Chapter 2

RC and RL Circuits with Piecewise Constant Sources

2.1 Background

Consider a circuit consisting of a single capacitor, resistors, dc independent sources, and controlled sources (assumed to be linear), as shown in Fig. 2.1 (a). Using Thevenin's theorem,

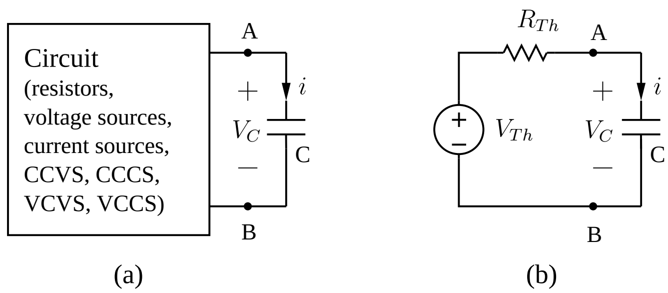

Figure 2.1: A linear circuit with a single capacitor.

the circuit can be reduced to that shown in Fig. 2.1 (b), where V_{Th} is a dc source. The behaviour of this circuit is governed by the Ordinary Differential Equation (ODE),

$$V_{Th} = i\,R_{Th} + V_C = R_{Th}\,C\,\frac{dV_C}{dt} + V_C, \tag{2.1}$$

$$\text{i.e.,}\quad \frac{dV_C}{dt} = \frac{1}{\tau}\,(V_{Th} - V_C)\;, \tag{2.2}$$

where $\tau = R_{Th}\,C$. The homogeneous part of Eq. 2.1 is,

$$\frac{dV_C}{dt} = -\frac{V_C}{\tau}\,, \tag{2.3}$$

for which the solution is $V_C^{(h)} = A\,e^{-t/\tau}$. The complete solution is $V_C(t) = V_C^{(h)}(t) + V_C^{(p)}(t)$, $V_C^{(p)}(t)$ being a particular solution. In Eq. 2.2, if we substitute $V_C = K$ (a constant), we get a particular solution, viz., $V_C = K = V_{Th}$. In general, the complete solution for $V_C(t)$ is therefore given by,

$$V_C(t) = A\,e^{-t/\tau} + B\,, \tag{2.4}$$

where A and B are constants, and $\tau = R_{Th}C$ is the circuit time constant.
In fact, *any* current or voltage in the circuit is given by the general form of Eq. 2.4, the constants A and B depending on the variable under consideration.
For a circuit containing a single inductor, resistors, dc independent sources, and controlled sources (assumed to be linear), we get (see Fig. 2.2),

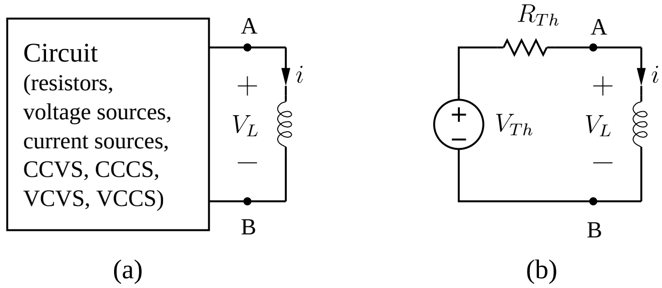

Figure 2.2: A linear circuit with a single inductor.

$$V_{Th} = R_{Th}\,i + L\,\frac{di}{dt}, \tag{2.5}$$

$$\text{i.e.,}\ \frac{di}{dt} = -\frac{i}{\tau} + \frac{V_{Th}}{L}, \tag{2.6}$$

where $\tau = L/R_{Th}$. The general solution for i (or any other current or voltage in the circuit) is of the form,

$$i(t) = A\,e^{-t/\tau} + B\,. \tag{2.7}$$

There are situations in which the independent sources in the circuit of Fig. 2.1 or Fig. 2.2 are not constant at all times but are piece-wise constant (see Fig. 2.3.) The general form, $x(t) = A\,e^{-t/\tau} + B$ is applicable in such cases as well except that the constants A and B need

