EE101: RLC Circuits (with DC sources)

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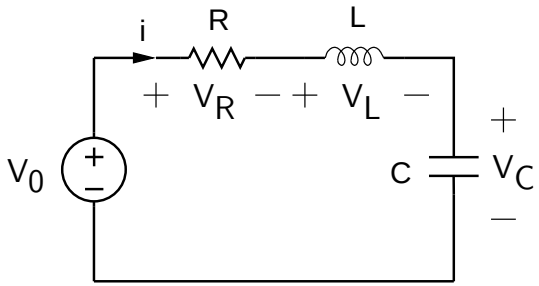
Series *RLC* circuit



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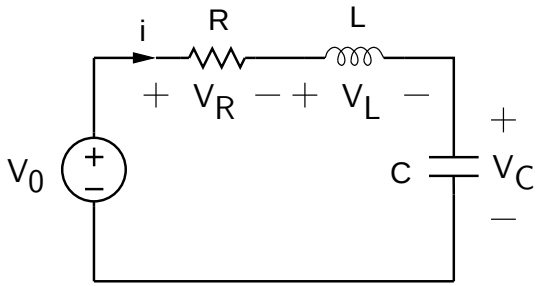

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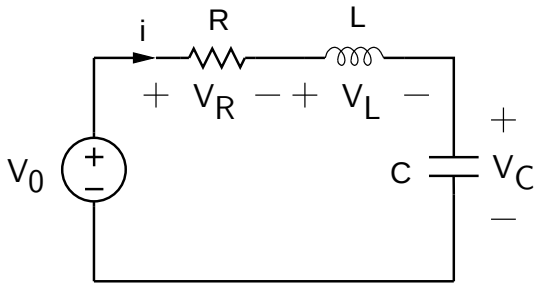

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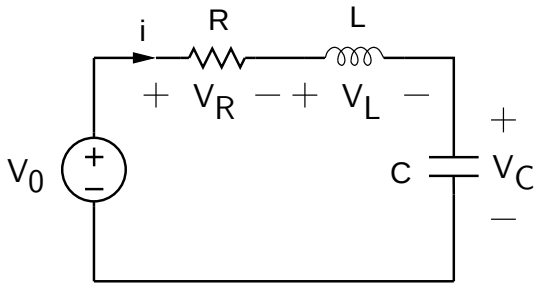

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a second-order ODE with constant coefficients.

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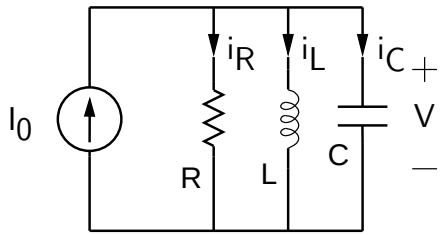

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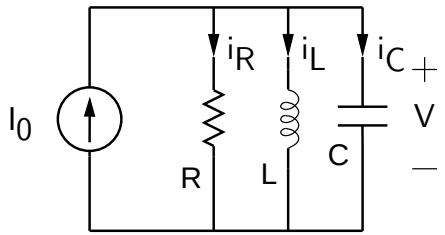

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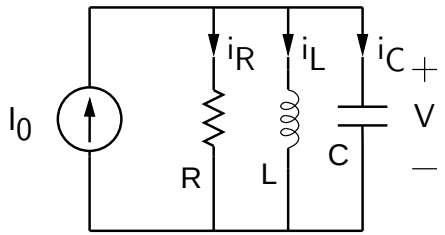

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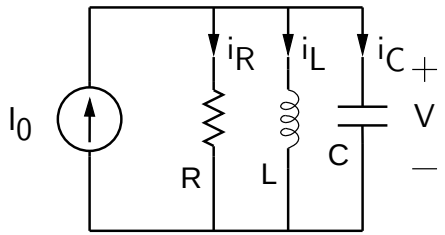

