,1\right]\right]^6$.

The results for examples 2.1 and 2.6 with the various forms are given[2] in Tables 2.1 to 2.6. Note that the results in all the Tables are rounded purely for display purposes. The average timings for all examples are reported in Table 2.7.

2.5 Discussion

>From the results given in the Tables, we observe that

1. With the Taylor model[3] as an inclusion function form, we obtain only *quadratic* convergence in all problems, irrespective of the chosen Taylor order m.

2. With the Lin and Rokne's TB form F_{LR} as an inclusion function form, in all cases (except for Taylor orders $m=2$) we are unable to proceed after just one subdivision, i.e., with $i>1$, due to the excessive memory requirements arising from high degrees of the associated Bernstein polynomials. For Taylor orders $m=2$, we are unable to proceed after just two subdivisions, i.e., with $i>2$, for the same reason. Therefore, as an inclusion form for obtaining higher order convergence, the practical utility of F_{LR} is found to be severely restricted.

3. With the proposed TB form F_{TB} as an inclusion function form, in problems of up to $4-\dim$ we quite easily obtain[4] higher order convergence of orders up to 9. In problems of 5 and $6-$ dim, we do obtain higher order convergence of orders up to 9; however, the computational demands are somewhat large for the $5-\dim$ problem, and become excessive for the $6-$ dim one.

[2] In the Tables, a starred entry denotes that the execution is aborted due to excessive memory requirements.

[3] In version 8.1 of COSY-INFINITY package made available to us, the range of the polynomial part is evaluated by simple interval arithmetic, see also [18].

[4] until we have overestimations of very small magnitudes (of order of $E-10$ or less).

2.6 Conclusions

We proposed a new inclusion function form for multidimensional functions. With the proposed form, we could quite easily obtain higher order convergence (of orders up to 9) for low to medium dimensional problems. To our knowledge, it is perhaps for the first time that higher order convergence of such high orders has actually been demonstrated on multidimensional problems. The new higher order convergent form can be constructed on a computer with the fully automated algorithm presented, without any need for hand computations.

For a problem of higher dimensions ($l = 6$), the proposed form was found to be computationally inefficient. This strongly suggests the need for further improvements in the proposed algorithm for dealing with higher dimensional ($l \geq 6$) problems.

TABLE 2.1. Overestimations and their reduction ratios for various Taylor orders obtained with Taylor model, Algorithm LR, and Algorithm TB in Example 2.1 Gritton ($1-dim$).

For Taylor order $m = 2$:

i	0	1	2	3	4	5	6	7
$w\left(\mathbf{X}^{(i)}\right)$	$2*2^{-0}$	$2*2^{-1}$	$2*2^{-2}$	$2*2^{-3}$	$2*2^{-4}$	$2*2^{-5}$	$2*2^{-6}$	$2*2^{-7}$
$\mathcal{H}_{TM}$	$1E+6$	$1E+5$	$2E+4$	$4E+3$	$9E+2$	$2E+2$	$6E+1$	$1E+1$
$\mathcal{H}_{LR}$	$1E+6$	$8E+4$	$5E+3$	$*$	$*$	$*$	$*$	$*$
$\mathcal{H}_{TB}$	$1E+6$	$8E+4$	$5E+3$	$5E+2$	$5E+1$	$6E+0$	$7E-1$	$1E-1$
$\mathcal{R}^*$	–	8	8	8	8	8	8	8
$\mathcal{R}_{TM}$	-	14.2	5.8	4.5	4.2	4.1	4.0	4.0
$\mathcal{R}_{LR}$	–	17.9	15.5	–	–	–	–	–
$\mathcal{R}_{TB}$	–	19.0	14.6	11.2	9.6	8.8	8.4	8.2

For Taylor order $m = 4$:

i	0	1	2	3	4	5	6	7
$w\left(\mathbf{X}^{(i)}\right)$	$2*2^{-0}$	$2*2^{-1}$	$2*2^{-2}$	$2*2^{-3}$	$2*2^{-4}$	$2*2^{-5}$	$2*2^{-6}$	$2*2^{-7}$
$\mathcal{H}_{TM}$	$8E+5$	$8E+4$	$1E+4$	$4E+3$	$9E+2$	$2E+2$	$2E+1$	$1E+1$
$\mathcal{H}_{LR}$	$7E+5$	$2E+4$	$*$	$*$	$*$	$*$	$*$	$*$
$\mathcal{H}_{TB}$	$7E+5$	$2E+4$	$6E+2$	$2E+1$	$5E-1$	$1E-2$	$5E-4$	$1E-5$
$\mathcal{R}^*$	–	32	32	32	32	32	32	32
$\mathcal{R}_{TM}$	–	10.8	5.0	4.1	4.0	4.0	4.0	4.0
$\mathcal{R}_{LR}$	–	38.0	–	–	–	–	–	–
$\mathcal{R}_{TB}$	–	38.0	35.9	33.9	32.9	32.4	32.2	32.1

Table 2.1 (contd.): For Taylor order $m = 6$:

i	0	1	2	3	4	5	6	7
$w\left(\mathbf{X}^{(i)}\right)$	$2*2^{-0}$	$2*2^{-1}$	$2*2^{-2}$	$2*2^{-3}$	$2*2^{-4}$	$2*2^{-5}$	$2*2^{-6}$	$2*2^{-7}$
$\mathcal{H}_{TM}$	$7E+5$	$7E+4$	$1E+4$	$4E+3$	$9E+2$	$2E+2$	$6E+1$	$1E+1$
$\mathcal{H}_{LR}$	$5E+5$	$3E+2$	$*$	$*$	$*$	$*$	$*$	$*$
$\mathcal{H}_{TB}$	$1E+5$	$3E+2$	$2E+0$	$1E-2$	$8E-5$	$6E-7$	$5E-9$	$2E-10$
$\mathcal{R}^*$	–	128	128	128	128	128	128	128
$\mathcal{R}_{TM}$	–	9.9	5.0	4.1	4.0	4.0	4.0	4.0
$\mathcal{R}_{LR}$	–	1375.0	–	–	–	–	–	–
$\mathcal{R}_{TB}$	–	291.0	179.6	158.8	145.3	137.2	128.5	22.0

For Taylor order $m = 8$:

i	0	1	2	3	4	5	6	7
$w\left(\mathbf{X}^{(i)}\right)$	$2*2^{-0}$	$2*2^{-1}$	$2*2^{-2}$	$2*2^{-3}$	$2*2^{-4}$	$2*2^{-5}$	$2*2^{-6}$	$2*2^{-7}$
$\mathcal{H}_{TM}$	$7E+5$	$7E+4$	$1E+4$	$4E+3$	$9E+2$	$2E+2$	$6E+1$	$1E+1$
$\mathcal{H}_{LR}$	$2E+5$	$3E+0$	$*$	$*$	$*$	$*$	$*$	$*$
$\mathcal{H}_{TB}$	$2E+4$	$3E+0$	$4E-3$	$6E-6$	$1E-8$	$2E-10$	$2E-10$	$2E-10$
$\mathcal{R}^*$	–	512	512	512	512	512	512	512
$\mathcal{R}_{TM}$	–	9.3	4.9	4.1	4.0	4.0	4.0	4.0
$\mathcal{R}_{LR}$	–	$6E+4$	–	–	–	–	–	–
$\mathcal{R}_{TB}$	–	5170.6	805.8	671.7	587.7	518.4	12.1	0.9

TABLE 2.2. Overestimations and their reduction ratios for various Taylor orders obtained with Taylor model, Algorithm LR, and Algorithm TB in Example 2.2 Jennrich and Sampson ($2-dim$).

For Taylor order $m=2$:

i	0	1	2	3	4	5	6	7
$w\left(\mathbf{X}^{(i)}\right)$	$2*2^{-0}$	$2*2^{-1}$	$2*2^{-2}$	$2*2^{-3}$	$2*2^{-4}$	$2*2^{-5}$	$2*2^{-6}$	$2*2^{-7}$
$\mathcal{H}_{TM}$	$3E+5$	$1E+2$	$4E+0$	$5E-1$	$1E-1$	$3E-2$	$7E-3$	$2E-3$
$\mathcal{H}_{LR}$	$3E+5$	$1E+2$	$2E+0$	$*$	$*$	$*$	$*$	$*$
$\mathcal{H}_{TB}$	$3E+5$	$1E+2$	$2E+0$	$8E-2$	$5E-3$	$3E-4$	$2E-5$	$2E-6$
$\mathcal{R}^*$	$-$	8	8	8	8	8	8	8
$\mathcal{R}_{TM}$	$-$	$3E+3$	25.6	6.7	4.5	4.1	4.0	4.0
$\mathcal{R}_{LR}$	$-$	$3E+3$	50.2	$-$	$-$	$-$	$-$	$-$
$\mathcal{R}_{TB}$	$-$	$3E+3$	48.3	22.6	17.1	14.6	12.7	11.1

For Taylor order $m=4$:

i	0	1	2	3	4	5	6	7
$w\left(\mathbf{X}^{(i)}\right)$	$2*2^{-0}$	$2*2^{-1}$	$2*2^{-2}$	$2*2^{-3}$	$2*2^{-4}$	$2*2^{-5}$	$2*2^{-6}$	$2*2^{-7}$
$\mathcal{H}_{TM}$	$4E+6$	$7E+1$	$2E+0$	$5E-1$	$1E-1$	$3E-2$	$7E-3$	$2E-3$
$\mathcal{H}_{LR}$	$4E+6$	$6E+1$	$*$	$*$	$*$	$*$	$*$	$*$
$\mathcal{H}_{TB}$	$4E+6$	$6E+1$	$2E-1$	$2E-3$	$3E-5$	$5E-7$	$9E-9$	$2E-10$
$\mathcal{R}^*$	$-$	32	32	32	32	32	32	32
$\mathcal{R}_{TM}$	$-$	$6E+4$	29.9	4.7	4.1	4.0	4.0	4.0
$\mathcal{R}_{LR}$	$-$	$7E+4$	$-$	$-$	$-$	$-$	$-$	$-$
$\mathcal{R}_{TB}$	$-$	$7E+4$	251.7	104.4	74.4	61.9	53.7	47.6

Table 2.2 (Contd.) For Taylor order $m = 6$:

i	0	1	2	3	4	5	6	7
$w\left(\mathbf{X}^{(i)}\right)$	$2*2^{-0}$	$2*2^{-1}$	$2*2^{-2}$	$2*2^{-3}$	$2*2^{-4}$	$2*2^{-5}$	$2*2^{-6}$	$2*2^{-7}$
$\mathcal{H}_{TM}$	$2E+7$	$4E+1$	$2E+0$	$5E-1$	$1E-1$	$3E-2$	$7E-3$	$2E-3$
$\mathcal{H}_{LR}$	$2E+7$	$2E+1$	$*$	$*$	$*$	$*$	$*$	$*$
$\mathcal{H}_{TB}$	$2E+7$	$2E+1$	$2E-2$	$4E-5$	$1E-7$	$5E-10$	$1E-12$	$3E-12$
$\mathcal{R}^*$	–	128	128	128	128	128	128	128
$\mathcal{R}_{TM}$	–	$5E+5$	16.7	4.4	4.1	4.0	4.0	4.0
$\mathcal{R}_{LR}$	–	$8E+5$	–	–	–	–	–	–
$\mathcal{R}_{TB}$	–	$8E+5$	$1E+3$	456.8	311.2	256.2	495.6	0.3

For Taylor order $m = 8$:

i	0	1	2	3	4	5	6	7
$w\left(\mathbf{X}^{(i)}\right)$	$2*2^{-0}$	$2*2^{-1}$	$2*2^{-2}$	$2*2^{-3}$	$2*2^{-4}$	$2*2^{-5}$	$2*2^{-6}$	$2*2^{-7}$
$\mathcal{H}_{TM}$	$4E+7$	$2E+1$	$2E+0$	$5E-1$	$1E-1$	$3E-2$	$7E-3$	$2E-3$
$\mathcal{H}_{LR}$	$4E+7$	$7E+0$	$*$	$*$	$*$	$*$	$*$	$*$
$\mathcal{H}_{TB}$	$4E+7$	$7E+0$	$1E-3$	$6E-7$	$5E-10$	$2E-12$	$2E-12$	$3E-12$
$\mathcal{R}^*$	–	512	512	512	512	512	512	512
$\mathcal{R}_{TM}$	–	$2E+6$	8.6	4.3	4.1	4.0	4.0	4.0
$\mathcal{R}_{LR}$	–	$5E+6$	–	–	–	–	–	–
$\mathcal{R}_{TB}$	–	$5E+6$	$5E+3$	$2E+3$	$1E+3$	205.6	1.0	0.7

TABLE 2.3. Overestimations and their reduction ratios for various Taylor orders obtained with Taylor model, Algorithm LR, and Algorithm TB in Example 2.3 Levy $(3 - dim)$.

For Taylor order $m = 2$:

i	0	1	2	3	4	5	6	7
$w\left(\mathbf{X}^{(i)}\right)$	$2*2^{-0}$	$2*2^{-1}$	$2*2^{-2}$	$2*2^{-3}$	$2*2^{-4}$	$2*2^{-5}$	$2*2^{-6}$	$2*2^{-7}$
$\mathcal{H}_{TM}$	$2E+2$	$3E+1$	$5E+0$	$1E+0$	$2E-1$	$5E-2$	$1E-2$	$3E-3$
$\mathcal{H}_{LR}$	$2E+2$	$2E+1$	$2E+0$	$*$	$*$	$*$	$*$	$*$
$\mathcal{H}_{TB}$	$2E+2$	$2E+1$	$2E+0$	$3E-1$	$3E-2$	$4E-3$	$5E-4$	$7E-5$
$\mathcal{R}^*$	–	8	8	8	8	8	8	8
$\mathcal{R}_{TM}$	–	6.7	5.8	5.0	4.6	4.3	4.1	4.0
$\mathcal{R}_{LR}$	–	9.3	8.7	–	–	–	–	–
$\mathcal{R}_{TB}$	–	9.3	8.8	8.4	8.2	8.1	8.1	8.0

For Taylor order $m = 4$:

i	0	1	2	3	4	5	6	7
$w\left(\mathbf{X}^{(i)}\right)$	$2*2^{-0}$	$2*2^{-1}$	$2*2^{-2}$	$2*2^{-3}$	$2*2^{-4}$	$2*2^{-5}$	$2*2^{-6}$	$2*2^{-7}$
$\mathcal{H}_{TM}$	$2E+2$	$3E+1$	$5E+0$	$1E+0$	$2E-1$	$5E-2$	$1E-2$	$3E-3$
$\mathcal{H}_{LR}$	$2E+1$	$1E-1$	$*$	$*$	$*$	$*$	$*$	$*$
$\mathcal{H}_{TB}$	$7E+0$	$1E-1$	$2E-3$	$4E-5$	$1E-6$	$3E-8$	$8E-10$	$2E-11$
$\mathcal{R}^*$	–	32	32	32	32	32	32	32
$\mathcal{R}_{TM}$	–	6.5	5.6	5.0	4.6	4.3	4.1	4.1
$\mathcal{R}_{LR}$	–	203.0	–	–	–	–	–	–
$\mathcal{R}_{TB}$	–	72.0	53.2	46.1	40.6	36.8	34.9	51.5

Table 2.3 (Contd.) For Taylor order $m = 6$:

i	0	1	2	3	4	5	6	7
$w\left(\mathbf{X}^{(i)}\right)$	$2*2^{-0}$	$2*2^{-1}$	$2*2^{=2}$	$2*2^{-3}$	$2*2^{-4}$	$2*2^{-5}$	$2*2^{-6}$	$2*2^{-7}$
$\mathcal{H}_{TM}$	$2E+2$	$3E+1$	$5E+0$	$1E+0$	$2E-1$	$5E-2$	$1E-2$	$3E-3$
$\mathcal{H}_{LR}$	$1E+0$	$8E-3$	$*$	$*$	$*$	$*$	$*$	$*$
$\mathcal{H}_{TB}$	$1E+0$	$8E-3$	$76E-5$	$5E-7$	$4E-9$	$2E-11$	$9E-12$	$9E-12$
$\mathcal{R}^*$	–	128	128	128	128	128	128	128
$\mathcal{R}_{TM}$	–	6.4	5.6	5.0	4.6	4.3	4.1	4.0
$\mathcal{R}_{LR}$	–	143.4	–	–	–	–	–	–
$\mathcal{R}_{TB}$	–	143.4	135.7	132.0	130.3	185.4	2.26	0.96

For Taylor order $m = 8$:

i	0	1	2	3	4	5	6	7
$w\left(\mathbf{X}^{(i)}\right)$	$2*2^{-0}$	$2*2^{-1}$	$2*2^{-2}$	$2*2^{-3}$	$2*2^{-4}$	$2*2^{-5}$	$2*2^{-6}$	$2*2^{-7}$
$\mathcal{H}_{TM}$	$2E+2$	$3E+1$	$5E+0$	$1E+0$	$2E-1$	$5E-2$	$1E-2$	$3E-3$
$\mathcal{H}_{LR}$	$1E-2$	$2E-5$	$*$	$*$	$*$	$*$	$*$	$*$
$\mathcal{H}_{TB}$	$1E-2$	$2E-5$	$3E-8$	$4E-11$	$9E-12$	$10E-12$	$9E-12$	$9E-12$
$\mathcal{R}^*$	–	512	512	512	512	512	512	512
$\mathcal{R}_{TM}$	–	6.4	5.6	5.0	4.6	4.3	4.1	4.1
$\mathcal{R}_{LR}$	–	738.8	–	–	–	–	–	–
$\mathcal{R}_{TB}$	–	738.9	704.2	643.3	4.6	0.9	1.0	0.9

TABLE 2.4. Overestimations and their reduction ratios for various Taylor orders obtained with Taylor model, Algorithm LR, and Algorithm TB in Example 2.4 Trigonometric ($4-dim$).

For Taylor order $m=2$:

i	0	1	2	3	4	5	6	7
$w\left(\mathbf{X}^{(i)}\right)$	$2*2^{-0}$	$2*2^{-1}$	$2*2^{-2}$	$2*2^{-3}$	$2*2^{-4}$	$2*2^{-5}$	$2*2^{-6}$	$2*2^{-7}$
$\mathcal{H}_{TM}$	$4E+2$	$9E+1$	$2E+1$	$5E+0$	$1E+0$	$3E-1$	$7E-2$	$2E-2$
$\mathcal{H}_{LR}$	$3E+2$	$3E+1$	$3E+0$	$*$	$*$	$*$	$*$	$*$
$\mathcal{H}_{TB}$	$3E+2$	$3E+1$	$3E+0$	$3E-1$	$3E-2$	$4E-3$	$5E-4$	$7E-5$
$\mathcal{R}^*$	–	8	8	8	8	8	8	8
$\mathcal{R}_{TM}$	–	4.9	4.5	4.2	4.1	4.1	4.0	4.0
$\mathcal{R}_{LR}$	–	10.5	9.5	–	–	–	–	–
$\mathcal{R}_{TB}$	–	10.5	9.5	8.8	8.4	8.2	8.1	8.1

For Taylor order $m=4$:

i	0	1	2	3	4	5	6	7
$w\left(\mathbf{X}^{(i)}\right)$	$2*2^{-0}$	$2*2^{-1}$	$2*2^{-2}$	$2*2^{-3}$	$2*2^{-4}$	$2*2^{-5}$	$2*2^{-6}$	$2*2^{-7}$
$\mathcal{H}_{TM}$	$4E+2$	$9E+1$	$2E+1$	$5E+0$	$1E+0$	$3E-1$	$7E-2$	$2E-2$
$\mathcal{H}_{LR}$	$1E+1$	$2E-1$	$*$	$*$	$*$	$*$	$*$	$*$
$\mathcal{H}_{TB}$	$1E+1$	$2E-1$	$5E-3$	$1E-4$	$3E-6$	$8E-8$	$2E-9$	$7E-11$
$\mathcal{R}^*$	–	32	32	32	32	32	32	32
$\mathcal{R}_{TM}$	–	4.9	4.4	4.2	4.1	4.1	4.0	4.0
$\mathcal{R}_{LR}$	–	56.3	–	–	–	–	–	–
$\mathcal{R}_{TB}$	–	56.3	50.4	44.5	39.8	36.5	34.3	30.6

Table 2.4 (Contd.) For Taylor order $m = 6$:

i	0	1	2	3	4	5	6	7
$w\left(\mathbf{X}^{(i)}\right)$	$2*2^{-0}$	$2*2^{-1}$	$2*2^{-2}$	$2*2^{-3}$	$2*2^{-4}$	$2*2^{-5}$	$2*2^{-6}$	$2*2^{-7}$
$\mathcal{H}_{TM}$	$4E+2$	$9E+1$	$2E+1$	$5E+0$	$1E+0$	$3E-1$	$7E-2$	$2E-2$
$\mathcal{H}_{LR}$	$2E+0$	$9E-3$	$*$	$*$	$*$	$*$	$*$	$*$
$\mathcal{H}_{TB}$	$2E+0$	$9E-3$	$6E-5$	$4E-7$	$3E-9$	$3E-11$	$7E-12$	$7E-12$
$\mathcal{R}^*$	–	128	128	128	128	128	128	128
$\mathcal{R}_{TM}$	–	4.9	4.4	4.2	4.1	4.1	4.0	4.0
$\mathcal{R}_{LR}$	–	189.0	–	–	–	–	–	–
$\mathcal{R}_{TB}$	–	189.0	167.6	151.3	140.5	99.2	3.6	1.0

For Taylor order $m = 8$:

i	0	1	2	3	4	5	6	7
$w\left(\mathbf{X}^{(i)}\right)$	$2*2^{-0}$	$2*2^{-1}$	$2*2^{-2}$	$2*2^{-3}$	$2*2^{-4}$	$2*2^{-5}$	$2*2^{-6}$	$2*2^{-7}$
$\mathcal{H}_{TM}$	$4E+2$	$9E+1$	$2E+1$	$5E+0$	$1E+0$	$3E-1$	$7E-2$	$2E-2$
$\mathcal{H}_{LR}$	$5E-1$	$6E-5$	$*$	$*$	$*$	$*$	$*$	$*$
$\mathcal{H}_{TB}$	$5E-1$	$6E-5$	$8E-8$	$1E-10$	$8E-12$	$7E-12$	$7E-12$	$7E-12$
$\mathcal{R}^*$	–	512	512	512	512	512	512	512
$\mathcal{R}_{TM}$	–	4.9	4.4	4.2	4.1	4.1	4.0	4.0
$\mathcal{R}_{LR}$	–	828.6	–	–	–	–	–	–
$\mathcal{R}_{TB}$	–	828.6	734.6	623.2	17.2	1.1	1.0	0.9

TABLE 2.5. Overestimations and their reduction ratios for various Taylor orders obtained with Taylor model, Algorithm LR, and Algorithm TB in Example 2.5 Griewank ($5 - dim$).

For Taylor order $m = 2$:

i	0	1	2	3	4	5	6	7
$w\left(\mathbf{X}^{(i)}\right)$	$2*2^{-0}$	$2*2^{-1}$	$2*2^{-2}$	$2*2^{-3}$	$2*2^{-4}$	$2*2^{-5}$	$2*2^{-6}$	$2*2^{-7}$
$\mathcal{H}_{TM}$	$9E-1$	$1E-1$	$2E-2$	$5E-3$	$1E-3$	$3E-4$	$7E-5$	$2E-5$
$\mathcal{H}_{LR}$	$9E-1$	$6E-2$	$5E-3$	$*$	$*$	$*$	$*$	$*$
$\mathcal{H}_{TB}$	$9E-1$	$6E-2$	$5E-3$	$6E-4$	$6E-5$	$7E-6$	$9E-7$	$1E-7$
$\mathcal{R}^*$	$-$	8	8	8	8	8	8	8
$\mathcal{R}_{TM}$	$-$	8.2	5.0	4.4	4.2	4.1	4.0	4.0
$\mathcal{R}_{LR}$	$-$	14.3	11.2	$-$	$-$	$-$	$-$	$-$
$\mathcal{R}_{TB}$	$-$	14.3	11.2	9.7	8.9	8.5	8.2	8.1

For Taylor order $m = 4$:

i	0	1	2	3	4	5	6	7
$w\left(\mathbf{X}^{(i)}\right)$	$2*2^{-0}$	$2*2^{-1}$	$2*2^{-2}$	$2*2^{-3}$	$2*2^{-4}$	$2*2^{-5}$	$2*2^{-6}$	$2*2^{-7}$
$\mathcal{H}_{TM}$	$9E-1$	$1E-1$	$2E-2$	$5E-3$	$1E-3$	$3E-4$	$7E-5$	$2E-5$
$\mathcal{H}_{LR}$	$4E-1$	$9E-3$	$*$	$*$	$*$	$*$	$*$	$*$
$\mathcal{H}_{TB}$	$4E-1$	$9E-3$	$2E-4$	$7E-6$	$2E-7$	$6E-9$	$2E-10$	$3E-12$
$\mathcal{R}^*$	$-$	32	32	32	32	32	32	32
$\mathcal{R}_{TM}$	$-$	8.3	5.0	4.4	4.2	4.1	4.0	4.0
$\mathcal{R}_{LR}$	$-$	43.0	$-$	$-$	$-$	$-$	$-$	$-$
$\mathcal{R}_{TB}$	$-$	43.0	37.8	35.1	33.6	32.9	32.6	67.3

Table 2.5 (Contd.) For Taylor order $m = 6$:

i	0	1	2	3	4	5	6	7
$w\left(\mathbf{X}^{(i)}\right)$	$2*2^{-0}$	$2*2^{-1}$	$2*2^{-2}$	$2*2^{-3}$	$2*2^{-4}$	$2*2^{-5}$	$2*2^{-6}$	$2*2^{-7}$
$\mathcal{H}_{TM}$	$9E-1$	$1E-1$	$2E-2$	$5E-3$	$1E-3$	$3E-4$	$7E-5$	$1E-5$
$\mathcal{H}_{LR}$	$6E-2$	$3E-4$	$*$	$*$	$*$	$*$	$*$	$*$
$\mathcal{H}_{TB}$	$6E-2$	$3E-4$	$2E-6$	$1E-8$	$9E-11$	$6E-12$	$1E-11$	$9E-12$
$\mathcal{R}^*$	–	128	128	128	128	128	128	128
$\mathcal{R}_{TM}$	–	8.1	5.1	4.4	4.2	4.1	4.0	4.0
$\mathcal{R}_{LR}$	–	196.1	–	–	–	–	–	–
$\mathcal{R}_{TB}$	–	196.1	166.2	148.6	143.4	14.3	0.53	1.3

For Taylor order $m = 8$:

i	0	1	2	3	4	5	6	7
$w\left(\mathbf{X}^{(i)}\right)$	$2*2^{-0}$	$2*2^{-1}$	$2*2^{-2}$	$2*2^{-3}$	$2*2^{-4}$	$2*2^{-5}$	$2*2^{-6}$	$2*2^{-7}$
$\mathcal{H}_{TM}$	$9E-1$	$1E-1$	$2E-2$	$5E-3$	$1E-3$	$3E-4$	$7E-5$	$1E-5$
$\mathcal{H}_{LR}$	$1E-2$	$2E-5$	$*$	$*$	$*$	$*$	$*$	$*$
$\mathcal{H}_{TB}$	$1E-2$	$2E-5$	$4E-8$	$7E-11$	$1E-11$	$7E-12$	$1E-11$	$9E-12$
$\mathcal{R}^*$	–	512	512	512	512	512	512	512
$\mathcal{R}_{TM}$	–	8.1	5.1	4.4	4.2	4.1	4.1	4.0
$\mathcal{R}_{LR}$	–	599.2	–	–	–	–	–	–
$\mathcal{R}_{TB}$	–	599.2	583.4	557.5	7.2	1.4	0.6	1.3

TABLE 2.6. Overestimations and their reduction ratios for various Taylor orders obtained with Taylor model, Algorithm LR, and Algorithm TB in Example 2.6 Trigonometric $(6-dim)$.

For Taylor order $m=2$:

i	0	1	2	3	4	5	6	7
$w\left(\mathbf{X}^{(i)}\right)$	$2*2^{-0}$	$2*2^{-1}$	$2*2^{-2}$	$2*2^{-3}$	$2*2^{-4}$	$2*2^{-5}$	$2*2^{-6}$	$2*2^{-7}$
$\mathcal{H}_{TM}$	$1E+3$	$3E+2$	$6E+1$	$2E+1$	$3E+0$	$1E+0$	$2E-1$	$5E-2$
$\mathcal{H}_{LR}$	$9E+2$	$9E+1$	$9E+0$	$*$	$*$	$*$	$*$	$*$
$\mathcal{H}_{TB}$	$9E+2$	$9E+1$	$9E+0$	$1E+0$	$1E-1$	$2E-2$	$2E-3$	$2E-4$
$\mathcal{R}^*$	$-$	8	8	8	8	8	8	8
$\mathcal{R}_{TM}$	$-$	5.0	4.5	4.3	4.1	4.0	4.0	4.0
$\mathcal{R}_{LR}$	$-$	10.1	10.1	$-$	$-$	$-$	$-$	$-$
$\mathcal{R}_{TB}$	$-$	10.1	9.3	8.7	8.4	8.2	8.1	8.1

For Taylor order $m=4$:

i	0	1	2	3	4	5	6	7
$w\left(\mathbf{X}^{(i)}\right)$	$2*2^{-0}$	$2*2^{-1}$	$2*2^{-2}$	$2*2^{-3}$	$2*2^{-4}$	$2*2^{-5}$	$2*2^{-6}$	$2*2^{-7}$
$\mathcal{H}_{TM}$	$1E+3$	$3E+2$	$6E+1$	$1E+1$	$3E+0$	$1E+0$	$2E-1$	$5E-2$
$\mathcal{H}_{LR}$	$4E+1$	$7E-1$	$*$	$*$	$*$	$*$	$*$	$*$
$\mathcal{H}_{TB}$	$4E+1$	$7E-1$	$1E-2$	$2E-4$	$5E-6$	$2E-7$	$3E-9$	$1E-10$
$\mathcal{R}^*$	-	32	32	32	32	32	32	32
$\mathcal{R}_{TM}$	-	5.0	4.5	4.2	4.1	4.1	4.0	4.0
$\mathcal{R}_{LR}$	$-$	61.3	$-$	$-$	$-$	$-$	$-$	$-$
$\mathcal{R}_{TB}$	$-$	61.3	56.9	51.6	46.2	41.2	37.1	28.5

Table 2.6 (Contd.) For Taylor order $m = 6$:

i	0	1	2	3	4	5	6	7
$w\left(\mathbf{X}^{(i)}\right)$	$2*2^{-0}$	$2*2^{-1}$	$2*2^{-2}$	$2*2^{-3}$	$2*2^{-4}$	$2*2^{-5}$	$2*2^{-6}$	$2*2^{-7}$
$\mathcal{H}_{TM}$	$1E+3$	$3E+2$	$6E+1$	$1E+1$	$3E+0$	$1E+0$	$2E-1$	$5E-2$
$\mathcal{H}_{LR}$	$7E+0$	$4E-2$	$*$	$*$	$*$	$*$	$*$	$*$
$\mathcal{H}_{TB}$	$7E+0$	$4E-2$	$2E-4$	$2E-6$	$1E-8$	$1E-10$	$3E-11$	$3E-11$
$\mathcal{R}^*$	–	128	128	128	128	128	128	128
$\mathcal{R}_{TM}$	–	5.0	4.5	4.3	4.1	4.1	4.0	4.0
$\mathcal{R}_{LR}$	–	176.4	–	–	–	–	–	–
$\mathcal{R}_{TB}$	–	176.4	157.8	144.9	136.8	103.7	4.4	1.0

For Taylor order $m = 8$:

i	0	1	2	3	4	5	6	7
$w\left(\mathbf{X}^{(i)}\right)$	$2*2^{-0}$	$2*2^{-1}$	$2*2^{-2}$	$2*2^{-3}$	$2*2^{-4}$	$2*2^{-5}$	$2*2^{-6}$	$2*2^{-7}$
$\mathcal{H}_{TM}$	$1E+3$	$3E+2$	$6E+1$	$1E+0$	$3E+0$	$1E+0$	$2E-1$	$5E-2$
$\mathcal{H}_{LR}$	$1E-1$	$2E-4$	$*$	$*$	$*$	$*$	$*$	$*$
$\mathcal{H}_{TB}$	$1E-1$	$2E-4$	$2E-7$	$3E-10$	$2E-11$	$3E-11$	$3E-11$	$3E-11$
$\mathcal{R}^*$	–	512	512	512	512	512	512	512
$\mathcal{R}_{TM}$	–	5.0	4.5	4.3	4.1	4.1	4.0	4.0
$\mathcal{R}_{LR}$	–	931.7	–	–	–	–	–	–
$\mathcal{R}_{TB}$	–	931.7	852.5	700.7	10.7	0.9	0.8	1.2

TABLE 2.7. Average execution times with various algorithms. The time is in seconds, unless otherwise stated. The average is taken over all i, and all Taylor orders m. Note that with Algorithm LR, mostly only one subdivision ($i = 1$) is found possible in the problems.