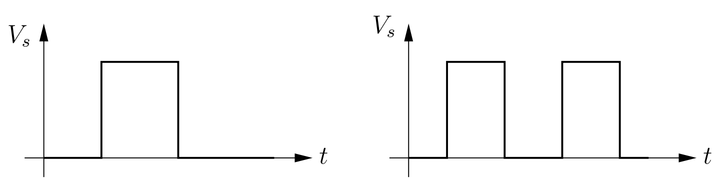

Figure 2.3: Examples of a piece-wise constant voltage source.

to be computed in each interval where the source is constant.
If there is a sudden change in the circuit (such as a voltage source changing from $0\,V$ to $5\,V$ or a switch opening or closing), we need to obtain new initial conditions for the quantity of interest, as illustrated in the following example.
Consider the circuit shown in Fig. 2.4. Suppose the change in the source voltage from $0\,V$ to $5\,V$ occurs at time t_0. If the capacitor voltage at t_0^- and t_0^+ are different, then $\dfrac{V_C(t_0^+) - V_C(t_0^-)}{t_0^+ - t_0^-}$ would be a large quantity. This quantity is in fact $\dfrac{dV_C}{dt}$, as $t_0^+ \rightarrow t_0^-$, which means that, if $V_C(t_0^+) \neq V_C(t_0^-)$, a large capacitor current would result. In the circuit shown in Fig. 2.4, for

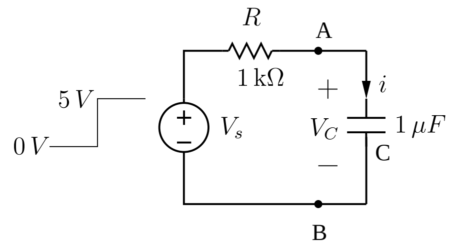

Figure 2.4: A capacitor circuit with a sudden change in the source voltage.

example, this large capacitor current would cause a large voltage drop across R, violating Kirchhoff's Voltage Law (KVL). For this reason, $V_C(t_0^+) = V_C(t_0^-)$, i.e., the voltage across a capacitor is a continuous function.
A similar comment applies to an inductor current i_L. If there is a sudden change in i_L, the voltage across the inductor would be very large, violating KVL for the circuit under consideration.
The condition of continuity of a capacitor voltage and an inductor current can be used to obtain variables (currents/voltages) of interest just after a sudden change in a source voltage or source current and a sudden opening or closing of a switch, as we shall see in the following examples.

2.2 Examples

1. For the circuit shown in Fig. 2.5, $R_1 = 10\,\text{k}\Omega$, $R_2 = 2\,\text{k}\Omega$, $R_3 = 10\,\text{k}\Omega$, and $C = 0.1\,\mu F$.

   (a) Find $V_C(t)$.

   (b) Find $i_3(t)$, using $V_C(t)$ computed in (a).

   (c) Find $i_3(t)$ directly, i.e., without using $V_C(t)$.

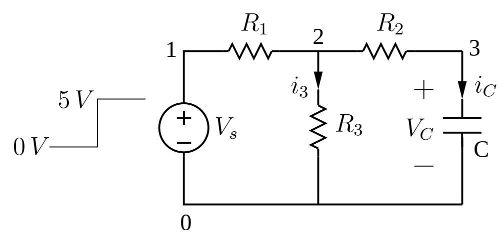

Figure 2.5: Circuit for Example 1.

We assume that the source voltage has been $0\,V$ for a long time which means that the circuit is in a steady state (dc conditions) just before $t=0$, i.e., at $t=0^-$, with no currents or voltages changing with time. The derivative, dV_C/dt is therefore zero, making the capacitor current equal to zero. Equivalently, the capacitor can be replaced with an open circuit at $t=0^-$. This is true in general about circuits involving dc sources and capacitors in steady state.