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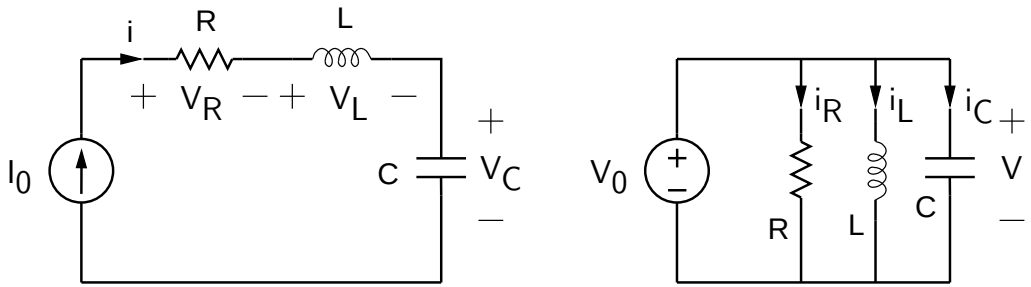

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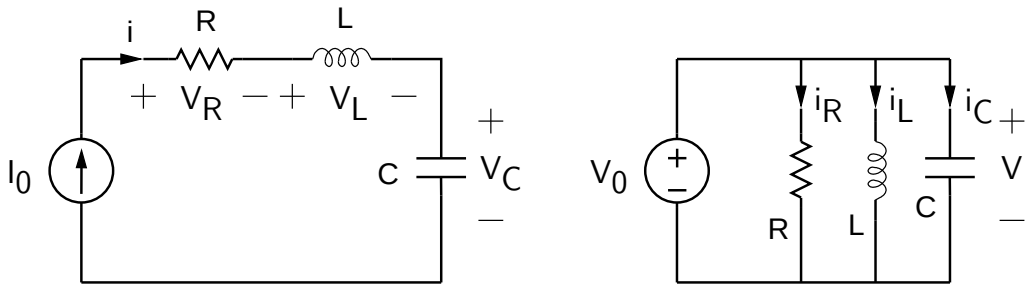

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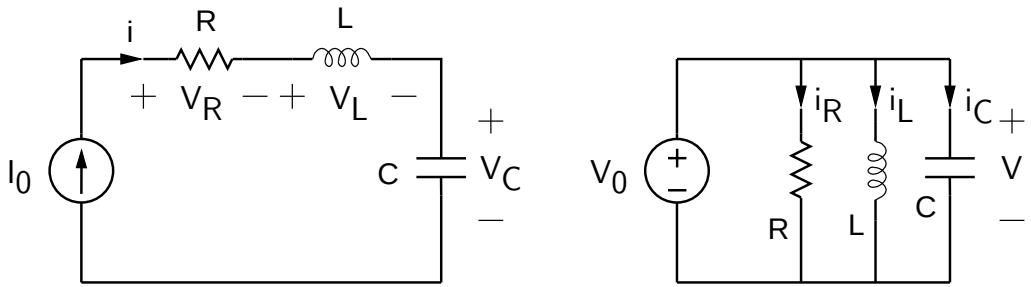

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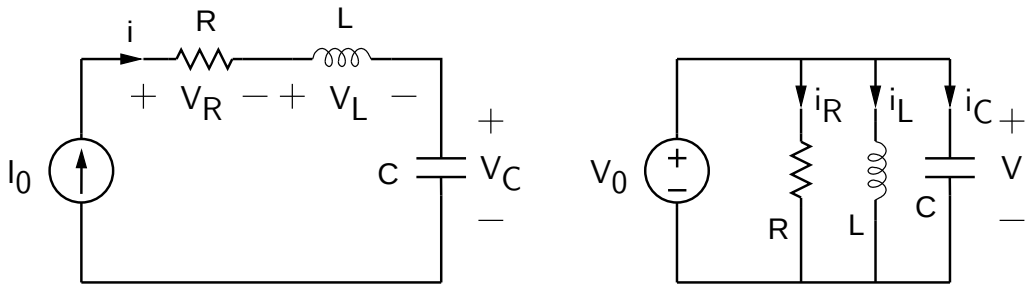