			Average Execution Time		
Example	Name	dim	Taylor model	Algorithm LR	Algorithm TB
2.1	Gritton	1	0.01	0.07	0.04
2.2	Jennrich & Sampson	2	0.11	0.01	0.15
2.3	Levy	3	0.12	0.09	0.20
2.4	Trigonometric	4	0.12	0.35	3.60
2.5	Griewank	5	0.20	3	183
2.6	Trigonometric	6	0.23	6	3 hours

3

Global optimization using the improved Taylor-Bernstein form

3.1 Introduction

Let $f : \mathbf{X} \subseteq \Re^l \to \Re$ be a $m+1$ times differentiable function for some positive integer m. We seek global optimization algorithms that are able to efficiently determine arbitrarily good lower bounds for the minimum of $\bar{f}\,(\mathbf{X})$.

Many algorithms based on interval analysis (IA) are available to solve this global optimization problem, see for example, [13], [17], [31] and the references cited therein. IA methods are usually based on branch and bound techniques, that is, they start from the initial box $\mathbf{X}$, subdivide $\mathbf{X}$ and store the subboxes in a list, discarding subboxes which are guaranteed not to contain a global minimizer until the desired accuracy in terms of the width of the intervals in the list is achieved. A basic branch and bound algorithm of IA is the Moore-Skelboe (MS) algorithm [31]. Although the MS algorithm is reliable, it is somewhat slow to converge in 'difficult' problems, when inclusion functions of first and sometimes even second orders are used. Faster convergence could possibly be obtained with higher order inclusion functions, and it is of interest in this work to investigate their effectiveness in some such 'difficult' problems.

Our proposed algorithm for global optimization uses the TB form proposed in Chapter 2 as an inclusion function form for the objective function f in the MS algorithm. As the TB form is an inclusion function form having high order convergence property, we expect to obtain faster convergence with this form. The TB form also allows us to make the cut-off test and termination condition more effective, and we incorporate these modifications in the proposed algorithm.

We can also have the Taylor model of Berz *et al.* as an inclusion function form in the MS algorithm as done, for instance, in the preliminary work in [18]. We call such an algorithm as Algorithm TMS below. We also test and compare the performance of the proposed algorithm with that of Algorithms TMS and MS on six 'difficult' examples.

3.2 Background

3.2.1 Algorithm MS

We first outline the well-known MS algorithm of IA. Actually, the algorithm below is the MS algorithm augmented with the monotonicity test and the cut-off test of Ichida and Fujii [16]. However, for convenience we refer to it as just the MS algorithm.

MS Algorithm for Global Optimization [31]

Inputs: The box $\mathbf{X}$, natural inclusion functions [24] F and F' for the function f and its Jacobian, respectively, and an accuracy parameter ε.

Output: A lower bound, of accuracy ε, on the global minimum of f over $\mathbf{X}$. This lower bound is output as the value of variable y in the last but one step below.

BEGIN Algorithm

1. Set $\mathbf{Y} = \mathbf{X}$.

2. Calculate $F(\mathbf{Y})$.

3. Set $y = \min F(\mathbf{Y})$.

4. Initialize the list $L = ((\mathbf{Y}, y))$ and the cut-off value $z = \max F(\mathbf{Y})$.

5. Choose a coordinate direction k parallel to which $\mathbf{Y}$ has an edge of maximum length[1], i.e., choose k as
$$k = \{i : w(\mathbf{Y}) = w(\mathbf{Y}_i)\}$$

6. Bisect $\mathbf{Y}$ in direction k getting boxes $\mathbf{V}^1$ and $\mathbf{V}^2$ such that $\mathbf{Y} = \mathbf{V}^1 \bigcup \mathbf{V}^2$.

7. Monotonicity test (see Remark 3.1): discard any box $\mathbf{V}^i$ if $0 \notin F_j'(\mathbf{V}^i)$ for any $j \in \{1, 2, ..., l\}$ and $i = 1, 2$.

8. Calculate $F(\mathbf{V}^1)$ and $F(\mathbf{V}^2)$.

[1] For other bisection strategies that have often been found more efficient, see, for instance, [7]. The same remark also holds for the bisection step in Algorithms TMS and TBMS described in the sequel.

9. Set $v^i = \min F\left(\mathbf{V}^i\right)$ for $i = 1, 2$.

10. Update the cut-off value z as

$$z = \min\left\{z, \max F\left(\mathbf{V}^1\right), \max F\left(\mathbf{V}^2\right)\right\}$$

11. Remove $(\mathbf{Y}, y)$ from the list L.

12. Add the pairs $\left(\mathbf{V}^1, v^1\right), \left(\mathbf{V}^2, v^2\right)$ to the list L such that the second members of all pairs of the list do not decrease.

13. Cut-off test: discard from the list all pairs whose second members are greater than z.

14. Denote the first pair of the list by $(\mathbf{Y}, y)$.

15. If the width of $F(\mathbf{Y})$ is less than ε, then print y and EXIT algorithm.

16. Go to step 5.

END Algorithm

The first pair $(\mathbf{Y}, y)$ of the list in each algorithmic iteration is called the leading pair, and $\mathbf{Y}$ the leading box.

Remark 3.1 *In the monotonicity test if $0 \notin F_j^{'}(\mathbf{V}^i)$ then the interior of $\mathbf{V}^i$ cannot contain a global minimizer. The edge of $\mathbf{V}^i$ still can contain global minimizer if that part of the edge which has the smallest function values is also part of the edge of $\mathbf{X}$. Otherwise, no global minimizer lies in $\mathbf{V}^i$. For details, see [31].*

3.2.2 Algorithm TMS

In this algorithm, we simply use the Taylor model of Berz *et al.* [3] as an inclusion function form for the objective function f in Algorithm MS. As this involves using **T**aylor model in the **M**oore-**S**kelboe algorithm, we call it as Algorithm TMS. Algorithm TMS is not new in the literature, and has been proposed and investigated, for instance, in [18].

3.3 Proposed optimization Algorithm TBMS

Consider the leading box $\mathbf{Y}$ in a given iteration of the MS algorithm, and apply Algorithm TB for finding an enclosure of $\overline{f}(\mathbf{Y})$ using $F_{TB}(\mathbf{Y})$. From (2.11) and Theorem 2.5,

$$\overline{f}(\mathbf{Y}) \subseteq F_{TB}(\mathbf{Y}) = \bar{p}(\mathbf{Y}) + R(\mathbf{Y}) = [\min \bar{p}(\mathbf{Y}), \max \bar{p}(\mathbf{Y})] + [\min R(\mathbf{Y}), \max R(\mathbf{Y})]$$

or

$$\overline{f}(\mathbf{Y}) \subseteq [\min \bar{p}(\mathbf{Y}) + \min R(\mathbf{Y}), \max \bar{p}(\mathbf{Y}) + \max R(\mathbf{Y})] \tag{3.1}$$

Since we obtain the exact range of $\bar{p}(\mathbf{Y})$ in Algorithm TB, we can also construct the so-called inner enclosure of $\overline{f}(\mathbf{Y})$ as

$$[\min \bar{p}(\mathbf{Y}) + \max R(\mathbf{Y}), \max \bar{p}(\mathbf{Y}) + \min R(\mathbf{Y})] \subseteq \overline{f}(\mathbf{Y}) \tag{3.2}$$

>From (3.1) and (3.2),

$$\begin{aligned} & [\min \bar{p}(\mathbf{Y}) + \max R(\mathbf{Y}), \max \bar{p}(\mathbf{Y}) + \min R(\mathbf{Y})] \\ \subseteq \quad & \overline{f}(\mathbf{Y}) \subseteq [\min \bar{p}(\mathbf{Y}) + \min R(\mathbf{Y}), \max \bar{p}(\mathbf{Y}) + \max R(\mathbf{Y})] \end{aligned}$$

which implies

$$\min \bar{p}(\mathbf{Y}) + \min R(\mathbf{Y}) \quad \leq \quad \min \overline{f}(\mathbf{Y}) \quad \leq \quad \min \bar{p}(\mathbf{Y}) + \max R(\mathbf{Y}) \tag{3.3}$$

Therefore, with F_{TB} as an inclusion function, we can redefine the cut-off level in the MS algorithm as $z = \min \bar{p}(\mathbf{Y}) + \max R(\mathbf{Y})$ which is obviously smaller and hence more effective than the original cut-off level of $\max F(\mathbf{Y}) = \max \bar{p}(\mathbf{Y}) + \max R(\mathbf{Y})$. Further, the error on $\min \overline{f}(\mathbf{Y})$ is seen from the above inequality to be no greater than

$$\{\min \bar{p}(\mathbf{Y}) + \max R(\mathbf{Y})\} - \{\min \bar{p}(\mathbf{Y}) + \min R(\mathbf{Y})\} = \max R(\mathbf{Y}) - \min R(\mathbf{Y}) = w(R(\mathbf{Y}))$$

This means that using F_{TB}, we can redefine the termination condition in MS algorithm based on the width of $R(\mathbf{Y})$, which is smaller and hence more effective than the original one based on $w(F(\mathbf{Y})) = w(\bar{p}(\mathbf{Y})) + w(R(\mathbf{Y}))$.

Based on these ideas, we make the following modifications to the MS algorithm:

1. The TB form F_{TB} is used as an inclusion function form for f. Using this form, an enclosure of the range of f over a given box can be obtained using Algorithm TB.

2. The cut-off value is now defined as $z = \min \bar{p}(\mathbf{Y}) + \max R(\mathbf{Y})$.

3. The termination criterion is modified, based on the width of the remainder interval $R(\mathbf{Y})$.

Since this global optimization algorithm involves using **T**aylor - **B**ernstein form in **M**oore-**S**kelboe algorithm, we call it as Algorithm TBMS.

Algorithm TBMS

Inputs: The box $\mathbf{X}$, order m of the Taylor form to be used, natural inclusion function F' for the Jacobian of the function f, and an accuracy parameter ε.

Output: A lower bound, of accuracy ε, on the global minimum of f over $\mathbf{X}$. This lower bound is output as the value of variable y in the last but one step below.

BEGIN Algorithm

1. Set $\mathbf{Y} = \mathbf{X}$.

2. Calculate $F_{TB}(\mathbf{Y})$ using Algorithm TB : $[F_{TB}(\mathbf{Y}), \bar{p}(\mathbf{Y}), R(\mathbf{Y})] = TB(\mathbf{Y}, f, m)$

3. Set $y = \min F_{TB}(\mathbf{Y})$.

4. Initialize the list $L = ((\mathbf{Y}, y))$ and the cut-off value z as
$$z = \min \bar{p}(\mathbf{Y}) + \max R(\mathbf{Y})$$

5. Choose a coordinate direction k parallel to which $\mathbf{Y}$ has an edge of maximum length, i.e., choose k as
$$k = \{i : w(\mathbf{Y}) = w(\mathbf{Y}_i)\}$$

6. Bisect $\mathbf{Y}$ in direction k getting boxes $\mathbf{V}^1$ and $\mathbf{V}^2$ such that $\mathbf{Y} = \mathbf{V}^1 \bigcup \mathbf{V}^2$.

7. Monotonicity test (see Remark 3.1): discard any box $\mathbf{V}^i$ if $0 \notin F_j^{'}(\mathbf{V}^i)$ for any $j \in \{1, 2, ..., l\}$ and $i = 1, 2$.

8. Calculate $F_{TB}\left(\mathbf{V}^1\right)$ and $F_{TB}\left(\mathbf{V}^2\right)$ using Algorithm TB.

9. Set $v^i = \min F\left(\mathbf{V}^i\right)$ for $i = 1, 2$.

10. Update the cut-off value z as
$$z = \min\left\{z, \min \bar{p}\left(\mathbf{V}^1\right) + \max R\left(\mathbf{V}^1\right), \min \bar{p}\left(\mathbf{V}^2\right) + \max R\left(\mathbf{V}^2\right)\right\}$$

11. Remove $(\mathbf{Y}, y)$ from the list L.

12. Add the pairs $\left(\mathbf{V}^1, v^1\right), \left(\mathbf{V}^2, v^2\right)$ to the list L such that the second members of all pairs of the list do not decrease.

13. Cut-off test: discard from the list all pairs whose second members are greater than z.

14. Denote the first pair of the list by $(\mathbf{Y}, y)$.

15. If the width of $R(\mathbf{Y})$ is less than ε, then print y and EXIT algorithm.

16. Go to step 5.

END Algorithm

The convergence properties of Algorithm TMS as well as that of Algorithm TBMS follow immediately from the convergence results for inclusion functions of higher order in the MS algorithm, as given by Moore and Ratschek in [25] and Ratschek in [29].

3.4 Numerical tests

We test and compare the performances of Algorithms TBMS, TMS, and MS on some examples. We use two values of accuracy, $\varepsilon = 10^{-03}$ and 10^{-05}. We choose the following measures for the tests: number of iterations, computational time, space-complexity in terms of maximum list length, and the final list length. For all our computations, we use a PC/Pentium III 800 MHz 256 MB RAM machine with a FORTRAN 90 compiler, and version 8.1 of the COSY-INFINITY package of Berz *et al. [2], [15].*

Example 3.1 *Gritton's second problem in Chemical Engineering [18]: The function is*

$$\begin{aligned} f(x) &= -371.93625 - 791.2465656 * x + 4044.944143 * x^2 + 978.1375167 * x^3 \\ &\quad -16547.8928 * x^4 + 22140.72827 * x^5 - 9326.549359 * x^6 - 3518.536872 * x^7 \\ &\quad +4782.532296 * x^8 - 1281.47944 * x^9 - 283.4435875 * x^{10} + 202.6270915 * x^{11} \\ &\quad -16.17913459 * x^{12} - 8.88303902 * x^{13} + 1.575580173 * x^{14} + 0.1245990848 * x^{15} \\ &\quad -0.03589148622 * x^{16} - 0.0001951095576 * x^{17} + 0.0002274682229 * x^{18} \end{aligned}$$

This is a unidimensional problem, with $l = 1$. We take the initial domain as $\mathbf{X} = ([1, 2])$.

Algorithm MS is unable to provide a solution, even after 1 hour, and is therefore aborted. The performances of Algorithms TMS and TBMS are given in the below Table[2].

[2] In the Tables to follow, m denotes Taylor order, ε - accuracy, It - iterations, t - time in seconds, MLL - maximum list length, FLL - final list length. Moreover, a starred entry denotes that the algorithm is aborted due to excessive computation time taken (greater than an hour).

		TBMS				TMS			
m	ε	It	t	MLL	FLL	It	t	MLL	FLL
2	10^{-03}	50	0.41	37	1	58	0.32	47	2
	10^{-05}	52	0.43	37	1	64	0.34	47	1
4	10^{-03}	7	0.08	6	1	18	0.14	8	2
	10^{-05}	9	0.11	6	1	24	0.18	8	1
6	10^{-03}	3	0.06	2	1	17	0.17	7	2
	10^{-05}	4	0.06	2	1	23	0.21	7	1
8	10^{-03}	2	0.04	2	1	17	0.20	7	2
	10^{-05}	3	0.06	2	1	23	0.26	7	1

Both Algorithms TMS and TBMS are able to find the global minimum fairly quickly, as $-0.11811......$ It may be noted that Kearfott and Arazyan report in [18] that the software GLOBSOL had some difficulty in tackling this problem.

Example 3.2 *Jennrich and Sampson function [26, problem 6]. The two dimensional function is*

$$f(x) = \sum_{i=1}^{10} f_i(x)^2, \quad f_i(x) = 2 + 2i - (\exp(ix_1) + \exp(ix_2))$$

We take the initial domain as $\mathbf{X} = ([-1, 1], [-1, 1])$.

The performances of the various Algorithms are as under.

		TBMS				TMS			
m	ε	It	t	MLL	FLL	It	t	MLL	FLL
2	10^{-03}	136	1.75	18	3	354	1.56	32	23
	10^{-05}	141	2.00	18	2	432	2.00	32	24
4	10^{-03}	62	1.20	14	1	349	3.11	32	23
	10^{-05}	65	1.45	14	1	427	3.95	32	24
6	10^{-03}	53	1.66	14	1	349	6.24	32	23
	10^{-05}	55	1.81	14	1	427	7.84	32	24
8	10^{-03}	29	1.66	10	1	349	11.35	32	23
	10^{-05}	31	1.83	10	1	427	14.12	32	24

MS				
ε	It	t	MLL	FLL
10^{-03}	1443	6.71	81	42
10^{-05}	1961	10.20	81	37

The global minima is found in each case as 124.36218.....

Example 3.3 *Bard function [26, problem 8]. The three dimensional function is*

$$f(x) = \sum_{i=1}^{15} f_i(x)^2, \quad f_i(x) = y_i - \left(x_1 + \frac{u_i}{v_i x_2 + w_i x_3}\right), u_i = i, v_i = 16 - i, w_i = \min(u_i, v_i)$$

where, the values of y_i for $i = 1, \ldots, 15$ are given in the cited paper. We take the initial domain as $\mathbf{X} = ([-0.25, 0.25], [0.01, 2.5], [0.01, 2.5])$.

The performances of the various Algorithms are as under.

		TBMS				TMS			
m	ε	It	t	MLL	FLL	It	t	MLL	FLL
2	10^{-03}	406	16.64	74	45	3145	76.13	822	772
	10^{-05}	520	32.13	74	7	$*$	$>$ 3600	$*$	$*$
4	10^{-03}	191	35.00	38	7	3124	86.13	818	772
	10^{-05}	202	60.65	38	1	$*$	$>$ 3600	$*$	$*$
6	10^{-03}	162	67.80	38	2	3123	122.81	818	772
	10^{-05}	165	90.22	38	1	$*$	$>$ 3600		
8	10^{-03}	157	79.90	38	2	3122	181.05	818	772
	10^{-05}	159	92.03	38	1	$*$	$>$ 3600	$*$	$*$

MS				
ε	It	t	MLL	FLL
10^{-03}	6122	466.56	1643	1622
10^{-05}	$*$	$>$ 3600	$*$	$*$

The global minima found using each of the algorithms is 8.21487....$E - 03$.

Example 3.4 *Multidimensional function of Makino and Berz [21, first example]. The function is*

$$\begin{aligned} f(x) &= \frac{4\tan(3x_2)}{3x_1 + x_1\sqrt{\frac{6x_1}{-7(x_1-8)}}} - 120 - 2x_1 - 7x_3(1 + 2x_2) - \\ &\quad \sinh(0.5 + \frac{6x_2}{8x_2 + 7}) + \frac{(3x_2 + 13)^2}{3x_3} - 20x_3(2x_3 - 5) + \frac{5x_1 \tanh(0.9x_3)}{\sqrt{5x_2}} - 20x_2 \sin(3x_3) \end{aligned}$$

This is a three dimensional problem with $l = 3$. We take the initial domain as given in the paper cited, i.e.,

$$\mathbf{X} = ([1.95, 2.05], [0.95, 1.05], [0.95, 1.05])$$

Algorithm MS is unable to provide a solution even after 1 hour, and is aborted. The performances of Algorithms TMS and TBMS are as under.

		TBMS				TMS			
m	ε	It	t	MLL	FLL	It	t	MLL	FLL
2	10^{-03}	5	0.12	2	1	44	0.17	15	12
	10^{-05}	15	0.30	2	1	64	0.30	15	11
4	10^{-03}	0	0.04	1	1	44	0.47	15	12
	10^{-05}	2	0.07	2	1	64	0.64	15	11
6	10^{-03}	0	0.04	1	1	44	0.95	15	12
	10^{-05}	0	0.05	1	1	64	1.35	15	11
8	10^{-03}	0	0.06	1	1	44	1.75	15	12
	10^{-05}	0	0.08	1	1	64	2.46	15	11

The global minimum is given in the cited paper as $-2.31166.....$ Algorithms TMS and TBMS are able to find this global minimum successfully and quickly.

Example 3.5 *Brown and Dennis function [26, problem 16]. The function is*

$$f(x) = \sum_{i=1}^{20} f_i(x)^2, \quad f_i(x) = (x_1 + t_i x_2 - \exp(t_i))^2 + (x_3 + x_4 \sin(t_i) - \cos(t_i))^2, \quad t_i = \frac{i}{5}$$

This is a $4-$*dimensional problem. Following [12], we take the initial domain as*

$$\mathbf{X} = ([-10, 0, -100, -20], [100, 15, 0, 0.2])$$

Algorithm MS is unable to provide a solution, even after 1 hour, and is aborted. The performances of Algorithms TMS and TBMS are as under.

		TBMS				TMS			
m	ε	It	t	MLL	FLL	It	t	MLL	FLL
2	10^{-03}	250	35.95	23	1	418	2.53	40	18
	10^{-05}	259	85.18	23	1	476	3.09	45	24
4	10^{-03}	66	7.08	15	1	397	4.38	40	18
	10^{-05}	66	7.46	15	1	455	5.35	45	24
6	10^{-03}	47	29.67	15	1	397	5.37	40	18
	10^{-05}	47	31.23	15	1	455	6.64	45	24
8	10^{-03}	40	113.36	15	1	397	6.67	40	18
	10^{-05}	40	116.57	15	1	455	8.14	45	24

The global minimum is given in the above cited paper as 88860.47976.... Algorithms TMS and TBMS are able to find this global minimum successfully and fairly quickly, for $m \geq 4$.

Example 3.6 *Kowalik and Osborne function [26, problem 15]. The function is*

$$f(x) = \sum_{i=1}^{11} f_i(x)^2, \quad f_i(x) = \ y_i - \frac{x_1(u_i^2 + u_i x_2)}{(u_i^2 + u_i x_3 + x_4)}$$

where, the values of y_i and u_i for $i = 1, \ldots, 11$ are given in the cited paper. We take the initial domain as

$$\mathbf{X} = ([0.1, 0.2], [0.1, 0.2], [0.1, 0.2], [0.1, 0.2])$$

The performances of the various Algorithms are as under.

		TBMS				TMS			
m	ε	It	t	MLL	FLL	It	t	MLL	FLL
2	10^{-03}	33	21.57	5	5	422	2.47	219	219
	10^{-05}	$*$	> 3600	$*$	$*$	$*$	> 3600	$*$	$*$
4	10^{-03}	6	16.49	5	5		5.60	211	211
	10^{-05}	23	571.60	14	1	$*$	> 3600	$*$	$*$
6	10^{-03}	0	7.16	1	1		12.28	211	211
	10^{-05}	8	822.73	6	2	$*$	> 3600	$*$	$*$
8	10^{-03}	0	11.60	1	1		24.31	211	211
	10^{-05}	6	1294.16	5	1	$*$	> 3600	$*$	$*$

MS				
ε	It	t	MLL	FLL
10^{-03}	204	0.25	174	173
10^{-05}	$*$	> 3600	$*$	$*$

The global minima of this function over the given domain is $1.02734E - 03$.

3.5 Discussion

Based on the results of the above tests, we make some general observations.

Algorithm MS: takes excessive computation time in all examples, except example 3.2, and for lower accuracy in examples 3.3 and 3.6. Algorithm MS generally requires more time, iterations and list lengths than Algorithms TMS and TBMS.

Algorithm TMS: gives the results in reasonable computational time in all examples for lower accuracy, but needs excessive time in two examples for higher accuracy (examples 3.3

and 3.6). The computational time increases as the order m increases. An interesting feature is that the number of iterations remains almost the same for $m \geq 4$, though in a few examples there is a drop in this number when m is increased from 2 to 4. The same holds for the space-complexity in terms of maximum list length and final list length.

Algorithm TBMS: the number of iterations decreases as the order m is increased. A considerable reduction is obtained between $m = 2$ and 4. The maximum list length also decreases considerably between $m = 2$ and 4, but decreases little thereafter. In most examples (examples 3.2, 3.4, 3.5, and 3.6), the computational time first decreases, then increases with m, with the least time required for $m = 4$.

Algorithm TBMS is much faster than Algorithm TMS in all examples, except example 3.5. The speed-up is about $3 - 4$ times in Examples 3.1, 3.2, and 3.6, and is as high as $10 - 40$ in examples 3.3 and 3.4. The speed-up gets better with accuracy. In all examples, Algorithm TBMS requires much smaller list lengths and much lesser number of iterations than Algorithm TMS.

3.6 Conclusions

In summary, the preliminary tests indicate that Algorithms TMS and TBMS are quite effective compared to Algorithm MS, for lower accuracy problems. For higher accuracy problems, Algorithm TBMS is the most effective one.

The best overall choice, in terms of the number of iterations, space-complexity, and speed seems to be Algorithm TBMS with a medium Taylor order $m = 4$.

4

A combined Taylor-Bernstein form for higher order convergence

4.1 Introduction

In Chapter 2, an improved TB form F_{TB} having the property of higher order convergence was presented. As the domains shrink to small widths, the form F_{TB} usually successfully computes the range enclosure with the property of higher order convergence, while the TB form F_{LR} of Lin and Rokne [19] 'fails' to compute the range enclosures, due to excessively high degrees of the involved Bernstein form (by 'fail' we mean that the computations are aborted because of excessive memory and /or time requirements).

On the other hand, for *large* domain widths we find that the form F_{TB} usually fails to compute the range enclosures, due to the excessive number of Bernstein subdivisions required. Now, there are many application problems, for example, global optimization problems, where the domain widths are initially quite large, but solution boxes of small widths are eventually required. For such problems, neither of these two TB forms may be really effective over the *entire* range of domain widths. However, it may be advantageous to introduce a new inclusion form that appropriately switches between these two TB forms depending on the domain widths, i.e., behaves as F_{LR} for 'large' domain widths, and as F_{TB} for 'small' domain widths. As the domain shrinks from large to small widths, the new TB form is likely to be more effective than the two existing TB forms F_{TB} and F_{LR}.

In this chapter, such a 'combined' TB form is proposed and numerically tested. In section 4.2, an algorithm to compute the combined TB form is presented. In section 4.3, the performance of the combined TB form is numerically tested and compared with those of the existing TB forms F_{TB} and F_{LR}, the Taylor model, and the simple natural inclusion form.

For the testing, six benchmark examples with dimensions varying from 1 to 6 are considered, and the higher order convergence property for orders up to 9 is examined. The results are discussed in section 4.4. The concluding remarks of the chapter are given in section 4.5.

4.2 Proposed combined TB form

In Chapter 2, we observed that typically the F_{TB} form requires excessive subdivisions for 'large' $w(\mathbf{X})$, whereas the F_{LR} form requires excessively high degrees of Bernstein form for 'small' $w(\mathbf{X})$. It may be advantageous to have a new inclusion form that switches between these two forms depending on the domain widths, i.e., behave as F_{LR} for 'large' domain widths, and as F_{TB} for 'small' domain widths.

Let D be as in (2.6) and recall that N is the tuple of maximum degrees of $\mathbf{x}$ in $p(\mathbf{x})$ given by (2.1). The basic idea of the combined form is as follows. From (2.6) and (2.7),

- for 'large' $w(\mathbf{X})$, $D << N$, so $N' = N$. Therefore, for such domain widths, it would be simpler and more efficient to use F_{LR} based on a Bernstein form of degree N, rather than F_{TB} that involves successive subdivisions till the vertex property is satisfied on every subdivision.
- for 'small' $w(\mathbf{X})$, $D >> N$, so $N' >> N$. Therefore, for such domain widths, it would be more efficient to use F_{TB} based on Bernstein form of degree N, rather than F_{LR} that involves Bernstein form of high to very high degree.

Therefore, if $N \geq D$, we invoke Algorithm LR given in section 2.2.3 except that S is used in place of S'. That is, we compute a non-sharp enclosure of the exact range of the polynomial part of Taylor expansion using Bernstein polynomials of degree N. Otherwise, we invoke Algorithm TB given in section 2.3, i.e., we compute the exact range of polynomial part of Taylor expansion using subdivision and a vertex condition check on every subdivision. The combined algorithm is called as Algorithm CTB and the resulting form as the combined TB form, denoted F_{CTB}.

As F_{CTB} uses either of the existing TB forms F_{LR} or F_{TB} to enclose the function range for any given domain width, and since F_{LR} and F_{TB} have the $(m+1)$- th convergence order property (see Chapter 2) , it follows that F_{CTB} also has the $(m+1)$- th convergence order property.

We next present Algorithm CTB to compute the combined TB form.

Algorithm CTB : $[F_{CTB}(\mathbf{X}), \bar{p}(\mathbf{X}), B^*, R(\mathbf{X}), i_f] = \mathrm{CTB}(\mathbf{X}, f, m)$

Inputs: The box $\mathbf{X}$, an expression for the function f, and the order m of Taylor form to be used.

Output: Enclosure $F_{CTB}(\mathbf{X})$ of the range of f on $\mathbf{X}$, the range $\bar{p}(\mathbf{X})$ of the polynomial part of the Taylor form of f, an enclosure B^* of the same, an enclosure $R(\mathbf{X})$ of the remainder part of the Taylor form, and a flag i_f that takes the value zero resp. unity depending on whether F_{LR} resp. F_{TB} form is used in the algorithm.

Note: Depending on whether F_{LR} resp. F_{TB} is used, the quantity $\bar{p}(\mathbf{X})$ resp. B^* is set to the empty interval.

1. For the given function f, compute Taylor coefficients of p in (2.3) in parallel with the remainder interval $R(\mathbf{X})$, using the Taylor model technique of Berz *et al.* [3].
2. Relate the obtained Taylor coefficients to those of the power form in (2.1), and denote the coefficients in this form as a_I.
3. Compute the $l-$tuple of indices D given by

$$D = (d_1, \ldots, d_l), \text{ where } d_1, \ldots, d_l \geq [1/w(\mathbf{X})]^{m+1}$$

   If $N \geq D$ then go to the following step, else go to step 8.
4. Set flag $i_f = 0$ and $\bar{p}(\mathbf{X})$ to the empty interval.
5. Find a patch $B(\mathbf{U})$ of Bernstein coefficients of p on $\mathbf{U}$ by executing Algorithm Patch:

$$B(\mathbf{U}) = \text{Patch}(\mathbf{X}, a_I)$$

   then compute an enclosure B^* for the range of $\bar{p}(\mathbf{X})$ as

$$B^* = [\min B(\mathbf{U}), \max B(\mathbf{U})]$$

6. Compute an enclosure for the range of f over $\mathbf{X}$ as

$$F_{CTB}(\mathbf{X}) = B^* + R(\mathbf{X})$$

7. Go to step 11.
8. Set flag $i_f = 1$ and B^* to the empty interval.
9. Compute the range $\bar{p}(\mathbf{X})$ using Algorithm Bounder :

$$\bar{p}(\mathbf{X}) = \text{Bounder}(\mathbf{X}, a_I)$$

10. Using $R(\mathbf{X})$ obtained in step 1 and $\bar{p}(\mathbf{X})$ obtained in above step, compute an enclosure for the range of f over $\mathbf{X}$ as

$$F_{CTB}(\mathbf{X}) = \bar{p}(\mathbf{X}) + R(\mathbf{X})$$

11. RETURN $F_{CTB}(\mathbf{X})$, $\bar{p}(\mathbf{X})$, B^*, $R(\mathbf{X})$, i_f and EXIT.

END Algorithm

4.3 Numerical tests

The performance of the proposed inclusion function form is tested on the same six benchmark examples chosen in Chapter 2. However, the centers of the domains are now taken differently in some examples. Moreover, the domains also vary from fairly large to small widths.

A PC/Pentium III 800 MHz 256 MB RAM machine with a FORTRAN 90 compiler, and version 8.1 of the COSY-INFINITY package of Berz *et al.* [15] are used for the testing.

In each example, the following are computed:

$F_{NIE}(\mathbf{X})$ - using the simple natural inclusion function.

$F_{TM}(\mathbf{X})$ - using Taylor model of Berz *et al.* [20], computed with the COSY-INFINITY package.

$F_{LR}(\mathbf{X})$ - using TB form of Lin and Rokne [19].

$F_{TB}(\mathbf{X})$ - using TB form in Chapter 2.

$F_{CTB}(\mathbf{X})$ - using proposed combined TB form.

$F_{inner}(\mathbf{X})$ - using *inner* estimates of the range computed with the well-known Moore-Skelboe optimization algorithm of interval analysis.

The examples are

Example 4.1 *Gritton's second problem in Chemical Engineering [18]: The* $1-\dim$ *function is*

$$\begin{aligned} f(x) \;=\; & -371.93625 - 791.2465656 * x + 4044.944143 * x^2 + 978.1375167 * x^3 \\ & -16547.8928 * x^4 + 22140.72827 * x^5 - 9326.549359 * x^6 - 3518.536872 * x^7 \\ & +4782.532296 * x^8 - 1281.47944 * x^9 - 283.4435875 * x^{10} + 202.6270915 * x^{11} \\ & -16.17913459 * x^{12} - 8.88303902 * x^{13} + 1.575580173 * x^{14} + 0.1245990848 * x^{15} \\ & -0.03589148622 * x^{16} - 0.0001951095576 * x^{17} + 0.0002274682229 * x^{18} \end{aligned}$$

The domain is $\mathbf{X}^{(i)} = \left[-1 + 2^{-i}[-1, 1]\right]$.

Example 4.2 *Jennrich and Sampson function [26, problem 6]. The* $2-\dim$ *function is*

$$f(x) = \sum_{i=1}^{10} f_i(x)^2, \qquad f_i(x) = 2 + 2i - (\exp(ix_1) + \exp(ix_2))$$

The domain is $\mathbf{X}^{(i)} = \left[-1 + 2^{-i}[-1, 1]\right]^2$.

Example 4.3 *Levy function [32, Problem L8, pp. 204] The 3− dim function is*

$$
\begin{aligned}
f(x) &= \sum_{i=1}^{2}(y_i-1)^2(1+10\sin^2(\pi y_{i+1}))+\sin^2(\pi y_1)+(y_3-1)^2, \\
y_i &= 1+\frac{(x_i-1)}{4},\ i=1...3
\end{aligned}
$$

The domain is $\mathbf{X}^{(i)} = \left[9.5 + 2^{-i}\left[-1,1\right]\right]^3$.

Example 4.4 *Trigonometric function [26, problem 26]. The* $4-\text{dim}$ *function is*

$$
f(x)=\sum_{i=1}^{4} f_i(x)^2, f_i(x)=4-\sum_{j=1}^{4}\cos x_j + i(1-\cos x_i)-\sin x_i
$$

The domain is $\mathbf{X}^{(i)} = \left[1.75 + 2^{-i}\left[-1,1\right]\right]^4$.

Example 4.5 *Griewank function [32, Problem Griew5, pp. 205] The* $5-\text{dim}$ *function is*

$$
f(x)=\sum_{i=1}^{5}\frac{x_i^2}{400}-\prod_{i=1}^{5}\cos\left(\frac{x_i}{\sqrt{i}}\right)+1
$$

The domain is $\mathbf{X}^{(i)} = \left[0.5 + 2^{-i}\left[-1,1\right]\right]^5$.

Example 4.6 *Trigonometric function [26, problem 26]. The* $6-\text{dim}$ *function is*

$$
f(x)=\sum_{i=1}^{6} f_i(x)^2, f_i(x)=6-\sum_{j=1}^{6}\cos x_j + i(1-\cos x_i)-\sin x_i
$$

The domain is $\mathbf{X}^{(i)} = \left[1.75 + 2^{-i}\left[-1,1\right]\right]^6$.

The results for Examples 4.1 and 4.6 with the various forms are given[1] in Tables 4.1 to 4.6. Table 4.7 gives domain width parameter i and time taken by Algorithms TB and CTB to reach an accuracy of $1E-10$ for various Taylor orders in these examples. The data for F_{LR}, F_{TM} are not given in the table because F_{LR} fails to produce results of this accuracy due to excessive memory demands, whereas F_{TM} requires the domain width parameter $i >> 7$, exceeding the scope of the present investigations.The results using the natural inclusion function for the examples are reported in Tables 4.8 to 4.13. In the Tables, the quantities $\mathcal{H}, \mathcal{R}$ are defined analogously to section 2.4. The new quantities MLL, t and SD denote the maximum list length, computational time in seconds and the number of subdivisions required in the Bernstein step. The subscripts for these quantities refer to the inclusion function form used. Note that the results in all the Tables are rounded purely for display purposes.

[1] In the Tables, a starred entry denotes that the execution is aborted due to excessive memory requirements.

4.4 Discussion

>From the results given in the Tables, we observe that

1. With the Taylor model[2] as an inclusion function form, only *quadratic* convergence is obtained in all problems, irrespective of the chosen Taylor order m. This is in line with our earlier observations given in section 2.5.

2. For large domains, the form F_{TB} requires comparatively large times and memory than F_{LR}, and fails to compute the range enclosure for higher dimensional problems; the form F_{LR} does not fail for any problem dimension. For small domains, the situation is the reverse: the form F_{LR} fails for any problem dimension, while the form F_{TB} does not fail for any problem dimension.

3. In view of the above, the practical utility of F_{LR} is severely restricted as an inclusion form for obtaining higher order convergence. Further, the form F_{TB} is not satisfactory for large domain widths. Hence, neither of existing TB forms is really satisfactory in application problems where the domain varies from large to small widths, due to any domain splitting or subdivision techniques that may be employed.

4. The proposed form F_{CTB} does not fail to compute the range enclosure for any domain width and for any problem dimension !

5. With the proposed form F_{CTB} as an inclusion function form, in problems of up to $4-$dim higher order convergence of orders up to 9 is quite easily obtained[3]. In problems of 5 and $6-$ dim, higher order convergence of orders up to 9 is again obtained; however, the computational demands are somewhat large for the $5-$ dim problem, and become excessive for the $6-$ dim one. This behavior is identical to that of F_{TB} as observed in section 2.5, and moreover is expected to be so, because F_{CTB} reduces to F_{TB} for small domains.

6. With the existing TB form F_{TB}, for large domains the maximum list length increases with the problem dimension, and becomes excessive for the $6-$ dim example. As the domain width decreases, the maximum list length for F_{TB} decreases and tends to zero. On the other hand, the maximum list length with the combined form F_{CTB} is *nil*, for any domain width and for any problem dimension !

[2] In version 8.1 of COSY-INFINITY package made available to us, the range of the polynomial part is evaluated by simple interval arithmetic, see also [18].

[3] until we have overestimations of very small magnitudes (of order of $E-10$ or less).

7. The computation time taken for F_{CTB} is smaller than that for F_{TB} by as much as $12,000$ times for large domain widths, by as much as 73 times for intermediate domain widths, and is the same for small domain widths.

8. The overestimation given by F_{CTB} is usually about the same as that given by F_{TB} - in those cases where F_{TB} succeeds in computing the range enclosures. A similar remark holds with respect to the overestimation given by F_{LR}.

9. As seen from the Table 4.7 the best performance in terms of computational time and number of iterations is given by the form F_{CTB} for Taylor order $m = 4$. This observation is in line with the findings given in section 3.6.

10. For large domain widths, the natural inclusion form F_{NIE} usually gives much less overestimation than any of the more sophisticated Taylor or TB forms. This behavior is perhaps expected. However, it is quite remarkable that the same is observed in a few problems, even for *small* domain widths.

4.5 Conclusions

In all the examples considered, the proposed combined TB form indeed numerically exhibited the higher order convergence property. The overestimation given by the combined TB form was usually about the same as that given by existing TB forms F_{TB} and F_{LR} - in all those cases where the latter forms successfully computed the range enclosures.

Whereas the existing TB forms failed to compute the range enclosures for some domain widths, the combined TB form did not fail to compute the same for any domain width and for any problem dimension (in the context of the low to medium dimensions considered for testing). The maximum list length needed by the combined TB form was *nil*, for any domain width and for any problem dimension. Moreover, for large to intermediate domain widths, the combined TB form was significantly faster than F_{TB}, by as much as $2 - 4$ orders of magnitude.

A rather surprising behavior was exhibited by the simple natural inclusion form. For small domain widths, the natural inclusion form sometimes gave much less overestimation than any of the more sophisticated Taylor or TB forms. Since computation of the natural inclusion form is relatively very cheap, the behavior suggests that it may be worthwhile computing the range enclosure with this form for *any* domain width, and then intersecting it with that found by a more sophisticated form. We pursue this point in the following chapter.

TABLE 4.1. Overestimations and their reduction ratios for various Taylor orders obtained with Taylor model, Algorithm LR, Algorithm TB and Algorithm CTB in Example 4.1 Gritton ($1-dim$).

For Taylor order $m = 2$:

i	-7	-6	-5	-4	-3	-2	-1
$w\left(\mathbf{X}^{(i)}\right)$	$2*2^7$	$2*2^6$	$2*2^5$	$2*2^4$	$2*2^3$	$2*2^2$	$2*2^1$
$\mathcal{H}_{TM}$	$2E+34$	$1E+29$	$6E+23$	$5E+18$	$2E+14$	$3E+10$	$8E+7$
$\mathcal{H}_{LR}$	$2E+34$	$1E+29$	$6E+23$	$5E+18$	$2E+14$	$3E+10$	$8E+7$
$\mathcal{H}_{TB}$	$2E+34$	$1E+29$	$6E+23$	$5E+18$	$2E+14$	$3E+10$	$8E+7$
$\mathcal{H}_{CTB}$	$2E+34$	$1E+29$	$6E+23$	$5E+18$	$2E+14$	$3E+10$	$8E+7$
$\mathcal{R}^*$	–	8	8	8	8	8	8
$\mathcal{R}_{TM}$	–	$2E+5$	$2E+5$	$1E+5$	$3E+4$	$5E+3$	426.4
$\mathcal{R}_{LR}$	–	$2E+5$	$2E+5$	$1E+5$	$3E+4$	$5E+3$	426.5
$\mathcal{R}_{TB}$	–	$2E+5$	$2E+5$	$1E+5$	$3E+4$	$5E+3$	426.5
$\mathcal{R}_{CTB}$	–	$2E+5$	$2E+5$	$1E+5$	$3E+4$	$5E+3$	426.5
t_{TM}	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$
t_{LR}	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$
t_{TB}	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$
t_{CTB}	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$
MLL_{TM}	–	–	–	–	–	–	–
MLL_{LR}	–	–	–	–	–	–	–
MLL_{TB}	3	3	3	3	3	3	3
MLL_{CTB}	0	0	0	0	0	0	0
SD_{TM}	–	–	–	–	–	–	–
SD_{LR}	–	–	–	–	–	–	–
SD_{TB}	36	35	34	33	32	31	30
SD_{CTB}	0	0	0	0	0	0	0

Table 4.1 (Contd.) For Taylor order $m = 2$:

i	0	1	2	3	4	5	6	7
$w\left(\mathbf{X}^{(i)}\right)$	$2*2^{-0}$	$2*2^{-1}$	$2*2^{-2}$	$2*2^{-3}$	$2*2^{-4}$	$2*2^{-5}$	$2*2^{-6}$	$2*2^{-7}$
$\mathcal{H}_{TM}$	$1E+6$	$1E+5$	$2E+4$	$4E+3$	$9E+2$	$2E+2$	$6E+1$	$1E+1$
$\mathcal{H}_{LR}$	$1E+6$	$8E+4$	$5E+3$	–	–	–	–	–
$\mathcal{H}_{TB}$	$1E+6$	$8E+4$	$5E+3$	$5E+2$	$5E+1$	$6E+0$	$7E-1$	$1E-1$
$\mathcal{H}_{CTB}$	$1E+6$	$8E+4$	$5E+3$	$5E+2$	$5E+1$	$6E+0$	$7E-1$	$1E-1$
$\mathcal{R}^*$	8	8	8	8	8	8	8	8
$\mathcal{R}_{TM}$	50.6	14.2	5.8	4.5	4.2	4.1	4.0	4.0
$\mathcal{R}_{LR}$	50.6	17.9	15.5	–	–	–	–	–
$\mathcal{R}_{TB}$	50.6	19.0	14.6	11.2	9.6	8.8	8.4	8.2
$\mathcal{R}_{CTB}$	50.6	19.0	14.6	11.2	9.6	8.8	8.4	8.2
t_{TM}	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$
t_{LR}	$1E-2$	$1E-2$	$1E-2$	–	–	–	–	–
t_{TB}	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$
t_{CTB}	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$
MLL_{TM}	–	–	–	–	–	–	–	–
MLL_{LR}	–	–	–	–	–	–	–	–
MLL_{TB}	3	3	0	0	0	0	0	0
MLL_{CTB}	0	0	0	0	0	0	0	0
SD_{TM}	–	–	–	–	–	–	–	–
SD_{LR}	–	–	–	–	–	–	–	–
SD_{TB}	29	28	0	0	0	0	0	0
SD_{CTB}	0	0	0	0	0	0	0	0

Table 4.1 (Contd.) For Taylor order $m = 4$:

i	-7	-6	-5	-4	-3	-2	-1
$w\left(\mathbf{X}^{(i)}\right)$	$2*2^7$	$2*2^6$	$2*2^5$	$2*2^4$	$2*2^3$	$2*2^2$	$2*2^1$
$\mathcal{H}_{TM}$	$2E+34$	$1E+29$	$6E+23$	$5E+18$	$1E+14$	$2E+10$	$5E+7$
$\mathcal{H}_{LR}$	$2E+34$	$1E+29$	$6E+23$	$5E+18$	$1E+14$	$2E+10$	$5E+7$
$\mathcal{H}_{TB}$	$2E+34$	$1E+29$	$6E+23$	$5E+18$	$1E+14$	$2E+10$	$5E+7$
$\mathcal{H}_{CTB}$	$2E+34$	$1E+29$	$6E+23$	$5E+18$	$1E+14$	$2E+10$	$5E+7$
$\mathcal{R}^*$	$-$	32	32	32	32	32	32
$\mathcal{R}_{TM}$	$-$	$2E+5$	$2E+5$	$1E+5$	$4E+4$	$6E+3$	$5E+2$
$\mathcal{R}_{LR}$	$-$	$2E+5$	$2E+5$	$1E+5$	$4E+4$	$6E+3$	$5E+2$
$\mathcal{R}_{TB}$	$-$	$2E+5$	$2E+5$	$1E+5$	$4E+4$	$6E+3$	$5E+2$
$\mathcal{R}_{CTB}$	$-$	$2E+5$	$2E+5$	$1E+5$	$4E+4$	$6E+3$	$5E+2$
t_{TM}	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$
t_{LR}	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$
t_{TB}	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$
t_{CTB}	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$
MLL_{TM}	$-$	$-$	$-$	$-$	$-$	$-$	$-$
MLL_{LR}	$-$	$-$	$-$	$-$	$-$	$-$	$-$
MLL_{TB}	5	5	5	5	5	5	4
MLL_{CTB}	0	0	0	0	0	0	0
SD_{TM}	$-$	$-$	$-$	$-$	$-$	$-$	$-$
SD_{LR}	$-$	$-$	$-$	$-$	$-$	$-$	$-$
SD_{TB}	43	39	37	35	33	31	29
SD_{CTB}	0	0	0	0	0	0	0

Table 4.1 (Contd.) For Taylor order $m = 4$:

i	0	1	2	3	4	5	6	7
$w\left(\mathbf{X}^{(i)}\right)$	$2*2^{-0}$	$2*2^{-1}$	$2*2^{-2}$	$2*2^{-3}$	$2*2^{-4}$	$2*2^{-5}$	$2*2^{-6}$	$2*2^{-7}$
$\mathcal{H}_{TM}$	$8E+5$	$8E+4$	$1E+4$	$4E+3$	$9E+2$	$2E+2$	$2E+1$	$1E+1$
$\mathcal{H}_{LR}$	$7E+5$	$2E+4$	–	–	–	–	–	–
$\mathcal{H}_{TB}$	$7E+5$	$2E+4$	$6E+2$	$2E+1$	$5E-1$	$1E-2$	$5E-4$	$1E-5$
$\mathcal{H}_{CTB}$	$7E+5$	$2E+4$	$6E+2$	$2E+1$	$5E-1$	$1E-2$	$5E-4$	$1E-5$
$\mathcal{R}^*$	32	32	32	32	32	32	32	32
$\mathcal{R}_{TM}$	57.7	10.8	5.0	4.1	4.0	4.0	4.0	4.0
$\mathcal{R}_{LR}$	71.2	38.0	–	–	–	–	–	–
$\mathcal{R}_{TB}$	77.2	38.0	35.9	33.9	32.9	32.4	32.2	32.1
$\mathcal{R}_{CTB}$	71.2	38.0	35.9	33.9	32.9	32.4	32.2	32.1
t_{TM}	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$
t_{LR}	$1E-2$	$1E-2$	–	–	–	–	–	–
t_{TB}	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$
t_{CTB}	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$
MLL_{TM}	–	–	–	–	–	–	–	–
MLL_{LR}	–	–	–	–	–	–	–	–
MLL_{TB}	3	0	0	0	0	0	0	0
MLL_{CTB}	0	0	0	0	0	0	0	0
SD_{TM}	–	–	–	–	–	–	–	–
SD_{LR}	–	–	–	–	–	–	–	–
SD_{TB}	27	0	0	0	0	0	0	0
SD_{CTB}	0	0	0	0	0	0	0	0

Table 4.1 (Contd.) For Taylor order $m = 6$:

i	-7	-6	-5	-4	-3	-2	-1
$w\left(\mathbf{X}^{(i)}\right)$	$2*2^7$	$2*2^6$	$2*2^5$	$2*2^4$	$2*2^3$	$2*2^2$	$2*2^1$
$\mathcal{H}_{TM}$	$2E+34$	$1E+29$	$6E+23$	$5E+18$	$1E+14$	$2E+10$	$3E+7$
$\mathcal{H}_{LR}$	$2E+34$	$1E+29$	$6E+23$	$5E+18$	$1E+14$	$2E+10$	$3E+7$
$\mathcal{H}_{TB}$	$2E+34$	$1E+29$	$6E+23$	$5E+18$	$1E+14$	$2E+10$	$3E+7$
$\mathcal{H}_{CTB}$	$2E+34$	$1E+29$	$6E+23$	$5E+18$	$1E+14$	$2E+10$	$3E+7$
$\mathcal{R}^*$	$-$	128	128	128	128	128	128
$\mathcal{R}_{TM}$	$-$	$2E+5$	$2E+5$	$1E+5$	$4E+4$	$6E+3$	574.0
$\mathcal{R}_{LR}$	$-$	$2E+5$	$2E+5$	$1E+5$	$4E+4$	$6E+3$	528.9
$\mathcal{R}_{TB}$	$-$	$2E+5$	$2E+5$	$1E+5$	$4E+4$	$6E+3$	619.3
$\mathcal{R}_{CTB}$	$-$	$2E+5$	$2E+5$	$1E+5$	$4E+4$	$6E+3$	619.3
t_{TM}	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$
t_{LR}	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$
t_{TB}	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$
t_{CTB}	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$
MLL_{TM}	$-$	$-$	$-$	$-$	$-$	$-$	$-$
MLL_{LR}	$-$	$-$	$-$	$-$	$-$	$-$	$-$
MLL_{TB}	5	5	5	5	5	5	4
MLL_{CTB}	0	0	0	0	0	0	0
SD_{TM}	$-$	$-$	$-$	$-$	$-$	$-$	$-$
SD_{LR}	$-$	$-$	$-$	$-$	$-$	$-$	$-$
SD_{TB}	41	39	36	36	34	32	30
SD_{CTB}	0	0	0	0	0	0	0

Table 4.1 (Contd.) For Taylor order $m = 6$:

i	0	1	2	3	4	5	6	7
$w\left(\mathbf{X}^{(i)}\right)$	$2*2^{-0}$	$2*2^{-1}$	$2*2^{-2}$	$2*2^{-3}$	$2*2^{-4}$	$2*2^{-5}$	$2*2^{-6}$	$2*2^{-7}$
$\mathcal{H}_{TM}$	$7E+5$	$7E+4$	$1E+4$	$4E+3$	$9E+2$	$2E+2$	$6E+1$	$1E+1$
$\mathcal{H}_{LR}$	$5E+5$	$3E+2$	–	–	–	–	–	–
$\mathcal{H}_{TB}$	$1E+5$	$3E+2$	$2E+0$	$1E-2$	$8E-5$	$6E-7$	$5E-9$	$2E-10$
$\mathcal{H}_{CTB}$	$5E+5$	$3E+2$	$2E+0$	$1E-2$	$8E-5$	$6E-7$	$5E-9$	$2E-10$
$\mathcal{R}^*$	128	128	128	128	128	128	128	128
$\mathcal{R}_{TM}$	41.8	9.9	5.0	4.1	4.0	4.0	4.0	4.0
$\mathcal{R}_{LR}$	71.1	1375.0	–	–	–	–	–	–
$\mathcal{R}_{TB}$	286.8	291.0	179.6	158.8	145.3	137.2	128.5	22.0
$\mathcal{R}_{CTB}$	71.1	1375.0	179.6	158.8	145.3	137.2	128.5	22.0
t_{TM}	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$
t_{LR}	$1E-2$	$1E-2$	–	–	–	–	–	–
t_{TB}	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$
t_{CTB}	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$
MLL_{TM}	–	–	–	–	–	–	–	–
MLL_{LR}	–	–	–	–	–	–	–	–
MLL_{TB}	3	0	0	0	0	0	0	0
MLL_{CTB}	0	0	0	0	0	0	0	0
SD_{TM}	–	–	–	–	–	–	–	–
SD_{LR}	–	–	–	–	–	–	–	–
SD_{TB}	28	0	0	0	0	0	0	0
SD_{CTB}	0	0	0	0	0	0	0	0

Table 4.1 (Contd.) For Taylor order $m = 8$:

i	-7	-6	-5	-4	-3	-2	-1
$w\left(\mathbf{X}^{(i)}\right)$	$2*2^7$	$2*2^6$	$2*2^5$	$2*2^4$	$2*2^3$	$2*2^2$	$2*2^1$
$\mathcal{H}_{TM}$	$2E+34$	$1E+29$	$5E+23$	$4E+18$	$1E+14$	$1E+10$	$2E+7$
$\mathcal{H}_{LR}$	$2E+34$	$1E+29$	$5E+23$	$4E+18$	$1E+14$	$1E+10$	$1E+7$
$\mathcal{H}_{TB}$	$2E+34$	$1E+29$	$5E+23$	$4E+18$	$1E+14$	$1E+10$	$1E+7$
$\mathcal{H}_{CTB}$	$2E+34$	$1E+29$	$5E+23$	$4E+18$	$1E+14$	$1E+10$	$1E+7$
$\mathcal{R}^*$	$-$	256	256	256	256	256	256
$\mathcal{R}_{TM}$	$-$	$2E+5$	$2E+5$	$1E+5$	$4E+4$	$7E+3$	$6E+2$
$\mathcal{R}_{LR}$	$-$	$2E+5$	$2E+5$	$1E+5$	$4E+4$	$7E+3$	$1E+3$
$\mathcal{R}_{TB}$	$-$	$2E+5$	$2E+5$	$1E+5$	$4E+5$	$7E+3$	$1E+3$
$\mathcal{R}_{CTB}$	$-$	$2E+5$	$2E+5$	$1E+5$	$4E+5$	$7E+3$	$1E+3$
t_{TM}	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$
t_{LR}	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$
t_{TB}	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$
t_{CTB}	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$
MLL_{TM}	$-$	$-$	$-$	$-$	$-$	$-$	$-$
MLL_{LR}	$-$	$-$	$-$	$-$	$-$	$-$	$-$
MLL_{TB}	5	5	5	5	4	5	3
MLL_{CTB}	0	0	0	0	0	0	0
SD_{TM}	$-$	$-$	$-$	$-$	$-$	$-$	$-$
SD_{LR}	$-$	$-$	$-$	$-$	$-$	$-$	$-$
SD_{TB}	40	38	36	33	31	29	2
SD_{CTB}	0	0	0	0	0	0	0

Table 4.1 (Contd.) For Taylor order $m = 8$:

i	0	1	2	3	4	5	6	7
$w\left(\mathbf{X}^{(i)}\right)$	$2*2^{-0}$	$2*2^{-1}$	$2*2^{-2}$	$2*2^{-3}$	$2*2^{-4}$	$2*2^{-5}$	$2*2^{-6}$	$2*2^{-7}$
$\mathcal{H}_{TM}$	$7E+5$	$7E+4$	$1E+4$	$4E+3$	$9E+2$	$2E+2$	$6E+1$	$1E+1$
$\mathcal{H}_{LR}$	$2E+5$	$3E+0$	–	–	–	–	–	–
$\mathcal{H}_{TB}$	$2E+4$	$3E+0$	$4E-3$	$6E-6$	$1E-8$	$2E-10$	$2E-10$	$2E-10$
$\mathcal{H}_{CTB}$	$2E+5$	$3E+0$	$4E-3$	$6E-6$	$1E-8$	$2E-10$	$2E-10$	$2E-10$
$\mathcal{R}^*$	512	512	512	512	512	512	512	512
$\mathcal{R}_{TM}$	35.8	9.3	4.9	4.1	4.0	4.0	4.0	4.0
$\mathcal{R}_{LR}$	69.0	$6E+4$	–	–	–	–	–	–
$\mathcal{R}_{TB}$	619.3	5170.6	805.8	671.7	587.7	518.4	12.1	0.9
$\mathcal{R}_{CTB}$	69.0	$6E+4$	805.8	671.7	587.7	518.4	12.1	0.9
t_{TM}	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$
t_{LR}	$1E-2$	$1E-2$	–	–	–	–	–	–
t_{TB}	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$
t_{CTB}	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$
MLL_{TM}	–	–	–	–	–	–	–	–
MLL_{LR}	–	–	–	–	–	–	–	–
MLL_{TB}	3	0	0	0	0	0	0	0
MLL_{CTB}	0	0	0	0	0	0	0	0
SD_{TM}	–	–	–	–	–	–	–	–
SD_{LR}	–	–	–	–	–	–	–	–
SD_{TB}	27	0	0	0	0	0	0	0
SD_{CTB}	0	0	0	0	0	0	0	0

TABLE 4.2. Overestimations and their reduction ratios for various Taylor orders obtained with Taylor model, Algorithm LR, Algorithm TB and Algorithm CTB in Example 4.2 Jennrich and Sampson $(2-dim)$.

For Taylor order $m = 2$:

i	-7	-6	-5	-4	-3	-2	-1
$w\left(\mathbf{X}^{(i)}\right)$	$2*2^7$	$2*2^6$	$2*2^5$	$2*2^4$	$2*2^3$	$2*2^2$	$2*2^1$
$\mathcal{H}_{TM}$	$-$	$-$	$2E+283$	$4E+142$	$2E+71$	$5E+34$	$4E+15$
$\mathcal{H}_{LR}$	$-$	$-$	$2E+283$	$4E+142$	$2E+71$	$5E+34$	$4E+15$
$\mathcal{H}_{TB}$	$-$	$-$	$2E+283$	$4E+142$	$2E+71$	$5E+34$	$4E+15$
$\mathcal{H}_{CTB}$	$-$	$-$	$2E+283$	$4E+142$	$2E+71$	$5E+34$	$4E+15$
$\mathcal{R}^*$	$-$	$-$	$-$	8	8	8	8
$\mathcal{R}_{TM}$	$-$	$-$	$-$	$6E+140$	$2E+71$	$4E+36$	$1E+19$
$\mathcal{R}_{LR}$	$-$	$-$	$-$	$6E+140$	$2E+71$	$4E+36$	$1E+19$
$\mathcal{R}_{TB}$	$-$	$-$	$-$	$6E+140$	$2E+71$	$4E+36$	$1E+19$
$\mathcal{R}_{CTB}$	$-$	$-$	$-$	$6E+140$	$2E+71$	$4E+36$	$1E+19$
t_{TM}	$-$	$-$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$
t_{LR}	$-$	$-$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$
t_{TB}	$-$	$-$	$3E-2$	$3E-2$	$2E-2$	$2E-2$	$1E-2$
t_{CTB}	$-$	$-$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$
MLL_{TM}	$-$	$-$	$-$	$-$	$-$	$-$	$-$
MLL_{LR}	$-$	$-$	$-$	$-$	$-$	$-$	$-$
MLL_{TB}	$-$	$-$	10	10	10	10	10
MLL_{CTB}	$-$	$-$	0	0	0	0	0
SD_{TM}	$-$	$-$	$-$	$-$	$-$	$-$	$-$
SD_{LR}	$-$	$-$	$-$	$-$	$-$	$-$	$-$
SD_{TB}	$-$	$-$	237	231	225	219	213
SD_{CTB}	$-$	$-$	0	0	0	0	0

Table 4.2 (Contd.) For Taylor order $m = 2$.

i	0	1	2	3	4	5	6	7
$w\left(\mathbf{X}^{(i)}\right)$	$2*2^{-0}$	$2*2^{-1}$	$2*2^{-2}$	$2*2^{-3}$	$2*2^{-4}$	$2*2^{-5}$	$2*2^{-6}$	$2*2^{-7}$
$\mathcal{H}_{TM}$	$3E+5$	$1E+2$	$4E+0$	$5E-1$	$1E-1$	$3E-2$	$7E-3$	$2E-3$
$\mathcal{H}_{LR}$	$3E+5$	$1E+2$	$2E+0$	–	–	–	–	–
$\mathcal{H}_{TB}$	$3E+5$	$1E+2$	$2E+0$	$8E-2$	$5E-3$	$3E-4$	$2E-5$	$2E-6$
$\mathcal{H}_{CTB}$	$3E+5$	$1E+2$	$2E+0$	$8E-2$	$5E-3$	$3E-4$	$2E-5$	$2E-6$
$\mathcal{R}^*$	8	8	8	8	8	8	8	8
$\mathcal{R}_{TM}$	$1E+10$	$3E+3$	25.6	6.7	4.5	4.1	4.0	4.0
$\mathcal{R}_{LR}$	$1E+10$	$3E+3$	50.2	–	–	–	–	–
$\mathcal{R}_{TB}$	$1E+10$	$3E+3$	48.3	22.6	17.1	14.6	12.7	11.1
$\mathcal{R}_{CTB}$	$1E+10$	$3E+3$	48.3	22.6	17.1	14.6	12.7	11.1
t_{TM}	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$
t_{LR}	$1E-2$	$1E-2$	$1E-2$	–	–	–	–	–
t_{TB}	$2E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$
t_{CTB}	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$
MLL_{TM}	–	–	–	–	–	–	–	–
MLL_{LR}	–	–	–	–	–	–	–	–
MLL_{TB}	10	10	0	0	0	0	0	0
MLL_{CTB}	0	0	0	0	0	0	0	0
SD_{TM}	–	–	–	–	–	–	–	–
SD_{LR}	–	–	–	–	–	–	–	–
SD_{TB}	204	191	0	0	0	0	0	0
SD_{CTB}	0	0	0	0	0	0	0	0

Table 4.2 (Contd.) For Taylor order $m = 4$:

i	-7	-6	-5	-4	-3	-2	-1
$w\left(\mathbf{X}^{(i)}\right)$	$2*2^7$	$2*2^6$	$2*2^5$	$2*2^4$	$2*2^3$	$2*2^2$	$2*2^1$
$\mathcal{H}_{TM}$	$-$	$-$	$6E+290$	$6E+148$	$2E+76$	$3E+38$	$1E+18$
$\mathcal{H}_{LR}$	$-$	$-$	$6E+290$	$6E+148$	$2E+76$	$3E+38$	$1E+18$
$\mathcal{H}_{TB}$	$-$	$-$	$6E+290$	$6E+148$	$2E+76$	$3E+38$	$1E+18$
$\mathcal{H}_{CTB}$	$-$	$-$	$6E+290$	$6E+148$	$2E+76$	$3E+38$	$1E+18$
$\mathcal{R}^*$	$-$	$-$	$-$	32	32	32	32
$\mathcal{R}_{TM}$	$-$	$-$	$-$	$1E+142$	$3E+72$	$6E+37$	$3E+20$
$\mathcal{R}_{LR}$	$-$	$-$	$-$	$1E+142$	$3E+72$	$6E+37$	$3E+20$
$\mathcal{R}_{TB}$	$-$	$-$	$-$	$1E+142$	$3E+72$	$6E+37$	$3E+20$
$\mathcal{R}_{CTB}$	$-$	$-$	$-$	$1E+142$	$3E+72$	$6E+37$	$3E+20$
t_{TM}	$-$	$-$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$
t_{LR}	$-$	$-$	$2E-2$	$2E-2$	$2E-2$	$2E-2$	$2E-2$
t_{TB}	$-$	$-$	$3E-2$	$3E-2$	$3E-2$	$3E-2$	$3E-2$
t_{CTB}	$-$	$-$	$2E-2$	$2E-2$	$2E-2$	$2E-2$	$2E-2$
MLL_{TM}	$-$	$-$	$-$	$-$	$-$	$-$	$-$
MLL_{LR}	$-$	$-$	$-$	$-$	$-$	$-$	$-$
MLL_{TB}	$-$	$-$	12	12	11	12	11
MLL_{CTB}	$-$	$-$	0	0	0	0	0
SD_{TM}	$-$	$-$	$-$	$-$	$-$	$-$	$-$
SD_{LR}	$-$	$-$	$-$	$-$	$-$	$-$	$-$
SD_{TB}	$-$	$-$	201	190	184	178	172
SD_{CTB}	$-$	$-$	0	0	0	0	0

Table 4.2 (Contd.) For Taylor order $m = 4$:

i	0	1	2	3	4	5	6	7
$w\left(\mathbf{X}^{(i)}\right)$	$2*2^{-0}$	$2*2^{-1}$	$2*2^{-2}$	$2*2^{-3}$	$2*2^{-4}$	$2*2^{-5}$	$2*2^{-6}$	$2*2^{-7}$
$\mathcal{H}_{TM}$	$4E+6$	$7E+1$	$2E+0$	$5E-1$	$1E-1$	$3E-2$	$7E-3$	$2E-3$
$\mathcal{H}_{LR}$	$4E+6$	$6E+1$	$-$	$-$	$-$	$-$	$-$	$-$
$\mathcal{H}_{TB}$	$4E+6$	$6E+1$	$2E-1$	$2E-3$	$3E-5$	$5E-7$	$9E-9$	$2E-10$
$\mathcal{H}_{CTB}$	$4E+6$	$6E+1$	$2E-1$	$2E-3$	$3E-5$	$5E-7$	$9E-9$	$2E-10$
$\mathcal{R}^*$	32	32	32	32	32	32	32	32
$\mathcal{R}_{TM}$	$3E+11$	$6E+4$	29.9	4.7	4.1	4.0	4.0	4.0
$\mathcal{R}_{LR}$	$3E+11$	$7E+4$	$-$	$-$	$-$	$-$	$-$	$-$
$\mathcal{R}_{TB}$	$3E+11$	$7E+4$	251.7	104.4	74.4	61.9	53.7	47.6
$\mathcal{R}_{CTB}$	$3E+11$	$7E+4$	253.9	104.4	74.1	61.9	53.7	47.6
t_{TM}	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$
t_{LR}	$2E-2$	$2E-2$	$-$	$-$	$-$	$-$	$-$	$-$
t_{TB}	$3E-2$	$2E-2$	$2E-2$	$2E-2$	$2E-2$	$1E-2$	$1E-2$	$1E-2$
t_{CTB}	$2E-2$	$2E-2$	$2E-2$	$2E-2$	$2E-2$	$1E-2$	$1E-2$	$1E-2$
MLL_{TM}	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$
MLL_{LR}	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$
MLL_{TB}	10	10	0	0	0	0	0	0
MLL_{CTB}	0	0	0	0	0	0	0	0
SD_{TM}	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$
SD_{LR}	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$
SD_{TB}	165	156	0	0	0	0	0	0
SD_{CTB}	0	0	0	0	0	0	0	0