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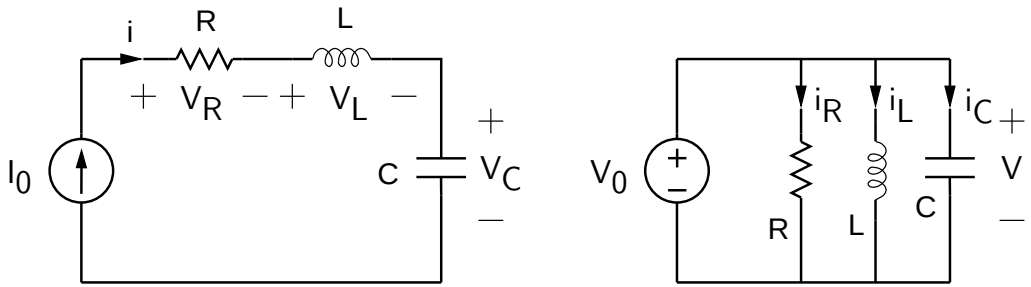

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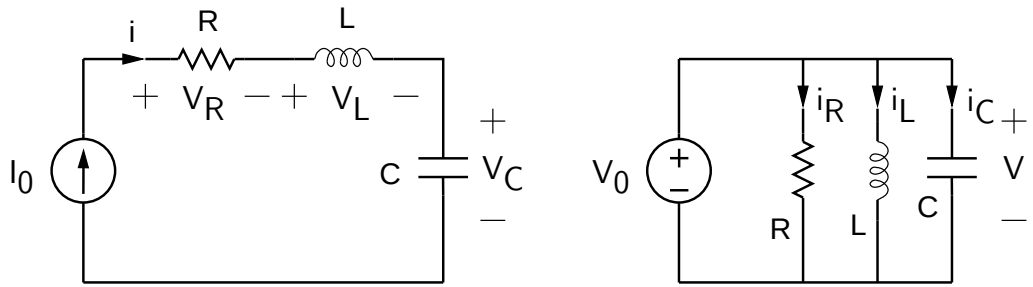

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* The above equations hold even if the applied voltage or current is not constant, and the variables of interest can still be easily obtained without solving a differential equation.

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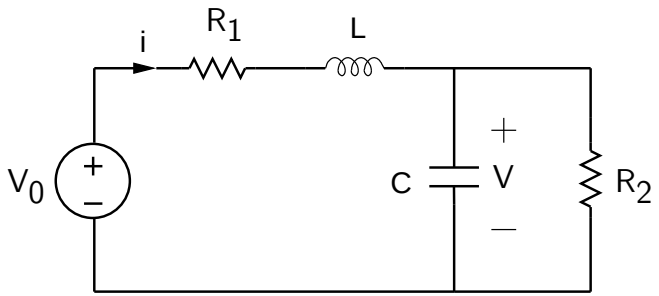

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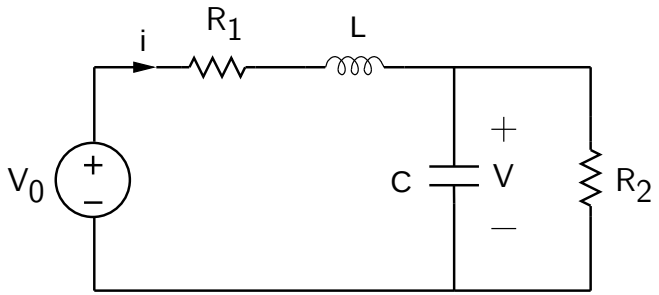

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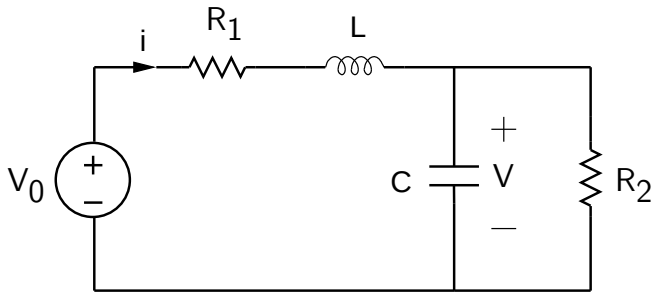