Table 4.2 (Contd.) For Taylor order $m = 6$:

i	-7	-6	-5	-4	-3	-2	-1
$w\left(\mathbf{X}^{(i)}\right)$	$2*2^7$	$2*2^6$	$2*2^5$	$2*2^4$	$2*2^3$	$2*2^2$	$2*2^1$
$\mathcal{H}_{TM}$	$-$	$-$	$3E+297$	$2E+154$	$4E+80$	$5E+41$	$1E+20$
$\mathcal{H}_{LR}$	$-$	$-$	$3E+297$	$2E+154$	$4E+80$	$5E+41$	$1E+20$
$\mathcal{H}_{TB}$	$-$	$-$	$3E+297$	$2E+154$	$4E+80$	$5E+41$	$1E+20$
$\mathcal{H}_{CTB}$	$-$	$-$	$3E+297$	$2E+154$	$4E+80$	$5E+41$	$1E+20$
$\mathcal{R}^*$	$-$	$-$	$-$	128	128	128	128
$\mathcal{R}_{TM}$	$-$	$-$	$-$	$2E+143$	$5E+73$	$9E+38$	$4E+21$
$\mathcal{R}_{LR}$	$-$	$-$	$-$	$2E+143$	$5E+73$	$9E+38$	$4E+21$
$\mathcal{R}_{TB}$	$-$	$-$	$-$	$2E+143$	$5E+73$	$9E+38$	$4E+21$
$\mathcal{R}_{CTB}$	$-$	$-$	$-$	$2E+143$	$5E+73$	$9E+38$	$4E+21$
t_{TM}	$-$	$-$	$2E-2$	$2E-2$	$2E-2$	$2E-2$	$2E-2$
t_{LR}	$-$	$-$	$3E-2$	$3E-2$	$3E-2$	$3E-2$	$3E-2$
t_{TB}	$-$	$-$	$6E-2$	$4E-2$	$5E-2$	$5E-2$	$4E-2$
t_{CTB}	$-$	$-$	$3E-2$	$3E-2$	$3E-2$	$3E-2$	$3E-2$
MLL_{TM}	$-$	$-$	$-$	$-$	$-$	$-$	$-$
MLL_{LR}	$-$	$-$	$-$	$-$	$-$	$-$	$-$
MLL_{TB}	$-$	$-$	23	17	12	12	12
MLL_{CTB}	$-$	$-$	0	0	0	0	0
SD_{TM}	$-$	$-$	$-$	$-$	$-$	$-$	$-$
SD_{LR}	$-$	$-$	$-$	$-$	$-$	$-$	$-$
SD_{TB}	$-$	$-$	223	206	192	177	170
SD_{CTB}	$-$	$-$	0	0	0	0	0

Table 4.2 (Contd.) For Taylor order $m = 6$:

i	0	1	2	3	4	5	6	7
$w\left(\mathbf{X}^{(i)}\right)$	$2*2^{-0}$	$2*2^{-1}$	$2*2^{-2}$	$2*2^{-3}$	$2*2^{-4}$	$2*2^{-5}$	$2*2^{-6}$	$2*2^{-7}$
$\mathcal{H}_{TM}$	$2E+7$	$4E+1$	$2E+0$	$5E-1$	$1E-1$	$3E-2$	$7E-3$	$2E-3$
$\mathcal{H}_{LR}$	$2E+7$	$2E+1$	–	–	–	–	–	–
$\mathcal{H}_{TB}$	$2E+7$	$2E+1$	$2E-2$	$4E-5$	$1E-7$	$5E-10$	$1E-12$	$3E-12$
$\mathcal{H}_{CTB}$	$2E+7$	$2E+1$	$2E-2$	$4E-5$	$1E-7$	$5E-10$	$1E-12$	$3E-12$
$\mathcal{R}^*$	128	128	128	128	128	128	128	128
$\mathcal{R}_{TM}$	$6E+12$	$5E+5$	16.7	4.4	4.1	4.0	4.0	4.0
$\mathcal{R}_{LR}$	$6E+12$	$8E+5$	–	–	–	–	–	–
$\mathcal{R}_{TB}$	$6E+12$	$8E+5$	$1E+3$	456.8	311.2	256.2	495.6	0.3
$\mathcal{R}_{CTB}$	$6E+12$	$8E+5$	$1E+3$	456.8	311.2	256.2	495.6	0.3
t_{TM}	$2E-2$	$2E-2$	$2E-2$	$2E-2$	$2E-2$	$2E-2$	$2E-2$	$2E-2$
t_{LR}	$3E-2$	$3E-2$	–	–	–	–	–	–
t_{TB}	$5E-2$	$3E-2$	$3E-2$	$3E-2$	$3E-2$	$3E-2$	$3E-2$	$3E-2$
t_{CTB}	$3E-2$	$3E-2$	$3E-2$	$3E-2$	$3E-2$	$3E-2$	$3E-2$	$3E-2$
MLL_{TM}	–	–	–	–	–	–	–	–
MLL_{LR}	–	–	–	–	–	–	–	–
MLL_{TB}	12	0	0	0	0	0	0	0
MLL_{CTB}	0	0	0	0	0	0	0	0
SD_{TM}	–	–	–	–	–	–	–	–
SD_{LR}	–	–	–	–	–	–	–	–
SD_{TB}	162	0	0	0	0	0	0	0
SD_{CTB}	0	0	0	0	0	0	0	0

Table 4.2 (Contd.) For Taylor order $m = 8$:

i	-7	-6	-5	-4	-3	-2	-1
$w\left(\mathbf{X}^{(i)}\right)$	$2*2^7$	$2*2^6$	$2*2^5$	$2*2^4$	$2*2^3$	$2*2^2$	$2*2^1$
$\mathcal{H}_{TM}$	–	–	$7E+303$	$3E+159$	$3E+84$	$2E+44$	$4E+21$
$\mathcal{H}_{LR}$	–	–	$7E+303$	$3E+159$	$3E+84$	$2E+44$	$4E+21$
$\mathcal{H}_{TB}$	–	–	$7E+303$	$3E+159$	$3E+84$	$2E+44$	$4E+21$
$\mathcal{H}_{CTB}$	–	–	$7E+303$	$3E+159$	$3E+84$	$2E+44$	$4E+21$
$\mathcal{R}^*$	–	–	–	512	512	512	512
$\mathcal{R}_{TM}$	–	–	–	$2E+144$	$8E+74$	$1E+40$	$6E+22$
$\mathcal{R}_{LR}$	–	–	–	$2E+144$	$8E+74$	$1E+40$	$6E+22$
$\mathcal{R}_{TB}$	–	–	–	$2E+144$	$8E+74$	$1E+40$	$6E+22$
$\mathcal{R}_{CTB}$	–	–	–	$2E+144$	$8E+74$	$1E+40$	$6E+22$
t_{TM}	–	–	$2E-2$	$2E-2$	$2E-2$	$2E-2$	$2E-2$
t_{LR}	–	–	$4E-2$	$4E-2$	$4E-2$	$4E-2$	$4E-2$
t_{TB}	–	–	$8E-2$	$8E-2$	$7E-2$	$8E-2$	$7E-2$
t_{CTB}	–	–	$4E-2$	$4E-2$	$4E-2$	$4E-2$	$4E-2$
MLL_{TM}	–	–	–	–	–	–	–
MLL_{LR}	–	–	–	–	–	–	–
MLL_{TB}	–	–	23	17	12	12	11
MLL_{CTB}	–	–	0	0	0	0	0
SD_{TM}	–	–	–	–	–	–	–
SD_{LR}	–	–	–	–	–	–	–
SD_{TB}	–	–	222	193	193	185	176
SD_{CTB}	–	–	0	0	0	0	0

Table 4.2 (Contd.) For Taylor order $m = 8$:

i	0	1	2	3	4	5	6	7
$w\left(\mathbf{X}^{(i)}\right)$	$2*2^{-0}$	$2*2^{-1}$	$2*2^{-2}$	$2*2^{-3}$	$2*2^{-4}$	$2*2^{-5}$	$2*2^{-6}$	$2*2^{-7}$
$\mathcal{H}_{TM}$	$4E+7$	$2E+1$	$2E+0$	$5E-1$	$1E-1$	$3E-2$	$7E-3$	$2E-3$
$\mathcal{H}_{LR}$	$4E+7$	$7E+0$	–	–	–	–	–	–
$\mathcal{H}_{TB}$	$4E+7$	$7E+0$	$1E-3$	$6E-7$	$5E-10$	$2E-12$	$2E-12$	$3E-12$
$\mathcal{H}_{CTB}$	$4E+7$	$7E+0$	$1E-3$	$6E-7$	$5E-10$	$2E-12$	$2E-12$	$3E-12$
$\mathcal{R}^*$	512	512	512	512	512	512	512	512
$\mathcal{R}_{TM}$	$1E+14$	$2E+6$	8.6	4.3	4.1	4.0	4.0	4.0
$\mathcal{R}_{LR}$	$1E+14$	$5E+6$	–	–	–	–	–	–
$\mathcal{R}_{TB}$	$1E+14$	$5E+6$	$5E+3$	$2E+3$	$1E+3$	205.6	1.0	0.7
$\mathcal{R}_{CTB}$	$1E+14$	$5E+6$	$5E+3$	$2E+3$	$1E+3$	205.6	1.0	0.7
t_{TM}	$2E-2$	$2E-2$	$2E-2$	$2E-2$	$2E-2$	$2E-2$	$2E-2$	$2E-2$
t_{LR}	$4E-2$	$4E-2$	–	–	–	–	–	–
t_{TB}	$8E-2$	$5E-2$	$5E-2$	$5E-2$	$4E-2$	$4E-2$	$4E-2$	$4E-2$
t_{CTB}	$4E-2$	$4E-2$	$5E-2$	$5E-2$	$4E-2$	$4E-2$	$4E-2$	$4E-2$
MLL_{TM}	–	–	–	–	–	–	–	–
MLL_{LR}	–	–	–	–	–	–	–	–
MLL_{TB}	11	0	0	0	0	0	0	0
MLL_{CTB}	0	0	0	0	0	0	0	0
SD_{TM}	–	–	–	–	–	–	–	–
SD_{LR}	–	–	–	–	–	–	–	–
SD_{TB}	168	0	0	0	0	0	0	0
SD_{CTB}	0	0	0	0	0	0	0	0

TABLE 4.3. Overestimations and their reduction ratios for various Taylor orders obtained with Taylor model, Algorithm LR, Algorithm TB and Algorithm CTB in Example 4.3 Levy $(3 - dim)$.

For Taylor order $m = 2$:

i	-7	-6	-5	-4	-3	-2	-1
$w\left(\mathbf{X}^{(i)}\right)$	$2*2^7$	$2*2^6$	$2*2^5$	$2*2^4$	$2*2^3$	$2*2^2$	$2*2^1$
$\mathcal{H}_{TM}$	$7E+14$	$4E+12$	$2E+10$	$1E+8$	$1E+6$	$2E+4$	$1E+3$
$\mathcal{H}_{LR}$	$7E+14$	$3E+12$	$2E+10$	$1E+8$	$1E+6$	$2E+4$	$9E+2$
$\mathcal{H}_{TB}$	$7E+14$	$3E+12$	$2E+10$	$1E+8$	$1E+6$	$2E+4$	$9E+2$
$\mathcal{H}_{CTB}$	$7E+14$	$3E+12$	$2E+10$	$1E+8$	$1E+6$	$2E+4$	$9E+2$
$\mathcal{R}^*$	$-$	8	8	8	8	8	8
$\mathcal{R}_{TM}$	$-$	221.1	193.7	153.3	103.0	53.9	20.9
$\mathcal{R}_{LR}$	$-$	221.1	193.7	153.3	103.0	53.9	20.9
$\mathcal{R}_{TB}$	$-$	221.1	193.7	153.3	103.0	53.9	20.9
$\mathcal{R}_{CTB}$	$-$	221.1	193.7	153.3	103.0	53.9	20.9
t_{TM}	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$
t_{LR}	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$
t_{TB}	$8E-1$	$8E-1$	$8E-1$	$8E-1$	$8E-1$	$8E-1$	$7E-1$
t_{CTB}	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$
MLL_{TM}	$-$	$-$	$-$	$-$	$-$	$-$	$-$
MLL_{LR}	$-$	$-$	$-$	$-$	$-$	$-$	$-$
MLL_{TB}	99	99	99	99	99	99	99
MLL_{CTB}	0	0	0	0	0	0	0
SD_{TM}	$-$	$-$	$-$	$-$	$-$	$-$	$-$
SD_{LR}	$-$	$-$	$-$	$-$	$-$	$-$	$-$
SD_{TB}	3686	3619	3550	3465	3343	3213	2905
SD_{CTB}	0	0	0	0	0	0	0

Table 4.3 (Contd.) For Taylor order $m = 2$:

i	0	1	2	3	4	5	6	7
$w\left(\mathbf{X}^{(i)}\right)$	$2*2^{-0}$	$2*2^{-1}$	$2*2^{-2}$	$2*2^{-3}$	$2*2^{-4}$	$2*2^{-5}$	$2*2^{-6}$	$2*2^{-7}$
$\mathcal{H}_{TM}$	$1E+2$	$2E+1$	$5E+0$	$1E+0$	$3E-1$	$6E-2$	$2E-2$	$4E-3$
$\mathcal{H}_{LR}$	$8E+1$	$7E+0$	$8E-1$	–	–	–	–	–
$\mathcal{H}_{TB}$	$7E+1$	$7E+0$	$8E-1$	$9E-2$	$1E-2$	$1E-3$	$2E-4$	$2E-5$
$\mathcal{H}_{CTB}$	$8E+1$	$7E+0$	$8E-1$	$9E-2$	$1E-2$	$1E-3$	$2E-4$	$2E-5$
$\mathcal{R}^*$	8	8	8	8	8	8	8	8
$\mathcal{R}_{TM}$	8.6	4.8	4.6	4.3	4.2	4.1	4.0	4.0
$\mathcal{R}_{LR}$	11.7	10.5	9.3	–	–	–	–	–
$\mathcal{R}_{TB}$	11.9	10.3	9.3	8.7	8.4	8.2	8.1	8.0
$\mathcal{R}_{CTB}$	11.7	10.5	9.3	8.7	8.4	8.2	8.1	8.0
t_{TM}	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$
t_{LR}	$1E-2$	$1E-2$	$1E-2$	–	–	–	–	–
t_{TB}	$2E-1$	$1E-1$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$
t_{CTB}	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$
MLL_{TM}	–	–	–	–	–	–	–	–
MLL_{LR}	–	–	–	–	–	–	–	–
MLL_{TB}	24	3	0	0	0	0	0	0
MLL_{CTB}	0	0	0	0	0	0	0	0
SD_{TM}	–	–	–	–	–	–	–	–
SD_{LR}	–	–	–	–	–	–	–	–
SD_{TB}	393	69	0	0	0	0	0	0
SD_{CTB}	0	0	0	0	0	0	0	0

Table 4.3 (Contd.) For Taylor order $m = 4$:

i	-7	-6	-5	-4	-3	-2	-1
$w\left(\mathbf{X}^{(i)}\right)$	$2*2^7$	$2*2^6$	$2*2^5$	$2*2^4$	$2*2^3$	$2*2^2$	$2*2^1$
$\mathcal{H}_{TM}$	$2E+20$	$5E+16$	$2E+13$	$8E+9$	$8E+6$	$3E+4$	$9E+2$
$\mathcal{H}_{LR}$	$2E+20$	$5E+16$	$2E+13$	$8E+9$	$8E+6$	$3E+4$	$5E+2$
$\mathcal{H}_{TB}$	$2E+20$	$5E+16$	$2E+13$	$8E+9$	$8E+6$	$3E+4$	$3E+2$
$\mathcal{H}_{CTB}$	$2E+20$	$5E+16$	$2E+13$	$8E+9$	$8E+6$	$3E+4$	$5E+2$
$\mathcal{R}^*$	$-$	32	32	32	32	32	32
$\mathcal{R}_{TM}$	$-$	$3E+3$	$3E+3$	$2E+3$	$1E+3$	273.8	32.5
$\mathcal{R}_{LR}$	$-$	$3E+4$	$3E+3$	$2E+3$	$1E+3$	284.4	55.9
$\mathcal{R}_{TB}$	$-$	$3E+3$	$3E+3$	$2E+3$	$1E+3$	313	77.7
$\mathcal{R}_{CTB}$	$-$	$3E+4$	$3E+3$	$2E+3$	$1E+3$	284.4	55.9
t_{TM}	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$
t_{LR}	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$
t_{TB}	$3E-1$	$3E-1$	$3E-1$	$3E-1$	$3E-1$	$2E-1$	$2E-1$
t_{CTB}	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$
MLL_{TM}	$-$	$-$	$-$	$-$	$-$	$-$	$-$
MLL_{LR}	$-$	$-$	$-$	$-$	$-$	$-$	$-$
MLL_{TB}	75	75	55	23	36	36	18
MLL_{CTB}	0	0	0	0	0	0	0
SD_{TM}	$-$	$-$	$-$	$-$	$-$	$-$	$-$
SD_{LR}	$-$	$-$	$-$	$-$	$-$	$-$	$-$
SD_{TB}	424	425	422	379	460	107	105
SD_{CTB}	0	0	0	0	0	0	0

Table 4.3 (Contd.) For Taylor order $m = 4$:

i	0	1	2	3	4	5	6	7
$w\left(\mathbf{X}^{(i)}\right)$	$2*2^{-0}$	$2*2^{-1}$	$2*2^{-2}$	$2*2^{-3}$	$2*2^{-4}$	$2*2^{-5}$	$2*2^{-6}$	$2*2^{-7}$
$\mathcal{H}_{TM}$	$1E+2$	$2E+1$	$5E+0$	$1E+0$	$3E-1$	$6E-2$	$2E-2$	$3E-3$
$\mathcal{H}_{LR}$	$2E+1$	$2E-1$	–	–	–	–	–	–
$\mathcal{H}_{TB}$	$6E+0$	$2E-1$	$5E-3$	$1E-4$	$4E-6$	$1E-7$	$4E-9$	$1E-10$
$\mathcal{H}_{CTB}$	$2E+1$	$2E-1$	$5E-3$	$1E-4$	$4E-6$	$1E-7$	$4E-9$	$1E-10$
$\mathcal{R}^*$	32	32	32	32	32	32	32	32
$\mathcal{R}_{TM}$	8.5	4.9	4.6	4.3	4.2	4.1	4.0	4.0
$\mathcal{R}_{LR}$	26.0	108.7	–	–	–	–	–	–
$\mathcal{R}_{TB}$	51.0	36.2	36.0	34.1	33.1	32.5	32.3	31.6
$\mathcal{R}_{CTB}$	26.0	108.7	36.0	34.1	33.1	32.5	32.3	31.6
t_{TM}	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$
t_{LR}	$1E-2$	$1E-2$	–	–	–	–	–	–
t_{TB}	$2E-1$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$
t_{CTB}	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$
MLL_{TM}	–	–	–	–	–	–	–	–
MLL_{LR}	–	–	–	–	–	–	–	–
MLL_{TB}	14	0	0	0	0	0	0	0
MLL_{CTB}	0	0	0	0	0	0	0	0
SD_{TM}	–	–	–	–	–	–	–	–
SD_{LR}	–	–	–	–	–	–	–	–
SD_{TB}	289	0	0	0	0	0	0	0
SD_{CTB}	0	0	0	0	0	0	0	0

Table 4.3 (Contd.) For Taylor order $m = 6$:

i	-7	-6	-5	-4	-3	-2	-1
$w\left(\mathbf{X}^{(i)}\right)$	$2*2^7$	$2*2^6$	$2*2^5$	$2*2^4$	$2*2^3$	$2*2^2$	$2*2^1$
$\mathcal{H}_{TM}$	$1E+25$	$2E+20$	$4E+15$	$2E+11$	$2E+7$	$3E+4$	$9E+2$
$\mathcal{H}_{LR}$	$1E+25$	$2E+20$	$4E+15$	$2E+11$	$2E+7$	$2E+4$	$2E+2$
$\mathcal{H}_{TB}$	$1E+25$	$2E+20$	$4E+15$	$2E+11$	$2E+7$	$2E+4$	$8E+1$
$\mathcal{H}_{CTB}$	$1E+25$	$2E+20$	$4E+15$	$2E+11$	$2E+7$	$2E+4$	$2E+2$
$\mathcal{R}^*$	$-$	128	128	128	128	128	128
$\mathcal{R}_{TM}$	$-$	$5E+4$	$4E+4$	$3E+4$	$8E+3$	$6E+2$	$4E+1$
$\mathcal{R}_{LR}$	$-$	$5E+4$	$4E+4$	$3E+4$	$8E+3$	$8E+2$	$2E+2$
$\mathcal{R}_{TB}$	$-$	$5E+4$	$4E+4$	$3E+4$	$8E+3$	$9E+2$	$3E+2$
$\mathcal{R}_{CTB}$	$-$	$5E+4$	$4E+4$	$3E+4$	$8E+3$	$8E+2$	$2E+2$
t_{TM}	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$
t_{LR}	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$
t_{TB}	$5E-1$	$5E-1$	$5E-1$	$5E-1$	$5E-1$	$5E-1$	$9E-1$
t_{CTB}	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$
MLL_{TM}	$-$	$-$	$-$	$-$	$-$	$-$	$-$
MLL_{LR}	$-$	$-$	$-$	$-$	$-$	$-$	$-$
MLL_{TB}	54	54	34	34	33	14	45
MLL_{CTB}	0	0	0	0	0	0	0
SD_{TM}	$-$	$-$	$-$	$-$	$-$	$-$	$-$
SD_{LR}	$-$	$-$	$-$	$-$	$-$	$-$	$-$
SD_{TB}	207	180	114	114	108	164	624
SD_{CTB}	0	0	0	0	0	0	0

Table 4.3 (Contd.) For Taylor order $m = 6$:

i	0	1	2	3	4	5	6	7
$w\left(\mathbf{X}^{(i)}\right)$	$2*2^{-0}$	$2*2^{-1}$	$2*2^{=2}$	$2*2^{-3}$	$2*2^{-4}$	$2*2^{-5}$	$2*2^{-6}$	$2*2^{-7}$
$\mathcal{H}_{TM}$	$1E+2$	$2E+1$	$4E+0$	$1E+0$	$3E-1$	$6E-2$	$2E-2$	$4E-3$
$\mathcal{H}_{LR}$	$9E+0$	$3E-3$	–	–	–	–	–	–
$\mathcal{H}_{TB}$	$8E-1$	$3E-3$	$2E-5$	$2E-7$	$1E-9$	$1E-11$	$3E-12$	$2E-12$
$\mathcal{H}_{CTB}$	$9E+0$	$3E-3$	$2E-5$	$2E-7$	$1E-9$	$1E-11$	$3E-12$	$2E-12$
$\mathcal{R}^*$	128	128	128	128	128	128	128	128
$\mathcal{R}_{TM}$	8.4	4.8	4.6	4.3	4.2	4.1	4.0	4.0
$\mathcal{R}_{LR}$	18.8	$3E+3$	–	–	–	–	–	–
$\mathcal{R}_{TB}$	94.8	273.6	142.6	135.2	131.7	100.3	4.5	1.1
$\mathcal{R}_{CTB}$	18.8	$3E+3$	142.6	135.2	131.7	100.3	4.5	1.1
t_{TM}	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$
t_{LR}	$1E-2$	$1E-2$	–	–	–	–	–	–
t_{TB}	$5E-1$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$
t_{CTB}	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$
MLL_{TM}	–	–	–	–	–	–	–	–
MLL_{LR}	–	–	–	–	–	–	–	–
MLL_{TB}	15	0	0	0	0	0	0	0
MLL_{CTB}	0	0	0	0	0	0	0	0
SD_{TM}	–	–	–	–	–	–	–	–
SD_{LR}	–	–	–	–	–	–	–	–
SD_{TB}	300	0	0	0	0	0	0	0
SD_{CTB}	0	0	0	0	0	0	0	0

Table 4.3 (Contd.) For Taylor order $m = 8$:

i	-7	-6	-5	-4	-3	-2	-1
$w\left(\mathbf{X}^{(i)}\right)$	$2*2^7$	$2*2^6$	$2*2^5$	$2*2^4$	$2*2^3$	$2*2^2$	$2*2^1$
$\mathcal{H}_{TM}$	$2E+29$	$3E+23$	$4E+17$	$1E+12$	$3E+7$	$3E+4$	$9E+2$
$\mathcal{H}_{LR}$	$2E+29$	$3E+23$	$4E+17$	$1E+12$	$3E+7$	$1E+4$	$6E+1$
$\mathcal{H}_{TB}$	$2E+29$	$3E+23$	$4E+17$	$1E+12$	$3E+7$	$1E+4$	$1E+1$
$\mathcal{H}_{CTB}$	$2E+29$	$3E+23$	$4E+17$	$1E+12$	$3E+7$	$1E+4$	$6E+1$
$\mathcal{R}^*$	$-$	512	512	512	512	512	512
$\mathcal{R}_{TM}$	$-$	$8E+5$	$6E+5$	$3E+5$	$4E+4$	$9E+3$	37.1
$\mathcal{R}_{LR}$	$-$	$8E+5$	$6E+5$	$3E+5$	$4E+4$	$2E+3$	211.5
$\mathcal{R}_{TB}$	$-$	$8E+5$	$6E+5$	$3E+5$	$5E+4$	$3E+3$	$1E+3$
$\mathcal{R}_{CTB}$	$-$	$8E+5$	$6E+5$	$3E+5$	$4E+4$	$2E+3$	211.5
t_{TM}	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$
t_{LR}	$3E-2$	$3E-2$	$3E-2$	$3E-2$	$3E-2$	$3E-2$	$3E-2$
t_{TB}	$2E-1$	$2E-1$	$2E-1$	$3E-1$	$3E-1$	$3E-1$	$2E+0$
t_{CTB}	$3E-2$	$3E-2$	$3E-2$	$3E-2$	$3E-2$	$3E-2$	$3E-2$
MLL_{TM}	$-$	$-$	$-$	$-$	$-$	$-$	$-$
MLL_{LR}	$-$	$-$	$-$	$-$	$-$	$-$	$-$
MLL_{TB}	5	5	5	6	4	18	45
MLL_{CTB}	0	0	0	0	0	0	0
SD_{TM}	$-$	$-$	$-$	$-$	$-$	$-$	$-$
SD_{LR}	$-$	$-$	$-$	$-$	$-$	$-$	$-$
SD_{TB}	4	4	4	82	71	97	627
SD_{CTB}	0	0	0	0	0	0	0

Table 4.3 (Contd.) For Taylor order $m = 8$:

i	0	1	2	3	4	5	6	7
$w\left(\mathbf{X}^{(i)}\right)$	$2*2^{-0}$	$2*2^{-1}$	$2*2^{-2}$	$2*2^{-3}$	$2*2^{-4}$	$2*2^{-5}$	$2*2^{-6}$	$2*2^{-7}$
$\mathcal{H}_{TM}$	$1E+2$	$2E+1$	$5E+0$	$1E+0$	$3E-1$	$6E-2$	$2E-2$	$4E-3$
$\mathcal{H}_{LR}$	$5E+0$	$4E-5$	$-$	$-$	$-$	$-$	$-$	$-$
$\mathcal{H}_{TB}$	$4E-1$	$4E-5$	$7E-8$	$1E-10$	$3E-13$	$3E-12$	$3E-12$	$2E-12$
$\mathcal{H}_{CTB}$	$5E+0$	$4E-5$	$7E-8$	$1E-10$	$3E-13$	$3E-12$	$3E-12$	$2E-12$
$\mathcal{R}^*$	512	512	512	512	512	512	512	512
$\mathcal{R}_{TM}$	8.4	4.8	4.6	4.3	4.2	4.1	4.0	4.0
$\mathcal{R}_{LR}$	11.5	$1E+5$	$-$	$-$	$-$	$-$	$-$	$-$
$\mathcal{R}_{TB}$	28.6	$9E+3$	539.4	518.7	47.3	1.1	1.1	1.0
$\mathcal{R}_{CTB}$	11.5	$1E+5$	539.4	518.7	47.3	1.1	1.1	1.0
t_{TM}	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$
t_{LR}	$5E-2$	$3E-2$	$-$	$-$	$-$	$-$	$-$	$-$
t_{TB}	$5E-1$	$3E-2$	$3E-2$	$3E-2$	$3E-2$	$3E-2$	$3E-2$	$3E-2$
t_{CTB}	$5E-2$	$3E-2$	$3E-2$	$3E-2$	$3E-2$	$3E-2$	$3E-2$	$3E-2$
MLL_{TM}	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$
MLL_{LR}	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$
MLL_{TB}	14	0	0	0	0	0	0	0
MLL_{CTB}	0	0	0	0	0	0	0	0
SD_{TM}	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$
SD_{LR}	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$
SD_{TB}	281	0	0	0	0	0	0	0
SD_{CTB}	0	0	0	0	0	0	0	0

TABLE 4.4. Overestimations and their reduction ratios for various Taylor orders obtained with Taylor model, Algorithm LR, Algorithm TB and Algorithm CTB in Example 4.4 Trigonometric ($4-dim$).

For Taylor order $m=2$:

i	-7	-6	-5	-4	-3	-2	-1
$w\left(\mathbf{X}^{(i)}\right)$	$2*2^7$	$2*2^6$	$2*2^5$	$2*2^4$	$2*2^3$	$2*2^2$	$2*2^1$
$\mathcal{H}_{TM}$	$3E+13$	$4E+11$	$7E+9$	$1E+8$	$2E+6$	$5E+4$	$3E+3$
$\mathcal{H}_{LR}$	$3E+13$	$4E+11$	$7E+9$	$1E+8$	$2E+6$	$5E+4$	$3E+3$
$\mathcal{H}_{TB}$	$3E+13$	$4E+11$	$7E+9$	$1E+8$	$2E+6$	$5E+4$	$3E+3$
$\mathcal{H}_{CTB}$	$3E+13$	$4E+11$	$7E+9$	$1E+8$	$2E+6$	$5E+4$	$3E+3$
$\mathcal{R}^*$	$-$	8	8	8	8	8	8
$\mathcal{R}_{TM}$	$-$	63.7	63.0	61.3	55.5	38.0	17.5
$\mathcal{R}_{LR}$	$-$	63.7	63.0	61.3	55.5	38.0	17.5
$\mathcal{R}_{TB}$	$-$	63.7	63.0	61.3	55.5	55.5	17.5
$\mathcal{R}_{CTB}$	$-$	63.7	63.0	61.3	55.5	38.0	17.5
t_{TM}	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$
t_{LR}	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$
t_{TB}	125	124	123	122	120	114	109
t_{CTB}	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$
MLL_{TM}	$-$	$-$	$-$	$-$	$-$	$-$	$-$
MLL_{LR}	$-$	$-$	$-$	$-$	$-$	$-$	$-$
MLL_{TB}	235	235	235	235	235	235	235
MLL_{CTB}	0	0	0	0	0	0	0
SD_{TM}	$-$	$-$	$-$	$-$	$-$	$-$	$-$
SD_{LR}	$-$	$-$	$-$	$-$	$-$	$-$	$-$
SD_{TB}	12745	12507	12266	11996	11671	11210	10646
SD_{CTB}	0	0	0	0	0	0	0

Table 4.4 (Contd.) For Taylor order $m = 2$:

i	0	1	2	3	4	5	6	7
$w\left(\mathbf{X}^{(i)}\right)$	$2*2^{-0}$	$2*2^{-1}$	$2*2^{-2}$	$2*2^{-3}$	$2*2^{-4}$	$2*2^{-5}$	$2*2^{-6}$	$2*2^{-7}$
$\mathcal{H}_{TM}$	$4E+2$	$9E+1$	$2E+1$	$5E+0$	$1E+0$	$3E-1$	$7E-2$	$2E-2$
$\mathcal{H}_{LR}$	$3E+2$	$3E+1$	$3E+0$	–	–	–	–	–
$\mathcal{H}_{TB}$	$3E+2$	$3E+1$	$3E+0$	$3E-1$	$3E-2$	$4E-3$	$5E-4$	$7E-5$
$\mathcal{H}_{CTB}$	$3E+2$	$3E+1$	$3E+0$	$3E-1$	$3E-2$	$4E-3$	$5E-4$	$7E-5$
$\mathcal{R}^*$	8	8	8	8	8	8	8	8
$\mathcal{R}_{TM}$	7.0	4.9	4.5	4.2	4.1	4.1	4.0	4.0
$\mathcal{R}_{LR}$	11.5	10.5	9.5	–	–	–	–	–
$\mathcal{R}_{TB}$	11.5	10.5	9.5	8.8	8.4	8.2	8.1	8.1
$\mathcal{R}_{CTB}$	11.5	10.5	9.5	8.8	8.4	8.2	8.1	8.1
t_{TM}	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$
t_{LR}	$1E-2$	$1E-2$	1.9	–	–	–	–	–
t_{TB}	$3E-2$	$3E-2$	$3E-2$	$3E-2$	$3E-2$	$3E-2$	$3E-2$	$3E-2$
t_{CTB}	$1E-2$	$1E-2$	$3E-2$	$3E-2$	$3E-2$	$3E-2$	$3E-2$	$3E-2$
MLL_{TM}	–	–	–	–	–	–	–	–
MLL_{LR}	–	–	–	–	–	–	–	–
MLL_{TB}	0	0	0	0	0	0	0	0
MLL_{CTB}	0	0	0	0	0	0	0	0
SD_{TM}	–	–	–	–	–	–	–	–
SD_{LR}	–	–	–	–	–	–	–	–
SD_{TB}	0	0	0	0	0	0	0	0
SD_{CTB}	0	0	0	0	0	0	0	0

Table 4.4 (Contd.) For Taylor order $m = 4$:

i	-7	-6	-5	-4	-3	-2	-1
$w\left(\mathbf{X}^{(i)}\right)$	$2*2^7$	$2*2^6$	$2*2^5$	$2*2^4$	$2*2^3$	$2*2^2$	$2*2^1$
$\mathcal{H}_{TM}$	$2E+19$	$2E+16$	$2E+13$	$2E+10$	$3E+7$	$1E+5$	$3E+3$
$\mathcal{H}_{LR}$	$2E+19$	$2E+16$	$2E+13$	$2E+10$	$3E+7$	$1E+5$	$2E+3$
$\mathcal{H}_{TB}$	$2E+19$	$2E+16$	$2E+13$	$2E+10$	$3E+7$	$1E+5$	$9E+2$
$\mathcal{H}_{CTB}$	$2E+19$	$2E+16$	$2E+13$	$2E+10$	$3E+7$	$1E+5$	$2E+3$
$\mathcal{R}^*$	$-$	32	32	32	32	32	32
$\mathcal{R}_{TM}$	$-$	$1E+3$	987.1	906.1	670.4	261.3	39.0
$\mathcal{R}_{LR}$	$-$	$1E+3$	987.1	906.1	671.5	276.3	68.6
$\mathcal{R}_{TB}$	$-$	$1E+3$	987.1	906.1	674.5	302.4	115.0
$\mathcal{R}_{CTB}$	$-$	$1E+3$	987.1	906.1	671.5	276.3	68.6
t_{TM}	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$
t_{LR}	$3E-2$	$3E-2$	$3E-2$	$3E-2$	$3E-2$	$3E-2$	$3E-2$
t_{TB}	16.0	14.6	14.2	13.9	16.6	14.5	62.5
t_{CTB}	$3E-2$	$3E-2$	$3E-2$	$3E-2$	$3E-2$	$3E-2$	$3E-2$
MLL_{TM}	$-$	$-$	$-$	$-$	$-$	$-$	$-$
MLL_{LR}	$-$	$-$	$-$	$-$	$-$	$-$	$-$
MLL_{TB}	87	84	82	84	117	112	120
MLL_{CTB}	0	0	0	0	0	0	0
SD_{TM}	$-$	$-$	$-$	$-$	$-$	$-$	$-$
SD_{LR}	$-$	$-$	$-$	$-$	$-$	$-$	$-$
SD_{TB}	1455	1322	1312	1300	1258	1250	2906
SD_{CTB}	0	0	0	0	0	0	0

Table 4.4 (Contd.) For Taylor order $m = 4$:

i	0	1	2	3	4	5	6	7
$w\left(\mathbf{X}^{(i)}\right)$	$2*2^{-0}$	$2*2^{-1}$	$2*2^{-2}$	$2*2^{-3}$	$2*2^{-4}$	$2*2^{-5}$	$2*2^{-6}$	$2*2^{-7}$
$\mathcal{H}_{TM}$	$4E+2$	$9E+1$	$2E+1$	$5E+0$	$1E+0$	$3E-1$	$7E-2$	$2E-2$
$\mathcal{H}_{LR}$	$1E+1$	$2E-1$	–	–	–	–	–	–
$\mathcal{H}_{TB}$	$1E+1$	$2E-1$	$5E-3$	$1E-4$	$3E-6$	$8E-8$	$2E-9$	$7E-11$
$\mathcal{H}_{CTB}$	$1E+1$	$2E-1$	$5E-3$	$1E-4$	$3E-6$	$8E-8$	$2E-9$	$7E-11$
$\mathcal{R}^*$	32	32	32	32	32	32	32	32
$\mathcal{R}_{TM}$	7.2	4.9	4.4	4.2	4.1	4.1	4.0	4.0
$\mathcal{R}_{LR}$	114.4	56.3	–	–	–	–	–	–
$\mathcal{R}_{TB}$	62.1	56.3	50.4	44.5	39.8	36.5	34.3	30.6
$\mathcal{R}_{CTB}$	114.4	56.3	50.4	44.5	39.8	36.5	34.3	30.6
t_{TM}	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$
t_{LR}	$3E-2$	$3E-2$	–	–	–	–	–	–
t_{TB}	$6E-2$	$6E-2$	$6E-2$	$6E-2$	$6E-2$	$6E-2$	$6E-2$	$6E-2$
t_{CTB}	$3E-2$	$3E-2$	$6E-2$	$6E-2$	$6E-2$	$6E-2$	$6E-2$	$6E-2$
MLL_{TM}	–	–	–	–	–	–	–	–
MLL_{LR}	–	–	–	–	–	–	–	–
MLL_{TB}	0	0	0	0	0	0	0	0
MLL_{CTB}	0	0	0	0	0	0	0	0
SD_{TM}	–	–	–	–	–	–	–	–
SD_{LR}	–	–	–	–	–	–	–	–
SD_{TB}	0	0	0	0	0	0	0	0
SD_{CTB}	0	0	0	0	0	0	0	0

Table 4.4 (Contd.) For Taylor order $m = 6$:

i	-7	-6	-5	-4	-3	-2	-1
$w\left(\mathbf{X}^{(i)}\right)$	$2*2^7$	$2*2^6$	$2*2^5$	$2*2^4$	$2*2^3$	$2*2^2$	$2*2^1$
$\mathcal{H}_{TM}$	$3E+24$	$2E+20$	$2E+16$	$9E+11$	$1E+8$	$1E+5$	$3E+3$
$\mathcal{H}_{LR}$	$3E+24$	$2E+20$	$2E+16$	$9E+11$	$1E+8$	$1E+5$	$4E+2$
$\mathcal{H}_{TB}$	$3E+24$	$2E+20$	$2E+16$	$9E+11$	$1E+8$	$1E+5$	$4E+2$
$\mathcal{H}_{CTB}$	$3E+24$	$2E+20$	$2E+16$	$9E+11$	$1E+8$	$1E+5$	$4E+2$
$\mathcal{R}^*$	$-$	128	128	128	128	128	128
$\mathcal{R}_{TM}$	$-$	$2E+4$	$2E+4$	$1E+4$	$7E+3$	$9E+2$	47.0
$\mathcal{R}_{LR}$	$-$	$2E+4$	$2E+4$	$1E+4$	$7E+3$	$1E+3$	270.3
$\mathcal{R}_{TB}$	$-$	$2E+4$	$2E+4$	$1E+4$	$7E+3$	$1E+3$	361.5
$\mathcal{R}_{CTB}$	$-$	$2E+4$	$2E+4$	$1E+4$	$7E+3$	$1E+3$	270.3
t_{TM}	$2E-2$	$2E-2$	$2E-2$	$2E-2$	$2E-2$	$2E-2$	$2E-2$
t_{LR}	$2E-1$	$2E-1$	$2E-1$	$2E-1$	$2E-1$	$2E-1$	$2E-1$
t_{TB}	226.6	224.5	244.7	397.9	186.1	201.0	229.3
t_{CTB}	$2E-1$	$2E-1$	$2E-1$	$2E-1$	$2E-1$	$2E-1$	$2E-1$
MLL_{TM}	$-$	$-$	$-$	$-$	$-$	$-$	$-$
MLL_{LR}	$-$	$-$	$-$	$-$	$-$	$-$	$-$
MLL_{TB}	214	216	285	319	224	274	109
MLL_{CTB}	0	0	0	0	0	0	0
SD_{TM}	$-$	$-$	$-$	$-$	$-$	$-$	$-$
SD_{LR}	$-$	$-$	$-$	$-$	$-$	$-$	$-$
SD_{TB}	2183	2164	2055	2253	1737	1839	3172
SD_{CTB}	0	0	0	0	0	0	0

Table 4.4 (Contd.) For Taylor order $m = 6$:

i	0	1	2	3	4	5	6	7
$w\left(\mathbf{X}^{(i)}\right)$	$2*2^{-0}$	$2*2^{-1}$	$2*2^{-2}$	$2*2^{-3}$	$2*2^{-4}$	$2*2^{-5}$	$2*2^{-6}$	$2*2^{-7}$
$\mathcal{H}_{TM}$	$4E+2$	$9E+1$	$2E+1$	$5E+0$	$1E+0$	$3E-1$	$7E-2$	$2E-2$
$\mathcal{H}_{LR}$	$2E+0$	$9E-3$	–	–	–	–	–	–
$\mathcal{H}_{TB}$	$2E+0$	$9E-3$	$6E-5$	$4E-7$	$3E-9$	$3E-11$	$7E-12$	$7E-12$
$\mathcal{H}_{CTB}$	$2E+0$	$9E-3$	$6E-5$	$4E-7$	$3E-9$	$3E-11$	$7E-12$	$7E-12$
$\mathcal{R}^*$	128	128	128	128	128	128	128	128
$\mathcal{R}_{TM}$	7.3	4.9	4.4	4.2	4.1	4.1	4.0	4.0
$\mathcal{R}_{LR}$	234.1	189.0	–	–	–	–	–	–
$\mathcal{R}_{TB}$	175.0	189.0	167.6	151.3	140.5	99.2	3.6	1.0
$\mathcal{R}_{CTB}$	234.1	189.0	167.6	151.3	140.5	99.2	3.6	1.0
t_{TM}	$2E-2$	$2E-2$	$2E-2$	$2E-2$	$2E-2$	$2E-2$	$2E-2$	$2E-2$
t_{LR}	$2E-1$	$2E-1$	–	–	–	–	–	–
t_{TB}	$3E-1$	$3E-1$	$3E-1$	$3E-1$	$3E-1$	$3E-1$	$3E-1$	$3E-1$
t_{CTB}	$2E-1$	$2E-1$	$3E-1$	$3E-1$	$3E-1$	$3E-1$	$3E-1$	$3E-1$
MLL_{TM}	–	–	–	–	–	–	–	–
MLL_{LR}	–	–	–	–	–	–	–	–
MLL_{TB}	0	0	0	0	0	0	0	0
MLL_{CTB}	0	0	0	0	0	0	0	0
SD_{TM}	–	–	–	–	–	–	–	–
SD_{LR}	–	–	–	–	–	–	–	–
SD_{TB}	0	0	0	0	0	0	0	0
SD_{CTB}	0	0	0	0	0	0	0	0

Table 4.4 (Contd.) For Taylor order $m = 8$:

i	-7	-6	-5	-4	-3	-2	-1
$w\left(\mathbf{X}^{(i)}\right)$	$2*2^7$	$2*2^6$	$2*2^5$	$2*2^4$	$2*2^3$	$2*2^2$	$2*2^1$
$\mathcal{H}_{TM}$	$2E+29$	$6E+23$	$3E+18$	$1E+13$	$3E+8$	$1E+5$	$3E+3$
$\mathcal{H}_{LR}$	$2E+29$	$6E+23$	$3E+18$	$1E+13$	$3E+8$	$7E+4$	$1E+2$
$\mathcal{H}_{TB}$	$2E+29$	$6E+23$	$3E+18$	$1E+13$	$3E+8$	$7E+4$	$1E+2$
$\mathcal{H}_{CTB}$	$2E+29$	$6E+23$	$3E+18$	$1E+13$	$3E+8$	$7E+4$	$1E+2$
$\mathcal{R}^*$	$-$	512	512	512	512	512	512
$\mathcal{R}_{TM}$	$-$	$3E+5$	$2E+5$	$2E+5$	$5E+4$	$2E+3$	43.7
$\mathcal{R}_{LR}$	$-$	$3E+5$	$2E+5$	$2E+5$	$5E+4$	$4E+3$	514.5
$\mathcal{R}_{TB}$	$-$	$3E+5$	$2E+5$	$2E+4$	$5E+4$	$5E+3$	$1E+3$
$\mathcal{R}_{CTB}$	$-$	$3E+5$	$2E+5$	$2E+5$	$5E+4$	$4E+3$	514.5
t_{TM}	$4E-2$	$4E-2$	$4E-2$	$4E-2$	$4E-2$	$4E-2$	$4E-2$
t_{LR}	1.4	1.4	1.4	1.4	1.4	1.4	1.4
t_{TB}	3960	4148	3721	3442	1039	760	457
t_{CTB}	1.4	1.4	1.4	1.4	1.4	1.4	1.4
MLL_{TM}	$-$	$-$	$-$	$-$	$-$	$-$	$-$
MLL_{LR}	$-$	$-$	$-$	$-$	$-$	$-$	$-$
MLL_{TB}	410	411	411	411	378	256	76
MLL_{CTB}	0	0	0	0	0	0	0
SD_{TM}	$-$	$-$	$-$	$-$	$-$	$-$	$-$
SD_{LR}	$-$	$-$	$-$	$-$	$-$	$-$	$-$
SD_{TB}	9446	9483	8734	8029	2214	3043	2533
SD_{CTB}	0	0	0	0	0	0	0

Table 4.4 (Contd.) For Taylor order $m = 8$:

i	0	1	2	3	4	5	6	7
$w\left(\mathbf{X}^{(i)}\right)$	$2*2^{-0}$	$2*2^{-1}$	$2*2^{-2}$	$2*2^{-3}$	$2*2^{-4}$	$2*2^{-5}$	$2*2^{-6}$	$2*2^{-7}$
$\mathcal{H}_{TM}$	$4E+2$	$9E+1$	$2E+1$	$5E+0$	$1E+0$	$3E-1$	$7E-2$	$2E-2$
$\mathcal{H}_{LR}$	$5E-1$	$6E-5$	–	–	–	–	–	–
$\mathcal{H}_{TB}$	$5E-1$	$6E-5$	$8E-8$	$1E-10$	$8E-12$	$7E-12$	$7E-12$	$7E-12$
$\mathcal{H}_{CTB}$	$5E-1$	$6E-5$	$8E-8$	$1E-10$	$8E-12$	$7E-12$	$7E-12$	$7E-12$
$\mathcal{R}^*$	512	512	512	512	512	512	512	512
$\mathcal{R}_{TM}$	7.3	4.9	4.4	4.2	4.1	4.1	4.0	4.0
$\mathcal{R}_{LR}$	$3E+3$	828.6	–	–	–	–	–	–
$\mathcal{R}_{TB}$	909.9	828.6	734.6	623.2	17.2	1.1	1.0	0.9
$\mathcal{R}_{CTB}$	$3E+3$	828.6	734.6	623.2	17.2	1.1	1.0	0.9
t_{TM}	$4E-2$	$4E-2$	$4E-2$	$4E-2$	$4E-2$	$4E-2$	$4E-2$	$4E-2$
t_{LR}	1.4	1.4	–	–	–	–	–	–
t_{TB}	1.5	1.5	1.5	1.5	1.5	1.5	1.5	1.5
t_{CTB}	1.4	1.4	1.5	1.5	1.5	1.5	1.5	1.5
MLL_{TM}	–	–	–	–	–	–	–	–
MLL_{LR}	–	–	–	–	–	–	–	–
MLL_{TB}	0	0	0	0	0	0	0	0
MLL_{CTB}	0	0	0	0	0	0	0	0
SD_{TM}	–	–	–	–	–	–	–	–
SD_{LR}	–	–	–	–	–	–	–	–
SD_{TB}	0	0	0	0	0	0	0	0
SD_{CTB}	0	0	0	0	0	0	0	0

TABLE 4.5. Overestimations and their reduction ratios for various Taylor orders obtained with Taylor model, Algorithm LR, Algorithm TB and Algorithm CTB in Example 4.5 Griewank ($5-dim$).

For Taylor order $m=2$:

i	-7	-6	-5	-4	-3	-2	-1
$w\left(\mathbf{X}^{(i)}\right)$	$2*2^7$	$2*2^6$	$2*2^5$	$2*2^4$	$2*2^3$	$2*2^2$	$2*2^1$
$\mathcal{H}_{TM}$	$5E+24$	$2E+20$	$6E+15$	$3E+11$	$3E+7$	$7E+3$	$3E+1$
$\mathcal{H}_{LR}$	$4E+24$	$1E+20$	$6E+15$	$3E+11$	$3E+7$	$7E+3$	$3E+1$
$\mathcal{H}_{TB}$	–	–	–	–	–	–	–
$\mathcal{H}_{CTB}$	$4E+24$	$1E+20$	$6E+15$	$3E+11$	$3E+7$	$7E+3$	$3E+1$
$\mathcal{R}^*$	–	8	8	8	8	8	8
$\mathcal{R}_{TM}$	–	$3E+4$	$3E+4$	$2E+4$	$1E+4$	$3E+3$	$3E+2$
$\mathcal{R}_{LR}$	–	$3E+4$	$2E+4$	$2E+4$	$1E+4$	$3E+3$	$3E+2$
$\mathcal{R}_{TB}$	–	–	–	–	–	–	–
$\mathcal{R}_{CTB}$	–	$3E+4$	$3E+4$	$2E+4$	$1E+4$	$3E+3$	$3E+2$
t_{TM}	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$
t_{LR}	$2E-2$	$2E-2$	$2E-2$	$2E-2$	$2E-2$	$2E-2$	$2E-2$
t_{TB}	–	–	–	–	–	–	–
t_{CTB}	$2E-2$	$2E-2$	$2E-2$	$2E-2$	$2E-2$	$2E-2$	$2E-2$
MLL_{TM}	–	–	–	–	–	–	–
MLL_{LR}	–	–	–	–	–	–	–
MLL_{TB}	–	–	–	–	–	–	–
MLL_{CTB}	0	0	0	0	0	0	0
SD_{TM}	–	–	–	–	–	–	–
SD_{LR}	–	–	–	–	–	–	–
SD_{TB}	–	–	–	–	–	–	–
SD_{CTB}	0	0	0	0	0	0	0

Table 4.5 (Contd.) For Taylor order $m = 2$:

i	0	1	2	3	4	5	6	7
$w\left(\mathbf{X}^{(i)}\right)$	$2*2^{-0}$	$2*2^{-1}$	$2*2^{-2}$	$2*2^{-3}$	$2*2^{-4}$	$2*2^{-5}$	$2*2^{-6}$	$2*2^{-7}$
$\mathcal{H}_{TM}$	$1E+0$	$1E-1$	$3E-2$	$7E-3$	$2E-3$	$4E-4$	$1E-4$	$3E-5$
$\mathcal{H}_{LR}$	$1E+0$	$6E-2$	$5E-3$	–	–	–	–	–
$\mathcal{H}_{TB}$	–	$6E-2$	$5E-3$	$6E-4$	$6E-5$	$7E-6$	$9E-7$	$1E-7$
$\mathcal{H}_{CTB}$	$1E+0$	$6E-2$	$5E-3$	$6E-4$	$6E-5$	$7E-6$	$9E-7$	$1E-7$
$\mathcal{R}^*$	8	8	8	8	8	8	8	8
$\mathcal{R}_{TM}$	26.3	7.0	4.7	4.3	4.1	4.1	4.0	4.0
$\mathcal{R}_{LR}$	30.4	14.2	11.2	–	–	–	–	–
$\mathcal{R}_{TB}$	–	–	11.2	9.7	8.9	8.5	8.2	8.1
$\mathcal{R}_{CTB}$	30.4	14.2	11.2	9.7	8.9	8.5	8.2	8.1
t_{TM}	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$
t_{LR}	$2E-2$	$2E-2$	81.0	–	–	–	–	–
t_{TB}	–	$5E-1$	$5E-1$	$5E-1$	$5E-1$	$5E-1$	$5E-1$	$5E-1$
t_{CTB}	$2E-2$	$2E-2$	$5E-1$	$5E-1$	$5E-1$	$5E-1$	$5E-1$	$5E-1$
MLL_{TM}	–	–	–	–	–	–	–	–
MLL_{LR}	–	–	–	–	–	–	–	–
MLL_{TB}	–	0	0	0	0	0	0	0
MLL_{CTB}	0	0	0	0	0	0	0	0
SD_{TM}	–	–	–	–	–	–	–	–
SD_{LR}	–	–	–	–	–	–	–	–
SD_{TB}	–	0	0	0	0	0	0	0
SD_{CTB}	0	0	0	0	0	0	0	0

Table 4.5 (Contd.) For Taylor order $m = 4$:

i	-7	-6	-5	-4	-3	-2	-1
$w\left(\mathbf{X}^{(i)}\right)$	$2*2^7$	$2*2^6$	$2*2^5$	$2*2^4$	$2*2^3$	$2*2^2$	$2*2^1$
$\mathcal{H}_{TM}$	$1E+37$	$5E+29$	$2E+22$	$2E+15$	$9E+8$	$1E+4$	$3E+1$
$\mathcal{H}_{LR}$	$1E+37$	$5E+29$	$2E+22$	$2E+15$	$9E+8$	$1E+4$	$2E+1$
$\mathcal{H}_{TB}$	$-$	$-$	$-$	$-$	$-$	$-$	$-$
$\mathcal{H}_{CTB}$	$1E+37$	$5E+29$	$2E+22$	$2E+15$	$9E+8$	$1E+4$	$2E+1$
$\mathcal{R}^*$	$-$	32	32	32	32	32	32
$\mathcal{R}_{TM}$	$-$	$3E+7$	$2E+7$	$1E+7$	$2E+6$	$6E+4$	$5E+2$
$\mathcal{R}_{LR}$	$-$	$3E+7$	$2E+7$	$1E+7$	$2E+6$	$6E+4$	$6E+2$
$\mathcal{R}_{TB}$	$-$	$-$	$-$	$-$	$-$	$-$	$-$
$\mathcal{R}_{CTB}$	$-$	$3E+7$	$2E+7$	$1E+7$	$2E+6$	$6E+4$	$6E+2$
t_{TM}	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$
t_{LR}	$3E-1$	$3E-1$	$3E-1$	$3E-1$	$3E-1$	$3E-1$	$3E-1$
t_{TB}	$-$	$-$	$-$	$-$	$-$	$-$	$-$
t_{CTB}	$3E-1$	$3E-1$	$3E-1$	$3E-1$	$3E-1$	$3E-1$	$3E-1$
MLL_{TM}	$-$	$-$	$-$	$-$	$-$	$-$	$-$
MLL_{LR}	$-$	$-$	$-$	$-$	$-$	$-$	$-$
MLL_{TB}	$-$	$-$	$-$	$-$	$-$	$-$	$-$
MLL_{CTB}	0	0	0	0	0	0	0
SD_{TM}	$-$	$-$	$-$	$-$	$-$	$-$	$-$
SD_{LR}	$-$	$-$	$-$	$-$	$-$	$-$	$-$
SD_{TB}	$-$	$-$	$-$	$-$	$-$	$-$	$-$
SD_{CTB}	0	0	0	0	0	0	0

Table 4.5 (Contd.) For Taylor order $m = 4$:

i	0	1	2	3	4	5	6	7
$w\left(\mathbf{X}^{(i)}\right)$	$2*2^{-0}$	$2*2^{-1}$	$2*2^{-2}$	$2*2^{-3}$	$2*2^{-4}$	$2*2^{-5}$	$2*2^{-6}$	$2*2^{-7}$
$\mathcal{H}_{TM}$	$1E+0$	$1E-1$	$3E-2$	$7E-3$	$2E-3$	$4E-4$	$1E-4$	$3E-5$
$\mathcal{H}_{LR}$	$4E-1$	$9E-3$	–	–	–	–	–	–
$\mathcal{H}_{TB}$	–	–	$2E-4$	$7E-6$	$2E-7$	$6E-9$	$2E-10$	$3E-12$
$\mathcal{H}_{CTB}$	$4E-1$	$9E-3$	$2E-4$	$7E-6$	$2E-7$	$6E-9$	$2E-10$	$3E-12$
$\mathcal{R}^*$	32	32	32	32	32	32	32	32
$\mathcal{R}_{TM}$	25.9	7.2	4.8	4.3	4.1	4.1	4.0	4.0
$\mathcal{R}_{LR}$	62.8	42.9	–	–	–	–	–	–
$\mathcal{R}_{TB}$	–	–	–	35.1	33.6	32.9	32.6	67.3
$\mathcal{R}_{CTB}$	62.8	42.9	37.8	35.1	33.6	32.9	32.6	67.3
t_{TM}	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$
t_{LR}	$3E-1$	$3E-1$	–	–	–	–	–	–
t_{TB}	–	–	$9E-1$	$9E-1$	$9E-1$	$9E-1$	$9E-1$	$9E-1$
t_{CTB}	$3E-1$	$3E-1$	$9E-1$	$9E-1$	$9E-1$	$9E-1$	$9E-1$	$9E-1$
MLL_{TM}	–	–	–	–	–	–	–	–
MLL_{LR}	–	–	–	–	–	–	–	–
MLL_{TB}	–	–	0	0	0	0	0	0
MLL_{CTB}	–	–	0	0	0	0	0	0
SD_{TM}	–	–	–	–	–	–	–	–
SD_{LR}	–	–	–	–	–	–	–	–
SD_{TB}	–	0	0	0	0	0	0	0
SD_{CTB}	0	0	0	0	0	0	0	0

Table 4.5 (Contd.) For Taylor order $m = 6$:

i	-7	-6	-5	-4	-3	-2	-1
$w\left(\mathbf{X}^{(i)}\right)$	$2*2^7$	$2*2^6$	$2*2^5$	$2*2^4$	$2*2^3$	$2*2^2$	$2*2^1$
$\mathcal{H}_{TM}$	$1E+48$	$5E+37$	$3E+27$	$5E+17$	$3E+9$	$2E+4$	$3E+1$
$\mathcal{H}_{LR}$	$1E+48$	$5E+37$	$3E+27$	$5E+17$	$3E+9$	$2E+4$	$2E+1$
$\mathcal{H}_{TB}$	$-$	$-$	$-$	$-$	$-$	$-$	$-$
$\mathcal{H}_{CTB}$	$1E+48$	$5E+37$	$3E+27$	$5E+17$	$3E+9$	$2E+4$	$2E+1$
$\mathcal{R}^*$	$-$	128	128	128	128	128	128
$\mathcal{R}_{TM}$	$-$	$3E+10$	$2E+10$	$5E+9$	$2E+8$	$2E+5$	$5E+2$
$\mathcal{R}_{LR}$	$-$	$2E+10$	$2E+10$	$7E+9$	$4E+8$	$2E+5$	$4E+2$
$\mathcal{R}_{TB}$	$-$	$-$	$-$	$-$	$-$	$-$	$-$
$\mathcal{R}_{CTB}$	$-$	$2E+10$	$2E+10$	$7E+9$	$4E+8$	$2E+5$	$4E+2$
t_{TM}	$2E-2$	$2E-2$	$2E-2$	$2E-2$	$2E-2$	$2E-2$	$2E-2$
t_{LR}	7.5	7.5	7.5	7.5	7.5	7.5	7.5
t_{TB}	$-$	$-$	$-$	$-$	$-$	$-$	$-$
t_{CTB}	7.5	7.5	7.5	7.5	7.5	7.5	7.5
MLL_{TM}	$-$	$-$	$-$	$-$	$-$	$-$	$-$
MLL_{LR}	$-$	$-$	$-$	$-$	$-$	$-$	$-$
MLL_{TB}	$-$	$-$	$-$	$-$	$-$	$-$	$-$
MLL_{CTB}	0	0	0	0	0	0	0
SD_{TM}	$-$	$-$	$-$	$-$	$-$	$-$	$-$
SD_{LR}	$-$	$-$	$-$	$-$	$-$	$-$	$-$
SD_{TB}	$-$	$-$	$-$	$-$	$-$	$-$	$-$
SD_{CTB}	0	0	0	0	0	0	0

Table 4.5 (Contd.) For Taylor order $m = 6$:

i	0	1	2	3	4	5	6	7
$w\left(\mathbf{X}^{(i)}\right)$	$2*2^{-0}$	$2*2^{-1}$	$2*2^{-2}$	$2*2^{-3}$	$2*2^{-4}$	$2*2^{-5}$	$2*2^{-6}$	$2*2^{-7}$
$\mathcal{H}_{TM}$	$1E+0$	$1E-1$	$3E-2$	$7E-3$	$2E-3$	$4E-4$	$1E-4$	$3E-5$
$\mathcal{H}_{LR}$	$6E-2$	$3E-4$	–	–	–	–	–	–
$\mathcal{H}_{TB}$	–	$3E-4$	$2E-6$	$1E-8$	$9E-11$	$6E-12$	$1E-11$	$9E-12$
$\mathcal{H}_{CTB}$	$6E-2$	$3E-4$	$2E-6$	$1E-8$	$9E-11$	$6E-12$	$1E-11$	$9E-12$
$\mathcal{R}^*$	128	128	128	128	128	128	128	128
$\mathcal{R}_{TM}$	29.7	7.4	4.8	4.3	4.1	4.1	4.0	4.0
$\mathcal{R}_{LR}$	255.2	196.1	–	–	–	–	–	–
$\mathcal{R}_{TB}$	–	–	166.2	148.6	143.4	14.3	0.5	1.3
$\mathcal{R}_{CTB}$	255.2	196.1	166.2	148.6	143.4	14.3	0.5	1.3
t_{TM}	$2E-2$	$2E-2$	$2E-2$	$2E-2$	$2E-2$	$2E-2$	$2E-2$	$2E-2$
t_{LR}	7.5	7.5	–	–	–	–	–	–
t_{TB}	–	8.0	8.0	8.0	8.0	8.0	8.0	8.0
t_{CTB}	7.5	7.5	8.0	8.0	8.0	8.0	8.0	8.0
MLL_{TM}	–	–	–	–	–	–	–	–
MLL_{LR}	–	–	–	–	–	–	–	–
MLL_{TB}	–	0	0	0	0	0	0	0
MLL_{CTB}	0	0	0	0	0	0	0	0
SD_{TM}	–	–	–	–	–	–	–	–
SD_{LR}	–	–	–	–	–	–	–	–
SD_{TB}	–	0	0	0	0	0	0	0
SD_{CTB}	0	0	0	0	0	0	0	0

Table 4.5 (Contd.) For Taylor order $m = 8$:

i	-7	-6	-5	-4	-3	-2	-1
$w\left(\mathbf{X}^{(i)}\right)$	$2*2^7$	$2*2^6$	$2*2^5$	$2*2^4$	$2*2^3$	$2*2^2$	$2*2^1$
$\mathcal{H}_{TM}$	$7E+57$	$3E+44$	$2E+31$	$1E+19$	$5E+9$	$2E+4$	$3E+1$
$\mathcal{H}_{LR}$	$7E+57$	$3E+44$	$2E+31$	$1E+19$	$5E+9$	$2E+4$	$1E+1$
$\mathcal{H}_{TB}$	$-$	$-$	$-$	$-$	$-$	$-$	$-$
$\mathcal{H}_{CTB}$	$7E+57$	$3E+44$	$2E+31$	$1E+19$	$5E+9$	$2E+4$	$1E+1$
$\mathcal{R}^*$	$-$	256	256	256	256	256	256
$\mathcal{R}_{TM}$	$-$	$2E+13$	$1E+13$	$2E+12$	$3E+9$	$3E+5$	$5E+2$
$\mathcal{R}_{LR}$	$-$	$2E+13$	$1E+13$	$2E+12$	$3E+9$	$3E+5$	$2E+3$
$\mathcal{R}_{TB}$	$-$	$-$	$-$	$-$	$-$	$-$	$-$
$\mathcal{R}_{CTB}$	$-$	$2E+13$	$1E+13$	$2E+12$	$3E+9$	$3E+5$	$2E+3$
t_{TM}	$3E-2$	$3E-2$	$3E-2$	$3E-2$	$3E-2$	$3E-2$	$3E-2$
t_{LR}	82.4	82.4	82.4	82.4	82.4	82.4	82.4
t_{TB}	$-$	$-$	$-$	$-$	$-$	$-$	$-$
t_{CTB}	82.4	82.4	82.4	82.4	82.4	82.4	82.4
MLL_{TM}	$-$	$-$	$-$	$-$	$-$	$-$	$-$
MLL_{LR}	$-$	$-$	$-$	$-$	$-$	$-$	$-$
MLL_{TB}	$-$	$-$	$-$	$-$	$-$	$-$	$-$
MLL_{CTB}	0	0	0	0	0	0	0
SD_{TM}	$-$	$-$	$-$	$-$	$-$	$-$	$-$
SD_{LR}	$-$	$-$	$-$	$-$	$-$	$-$	$-$
SD_{TB}	$-$	$-$	$-$	$-$	$-$	$-$	$-$
SD_{CTB}	0	0	0	0	0	0	0

Table 4.5 (Contd.) For Taylor order $m = 8$:

i	0	1	2	3	4	5	6	7
$w\left(\mathbf{X}^{(i)}\right)$	$2*2^{-0}$	$2*2^{-1}$	$2*2^{-2}$	$2*2^{-3}$	$2*2^{-4}$	$2*2^{-5}$	$2*2^{-6}$	$2*2^{-7}$
$\mathcal{H}_{TM}$	$1E+0$	$1E-1$	$3E-2$	$7E-3$	$2E-3$	$4E-4$	$1E-4$	$3E-5$
$\mathcal{H}_{LR}$	$1E-2$	$2E-5$	–	–	–	–	–	–
$\mathcal{H}_{TB}$	–	$2E-5$	$4E-8$	$7E-11$	$1E-11$	$7E-12$	$1E-11$	$9E-12$
$\mathcal{H}_{CTB}$	$1E-2$	$2E-5$	$4E-8$	$7E-11$	$1E-11$	$7E-12$	$1E-11$	$9E-12$
$\mathcal{R}^*$	512	512	512	512	512	512	512	512
$\mathcal{R}_{TM}$	31.6	7.4	4.8	4.3	4.1	4.1	4.1	4.0
$\mathcal{R}_{LR}$	700	599.2	–	–	–	–	–	–
$\mathcal{R}_{TB}$	–	–	557.5	583.4	7.1	1.3	0.7	1.3
$\mathcal{R}_{CTB}$	700	599.2	557.5	583.4	7.1	1.3	0.7	1.3
t_{TM}	$3E-2$	$3E-2$	$3E-2$	$3E-2$	$3E-2$	$3E-2$	$3E-2$	$3E-2$
t_{LR}	82.4	82.4	–	–	–	–	–	–
t_{TB}	–	83.6	83.6	83.6	83.6	83.6	83.6	83.6
t_{CTB}	82.4	82.4	83.6	83.6	83.6	83.6	83.6	83.6
MLL_{TM}	–	–	–	–	–	–	–	–
MLL_{LR}	–	–	–	–	–	–	–	–
MLL_{TB}	–	0	0	0	0	0	0	0
MLL_{CTB}	0	0	0	0	0	0	0	0
SD_{TM}	–	–	–	–	–	–	–	–
SD_{LR}	–	–	–	–	–	–	–	–
SD_{TB}	–	0	0	0	0	0	0	0
SD_{CTB}	0	0	0	0	0	0	0	0

TABLE 4.6. Overestimations and their reduction ratios for various Taylor orders obtained with Taylor model, Algorithm LR, Algorithm TB and Algorithm CTB in Example 4.6 Trigonometric ($6 - dim$).