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Substituting (2) in (1), we get

$$V_0 = R_1\left[CV' + V/R_2\right] + L\left[CV'' + V'/R_2\right] + V\,, \tag{3}$$

$$V''\left[LC\right] + V'\left[R_1 C + L/R_2\right] + V\left[1 + R_1/R_2\right] = V_0\,. \tag{4}$$

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$$\frac{d^2y}{dt^2} + a\,\frac{dy}{dt} + b\,y = K \text{ (constant)}\,.$$

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$$y(t) = y^{(h)}(t) + y^{(p)}(t)\,,$$

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In the context of *RLC* circuits, $y^{(p)}(t)$ is the steady-state value of the variable of interest, i.e.,

$$y^{(p)} = \lim_{t\to\infty} y(t),$$

which can be often found by inspection.

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For the homogeneous equation,

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$$r^2 + a\,r + b = 0\,.$$

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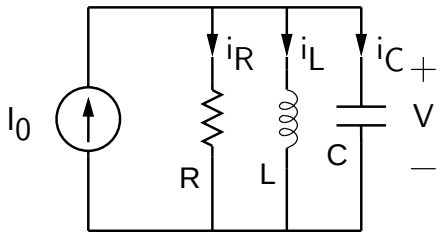

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L=0.44 mH

$I_0 = 100$ mA

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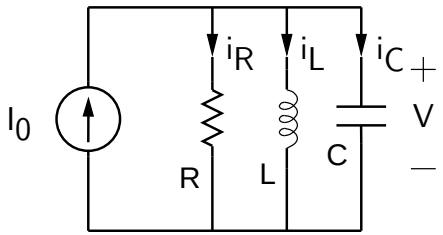

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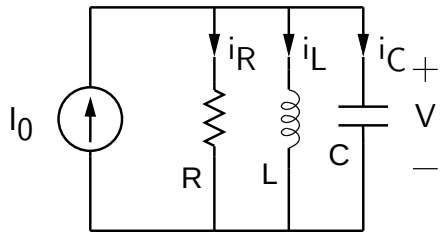

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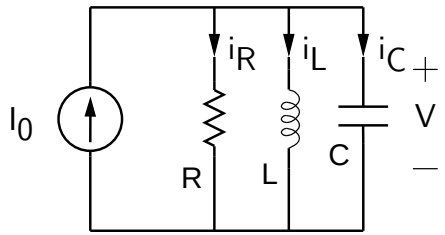

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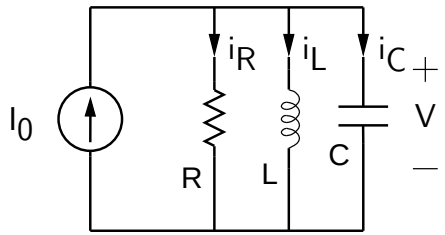

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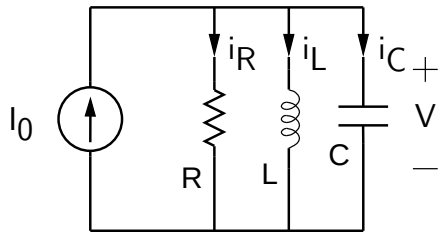

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where $\tau_1 = -1/r_1 = 15.4\,\mu s$, $\tau_2 = -1/r_1 = 28.6\,\mu s$.

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$$(A/\tau_1) + (B/\tau_2) = -I_0/C\,. \tag{2}$$

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$i_L(0^+) = I_0 - \dfrac{1}{R}\ V(0^+) - C\,\dfrac{dV}{dt}(0^+) = 0\ A$, which gives

$$(A/\tau_1) + (B/\tau_2) = -I_0/C\,. \tag{2}$$

From (1) and (2), we get the values of A and B, and

$$V(t) = -3.3\,[\exp(-t/\tau_1) - \exp(-t/\tau_2)]\ \ V\,. \tag{3}$$

(SEQUEL file: `ee101_rlc_1.sqproj`)

Parallel *RLC* circuit

As $t \to \infty$, $V = L\,\dfrac{di_L}{dt} = 0\ V \Rightarrow V(\infty) = 0\ V$.