For Taylor order $m = 2$:

i	-7	-6	-5	-4	-3	-2	-1
$w\left(\mathbf{X}^{(i)}\right)$	$2*2^7$	$2*2^6$	$2*2^5$	$2*2^4$	$2*2^3$	$2*2^2$	$2*2^1$
$\mathcal{H}_{TM}$	$8E+13$	$1E+12$	$2E+10$	$3E+8$	$6E+6$	$2E+5$	$1E+4$
$\mathcal{H}_{LR}$	$8E+13$	$1E+12$	$2E+10$	$3E+8$	$6E+6$	$2E+5$	$1E+4$
$\mathcal{H}_{TB}$	–	–	–	–	–	–	–
$\mathcal{H}_{CTB}$	$8E+13$	$1E+12$	$2E+10$	$3E+8$	$6E+6$	$2E+5$	$1E+4$
$\mathcal{R}^*$	–	8	8	8	8	8	8
$\mathcal{R}_{TM}$	–	63.6	63.0	61.2	55.8	37.7	16.7
$\mathcal{R}_{LR}$	–	63.6	63.0	61.2	55.8	37.7	16.7
$\mathcal{R}_{TB}$	–	–	–	–	–	–	–
$\mathcal{R}_{CTB}$	–	63.6	63.0	61.2	55.8	37.7	16.7
t_{TM}	$2E-2$	$2E-2$	$2E-2$	$2E-2$	$2E-2$	$2E-2$	$2E-2$
t_{LR}	$1E-1$	$1E-1$	$1E-1$	$1E-1$	$1E-1$	$1E-1$	$1E-1$
t_{TB}	–	–	–	–	–	–	–
t_{CTB}	$1E-1$	$1E-1$	$1E-1$	$1E-1$	$1E-1$	$1E-1$	$1E-1$
MLL_{TM}	–	–	–	–	–	–	–
MLL_{LR}	–	–	–	–	–	–	–
MLL_{TB}	–	–	–	–	–	–	–
MLL_{CTB}	0	0	0	0	0	0	0
SD_{TM}	–	–	–	–	–	–	–
SD_{LR}	–	–	–	–	–	–	–
SD_{TB}	–	–	–	–	–	–	–
SD_{CTB}	0	0	0	0	0	0	0

Table 4.6 (Contd.) For Taylor order $m = 2$:

i	0	1	2	3	4	5	6	7
$w\left(\mathbf{X}^{(i)}\right)$	$2*2^{-0}$	$2*2^{-1}$	$2*2^{-2}$	$2*2^{-3}$	$2*2^{-4}$	$2*2^{-5}$	$2*2^{-6}$	$2*2^{-7}$
$\mathcal{H}_{TM}$	$1E+3$	$3E+2$	$6E+1$	$2E+1$	$3E+0$	$1E+0$	$2E-1$	$6E-2$
$\mathcal{H}_{LR}$	$9E+2$	$9E+1$	$9E+0$	–	–	–	–	–
$\mathcal{H}_{TB}$	$9E+2$	$9E+1$	$9E+0$	$1E+0$	$1E-1$	$2E-2$	$2E-3$	$2E-4$
$\mathcal{H}_{CTB}$	$9E+2$	$9E+1$	$9E+0$	$1E+0$	$1E-1$	$2E-2$	$2E-3$	$2E-4$
$\mathcal{R}^*$	8	8	8	8	8	8	8	8
$\mathcal{R}_{TM}$	6.7	5.0	4.5	4.3	4.1	4.1	4.0	4.0
$\mathcal{R}_{LR}$	10.9	10.1	9.3	–	–	–	–	–
$\mathcal{R}_{TB}$	–	10.1	9.3	8.7	8.4	8.2	8.1	8.1
$\mathcal{R}_{CTB}$	10.9	10.1	9.3	8.7	8.4	8.1	8.1	8.1
t_{TM}	$2E-2$	$2E-2$	$2E-2$	$2E-2$	$2E-2$	$2E-2$	$2E-2$	$2E-2$
t_{LR}	$1E-1$	$1E-1$	9.4	–	–	–	–	–
t_{TB}	1.2	1.2	1.2	1.2	1.2	1.2	1.2	1.2
t_{CTB}	$1E-1$	$1E-1$	1.1	1.1	1.1	1.1	1.1	1.1
MLL_{TM}	–	–	–	–	–	–	–	–
MLL_{LR}	–	–	–	–	–	–	–	–
MLL_{TB}	0	0	0	0	0	0	0	0
MLL_{CTB}	0	0	0	0	0	0	0	0
SD_{TM}	–	–	–	–	–	–	–	–
SD_{LR}	–	–	–	–	–	–	–	–
SD_{TB}	0	0	0	0	0	0	0	0
SD_{CTB}	0	0	0	0	0	0	0	0

Table 4.6 (Contd.) For Taylor order $m = 4$:

i	-7	-6	-5	-4	-3	-2	-1
$w\left(\mathbf{X}^{(i)}\right)$	$2*2^7$	$2*2^6$	$2*2^5$	$2*2^4$	$2*2^3$	$2*2^2$	$2*2^1$
$\mathcal{H}_{TM}$	$6E+19$	$6E+16$	$6E+13$	$6E+10$	$1E+8$	$4E+5$	$1E+4$
$\mathcal{H}_{LR}$	$6E+19$	$6E+16$	$6E+13$	$6E+10$	$1E+8$	$4E+5$	$5E+3$
$\mathcal{H}_{TB}$	$-$	$-$	$-$	$-$	$-$	$-$	$-$
$\mathcal{H}_{CTB}$	$6E+19$	$6E+16$	$6E+13$	$6E+10$	$1E+8$	$4E+5$	$5E+3$
$\mathcal{R}^*$	$-$	32	32	32	32	32	32
$\mathcal{R}_{TM}$	$-$	$1E+3$	983.9	899.3	659.1	251.0	37.1
$\mathcal{R}_{LR}$	$-$	$1E+3$	983.9	899.3	661.1	271.4	66.5
$\mathcal{R}_{TB}$	$-$	$-$	$-$	$-$	$-$	$-$	$-$
$\mathcal{R}_{CTB}$	$-$	$1E+3$	983.9	899.3	661.1	271.4	66.5
t_{TM}	$4E-2$	$4E-2$	$4E-2$	$4E-2$	$4E-2$	$4E-2$	$4E-2$
t_{LR}	6.0	6.0	6.0	6.0	6.0	6.0	6.0
t_{TB}	$-$	$-$	$-$	$-$	$-$	$-$	$-$
t_{CTB}	6.0	6.0	6.0	6.0	6.0	6.0	6.0
MLL_{TM}	$-$	$-$	$-$	$-$	$-$	$-$	$-$
MLL_{LR}	$-$	$-$	$-$	$-$	$-$	$-$	$-$
MLL_{TB}	$-$	$-$	$-$	$-$	$-$	$-$	$-$
MLL_{CTB}	0	0	0	0	0	0	0
SD_{TM}	$-$	$-$	$-$	$-$	$-$	$-$	$-$
SD_{LR}	$-$	$-$	$-$	$-$	$-$	$-$	$-$
SD_{TB}	$-$	$-$	$-$	$-$	$-$	$-$	$-$
SD_{CTB}	0	0	0	0	0	0	0

Table 4.6 (Contd.) For Taylor order $m = 4$:

i	0	1	2	3	4	5	6	7
$w\left(\mathbf{X}^{(i)}\right)$	$2*2^{-0}$	$2*2^{-1}$	$2*2^{-2}$	$2*2^{-3}$	$2*2^{-4}$	$2*2^{-5}$	$2*2^{-6}$	$2*2^{-7}$
$\mathcal{H}_{TM}$	$1E+3$	$3E+2$	$6E+1$	$1E+1$	$3E+0$	$1E+0$	$2E-1$	$5E-2$
$\mathcal{H}_{LR}$	$4E+1$	$7E-1$	–	–	–	–	–	–
$\mathcal{H}_{TB}$	$4E+1$	$7E-1$	$1E-2$	$2E-4$	$5E-6$	$2E-7$	$3E-9$	$1E-10$
$\mathcal{H}_{CTB}$	$4E+1$	$7E-1$	$1E-2$	$2E-4$	$5E-6$	$2E-7$	$3E-9$	$1E-10$
$\mathcal{R}^*$	32	32	32	32	32	32	32	32
$\mathcal{R}_{TM}$	7.1	5.0	4.5	4.3	4.1	4.1	4.0	4.0
$\mathcal{R}_{LR}$	128.2	61.3	–	–	–	–	–	–
$\mathcal{R}_{TB}$	–	61.3	56.9	51.6	46.2	41.2	37.1	28.5
$\mathcal{R}_{CTB}$	128.2	61.3	56.9	51.6	46.2	41.2	37.1	28.5
t_{TM}	$4E-2$	$4E-2$	$4E-2$	$4E-2$	$4E-2$	$4E-2$	$4E-2$	$4E-2$
t_{LR}	6.0	6.0	–	–	–	–	–	–
t_{TB}	7.1	7.1	7.1	7.1	7.1	7.1	7.1	7.1
t_{CTB}	6.0	6.0	7.0	7.0	7.0	7.0	7.0	7.0
MLL_{TM}	–	–	–	–	–	–	–	–
MLL_{LR}	–	–	–	–	–	–	–	–
MLL_{TB}	0	0	0	0	0	0	0	0
MLL_{CTB}	0	0	0	0	0	0	0	0
SD_{TM}	–	–	–	–	–	–	–	–
SD_{LR}	–	–	–	–	–	–	–	–
SD_{TB}	0	0	0	0	0	0	0	0
SD_{CTB}	0	0	0	0	0	0	0	0

Table 4.6 (Contd.) For Taylor order $m = 6$:

i	-7	-6	-5	-4	-3	-2	-1
$w\left(\mathbf{X}^{(i)}\right)$	$2*2^7$	$2*2^6$	$2*2^5$	$2*2^4$	$2*2^3$	$2*2^2$	$2*2^1$
$\mathcal{H}_{TM}$	$9E+24$	$5E+20$	$4E+16$	$3E+12$	$4E+8$	$5E+5$	$1E+4$
$\mathcal{H}_{LR}$	$9E+24$	$5E+20$	$4E+16$	$3E+12$	$4E+8$	$4E+5$	$1E+3$
$\mathcal{H}_{TB}$	$-$	$-$	$-$	$-$	$-$	$-$	$-$
$\mathcal{H}_{CTB}$	$9E+24$	$5E+20$	$4E+16$	$3E+12$	$4E+8$	$4E+5$	$1E+3$
$\mathcal{R}^*$	$-$	128	128	128	128	128	128
$\mathcal{R}_{TM}$	$-$	$2E+4$	$2E+4$	$1E+4$	$7E+3$	$9E+2$	$4E+1$
$\mathcal{R}_{LR}$	$-$	$2E+4$	$2E+4$	$1E+4$	$7E+3$	$1E+3$	$3E+2$
$\mathcal{R}_{TB}$	$-$	$-$	$-$	$-$	$-$	$-$	$-$
$\mathcal{R}_{CTB}$	$-$	$2E+4$	$2E+4$	$1E+4$	$7E+3$	$1E+3$	$3E+2$
t_{TM}	$6E-2$	$4E-2$	$4E-2$	$4E-2$	$4E-2$	$4E-2$	$4E-2$
t_{LR}	283	283	283	283	283	283	283
t_{TB}	$-$	$-$	$-$	$-$	$-$	$-$	$-$
t_{CTB}	283	283	283	283	283	283	283
MLL_{TM}	$-$	$-$	$-$	$-$	$-$	$-$	$-$
MLL_{LR}	$-$	$-$	$-$	$-$	$-$	$-$	$-$
MLL_{TB}	$-$	$-$	$-$	$-$	$-$	$-$	$-$
MLL_{CTB}	0	0	0	0	0	0	0
SD_{TM}	$-$	$-$	$-$	$-$	$-$	$-$	$-$
SD_{LR}	$-$	$-$	$-$	$-$	$-$	$-$	$-$
SD_{TB}	$-$	$-$	$-$	$-$	$-$	$-$	$-$
SD_{CTB}	0	0	0	0	0	0	0

Table 4.6 (Contd.) For Taylor order $m = 6$:

i	0	1	2	3	4	5	6	7
$w\left(\mathbf{X}^{(i)}\right)$	$2*2^{-0}$	$2*2^{-1}$	$2*2^{-2}$	$2*2^{-3}$	$2*2^{-4}$	$2*2^{-5}$	$2*2^{-6}$	$2*2^{-7}$
$\mathcal{H}_{TM}$	$1E+3$	$3E+2$	$6E+1$	$1E+1$	$3E+0$	$1E+0$	$2E-1$	$5E-2$
$\mathcal{H}_{LR}$	$7E+0$	$4E-2$	–	–	–	–	–	–
$\mathcal{H}_{TB}$	$7E+0$	$4E-2$	$2E-4$	$2E-6$	$1E-8$	$1E-10$	$3E-11$	$3E-11$
$\mathcal{H}_{CTB}$	$7E+0$	$4E-2$	$2E-4$	$2E-6$	$1E-8$	$1E-10$	$3E-11$	$3E-11$
$\mathcal{R}^*$	128	128	128	128	128	128	128	128
$\mathcal{R}_{TM}$	7.5	5.0	4.5	4.3	4.1	4.1	4.0	4.0
$\mathcal{R}_{LR}$	206.1	176.4	–	–	–	–	–	–
$\mathcal{R}_{TB}$	–	176.4	157.8	144.9	136.8	103.7	4.4	1.0
$\mathcal{R}_{CTB}$	206.1	176.4	157.8	144.9	136.8	103.7	4.4	1.0
t_{TM}	$4E-2$	$4E-2$	$4E-2$	$4E-2$	$4E-2$	$4E-2$	$4E-2$	$4E-2$
t_{LR}	283	283	–	–	–	–	–	–
t_{TB}	284.2	284.2	284.2	284.2	284.2	284.2	284.2	284.2
t_{CTB}	283	283	284.2	284.2	284.2	284.2	284.2	284.2
MLL_{TM}	–	–	–	–	–	–	–	–
MLL_{LR}	–	–	–	–	–	–	–	–
MLL_{TB}	0	0	0	0	0	0	0	0
MLL_{CTB}	0	0	0	0	0	0	0	0
SD_{TM}	–	–	–	–	–	–	–	–
SD_{LR}	–	–	–	–	–	–	–	–
SD_{TB}	0	0	0	0	0	0	0	0
SD_{CTB}	0	0	0	0	0	0	0	0

Table 4.6 (Contd.) For Taylor order $m = 8$:

i	-7	-6	-5	-4	-3	-2	-1
$w\left(\mathbf{X}^{(i)}\right)$	$2*2^7$	$2*2^6$	$2*2^5$	$2*2^4$	$2*2^3$	$2*2^2$	$2*2^1$
$\mathcal{H}_{TM}$	$4E+29$	$2E+24$	$8E+18$	$4E+13$	$9E+8$	$5E+5$	$1E+4$
$\mathcal{H}_{LR}$	$4E+29$	$2E+24$	$8+18$	$4E+13$	$9E+8$	$2E+5$	$3E+3$
$\mathcal{H}_{TB}$	$-$	$-$	$-$	$-$	$-$	$-$	$-$
$\mathcal{H}_{CTB}$	$4E+29$	$2E+24$	$8E+18$	$4E+13$	$9E+8$	$2E+5$	$3E+3$
$\mathcal{R}^*$	$-$	256	256	256	256	256	256
$\mathcal{R}_{TM}$	$-$	$2E+5$	$2E+5$	$2E+5$	$5E+4$	$2E+3$	$4E+1$
$\mathcal{R}_{LR}$	$-$	$2E+5$	$2E+5$	$2E+5$	$5E+4$	$4E+3$	$8E+1$
$\mathcal{R}_{TB}$	$-$	$-$	$-$	$-$	$-$	$-$	$-$
$\mathcal{R}_{CTB}$	$-$	$2E+5$	$2E+5$	$2E+5$	$5E+4$	$4E+3$	$8E+1$
t_{TM}	$8E-2$	$8E-2$	$8E-2$	$8E-2$	$8E-2$	$8E-2$	$8E-2$
t_{LR}	5397.7	5397.7	5397.7	5397.7	5397.7	5397.7	5397.7
t_{TB}	$-$	$-$	$-$	$-$	$-$	$-$	$-$
t_{CTB}	5397.7	5397.7	5397.7	5397.7	5397.7	5397.7	5397.7
MLL_{TM}	$-$	$-$	$-$	$-$	$-$	$-$	$-$
MLL_{LR}	$-$	$-$	$-$	$-$	$-$	$-$	$-$
MLL_{TB}	$-$	$-$	$-$	$-$	$-$	$-$	$-$
MLL_{CTB}	0	0	0	0	0	0	0
SD_{TM}	$-$	$-$	$-$	$-$	$-$	$-$	$-$
SD_{LR}	$-$	$-$	$-$	$-$	$-$	$-$	$-$
SD_{TB}	$-$	$-$	$-$	$-$	$-$	$-$	$-$
SD_{CTB}	0	0	0	0	0	0	0

Table 4.6 (Contd.) For Taylor order $m = 8$:

i	0	1	2	3	4	5	6	7
$w\left(\mathbf{X}^{(i)}\right)$	$2*2^{-0}$	$2*2^{-1}$	$2*2^{-2}$	$2*2^{-3}$	$2*2^{-4}$	$2*2^{-5}$	$2*2^{-6}$	$2*2^{-7}$
$\mathcal{H}_{TM}$	$1E+3$	$3E+2$	$6E+1$	$1E+0$	$3E+0$	$1E+0$	$2E-1$	$5E-2$
$\mathcal{H}_{LR}$	$1E-1$	$2E-4$	–	–	–	–	–	–
$\mathcal{H}_{TB}$	$1E-1$	$2E-4$	$2E-7$	$3E-10$	$2E-11$	$3E-11$	$3E-11$	$3E-11$
$\mathcal{H}_{CTB}$	$1E-1$	$2E-4$	$2E-7$	$3E-10$	$2E-11$	$3E-11$	$3E-11$	$3E-11$
$\mathcal{R}^*$	512	512	512	512	512	512	512	512
$\mathcal{R}_{TM}$	7.4	5.0	4.5	4.3	4.1	4.1	4.0	4.0
$\mathcal{R}_{LR}$	$2E+4$	931.7	–	–	–	–	–	–
$\mathcal{R}_{TB}$	–	931.7	852.5	700.7	10.7	0.9	0.8	1.2
$\mathcal{R}_{CTB}$	$2E+4$	931.7	852.5	700.7	10.7	0.9	0.8	1.2
t_{TM}	$8E-2$	$8E-2$	$8E-2$	$8E-2$	$8E-2$	$8E-2$	$8E-2$	$8E-2$
t_{LR}	5397.7	5397.7	–	–	–	–	–	–
t_{TB}	5400	5400	5400	5400	5400	5400	5400	5400
t_{CTB}	5397.7	5397.7	5400	5400	5400	5400	5400	5400
MLL_{TM}	–	–	–	–	–	–	–	–
MLL_{LR}	–	–	–	–	–	–	–	–
MLL_{TB}	0	0	0	0	0	0	0	0
MLL_{CTB}	0	0	0	0	0	0	0	0
SD_{TM}	–	–	–	–	–	–	–	–
SD_{LR}	–	–	–	–	–	–	–	–
SD_{TB}	0	0	0	0	0	0	0	0
SD_{CTB}	0	0	0	0	0	0	0	0

TABLE 4.7. Domain width parameter i and time taken by Algorithms TB and CTB to reach an accuracy of $1E-10$ for various Taylor orders in Examples 4.1 to 4.6.

			F_{TB}		F_{CTB}	
Example	Name	Order, m	i	time	i	time
4.1	Gritton	2	>7	–	>7	–
		4	>7	–	>7	–
		6	7	0.15	7	0.15
		8	5	0.13	5	0.13
4.2	Jennrich & Sampson	2	>7	–	>7	–
		4	7	0.27	7	0.23
		6	5	0.44	5	0.33
		8	4	0.65	4	0.42
4.3	Levy	2	>7	–	>7	–
		4	7	2.17	7	0.15
		6	5	4.45	5	0.13
		8	3	4.09	3	0.35
4.4	Trigonometric	2	>7	–	>7	–
		4	7	153	7	0.39
		6	5	1712	5	3.0
		8	3	4.87 hr	3	15.6
4.5	Griewank	2	>7	∞	>7	–
		4	6	∞	6	7.2
		6	4	∞	4	91.5
		8	3	∞	3	909
4.6	Trigonometric	2	>7	∞	>7	–
		4	7	∞	7	96
		6	5	∞	5	3679
		8	3	∞	3	16.5 hr

TABLE 4.8. Overestimations and their reduction ratios obtained with the natural inclusion function form in Example 4.1 Gritton ($1-dim$).

i	-7	-6	-5	-4	-3	-2	-1
$w\left(\mathbf{X}^{(i)}\right)$	$2*2^7$	$2*2^6$	$2*2^5$	$2*2^4$	$2*2^3$	$2*2^2$	$2*2^1$
$\mathcal{H}_{NIE}$	$2E+34$	$1E+29$	$6E+23$	$5E+18$	$1E+14$	$3E+10$	$7E+7$
$\mathcal{R}_{NIE}$	$-$	$2E+5$	$2E+5$	$1E+5$	$3E+4$	$5E+3$	425.1
t_{NIE}	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$

i	0	1	2	3	4	5	6	7
$w\left(\mathbf{X}^{(i)}\right)$	$2*2^{-0}$	$2*2^{-1}$	$2*2^{-2}$	$2*2^{-3}$	$2*2^{-4}$	$2*2^{-5}$	$2*2^{-6}$	$2*2^{-7}$
$\mathcal{H}_{NIE}$	$1E+6$	$2E+5$	$3E+4$	$8E+3$	$3E+3$	$1E+3$	$5E+2$	$3E+2$
$\mathcal{R}_{NIE}$	47.2	8.5	5.8	3.7	2.8	2.4	2.2	2.1
t_{NIE}	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$

TABLE 4.9. Overestimations and their reduction ratios obtained with the natural inclusion function form in Example 4.2 Jennrich and Sampson ($2-dim$).

i	-7	-6	-5	-4	-3	-2	-1
$w\left(\mathbf{X}^{(i)}\right)$	$2*2^7$	$2*2^6$	$2*2^5$	$2*2^4$	$2*2^3$	$2*2^2$	$2*2^1$
$\mathcal{H}_{NIE}$	-	-	$1E+258$	$7E+118$	$1E+49$	$1E+19$	132.1
$\mathcal{R}_{NIE}$	-	-	$-$	$2E+139$	$7E+69$	$8E+29$	$9E+16$
t_{NIE}	-	-	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$

i	0	1	2	3	4	5	6	7
$w\left(\mathbf{X}^{(i)}\right)$	$2*2^{-0}$	$2*2^{-1}$	$2*2^{-2}$	$2*2^{-3}$	$2*2^{-4}$	$2*2^{-5}$	$2*2^{-6}$	$2*2^{-7}$
$\mathcal{H}_{NIE}$	$2E-13$	$2E-13$	$2E-13$	$4E-13$	$4E-13$	$2E-13$	$2E-13$	$4E-13$
$\mathcal{R}_{NIE}$	$6E+14$	$1E+0$	$1E+0$	$5E-1$	$1E+0$	$2E+0$	$1E+0$	$5E-1$
t_{NIE}	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$

TABLE 4.10. Overestimations and their reduction ratios obtained with the natural inclusion function form in Example 4.3 Levy $(3-dim)$.

i	-7	-6	-5	-4	-3	-2	-1
$w\left(\mathbf{X}^{(i)}\right)$	$2*2^7$	$2*2^6$	$2*2^5$	$2*2^4$	$2*2^3$	$2*2^2$	$2*2^1$
$\mathcal{H}_{NIE}$	$1E+3$	$6E+2$	$4E+2$	$1E+2$	$9E+1$	$7E+1$	$2E+1$
$\mathcal{R}_{NIE}$	$-$	1.9	1.7	3.0	1.4	1.2	3.3
t_{NIE}	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$

i	0	1	2	3	4	5	6	7
$w\left(\mathbf{X}^{(i)}\right)$	$2*2^{-0}$	$2*2^{-1}$	$2*2^{-2}$	$2*2^{-3}$	$2*2^{-4}$	$2*2^{-5}$	$2*2^{-6}$	$2*2^{-7}$
$\mathcal{H}_{NIE}$	$2E-3$	0.0	0.0	$4E-14$	$3E-14$	$6E-14$	$7E-14$	$6E-14$
$\mathcal{R}_{NIE}$	$9E+3$	$-$	$-$	$-$	$-$	$-$	$-$	$-$
t_{NIE}	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$

TABLE 4.11. Overestimations and their reduction ratios obtained with the natural inclusion function form in Example 4.4 Trigonometric $(4-dim)$.

i	-7	-6	-5	-4	-3	-2	-1
$w\left(\mathbf{X}^{(i)}\right)$	$2*2^7$	$2*2^6$	$2*2^5$	$2*2^4$	$2*2^3$	$2*2^2$	$2*2^1$
$\mathcal{H}_{NIE}$	$1E+2$	$1E+2$	$1E+2$	$1E+2$	$1E+2$	$1E+2$	$5E+1$
$\mathcal{R}_{NIE}$	$-$	1.0	1.0	1.0	1.0	1.0	1.9
t_{NIE}	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$

i	0	1	2	3	4	5	6	7
$w\left(\mathbf{X}^{(i)}\right)$	$2*2^{-0}$	$2*2^{-1}$	$2*2^{-2}$	$2*2^{-3}$	$2*2^{-4}$	$2*2^{-5}$	$2*2^{-6}$	$2*2^{-7}$
$\mathcal{H}_{NIE}$	$2E+0$	$1E+0$	$1E-1$	$1E-11$	$1E-11$	$1E-11$	$1E-11$	$1E-11$
$\mathcal{R}_{NIE}$	23.0	1.6	14.0	$7E+9$	1.0	1.0	1.0	1.0
t_{NIE}	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$

TABLE 4.12. Overestimations and their reduction ratios obtained with the natural inclusion function form in Example 4.5 Griewank ($5 - dim$).

i	-7	-6	-5	-4	-3	-2	-1
$w\left(\mathbf{X}^{(i)}\right)$	$2*2^{7}$	$2*2^{6}$	$2*2^{5}$	$2*2^{4}$	$2*2^{3}$	$2*2^{2}$	$2*2^{1}$
$\mathcal{H}_{NIE}$	$1E+0$	$1E+0$	$1E+0$	$1E+0$	$1E+0$	$1E+0$	$9E-1$
$\mathcal{R}_{NIE}$	$-$	1.3	0.87	0.96	1.2	1.1	1.1
t_{NIE}	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$

i	0	1	2	3	4	5	6	7
$w\left(\mathbf{X}^{(i)}\right)$	$2*2^{-0}$	$2*2^{-1}$	$2*2^{-2}$	$2*2^{-3}$	$2*2^{-4}$	$2*2^{-5}$	$2*2^{-6}$	$2*2^{-7}$
$\mathcal{H}_{NIE}$	$3E-1$	$9E-2$	$2E-2$	$4E-3$	$5E-4$	$2E-5$	$8E-6$	$3E-6$
$\mathcal{R}_{NIE}$	3.0	3.4	4.2	5.0	8.8	23.6	2.6	2.3
t_{NIE}	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$

TABLE 4.13. Overestimations and their reduction ratios obtained with the natural inclusion function form in Example 4.6 Trigonometric ($6 - dim$).

i	-7	-6	-5	-4	-3	-2	-1
$w\left(\mathbf{X}^{(i)}\right)$	$2*2^{7}$	$2*2^{6}$	$2*2^{5}$	$2*2^{4}$	$2*2^{3}$	$2*2^{2}$	$2*2^{1}$
$\mathcal{H}_{NIE}$	$2E+3$	$2E+3$	$2E+3$	$2E+3$	$2E+3$	$2E+3$	$2E+3$
$\mathcal{R}_{NIE}$	$-$	1.0	1.0	1.0	1.0	1.0	1.0
t_{NIE}	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$

i	0	1	2	3	4	5	6	7
$w\left(\mathbf{X}^{(i)}\right)$	$2*2^{-0}$	$2*2^{-1}$	$2*2^{-2}$	$2*2^{-3}$	$2*2^{-4}$	$2*2^{-5}$	$2*2^{-6}$	$2*2^{-7}$
$\mathcal{H}_{NIE}$	$7E+0$	$3E+0$	$2E-1$	$5E-11$	$5E-11$	$5E-11$	$5E-11$	$5E-11$
$\mathcal{R}_{NIE}$	255.1	1.9	14.4	$5E+9$	1.0	1.0	1.0	1.0
t_{NIE}	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$

5

Global optimization using the natural inclusion, Taylor model, and combined Taylor-Bernstein forms

5.1 Introduction

In the previous chapter, we presented a combined TB form F_{CTB} that is more effective than either of its two constituent TB forms F_{TB}, F_{LR} when the domain shrinks from large to small widths. The combined form F_{CTB}, of course, inherits the useful property of higher order convergence from its constituent forms. Moreover, we also saw in the previous chapter that the simple natural inclusion form sometimes yields tighter enclosures of the range than the more sophisticated Taylor and TB forms.

In this chapter, we propose an improved algorithm for unconstrained global optimization in the framework of the MS algorithm. A novel and powerful feature of the proposed algorithm is that a variety of inclusion function forms for the objective function are incorporated into it - the combined TB form, the Taylor model, and the simple natural inclusion form. Several improvements are also made in the Bernstein step of the combined TB form, such as selection of a more efficient direction for subdivision, and the use of cut-off and monotonicity tests to discard those boxes where the global minimizer cannot lie. Further, the incorporation of several inclusion function forms allows the cut-off test and termination condition in the MS algorithm to be made even more effective than in our earlier proposed optimization algorithm in Chapter 3. The performance of the proposed Algorithm is then numerically tested and compared with those of Algorithms MS, TMS, and TBMS on several benchmark examples.

The rest of this chapter is organized as follows. In section 5.2, some initial developments and improvements are given. In section 5.3, the proposed algorithm for global optimization is presented. In section 5.4, the performance of the proposed algorithm is numerically tested and compared with those of the MS algorithm, the MS algorithm with the Taylor model as an inclusion function form, and our earlier proposed optimization algorithm. The obtained test results are discussed in section 5.5, while the conclusions of the chapter are given in section 5.6.

5.2 Initial developments

In section 2.2.1, we presented Algorithm Bounder to compute the range $\bar{p}(\mathbf{X})$ of the polynomial part p on $\mathbf{X}$. For global optimization problems, where the computation of $\min f(\mathbf{X})$ is of interest, this algorithm can be tailored and improved as given below. We call the resulting improved algorithm as Algorithm NewBounder.

A study of the optimization Algorithm TBMS presented in section 3.3 reveals that the quantity $\max \bar{p}(\mathbf{Y})$, where $\mathbf{Y}$ is the current leading box, is never used in the algorithm and is therefore not of interest. Note that the quantity $\max \bar{p}(\mathbf{Y})$ required in step 8 of Algorithm TBMS is actually found in step 4 of Algorithm Bounder, through the application of the vertex condition to $\max B(\mathbf{D})$.