$\Rightarrow V(t) = A\ \exp(-t/\tau_1) + B\ \exp(-t/\tau_2)$,

Since $V(0^+) = 0\ V$, we have,

$$A + B = 0\,. \tag{1}$$

Our other initial condition is $i_L(0^+) = 0\ A$, which can be used to obtain $\dfrac{dV}{dt}(0^+)$.

$i_L(0^+) = I_0 - \dfrac{1}{R}\,V(0^+) - C\,\dfrac{dV}{dt}(0^+) = 0\ A$, which gives

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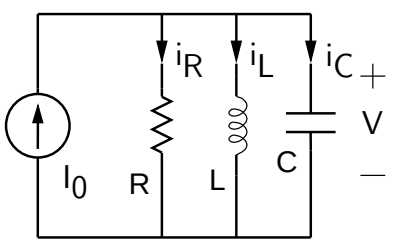
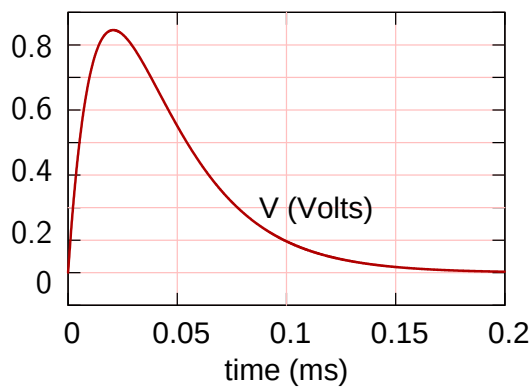
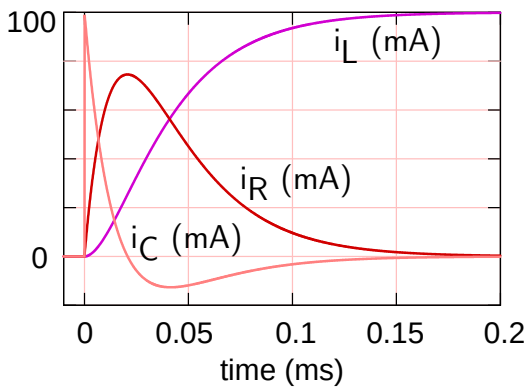
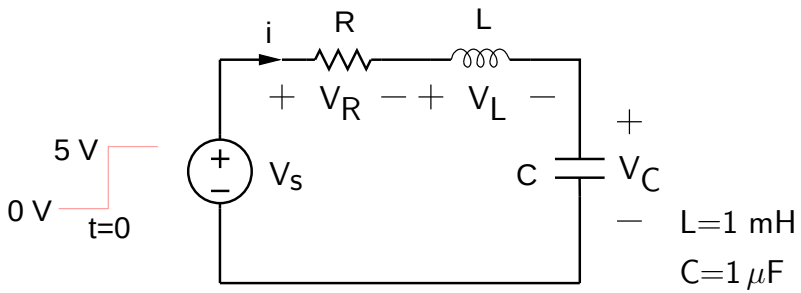