Since $\max \bar{p}(\mathbf{Y})$ is not of interest in Algorithm TBMS, in step 4 of Algorithm Bounder we may avoid applying the vertex condition to $\max B(\mathbf{D})$ and instead apply it only to $\min B(\mathbf{D})$. With this modification, a new Algorithm NewBounder arises from Algorithm Bounder. It computes an enclosure $P(\mathbf{X})$ of the range $\bar{p}(\mathbf{X})$, with $P(\mathbf{X})$ such that

$$\min P(\mathbf{X}) = \min \bar{p}(\mathbf{X})\,; \quad \max P(\mathbf{X}) \geq \min \bar{p}(\mathbf{X}) \tag{5.1}$$

We can also incorporate some more improvements into Algorithm NewBounder:

- A monotonicity test similar to that used in the MS algorithm, for discarding boxes where surely no global minimizer of p lies.

- A cut-off test, similar to that used in the MS algorithm, for discarding boxes where surely no global minimizer of p lies.

- An improved strategy for selection of subdivision direction of boxes.

We discuss below each of these improvements.

5.2.1 Monotonicity test for Bernstein patches

On a box $\mathbf{D} \subseteq \mathbf{U}$, the partial derivative with respect to x_r of a polynomial $p(\mathbf{x})$ in Bernstein form is [11]

$$\frac{\partial p}{\partial x_r}(\mathbf{x}) = n_r \sum_{I \leq N_{r,-1}} \left[b_{I_{r,1}}(\mathbf{D}) - b_I(\mathbf{D}) \right] B_{N_{r,-1},I}(\mathbf{x}), \quad 1 \leq r \leq l, \quad \mathbf{x} \in \mathbf{D} \tag{5.2}$$

Remark 5.1 *Let $P'_r(\mathbf{D})$ denote an enclosure of the range of the above partial derivative on $\mathbf{D}$. In the monotonicity test if $0 \notin P'_r(\mathbf{D})$ then the interior of $\mathbf{D}$ cannot contain a global minimizer of p on $\mathbf{U}$. The edge of $\mathbf{D}$ can still contain global minimizer if that part of the edge which has the smallest polynomial value is also part of $\mathbf{U}$. Otherwise, no global minimizer of p lies in $\mathbf{D}$, and $\mathbf{D}$ can be discarded.*

The enclosure $P'_r(\mathbf{D})$ can be found by evaluating the natural interval inclusion of the RHS expression in (5.2). However, in some cases, the evaluation can be avoided:

Remark 5.2 *From the fact that the Bernstein polynomials $B_{N_{r,-1},I}$ are always non-negative, it is easy to see from (5.2) that if all $\left[b_{I_{r,1}}(\mathbf{D}) - b_I(\mathbf{D}) \right]$ are positive resp. negative, then $P'_r(\mathbf{D}) > 0$ resp. $P'_r(\mathbf{D}) < 0 \Rightarrow p$ is monotonic with respect to direction r on box $\mathbf{D} \Rightarrow$ the interior of $\mathbf{D}$ cannot contain global minimizer of p.*

5.2.2 Direction selection for Bernstein patches

In step 5 of Algorithm Bounder in section 2.2.1, the direction in which the boxes are subdivided is varied cyclically from 1 to l. A more efficient strategy for selection of the subdivision direction could result in considerably fewer boxes being created and significant overall speed up of this algorithm.

Zettler and Garloff [33] suggest selection of the subdivision direction as the one along which the maximum absolute value of the partial derivatives of p occurs. Applying the triangle inequality and properties of Bernstein polynomials to (5.2), the authors show that the quantity

$$\max_{xsD} \left| \frac{\partial p}{\partial x_r}(\mathbf{x}) \right|$$

can be estimated as

$$\tilde{I}_r = \max_{I \leq N_{r,-1}} \left| b_{I_{r,1}}(\mathbf{D}) - b_I(\mathbf{D}) \right|$$

The direction selected for subdivision r_0 is such that

$$\tilde{I}_{r_0} = \max_{1 \leq r \leq l} \tilde{I}_r \tag{5.3}$$

A similar strategy can be given based on the second partial derivatives. In Algorithm New-Bounder, we use the above direction selection strategy based on the first partial derivatives.

5.2.3 Cut-off test for Bernstein patches

The list $\mathcal{L}$ in Algorithm Bounder consists of pairs $(\mathbf{D}, B(\mathbf{D}))$. Suppose we arrange this list at every iteration such that the minimums of the second members, i.e., $\min B(\mathbf{D})$, of all pairs of the list do not decrease.

Now, consider the leading box $\mathbf{D}$ of the list $\mathcal{L}$ at any given iteration of the algorithm. If $\min B(\mathbf{D})$ satisfies the vertex condition, then by the range enclosure property of Bernstein coefficients, $\min \bar{p}(\mathbf{D}) = \min B(\mathbf{D})$. As $\min \bar{p}(\mathbf{D}) \geq \min \bar{p}(\mathbf{X})$, we may discard all boxes $\mathbf{D}'$ in the list $\mathcal{L}$ for which $\min B(\mathbf{D}') > \min B(\mathbf{D})$.

Suppose instead that $\min B(\mathbf{D})$ does not satisfy the vertex condition. Then, $\mathbf{D}$ is subdivided into two subboxes $\mathbf{D}_A, \mathbf{D}_B$ and the patches $B(\mathbf{D}_A)$, $B(\mathbf{D}_B)$ computed. By the range enclosure property of Bernstein coefficients given in section 2.2.1,

$$\bar{p}(\mathbf{D}_A) \subseteq [\min B(\mathbf{D}_A), \max B(\mathbf{D}_A)]\,;\; \bar{p}(\mathbf{D}_B) \subseteq [\min B(\mathbf{D}_B), \max B(\mathbf{D}_B)]$$

So, if $\min B(\mathbf{D}_B) > \max B(\mathbf{D}_A)$ then the box $\mathbf{D}_B$ can be discarded in the search for the global minimum. In fact, we may also discard all other boxes $\mathbf{D}'$ in the list $\mathcal{L}$ for which $\min B(\mathbf{D}') > \max B(\mathbf{D}_A)$.

5.2.4 Algorithm for bounding polynomial range

We are now ready to present the improved Algorithm NewBounder. This algorithm is specially meant for global optimization problems where the primary interest is in obtaining sharp values for $\min \bar{p}(\mathbf{X})$ whereas the quantity $\max \bar{p}(\mathbf{X})$ can be overestimated.

Algorithm NewBounder : $P(\mathbf{X}) = \text{NewBounder}(\mathbf{X}, a_I)$

Inputs: A box $\mathbf{X}$, a polynomial p as in (2.1) of degree N in l-variables and having coefficients a_I.

Output: An enclosure $P(\mathbf{X})$ of the range $\bar{p}(\mathbf{X})$, where $P(\mathbf{X})$ is as in (5.1).

BEGIN Algorithm

1. (Compute patch $B(\mathbf{U})$) Execute Algorithm Patch
$$B(\mathbf{U}) = \text{Patch}(\mathbf{X}, a_I)$$
2. (Initialize lists) Set $\mathcal{L} \leftarrow \{(\mathbf{U}, B(\mathbf{U}))\}$, $\mathcal{L}^{sol} \leftarrow \{\}$. Set cut-off value $z' = \max B(\mathbf{U})$.
3. (Select item for processing) If $\mathcal{L}$ is empty, go to step 11. Otherwise, pick the first item from $\mathcal{L}$, denote it as $(\mathbf{D}, B(\mathbf{D}))$, and delete the item entry from $\mathcal{L}$.
4. (Check vertex condition for the min on patch) If $(\mathbf{D}, B(\mathbf{D}))$ is such that $\min B(\mathbf{D})$ satisfies the vertex condition in Lemma 2.2, that is, $\min B(\mathbf{D})$ occurs at some $I \in S_0$, then

(a) Update the cut-off value as $z' = \min\{z', \min B(\mathbf{D})\}$.

(b) Enter the item in list $\mathcal{L}^{sol}$ and return to previous step.

5. (Subdivide and find new patches) Execute Algorithm Subdivision

$$[B(\mathbf{D}_A), B(\mathbf{D}_B), \mathbf{D}_A, \mathbf{D}_B] = \mathrm{SD}(\mathbf{D}, B(\mathbf{D}), r_0)$$

where, r_0 is chosen as in (5.3).

6. (Monotonicity test, see Remarks 5.1 and 5.2): discard box $\mathbf{D}_A$ if $0 \notin P'_r(\mathbf{D}_A)$ for any $r \in \{1, 2, \ldots, l\}$. Do likewise for box $\mathbf{D}_B$.

7. Update the cut-off value as $z' = \min\{z', \max B(\mathbf{D}_A), \max B(\mathbf{D}_B)\}$.

8. (Add new entries to list) Enter the new items $(\mathbf{D}_A, B(\mathbf{D}_A))$ and $(\mathbf{D}_B, B(\mathbf{D}_B))$ to the list $\mathcal{L}$ such that minimums of the second members, i.e., $\min B(\mathbf{D})$, of all pairs of the list do not decrease.

9. Cut-off test: discard from the list all pairs whose minimums of the second members are greater than z'.

10. Return to step 3.

11. Compute an enclosure $P(\mathbf{X})$ of the range $\bar{p}(\mathbf{X})$ as the minimum to maximum over all the second entries of the items present in list $\mathcal{L}^{sol}$.

12. RETURN $P(\mathbf{X})$.

END Algorithm

5.2.5 A tighter enclosure of the function minimum

In the proposed algorithm for optimization given below, we are interested in computing an enclosure that is as tight as possible for the global minimum of the objective function f. This quantity is given by $\min \bar{f}(\mathbf{Y})$, where $\mathbf{Y}$ is the leading box at any given iteration of the algorithm. Suppose we compute the combined TB form using Algorithm CTB and obtain an enclosure of $\bar{f}(\mathbf{Y})$. If $w(\mathbf{Y})$ happens to be small enough, then the improved TB form $F_{TB}(\mathbf{Y})$ is in turn invoked in Algorithm CTB. In this case, we can use the interval

$$[\min \bar{p}(\mathbf{Y}) + \min R(\mathbf{Y}), \min \bar{p}(\mathbf{Y}) + \max R(\mathbf{Y})]$$

instead of the interval $F_{CTB}(\mathbf{Y})$ to get a tighter enclosure of $\min \bar{f}(\mathbf{Y})$.

Algorithm MIN_CTB below encapsulates this idea.

Algorithm MIN_CTB : $F_{\min,CTB}(\mathbf{X}) = \text{MIN\_CTB}(\mathbf{X}, f, m)$

Inputs: The box $\mathbf{X}$, an expression for the function f, and the order m of Taylor form to be used.

Output: An enclosure $F_{\min,CTB}(\mathbf{X})$ for $\min \bar{f}(\mathbf{X})$.

BEGIN Algorithm

1. Call Algorithm CTB:
$$[F_{CTB}(\mathbf{X}), \bar{p}(\mathbf{X}), B^*, R(\mathbf{X}), i_f] = CTB(\mathbf{X}, f, m)$$
2. If $i_f = 0$ set
$$F_{\min,CTB}(\mathbf{X}) = F_{CTB}(\mathbf{X})$$
else set
$$F_{\min,CTB}(\mathbf{X}) = [\min \bar{p}(\mathbf{X}) + \min R(\mathbf{X}), \min \bar{p}(\mathbf{X}) + \max R(\mathbf{X})]$$
3. RETURN $F_{\min,CTB}(\mathbf{X})$ and EXIT.

END Algorithm

5.3 Proposed optimization Algorithm CTBMS

The proposed algorithm is based on the following ideas. As before, let $\mathbf{Y}$ be the leading box at any given iteration of the MS algorithm. Then,

1. Since the computation of $F_{NIE}(\mathbf{Y})$ is relatively inexpensive, and since sometimes $F_{NIE}(\mathbf{Y})$ gives sharper enclosures than the sophisticated TB forms even for small domains (see previous chapter), we always compute $F_{NIE}(\mathbf{Y})$.
2. If $w(R(\mathbf{Y})) > w(F_{NIE}(\mathbf{Y}))$, then $F_{NIE}(\mathbf{Y})$ gives a sharper enclosure of the range than the TB forms. Since the effort to bound the polynomial range $\bar{p}(\mathbf{Y})$ may not be worthwhile in these cases, we do not use the TB forms and instead use $F_{NIE}(\mathbf{Y})$.
3. If the Taylor model technique of Berz *et al.* [3], [20] is used for computing $R(\mathbf{Y})$ needed in the above step, then we concurrently also obtain the Taylor model $F_{TM}(\mathbf{Y})$. Then, as anyway the cost of computing $F_{TM}(\mathbf{Y})$ is incurred in the Taylor model technique, instead of using only $F_{NIE}(\mathbf{Y})$ we can use $F_{NIE}(\mathbf{Y}) \bigcap F_{TM}(\mathbf{Y})$.

4. If $w(R(\mathbf{Y})) \leq w(F_{NIE}(\mathbf{Y}))$, we also use the combined TB form and get an enclosure of the global minimum $\min \bar{f}(\mathbf{Y})$ using $F_{\min,CTB}(\mathbf{Y})$. We then intersect the result with $F_{NIE}(\mathbf{Y}) \bigcap F_{TM}(\mathbf{Y})$ to obtain a (hopefully) sharper enclosure $F(\mathbf{Y})$ of the global minimum $\min \bar{f}(\mathbf{Y})$.

5. A lower bound on the global minimum $\min \bar{f}(\mathbf{Y})$ is obtained as $y = \min F(\mathbf{Y})$.

6. The global minimum $\min \bar{f}(\mathbf{Y})$ cannot exceed $\max F(\mathbf{Y})$. Hence, the cut-off value z in the MS algorithm can be updated accordingly.

7. Thus, the global minimum $\min \bar{f}(\mathbf{Y})$ is bounded by $[\min F(\mathbf{Y}), \max F(\mathbf{Y})]$. The maximum possible error in computing the global minimum is therefore given by $w(F(\mathbf{Y}))$.

8. This leads to the termination condition for the algorithm as $w(F(\mathbf{Y})) < \varepsilon$.

We can now present our algorithm for global optimization. Since our global optimization algorithm involves the **C**ombined **T**aylor - **B**ernstein form in **M**oore-**S**kelboe type algorithm, we call it as Algorithm CTBMS.

Algorithm CTBMS

Inputs: The box $\mathbf{X}$, order m of the Taylor form to be used, natural inclusion function F_{NIE} for the function $f : \mathbf{X} \to \Re$, an inclusion function F' for the Jacobian of f, and an accuracy parameter ε.

Output: A lower bound, of accuracy ε, on the global minimum of f over $\mathbf{X}$. This lower bound is output as the value of variable y in the last but one step below.

BEGIN Algorithm

1. Set $\mathbf{Y} = \mathbf{X}$.

2. Calculate $F_{NIE}(\mathbf{Y})$ and $F_{TM}(\mathbf{Y}) = p(\mathbf{Y}) + R(\mathbf{Y})$.

   (a) If $w(R(\mathbf{Y})) > w(F_{NIE}(\mathbf{Y}))$, set

   $$F(\mathbf{Y}) = F_{NIE}(\mathbf{Y}) \bigcap F_{TM}(\mathbf{Y})$$

   and go to the following step, else compute $F_{\min,CTB}(\mathbf{Y})$ using Algorithm MIN_CTB

   $$F_{\min,CTB}(\mathbf{Y}) = \text{MIN\_CTB}(\mathbf{Y}, f, m)$$

   and set

   $$F(\mathbf{Y}) = F_{NIE}(\mathbf{Y}) \bigcap F_{TM}(\mathbf{Y}) \bigcap F_{\min,CTB}(\mathbf{Y})$$

3. Set $y = \min F(\mathbf{Y})$.

4. Initialize the list $L = ((\mathbf{Y}, y))$ and the cut-off value $z = \max F(\mathbf{Y})$.

5. Choose a coordinate direction k parallel to which $\mathbf{Y}$ has an edge of maximum length, i.e., choose k as
$$k = \{i : w(\mathbf{Y}) = w(\mathbf{Y}_i)\}$$

6. Bisect $\mathbf{Y}$ in direction k getting boxes $\mathbf{V}^1$ and $\mathbf{V}^2$ such that $\mathbf{Y} = \mathbf{V}^1 \bigcup \mathbf{V}^2$.

7. Monotonicity test (see Remark 3.1): discard any box $\mathbf{V}^i$ if $0 \notin F_j'(\mathbf{V}^i)$ for any $j \in \{1, 2, ..., l\}$ and $i = 1, 2$.

8. For $i = 1, 2$ do the following: Calculate $F_{NIE}\left(\mathbf{V}^i\right)$ and $F_{TM}\left(\mathbf{V}^i\right) = p\left(\mathbf{V}^i\right) + R\left(\mathbf{V}^i\right)$.

    (a) If $w\left(R\left(\mathbf{V}^i\right)\right) > w\left(F_{NIE}\left(\mathbf{V}^i\right)\right)$, set
    $$F\left(\mathbf{V}^i\right) = F_{NIE}\left(\mathbf{V}^i\right) \bigcap F_{TM}\left(\mathbf{V}^i\right)$$
    else compute $F_{\min,CTB}\left(\mathbf{V}^i\right)$ using Algorithm MIN_CTB :
    $$F_{\min,CTB}\left(\mathbf{V}^i\right) = \text{MIN\_CTB}\left(\mathbf{V}^i, f, m\right)$$
    and set
    $$F\left(\mathbf{V}^i\right) = F_{NIE}\left(\mathbf{V}^i\right) \bigcap F_{TM}\left(\mathbf{V}^i\right) \bigcap F_{\min,CTB}\left(\mathbf{V}^i\right)$$

9. Set $v^i = \min F\left(\mathbf{V}^i\right)$ for $i = 1, 2$.

10. Update the cut-off value z as
$$z = \min\left\{z, \max F\left(\mathbf{V}^1\right), \max F\left(\mathbf{V}^2\right)\right\}$$

11. Remove $(\mathbf{Y}, y)$ from the list L.

12. Add the pairs $\left(\mathbf{V}^1, v^1\right), \left(\mathbf{V}^2, v^2\right)$ to the list L such that the second members of all pairs of the list do not decrease.

13. Cut-off test: discard from the list all pairs whose second members are greater than z.

14. Denote the first pair of the list by $(\mathbf{Y}, y)$.

15. If $w(F(\mathbf{Y})) < \varepsilon$ then print y and EXIT algorithm.

16. Go to step 5.

END Algorithm

From section 3.3, especially (3.3), it is straightforward to prove that y is a lower bound on the global minimum of f over $\mathbf{X}$. The convergence properties of Algorithm CTBMS follows immediately from the convergence results for inclusion functions of higher order in the MS algorithm, as given by Moore and Ratschek in [25] and Ratschek in [29].

5.4 Numerical tests

We test and compare the performances of Algorithms CTBMS, TBMS, TMS, and MS on eleven benchmark examples. We set the accuracy $\varepsilon = 1e - 05$ and the Taylor order $m = 4$. For all computations, we use a PC/Pentium III 800 MHz 256 MB RAM machine with a FORTRAN 90 compiler, and version 8.1 of the COSY-INFINITY package of Berz *et al. [2], [15].*

To compare the performances of the various Algorithms, we use the following performance metrics:

- Number of algorithmic iterations
- Computational time, seconds
- Maximum list length
- Final list length

The examples are as under:

Example 5.1 *Jennrich and Sampson function [26, problem 6]. The* $2-\dim$ *function is*

$$f(x) = \sum_{i=1}^{10} f_i(x)^2, \qquad f_i(x) = 2 + 2i - (\exp(ix_1) + \exp(ix_2))$$

We take the initial domain as $\mathbf{X} = \left([-1, 1]^2\right)$.

Example 5.2 *Bard function [26, problem 8]. The* $3-$ dim *function is*

$$f(x) = \sum_{i=1}^{15} f_i(x)^2, \quad f_i(x) = y_i - \left(x_1 + \frac{u_i}{v_i x_2 + w_i x_3}\right), u_i = i, v_i = 16 - i, w_i = \min(u_i, v_i)$$

where, the values of y_i *for* $i = 1, \dots, 15$ *are given in the cited paper. We take the initial domain as* $\mathbf{X} = \left([-0.25, 0.25], [0.01, 2.5]^2\right)$.

Example 5.3 *Box* $3-\dim$ *function [26, problem 12]. The function is*

$$f(x)=\sum_{i=1}^{10} f_i(x)^2, f_i(x)=\exp(-t_ix_1)-\exp(-t_ix_2)-x_3[\exp(-t_i)-\exp(-10t_i)], t_i=\frac{i}{10}$$

We take the initial domain as $\mathbf{X}=\left([-20,20],[1,20]^2\right)$.

Example 5.4 *Brown and Dennis function [26, problem 16]. The* $4-\dim$ *function is*

$$f(x)=\sum_{i=1}^{20} f_i(x)^2, \quad f_i\left(x\right)=\left(x_1+t_ix_2-\exp\left(t_i\right)\right)^2+\left(x_3+x_4\sin\left(t_i\right)-\cos\left(t_i\right)\right)^2, \quad t_i=\frac{i}{5}$$

We take the initial domain as $\mathbf{X}=\left([-10,0,-100,-20],[100,15,0,0.2]\right)$.

Example 5.5 *Variably dimensioned function [26, problem 25]. The* $2-\dim$ *function is*

$$f(x)=\sum_{i=1}^{4} f_i(x)^2,\ f_1(x)=x_1-1,\ f_2(x)=x_2-1,\ f_3(x)=\sum_{j=1}^{2} j(x_j-1),\ f_4(x)=\left(\sum_{j=1}^{2} j(x_j-1)\right)^2$$

We take the initial domain as $\mathbf{X}=\left([-1.5,1.5]^2\right)$.

Example 5.6 *Linear - rank 1 with zero columns and rows, [26, problem 34]. The* $2-\dim$ *function is*

$$f(x)=\sum_{i=1}^{4} f_i(x)^2, f_1(x)=-1,\ f_2(x)=(2x_1+3x_2)-1,\ f_3(x)=2(2x_1+3x_2)-1,\ f_4(x)=-1$$

We take the initial domain as $\mathbf{X}=\left([-10,10]^2\right)$.

Example 5.7 *Linear function - full rank [26, problem 32]. The* $4-\dim$ *function is*

$$f(x)=\sum_{i=1}^{4} f_i(x)^2,\ f_i(x)=x_i-\frac{1}{2}\left(\sum_{j=1}^{4} x_j\right)-1$$

We take the initial domain as $\mathbf{X}=\left([-1,1]^4\right)$.

Example 5.8 *Extended Rosenbrock function [26, problem 21]. The* $2-\dim$ *function is*

$$f(x)=\sum_{i=1}^{2} f_i(x)^2,\ f_1(x)=10(x_2-x_1^2),\ f_2(x)=1-x_1$$

We take the initial domain as $\mathbf{X}=\left([-12,12]^2\right)$.

Example 5.9 *Discrete boundary value function [26, problem 28]. The* $2-\dim$ *function is*

$$f(x)=\sum_{i=1}^{2} f_i(x)^2,\ \ f_i(x)=2x_i-x_{i-1}-x_{i+1}+\frac{h^2(x_i+t_i+1)^3}{2},\ \ h=\frac{1}{3},\ \ t_i=ih,\ \ x_0=x_3=0$$

We take the initial domain as $\mathbf{X}=\left([-5,5]^2\right)$.

Example 5.10 *Brown almost - linear function [26, problem 27]. The* $4-\dim$ *function is*

$$f(x)=\sum_{i=1}^{4} f_i(x)^2,\ \ f_i(x)=x_i+\sum_{j=1}^{4} x_j-5, i=1..3, \text{ and } f_4(x)=\left(\prod_{j=1}^{4} x_j\right)-1$$

We take the initial domain as $\mathbf{X}=\left([-2.5,2.5]^4\right)$.

Example 5.11 *Chebyquad function [26, problem 35]. The* $4-\dim$ *function is*

$$f(x)=\sum_{i=1}^{4} f_i(x)^2,\ \ f_i(x)=\frac{1}{4}\sum_{j=1}^{4} T_i(x_j)-\int_0^1 T_i(x)dx$$

where T_i *is the* i^{th} *Chebyshev polynomial shifted to the interval* $[0,1]$. *Hence,*

$$\begin{aligned}\int_0^1 T_i(x)dx &= 0 \text{ for } i \text{ odd}\\ \int_0^1 T_i(x)dx &= \frac{-1}{(i^2-1)} \text{ for } i \text{ even}\end{aligned}$$

We take the initial domain as $\mathbf{X}=\left([-2,2]^4\right)$.

5.5 Discussion

Table 5.4 lists the global minimum obtained using Algorithm CTBMS in each example. Tables 5.5 to 5.8 give the obtained results in terms of these performance metrics for the various test examples[1]. For each metric, we give the values of ratio and the percent reduction computed as

$$\begin{aligned}\text{Ratio} &= \frac{\text{Perf. metric with basic algorithm}}{\text{Perf. metric with proposed algorithm}}\\ \%\text{ reduction} &= \frac{\text{Perf. metric with basic algorithm - Perf. metric with proposed algorithm}}{\text{Perf. metric with basic algorithm}}\times 100\end{aligned}$$

[1] A starred entry in the following Tables indicates that a solution is not obtained with the corresponding algorithm for the prescribed accuracy, due to excessive time and /or memory requirements.

TABLE 5.1. Rankings obtained by proposed Algorithm CTBMS.

	Number of problems			
Performance metric	1^{st} Rank	2^{nd} Rank	3^{rd} Rank	4^{th} Rank
Iterations	9	2	0	0
Computational time	8	3	0	0
Maximum list length	10	1	0	0
Final list length	8	3	0	0

Based on the data in these Tables, we compare the performance of the four algorithms using different evaluation methods: ranking, statistical measures, average and other measures, and performance profiles. These various methods are incorporated in the analysis to avoid dominance of any one test function on the final conclusions about the relative performance of the algorithms.

At the outset, we note that for the considered domains and accuracy, the proposed Algorithm CTBMS is able to solve all the test examples, whereas Algorithms MS, TMS, and TBMS are able to solve only 54.45%, 45.45%, and 90.90% of the test examples, respectively.

5.5.1 *Ranking*

Ranking of algorithms has been used for performance comparison, see, for instance, [4], [27]. Ranking is based on the number of times an algorithm comes in the k^{th} place, here $k = 1, \ldots, 4$. A higher rank is assigned to the algorithm with lesser performance metric value. Table 5.1 gives the ranking of the CTBMS algorithm for the various performance metrics.

Table 5.1 shows that in a majority of the solved test examples, proposed Algorithm CTBMS is superior to the rest in terms of number of iterations, computational time, maximum list length and final list length metrics. Especially, the proposed algorithm is able to achieve the 1^{st} rank in 81.81% of the test examples for the iterations metric, in 72.72% of the test examples for the computational time metric, in 90.90% of the test examples for the maximum list length metric, and in 72.72% of the test examples for the final list length metric.

5.5.2 *Statistical measures*

Next, we compare the performance of the algorithms based on the *distribution* of the difference between the performance metrics. Such a comparison has been done, for instance, in [4]. The minimum, first quartile, median, third quartile and maximum of this distribution are reported in Table 5.2.

TABLE 5.2. Comparison of performance of various algorithms using statistical measures.

Perf. metric	Alg.	Minimum	1^{st} Quartile	Median	2^{nd} Quartile	Maximum
Iterations	TMS	−2820	−2147	−96	1134.50	1534
	TBMS	4	16	403	2569.75	4591
	CTBMS	18	42	582.50	2584.50	4635
Computational time	TMS	−10.74	−8.10	−0.06	4.66	6.2
	TBMS	−17.8	−5.19	−0.13	63.50	228.10
	CTBMS	−0.35	−0.13	3.54	67	241.3
Maximum list length	TMS	−320	−247.75	−15	37	49
	TBMS	−23	−2	36.5	517	1465
	CTBMS	2	5.75	38	560.75	1484
Final list length	TMS	−185	−139	1	10.50	13
	TBMS	4	8.5	25	475.5	1143
	CTBMS	4	8.5	25	465	1143

- Number of iterations: Largest positive minimum value is obtained for Algorithm CTBMS, indicating that the number of iterations are least with Algorithm CTBMS in a majority of the test examples.
- Computational time: The only positive median value is obtained for Algorithm CTBMS indicating that the least computational time is taken by Algorithm CTBMS in more than half the test examples.
- Maximum list length: The only positive minimum value is obtained for Algorithm CTBMS, indicating that it requires the least maximum list length in a majority of the test examples.
- Final list length: Equal positive minimum values are obtained for Algorithms CTBMS and TBMS, indicating that these algorithms require the least final list length in a majority of the test examples.

5.5.3 Minimum, mean, and maximum measures

In Table 5.3, we give the averages of the ratio and percent reduction over all the test examples. The Table shows that for a majority of the test problems, Algorithm CTBMS gives large reduction in maximum list length, number of iterations, and final list length, while it gives improvement in computation time for more than half the test problems.

TABLE 5.3. Minimum, mean, and maximum of ratios and reductions, with respect to Algorithm MS.

		Ratio			% Reduction		
Perf. metric	Alg.	Min.	Mean	Max.	Min.	Mean	Max.
Iterations	TMS	0.03	1.32	4.59	−556.52	−224.74	78.23
	TBMS	1.05	12.28	30.17	4.65	69.76	96.69
	CTBMS	3.58	15.83	38.99	72.09	84.75	97.44
Computational time	TMS	0.002	1.61	3.68	$-5E4$	$-1E4$	72.86
	TBMS	0.02	2.27	6.87	−4900	−961.18	85.45
	CTBMS	0.05	3.55	6.25	−1750	−268.32	88.12
Maximum list length	TMS	0.06	0.97	2.53	−1600	−493.51	60.49
	TBMS	0.47	6.16	18.87	−115	42.99	94.70
	CTBMS	1.4	7.21	24.56	28.57	65.43	95.93
Final list length	TMS	0.08	0.96	1.54	−20	7.52	35.13
	TBMS	5	244.34	1144	80	93.51	99.91
	CTBMS	5	204.82	1144	80	92.59	99.91

5.5.4 Performance profiles

Performance profile is proposed as a tool for evaluating and comparing performance of algorithms in [8]. The performance profile for an algorithm is the cumulative distribution function for a given performance metric. Performance profiles eliminate the influence of a small number of problems on the final evaluation conclusions.

For computational time as the performance metric, performance profiles can be generated as follows. Let $\mathcal{P}$ be the test set of examples, n_s be the number of algorithms and n_p be the number of examples. For each test function p and algorithm s, define

$$t_{p,s} = \text{computing time required to solve a test function } p \text{ by algorithm } s$$

The performance ratio for computation time is calculated as

$$r_{p,s} = \frac{t_{p,s}}{\min\{t_{p,s} : 1 \leq s \leq n_s\}}$$

We choose a parameter $r_M \geq r_{p,s}$ for all p, s, such that $r_{p,s} = r_M$ if and only if algorithm does not solve the test function p. Now, the performance profile for computing time can be defined as

$$\rho_s(\tau) = \frac{1}{n_p}\text{size}\{p \in \mathcal{P} : r_{p,s} \leq \tau\}$$

Similarly, performance profiles for other performance metrics can be defined.

The following observations are made from the performance profile plots computed for various performance metrics.

Number of iterations: Performance profile plots for the number of iterations are given in Figures 5.1 and 5.2. These show that

- Within a factor $\tau = 1$ of the best algorithm, Algorithm MS, TMS, TBMS, and CTBMS are able to solve $0\%, 0\%, 18\%, 82\%$ of the test examples.
- Algorithm MS, TMS, TBMS, and CTBMS are able to solve $54.54\%, 45.45\%, 90.90\%, 100\%$ of the test examples, within a factor of $\tau = 39, 152, 23, 2$ of the best algorithm.
- The proposed Algorithm CTBMS is able to solve all the test examples for $\tau < 2$, and that it requires less number of iterations for 81.81% of the test examples compared to Algorithms MS, TMS and TBMS.

Computational time: Performance profile plots for the computational time are shown in Figures 5.3 and 5.4. These show that

- Within a factor $\tau = 1$ of the best algorithm, Algorithm MS, TMS, TBMS, and CTBMS are able to solve $18.18\%, 9.09\%, 0\%, 72.72\%$ of the test examples.
- Algorithm MS, TMS, TBMS, and CTBMS are able to solve $54.54\%, 45.45\%, 90.90\%, 100\%$ of the test examples, within a factor of $\tau = 9, 540, 51, 19$ of the best algorithm.
- The proposed algorithm CTBMS is able to solve all the test examples for $\tau < 19$, and that it requires less computational time for 72.72% of the test examples compared to Algorithms MS, TMS, and TBMS.

Maximum list length: Performance profile plots for the maximum list length are given in Figures 5.5 and 5.6. These show that

- Within a factor $\tau = 1$ of the best algorithm, Algorithm MS, TMS, TBMS, and CTBMS are able to solve $0\%, 0\%, 18.18\%, 90.90\%$ of the test examples.
- Algorithm MS, TMS, TBMS, and CTBMS are able to solve $54.54\%, 45.45\%, 90.90\%, 100\%$ of the test examples, within a factor of $\tau = 25, 39, 8, 6$ of the best algorithm.
- The proposed algorithm CTBMS is able to solve all the test examples for $\tau < 6$, and that it requires less computational time for 90.90% of the test examples compared to Algorithms MS, TMS, and TBMS.

Final list length: Performance profile plots for the final list length are shown in Figures 5.7 and 5.8. These show that

- Within a factor $\tau = 1$ of the best algorithm, Algorithm MS, TMS, TBMS, and CTBMS are able to solve $0\%, 0\%, 90.90\%, 72.72\%$ of the test examples.
- Algorithm MS, TMS, TBMS, and CTBMS are able to solve $54.54\%, 45.45\%, 90.90\%, 100\%$ of the test examples, within a factor of $\tau = 1145, 210, 16$ of the best algorithm.
- The proposed algorithm CTBMS is able to solve all the test examples for $\tau < 16$, and that it requires less computational time for 72.72% of the test examples compared to Algorithms MS, TMS, and TBMS.