R=10 Ω
C=1 μF
L=0.44 mH
$I_0 = 100$ mA

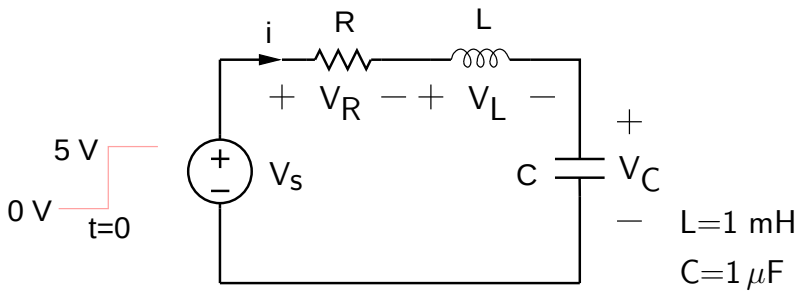




Series *RLC* circuit: home work

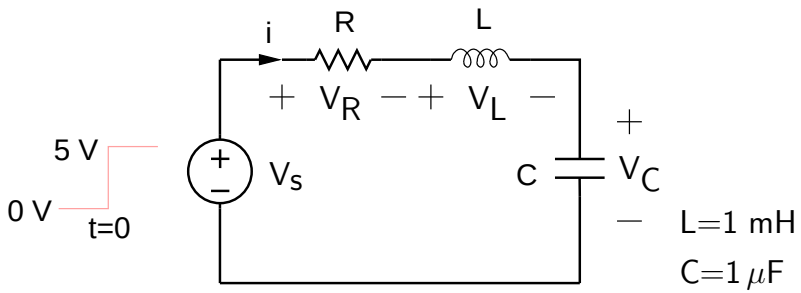


Series *RLC* circuit: home work



(a) Show that the condition for critically damped response is $R = 63.2\,\Omega$.

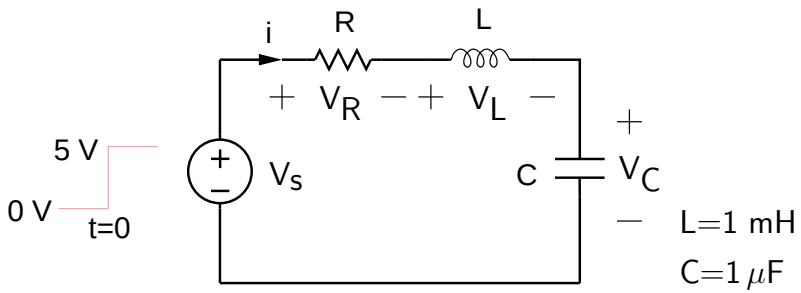
Series *RLC* circuit: home work



(a) Show that the condition for critically damped response is $R = 63.2\,\Omega$.

(b) For $R = 20\,\Omega$, derive expressions for $i(t)$ and $V_L(t)$ for $t > 0$ (Assume that $V_C(0^-) = 0\,V$ and $i_L(0^-) = 0\,A$). Plot them versus time.

Series *RLC* circuit: home work



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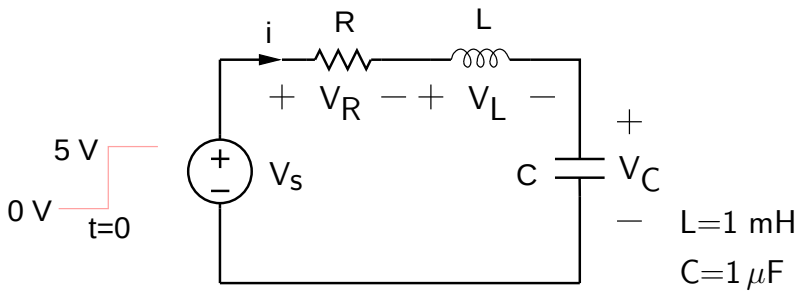

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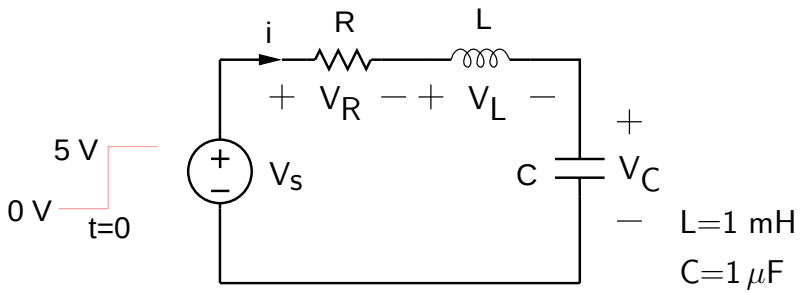
(SEQUEL file: `ee101_rlc_2.sqproj`)

Series *RLC* circuit: home work



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