Summary of the performance profile studies:

- If we are interested in an algorithm that can solve a given problem *successfully*, then the proposed Algorithm CTBMS stands out, as displayed by the height of its performance profile plot for every performance metric.
- If we are interested in an algorithm that can solve 90% of the problems with greatest efficiency, then the proposed Algorithm CTBMS stands out, for every performance metric except the final list length metric for which Algorithm TBMS is better.
- If we are interested in an algorithm that can solve all the problems with greatest efficiency, then the proposed Algorithm CTBMS again stands out for every performance metric.
- The probability that the proposed Algorithm CTBMS is a winner in a given problem is about 0.91 for the number of iterations, 0.82 for the computational time, 0.90 for the maximum list length, and 0.72 for the final list length metrics.
- The probability that the proposed Algorithm CTBMS can solve a given problem within a factor of 2 of the best algorithm (among the four considered) is about 100% for the number of iterations, 90% for the computational time, 92% for the maximum list length, and 75% for the final list length metrics.
- Algorithm CTBMS gives improvements in number of iterations, computation time, maximum list length and final list length in $81.81\%, 72.72\%, 90.90\%$ and 72.72% of test examples, respectively.

TABLE 5.4. Domains used, dimensions and the global minimum over the given domain.

Ex.	Test Function	dim	Domain	Global minimum
5.1	Jennrich & Sampson	2	$[-1, 1]^2$	$124.36217\ldots$
5.2	Bard	3	$[-0.25, 0.25][0.01, 2.5]^2$	$8.213\ldots E-3$
5.3	Box 3 – dim	3	$[-20, 20][1, 20]^2$	$0.00000\ldots$
5.4	Brown & Dennis	4	$[-10, 100][0, 15][-100, 0][-20, 0.2]$	$88860.47976\ldots$
5.5	Variably dim.	2	$[-1.5, 1.5]^2$	$0.00000\ldots$
5.6	Linear - rank1	2	$[-10, 10]^2$	$2.19999\ldots$
5.7	Linear- full rank	4	$[-1, 1]^4$	$0.00000\ldots$
5.8	Extended Rosenbrock	2	$[-12, 12]^2$	$0.00000\ldots$
5.9	Discrete boundary	2	$[-5, 5]^2$	$0.00000\ldots$
5.10	Brown almost -linear	4	$[-2.5, 2.5]^4$	$0.00000\ldots$
5.11	Chebyquad	4	$[-2, 2]^4$	$0.00000\ldots$

5.6 Conclusions

We presented a novel algorithm for global optimization that combines the sophisticated TB forms, Taylor model, and the simple natural inclusion function. The performance of the proposed algorithm was tested and compared with those of existing MS algorithms on a collection of eleven benchmark problems. The proposed algorithm stands out for every performance metric, as the one that can solve all considered problems with the greatest efficiency.

TABLE 5.5. Comparison of number of iterations required by various algorithms.

Ex.	Test Function	dim	Iterations	MS	TMS	TBMS	CTBMS
5.1	Jennrich and Sampson	2	Number:	1961	427	65	60
			Ratio:	–	4.59	30.17	32.68
			% Reduction:	–	78.23	96.69	96.94
5.2	Bard	3	Number:	$*$	$*$	202	46
			Ratio:	–	–	–	–
			% Reduction:	–	–	–	–
5.3	Box 3 – dim	3	Number:	1208	$*$	451	105
			Ratio:	–	–	2.68	11.5
			% Reduction:	–	–	62.67	91.31
5.4	Brown and Dennis	4	Number:	$*$	455	66	3
			Ratio:	–	–	–	–
			% Reduction:	–	–	–	–
5.5	Variably dimensioned	2	Number:	23	151	3	5
			Ratio:	–	0.15	7.67	4.6
			% Reduction:	–	–556.52	86.96	78.26
5.6	Linear - rank1	2	Number:	$*$	$*$	133	134
			Ratio:	–	–	–	–
			% Reduction:	–	–	–	–
5.7	Linear - full rank	4	Number:	4757	$*$	166	122
			Ratio:	–	–	28.66	38.99
			% Reduction:	–	–	96.51	97.44
5.8	Extended Rosenbrock	2	Number:	86	2906	82	24
			Ratio:	–	0.03	1.05	3.58
			% Reduction:	–	–327.91	4.65	72.09
5.9	Discrete boundary value	2	Number:	69	133	20	19
			Ratio:	–	0.52	3.45	3.63
			% Reduction:	–	–92.75	71.01	72.46
5.10	Brown -almost linear	4	Number:	$*$	$*$	669	331
			Ratio:	–	–	–	–
			% Reduction:	–	–	–	–
5.11	Chebyquad	4	Number:	$*$	$*$	$*$	1448
			Ratio:	–	–	–	–
			% Reduction:	–	–	–	–

TABLE 5.6. Comparison of computation time required by various algorithms.

Ex.	Test Function	dim	Time	MS	TMS	TBMS	CTBMS
5.1	Jennrich and Sampson	2	Number:	10.1	3.9	1.47	1.2
			Ratio:	–	2.59	6.87	8.42
			% Reduction:	–	61.39	85.45	88.12
5.2	Bard	3	Number:	> 1 hr	> 1 hr	60.1	48.2
			Ratio:	–	–	–	–
			% Reduction:	–	–	–	–
5.3	Box $3 - \text{dim}$	3	Number:	11.1	$*$	28.9	4.08
			Ratio:	–	–	0.38	2.72
			% Reduction:	–	–	-160.36	63.24
5.4	Brown and Dennis	4	Number:	> 1 hr	5.31	7.35	2.88
			Ratio:	–	–	–	–
			% Reduction:	–	–	–	–
5.5	Variably dimensioned	2	Number:	$7E-2$	$1.9E-2$	$5E-2$	$2E-2$
			Ratio:	-	3.68	1.4	3.5
			% Reduction:	-	72.86	28.57	71.43
5.6	Linear - rank1	2	Number:	> 10 hr	> 10 hr	3559.9	3011.9
			Ratio:	–	–	–	–
			% Reduction:	–	–	–	–
5.7	Linear - full rank	4	Number:	287.3	$*$	59.2	46.0
			Ratio:	–	–	4.85	6.25
			% Reduction:	–		79.39	83.99
5.8	Extended Rosenbrock	2	Number:	$2E-2$	10.76	1.0	0.37
			Ratio:	–	0.002	0.02	0.05
			% Reduction:	–	$-5E4$	-4900	-1750
5.9	Discrete boundary value	2	Number:	$3E-2$	$2E-1$	$3E-1$	$8E-2$
			Ratio:	–	0.15	0.1	0.38
			% Reduction:	–	-566.67	-900.1	-166.67
5.10	Brown almost linear	4	Number	> 10 hr	> 10 hr	4914.0	3112.72
			Ratio:	–	–	–	–
			% Reduction:	–	–	–	–
5.11	Chebyquad	4	Number	> 10 hr	> 10 hr	> 10 hr	2963.6
			Ratio:	–	–	–	–
			% Reduction:	–	–	–	–

TABLE 5.7. Comparison of maximum list length required by various algorithms.

Ex.	Test Function	dim	Max. list length	MS	TMS	TBMS	CTBMS
5.1	Jennrich and Sampson	2	Number:	81	32	14	13
			Ratio:	–	2.53	5.79	6.23
			% Reduction:	–	60.49	82.72	83.95
5.2	Bard	3	Number:	$*$	$*$	38	15
			Ratio:	–	–	–	–
			% Reduction:	–	–	–	–
5.3	Box $3 - \text{dim}$	3	Number:	295	$*$	94	42
			Ratio:	–	–	3.14	7.02
			% Reduction:	–	–	68.14	85.76
5.4	Brown and Dennis	4	Number:	$*$	44	15	2
			Ratio:	–	–	–	–
			% Reduction:	–	–	–	–
5.5	Variably dimensioned	2	Number:	7	38	1	5
			Ratio:	–	0.18	7.0	1.4
			% Reduction:	–	−442.86	85.71	28.57
5.6	Linear - rank1	2	Number:	$*$	$*$	53	53
			Ratio:	–	–	–	–
			% Reduction:	–	–	–	–
5.7	Linear - full rank	4	Number:	1547	$*$	82	63
			Ratio:	–	–	18.87	24.56
			% Reduction:	–	–	94.70	95.93
5.8	Extended Rosenbrock	2	Number:	20	340	43	12
			Ratio:	–	0.06	0.47	1.67
			% Reduction:	–	−1600	−115	40
5.9	Discrete boundary value	2	Number:	12	11	7	5
			Ratio:	–	1.1	1.71	2.4
			% Reduction:	–	8.33	41.67	58.34
5.10	Brown almost linear	4	Number:	$*$	$*$	370	60
			Ratio:	–	–	–	–
			% Reduction:	–	–	–	–
5.11	Chebyquad	4	Number:	$*$	$*$	$*$	305
			Ratio:	–	–	–	–
			% Reduction:	–	–	–	–

TABLE 5.8. Comparison of final list length required by various algorithms.

Ex.	Test Function	dim	Final list length	MS	TMS	TBMS	CTBMS
5.1	Jennrich and Sampson	2	Number:	37	24	1	1
			Ratio:	–	1.54	37	37
			% Reduction:	–	35.13	97.30	97.30
5.2	Bard	3	Number:	∗	∗	1	3
			Ratio:	–	–	–	–
			% Reduction:	–	–	–	–
5.3	Box 3 – dim	3	Number:	254	∗	1	15
			Ratio:	–	–	254	16.9
			% Reduction:	–	–	99.60	94.09
5.4	Brown and Dennis	4	Number:	∗	24	1	1
			Ratio:	–	–	–	–
			% Reduction:	–	–	–	–
5.5	Variably dimensioned	2	Number:	5	6	1	1
			Ratio:	–	0.83	5	5
			% Reduction:	–	–20	80	80
5.6	Linear - rank1	2	Number:	∗	∗	11	11
			Ratio:	–	–	–	–
			% Reduction:	–	–	–	–
5.7	Linear - full rank	4	Number:	1144	∗	1	1
			Ratio:	–	–	1144	1144
			% Reduction:	–	–	99.91	99.91
5.8	Extended Rosenbrock	2	Number:	15	200	1	1
			Ratio:	–	0.08	15	15
			% Reduction:	–	–12.33	93.33	93.33
5.9	Discrete boundary value	2	Number:	11	8	1	1
			Ratio:	–	1.38	11	11
			% Reduction:	–	27.27	90.90	90.90
5.10	Brown almost linear	4	Number:	∗	∗	2	7
			Ratio:	–	–	–	–
			% Reduction:	–	–	–	–
5.11	Chebyquad	4	Number	∗	∗	∗	1
			Ratio:	–	–	–	–
			% Reduction:	–	–	–	–

FIGURE 5.1. Performance profile plot of the number of iterations, for $[1, 25]$.

FIGURE 5.2. Performance profile plot of the number of iterations, for $[1, 160]$.

FIGURE 5.3. Performance profile plot of computational time, for $[1, 50]$.

FIGURE 5.4. Performance profile plot of computation time, for $[1, 550]$.

FIGURE 5.5. Performance profile plot of maximum list length, for $[1, 10]$.

FIGURE 5.6. Performance profile plot of maximum list length, for $[1, 40]$.

FIGURE 5.7. Performance profile plot of final list length, for $[1, 50]$.

FIGURE 5.8. Performance profile plot of final list length, for $[1, 1150]$.

6

Conclusions

A problem of great theoretical and practical interest in the area of interval analysis of Moore [24] is the construction of inclusion functions having the property of higher order convergence for multidimensional functions. Higher order inclusion functions have applications, for example, in the solution of equations, quadrature, and global optimization, where faster convergence could possibly be obtained with their aid. A study of the interval analysis literature reveals the lack of higher order inclusion function forms that are practically effective, even for low to medium (i.e., even up to say, six) dimensional problems.

Motivated by this concern, we set the following two objectives for the present work:

1. To develop higher order inclusion function form for multidimensional functions that are practically effective, and

2. To develop unconstrained global optimization algorithm with the developed higher order inclusion function form, for efficient determination of arbitrarily good lower bounds on the minimum of $\bar{f}(\mathbf{X})$.

In each case, we desire to numerically test and compare the practical effectiveness of the proposed tool with existing techniques, on several 'difficult' problems of different dimensions.

In this work, we first presented the improved TB form as a higher order inclusion function form for multidimensional functions. The improved TB form uses Bernstein polynomials for bounding the range of the polynomial obtained from the Taylor form of the given function f. The improved TB form has some important differences from Lin and Rokne's TB form [19] in the way it is constructed, and makes it more effective in practice. The higher order convergence behavior of the proposed form was numerically tested and compared with that of Lin and Rokne's TB form and also with that of the Taylor model of Berz *et al.* [3]. For the testing, we considered six benchmark examples with dimensions varying from 1 to 6. In all examples, unlike with the Taylor model and Lin and Rokne's TB form, we indeed obtained higher order convergence of orders up to 9 with the improved TB form. Moreover, with the improved form we could quite easily obtain these high orders of convergence for up to $5-$dim problems.

Then, we investigated the performance of the improved TB form in the framework of the Moore-Skelboe (MS) algorithm of interval analysis for unconstrained global optimization. We used the improved TB form as an inclusion function in a prototype or basic MS algorithm, and also modified the cut-off test and termination condition in the algorithm. We numerically tested and compared the performances of the proposed algorithm, the MS algorithm, and the MS algorithm with the Taylor model as inclusion function on six benchmark examples. The results of these tests indicated that the proposed global optimization algorithm with the improved TB form as inclusion function is superior to the rest, for the low to medium dimension problems studied.

In several application problems, the typically large initial domain is sooner or later reduced to small solution domains through domain splitting or subdivision techniques. In such applications, both Lin and Rokne's TB form and the improved TB form were found to be unsatisfactory for some domain widths, due to excessive memory and /or time requirements. We therefore proposed next the so-called 'combined' TB form that is more effective than either of the above TB forms when the domain shrinks from large to small widths. The combined form inherits the key property of higher order convergence from its constituent forms. We numerically tested and compared the performance of the combined TB form with those of the existing TB forms, the Taylor model, and the simple natural inclusion function. For the testing, we considered six benchmark examples with dimensions varying from 1 to 6. The results of the tests showed that the new combined form was indeed more effective than either of the existing TB forms over the entire range of domain widths considered. Test results also revealed that the simple natural inclusion form sometimes surprisingly yielded tighter range enclosures than the more sophisticated Taylor and TB forms, even for small domain widths.

Lastly, we proposed an improved algorithm for unconstrained global optimization in the framework of the Moore-Skelboe algorithm. A novel and powerful feature of the proposed algorithm was that a variety of inclusion function forms for the objective function were incorporated into it - the combined TB form, the Taylor model, and the simple natural inclusion form (the surprising observation referred above regarding the natural inclusion form led to its incorporation in to the proposed algorithm). Several improvements were also proposed in the Bernstein step of the combined TB form, such as an improved direction selection strategy for subdivision, and use of cut-off and monotonicity tests. Moreover, the cut-off test and termination condition in the MS algorithm were refined. The performance of the proposed Algorithm was then numerically tested and compared with those of the MS algorithm, the MS algorithm with the Taylor model as inclusion function, and our earlier proposed optimization algorithm, on a collection of eleven benchmark examples. The proposed algorithm was found to be superior for every performance metric as the one that could solve all considered test examples with the greatest efficiency.

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Appendix A

Bernstein approach - univariate case

We outline the Bernstein approach for the unidimensional case. Consider the nth degree polynomial p in a single variable $x \in \mathbf{U} = [0, 1]$

$$p(x) = \sum_{i=0}^{n} a_i x^i$$

The Bernstein form of order k for p is

$$p(x) = \sum_{j=0}^{k} b_j^k B_j^k(x)$$

where k is any integer $\geq n$, and where $B_j^k(x)$ are the Bernstein polynomials of degree k defined as

$$B_j^k(x) := \binom{k}{j} x^j (1-x)^{k-j}$$

and b_j^k are the Bernstein coefficients defined as

$$b_j^k := \sum_{i=0}^{j} a_i \frac{\binom{j}{i}}{\binom{k}{i}}$$

Note that the Bernstein coefficients can also be expressed in the form

$$b_j^k = \sum_{i=0}^{j} a_i \frac{j(j-1)\ldots(j-(i-1))}{k(k-1)\ldots(k-(i-1))} \tag{A.1}$$

The Bernstein polynomials $B_0^k(x), \ldots, B_k^k(x)$ span the space of all polynomials of degree smaller than or equal to k.

The Bernstein coefficients provide bounds for range of p over $\mathbf{U} = [0,1]$. The unit interval is not really a restriction as any finite interval $\mathbf{X}$ can be linearly transformed to it. Let $\bar{p}$ denote the range of p. Then,

Lemma A.1 (Range lemma) *[5]The range $\bar{p}([0,1])$ is bounded by the Bernstein coefficients as:*

$$\bar{p}([0,1]) \subseteq \left[\min_j b_j^k, \max_j b_j^k\right]$$

The proof of the range lemma relies manly on the fact that the Bernstein form shows $p(x)$ as a convex combination of the Bernstein coefficients, because

$$\sum_{j=0}^{k} B_j^k(x) = \sum_{j=0}^{k} \binom{k}{j} x^j (1-x)^{k-j} = 1$$

and all the terms in the sum are positive.

Example A.1 *To illustrate the Bernstein approach for bounding the ranges of polynomials, consider the simple polynomial*

$$p(x) = x(1-x)$$

whose range $\bar{p}([0,1])$ is easily found (say, using basic calculus) to be $\left[0, \frac{1}{4}\right]$.

In the Bernstein approach, we first express the above polynomial in standard sums of power form

$$p(x) = \sum_{i=0}^{n} a_i x_i$$

to get

$$n = 2, a_0 = 0, a_1 = 1, a_2 = -1$$

The formula (A.1) for the Bernstein coefficients now simplifies to

$$b_0^k = 0; \quad b_j^k = \frac{j}{k} a_1 + \frac{j(j-1)}{k(k-1)} a_2 = \frac{j}{k}\left(1 - \frac{j-1}{k-1}\right)$$

For $k = 2$ this gives

$$b_0^2 = 0, \quad b_1^2 = \frac{1}{2}, \quad b_2^2 = 0$$

so that

$$\min_j b_j^2 = 0, \quad \max_j b_j^2 = \frac{1}{2}$$

and the range lemma implies

$$\bar{p}([0,1]) \subseteq \left[0, \frac{1}{2}\right]$$

Tighter bounds on $\bar{p}([0,1])$ can be obtained by elevating the degree k of the Bernstein polynomial. For instance, for $k=3$ the formula (A.1) gives

$$b_0^3 = 0, \quad b_1^3 = \frac{1}{3}, \quad b_2^3 = \frac{2}{3}\left(1 - \frac{1}{2}\right) = \frac{1}{3}, \quad b_3^3 = 0$$

and

$$\min_j b_j^3 = 0, \quad \max_j b_j^3 = \frac{1}{3}$$

By the range lemma,

$$\bar{p}([0,1]) \subseteq \left[0, \frac{1}{3}\right]$$

Ratschek and Rokne [30] prove that the range overestimation can be made small as the degree k is elevated:

$$w\left(\left[\min_j b_j^k, \max_j b_j^k\right]\right) - w\left(\bar{p}([0,1])\right) \le \frac{A}{k}$$

where

$$A = 2\sum_{s=2}^{n} (s-1)^2 \left|p^{(s)}(0)\right| / s!$$

with $p^{(s)}$ denoting the sth derivative of p. So, while the degree of $p(x)$ stays fixed at n, we can use the Bernstein polynomial forms of higher degree $k \ge n$, and even get convergence of $\left[\min_j b_j^k, \max_j b_j^k\right]$ to the range $\bar{p}([0,1])$. For instance, if k is an odd integer $k = 2p+1$, we obtain

$$\max_j b_j^{2p+1} = \frac{1}{4}\left(\frac{p+1}{p+\frac{1}{2}}\right) > \frac{1}{4}$$

with the range overestimation being bounded by $\frac{1}{8}\left(\frac{1}{p+\frac{1}{2}}\right)$. In general, for a given k the computation takes about k^2 arithmetic operations, and no evaluations of the given polynomial $p(x)$ are required. Table A.1 gives the range enclosures in Example A.1 for various value of k, up to $k = 1000$. Because the convergence is linear in the degree k, the degree elevation approach is not really efficient when tight bounds on the range are desired.

A.1 Vertex property

A remarkable feature of the Bernstein method is that criteria can be obtained to indicate whether the calculated estimation is the range or not. Cargo and Shisha [5] give such a criteria based on the *vertex* property.

TABLE A.1. Range enclosures obtained with Bernstein form of various degrees in Example A.1.

Degree k	Range Enclosure	index j for $\min b_j^k$	index j for $\max b_j^k$	Range overestimation
2	$[0, 0.5]$	0	1	0.2500
3	$\left[0, \frac{1}{3}\right]$	0	1	0.0833
4	$\left[0, \frac{1}{3}\right]$	0	2	0.0833
5	$[0, 0.3]$	0	2	0.0500
6	$[0, 0.3]$	0	3	0.0500
7	$[0, 0.2857]$	0	3	0.0357
10	$[0, 0.2778]$	0	5	0.0278
20	$[0, 0.2632]$	0	10	0.0132
30	$[0, 0.2586]$	0	15	0.0086
100	$[0, 0.2525]$	0	50	0.0025
1000	$[0, 0.2503]$	0	500	0.00025

Lemma A.2 (Vertex lemma) *[5] :* $\bar{p}([0,1]) = \left[\min_j b_j^k, \max_j b_j^k\right]$ *if and only if* $\min_j b_j^k = \min\left\{b_0^k, b_k^k\right\}$ *and* $\max_j b_j^k = \max\left\{b_0^k, b_k^k\right\}$.

An elegant proof of the vertex lemma is given by Ratschek and Rokne [30], based on the relations

$$b_0^k = a_0 = p(0); \quad b_k^k = \sum_{i=0}^{k} a_i = p(1) \tag{A.2}$$

The vertex lemma also holds for any subinterval of $[0,1]$, see [22].

Consider again Example A.1. For $k = 4$ the formula (A.1) gives

$$b_0^4 = 0, \quad b_1^4 = \frac{1}{4}, \quad b_2^4 = \frac{1}{3}, \quad b_3^4 = \frac{1}{4}, \quad b_4^4 = 0 \tag{A.3}$$

and the range lemma implies

$$\bar{p}([0,1]) \subseteq \left[0, \frac{1}{3}\right] \tag{A.4}$$

Now, we may also apply the vertex lemma to ascertain if the enclosure given by (A.4) is the range itself or not. From (A.3), the minimum Bernstein coefficient is b_0^4 or b_4^4 which occurs at vertices $j \in \{0, 4\}$, while the maximum Bernstein coefficient is b_2^4 which occurs at $j = 2$. The property in the vertex lemma is thus satisfied for the minimum, but is not satisfied for the maximum as $\max_j b_j^k \neq \max\left\{b_0^4, b_4^4\right\}$. Therefore, by the vertex lemma, the enclosure given by (A.4) is *not* the range.

It is interesting to apply the vertex lemma to Table A.1 and ascertain if any of the range enclosures given there is the range or not. We find from the table that for any k, the index j

for $\max b_j^k$ (in column 4) is not from the vertex set $\{0, k\}$. By the vertex lemma, *none* of the enclosures in column 2 is the range.

A.2 Bernstein Subdivision

A generally more efficient approach than degree elevation of the Bernstein form is subdivision [9]. Let $\mathbf{D} = [\underline{d}, \bar{d}] \subseteq \mathbf{U}$ and assume we have already the Bernstein coefficients on $\mathbf{D}$. Suppose $\mathbf{D}$ is bisected to produce two subintervals $\mathbf{D}_A$ and $\mathbf{D}_B$ given by

$$\mathbf{D}_A = [\underline{d}, m(\mathbf{D})]\,; \mathbf{D}_B = [m(\mathbf{D}), \bar{d}]$$

Then, the Bernstein coefficients on the subintervals $\mathbf{D}_A$ and $\mathbf{D}_B$ can be obtained from those on $\mathbf{D}$, by executing the following algorithm.

Algorithm Subdivision:

Inputs: The interval $\mathbf{D} \subseteq \mathbf{U}$ and the associated set of Bernstein coefficients $\left\{\bar{b}_j^k\right\}$.

Outputs: Subintervals $\mathbf{D}_A$ and $\mathbf{D}_B$ and the associated set of Bernstein coefficients $\left\{\tilde{b}_j^k\right\}$ and $\left\{\hat{b}_j^k\right\}$, respectively.

BEGIN Algorithm

1. Bisect $\mathbf{D}$ to produce the two subintervals $\mathbf{D}_A$ and $\mathbf{D}_B$.
2. Compute Bernstein coefficients on subinterval $\mathbf{D}_A$ as follows.
   (a) Set : $b_j^0 \leftarrow \bar{b}_j^k$, for $j = 0, \dots, k$.
   (b) FOR $i = 1, \dots, k$ DO

$$b_j^i = \begin{cases} b_j^{i-1} & \text{for} \quad j < i \\ \frac{1}{2}\left\{b_{j-1}^{i-1} + b_j^{i-1}\right\} & \text{for} \quad j \geq i \end{cases} \tag{A.5}$$

   To obtain the new coefficients, apply the formula given above for $j = 0, \dots, k$.

   (c) Find the Bernstein coefficients on subinterval $\mathbf{D}_A$ as

$$\tilde{b}_j^k = b_j^k, \quad \text{for } j = 0, \dots, k$$

3. Find Bernstein coefficients on subinterval $\mathbf{D}_B$ from intermediate values in above step, as follows.

$$\hat{b}_j^k = b_k^j, \quad \text{for } j = 0, \dots, k$$

4. RETURN $\mathbf{D}_A$, $\mathbf{D}_B$, and the associated Bernstein coefficients $\left\{\tilde{b}_j^k\right\}$ and $\left\{\hat{b}_j^k\right\}$.

END Algorithm

Let us run through Algorithm Subdivision for Example A.1. For $k = 4$, we have already the Bernstein coefficients $\bar{b}_j^k$ given in (A.3) for the interval $\mathbf{D} = [0, 1]$. With these as the inputs to Algorithm subdivision, the results at the various steps are

- step 1: $\mathbf{D}$ is bisected to produce two subintervals $\mathbf{D}_A = [0, 0.5]$ and $\mathbf{D}_B = [0.5, 1]$.
- step 2: The Bernstein coefficients on subinterval $\mathbf{D}_A$ are computed as follows.
  - step 2a: Set : $b_j^0 \leftarrow \bar{b}_j^4$, for $j = 0, \ldots, 4$, to get

$$b_0^0 = \bar{b}_0^4 = 0; \quad b_1^0 = \bar{b}_1^4 = \frac{1}{4}; \quad b_2^0 = \bar{b}_2^4 = \frac{1}{3};$$
$$b_3^0 = \bar{b}_3^4 = \frac{1}{4}; \quad b_4^0 = \bar{b}_4^4 = 0$$

  - step 2b: Applying formula (A.5), we obtain

    for $i = 1$,

$$b_0^1 = b_0^0 = 0$$
$$b_1^1 = \frac{1}{2}\left(b_0^0 + b_1^0\right) = \frac{1}{2}\left(0 + \frac{1}{4}\right) = \frac{1}{8}$$
$$b_2^1 = \frac{1}{2}\left(b_1^0 + b_2^0\right) = \frac{1}{2}\left(\frac{1}{4} + \frac{1}{3}\right) = \frac{7}{24}$$
$$b_3^1 = \frac{1}{2}\left(b_2^0 + b_3^0\right) = \frac{1}{2}\left(\frac{1}{3} + \frac{1}{4}\right) = \frac{7}{24}$$
$$b_4^1 = \frac{1}{2}\left(b_3^0 + b_4^0\right) = \frac{1}{2}\left(\frac{1}{4} + 0\right) = \frac{1}{8}$$

    for $i = 2$,

$$b_0^2 = b_0^1 = 0$$
$$b_1^2 = b_1^1 = \frac{1}{8}$$
$$b_2^2 = \frac{1}{2}\left(b_1^1 + b_2^1\right) = \frac{1}{2}\left(\frac{1}{8} + \frac{7}{24}\right) = \frac{10}{48}$$
$$b_3^2 = \frac{1}{2}\left(b_2^1 + b_3^1\right) = \frac{1}{2}\left(\frac{7}{24} + \frac{7}{24}\right) = \frac{7}{24}$$
$$b_4^2 = \frac{1}{2}\left(b_3^1 + b_4^1\right) = \frac{1}{2}\left(\frac{7}{24} + \frac{1}{8}\right) = \frac{10}{48}$$

for $i = 3$,

$$\begin{aligned} b_0^3 &= b_0^2 = 0 \\ b_1^3 &= b_1^2 = \frac{1}{8} \\ b_2^3 &= b_2^2 = \frac{10}{48} \\ b_3^3 &= \frac{1}{2}\left(b_2^2 + b_3^2\right) = \frac{1}{2}\left(\frac{10}{48} + \frac{7}{24}\right) = \frac{1}{4} \\ b_4^3 &= \frac{1}{2}\left(b_3^2 + b_4^2\right) = \frac{1}{2}\left(\frac{7}{24} + \frac{10}{48}\right) = \frac{1}{4} \end{aligned}$$

and for $i = 4$,

$$\begin{aligned} b_0^4 &= b_0^3 = 0 \\ b_1^4 &= b_1^3 = \frac{1}{8} \\ b_2^4 &= b_2^3 = \frac{10}{48} \\ b_3^4 &= b_3^3 = \frac{1}{4} \\ b_4^4 &= \frac{1}{2}\left(b_3^3 + b_4^3\right) = \frac{1}{2}\left(\frac{1}{4} + \frac{1}{4}\right) = \frac{1}{4} \end{aligned}$$

– Step 2c: The Bernstein coefficients on the subinterval $\mathbf{D}_A$ are therefore

$$\tilde{b}_0^4 = b_0^4 = 0; \quad \tilde{b}_1^4 = b_1^4 = \frac{1}{8}; \quad \tilde{b}_2^4 = b_2^4 = \frac{10}{48}$$
$$\tilde{b}_3^4 = b_3^4 = \frac{1}{4}; \quad \tilde{b}_4^4 = b_4^4 = \frac{1}{4}$$

- step 3: The Bernstein coefficients on the neighboring subinterval $\mathbf{D}_B$ are obtained as

$$\hat{b}_0^4 = b_4^0 = 0; \quad \hat{b}_1^4 = b_4^1 = \frac{1}{8}; \quad \hat{b}_2^4 = b_4^2 = \frac{10}{48};$$
$$\hat{b}_3^4 = b_4^3 = \frac{1}{4}; \quad \hat{b}_4^4 = b_4^4 = \frac{1}{4}$$

Finally,

$$\begin{aligned} &\text{For subinterval } \mathbf{D}_A \ : \ \text{Bernstein coefficients are } \left(0, \frac{1}{8}, \frac{10}{48}, \frac{1}{4}, \frac{1}{4}\right) \\ &\text{For subinterval } \mathbf{D}_B \ : \ \text{Bernstein coefficients are } \left(0, \frac{1}{8}, \frac{10}{48}, \frac{1}{4}, \frac{1}{4}\right) \end{aligned} \tag{A.6}$$

(it is coincidental here that the Bernstein coefficients for both the subintervals are the same). By the range lemma

$$\bar{p}\left(\mathbf{D}_A\right) \subseteq \left[0, \frac{1}{4}\right]; \quad \bar{p}\left(\mathbf{D}_B\right) \subseteq \left[0, \frac{1}{4}\right] \tag{A.7}$$

A.3 Combining vertex property and subdivision

By successively subdividing $\mathbf{U}$, we can eventually make the polynomial to be monotonic (within machine precision) on every subinterval. Then, the range of p on any such subinterval $\mathbf{D} = \left[\underline{d}, \bar{d}\right] \subseteq \mathbf{U}$ is obtainable from the endpoint values $p\left(\underline{d}\right)$ and $p\left(\bar{d}\right)$. >From (A.2), the latter values are nothing but the Bernstein coefficients at the vertices $\{0, k\}$. Thus, we see that by combining the tool of Bernstein subdivision and the vertex property, we can repeatedly improve the bounds till the vertex condition is satisfied (within machine precision) on every subdivision.

Consider the Bernstein coefficients given in (A.6). For subinterval $\mathbf{D}_A$, the minimum Bernstein coefficient is $\tilde{b}_0^4$ while the maximum Bernstein coefficient is $\hat{b}_4^4$. Both these occur at the vertices, i.e., for $j \in \{0, 4\}$. By the vertex lemma, the range of $\bar{p}\left(\mathbf{D}_A\right)$ is given by (A.7) and is $\left[0, \frac{1}{4}\right]$. An identical situation holds for the other subinterval $\mathbf{D}_B$. Thus, we obtain the range $\bar{p}\left([0,1]\right) = \left[0, \frac{1}{4}\right]$.

In this example, using just *one* subdivision and application of the vertex lemma to the subintervals, we have been able to obtain the range of the given polynomial. Further, we are also able to assert that the obtained enclosure is indeed the range. Note that it was not possible to get the range through degree elevation, even with Bernstein form of as high a degree as $k = 1000$, that produced an overestimation of about $2.5e - 04$!