The situation at $t=0^-$ is shown in Fig. 2.6 (a). Clearly, the capacitor voltage at $t=0^-$ is

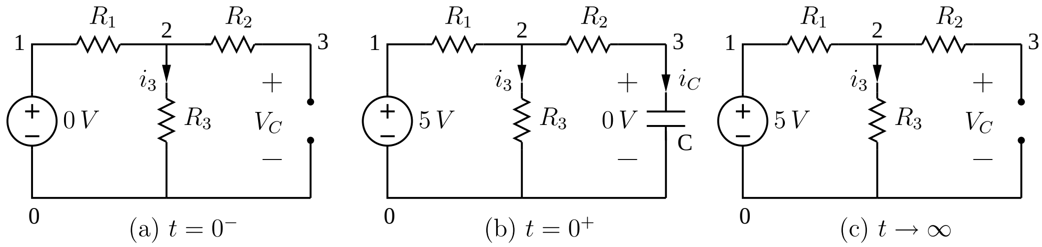

Figure 2.6: Circuit for Example 1 at $t = 0^-$, $t = 0^+$, and $t \to \infty$.

$0\,V$, and $V_C(t\!=\!0^+) = V_C(t\!=\!0^-) = 0\,V$, as shown in Fig. 2.6 (b). As $t \to \infty$, the circuit reaches steady state, the capacitor is replaced by an open circuit, and

$V_C(\infty) = 5\,V \times \dfrac{R_3}{R_1 + R_3} = 2.5\,V$ (see Fig. 2.6 (c)).

Next, we obtain the circuit time constant $\tau = R_{Th}\,C$ for $t > 0$ (see Fig. 2.7). To obtain

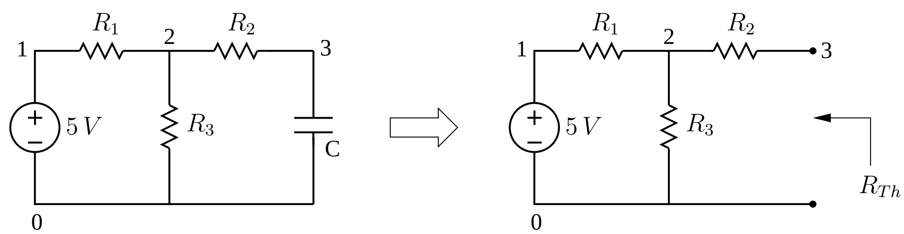

Figure 2.7: Computation of time constant for the circuit of Example 1 for $t > 0$.

the Thevenin resistance as seen by the capacitor, we deactivate the independent source (i.e., short the voltage source), and get $R_{Th} = R_2 + (R_1 \parallel R_3) = 7\,\text{k}\Omega$. The time constant of the circuit for $t > 0$ is therefore $\tau = R_{Th}\,C = 0.7\,\text{m}s$.

We are now in a position to obtain an analytical expression for $V_C(t)$ for $t > 0$. Let

$$V_C(t) = A\,e^{-t/\tau} + B\,. \tag{2.8}$$

Using $V_C(0^+) = 0\,V$ and $V_C(\infty) = 2.5\,V$, we get $A = -2.5$, $B = 2.5$. The result of Eq. 2.8 is plotted in Fig. 2.8 (a). Note that, in about five time constants (i.e., 3.5 ms), V_C becomes a constant. This is generally true – if there are no further changes in the source voltages (or switch conditions), all transients vanish in about five time constants because $e^{-5} = 0.0067 \approx 0$, or $(1 - e^{-5}) = 0.9933 \approx 1$.

Using the above $V_C(t)$, we can find i_3 for the circuit as (see Fig. 2.5),

$$i_3 = \frac{1}{R_3}\,(i_C\,R_2 + V_C) = \frac{1}{R_3}\left(C\,\frac{dV_C}{dt}\,R_2 + V_C\right)\,. \tag{2.9}$$

Fig. 2.8 (b) shows a plot of i_3 versus time using Eq. 2.9.

We can also compute i_3 *directly*, i.e., *without* using $V_C(t)$. Let

$$i_3(t) = A_1\,e^{-t/\tau} + B_1\,. \tag{2.10}$$

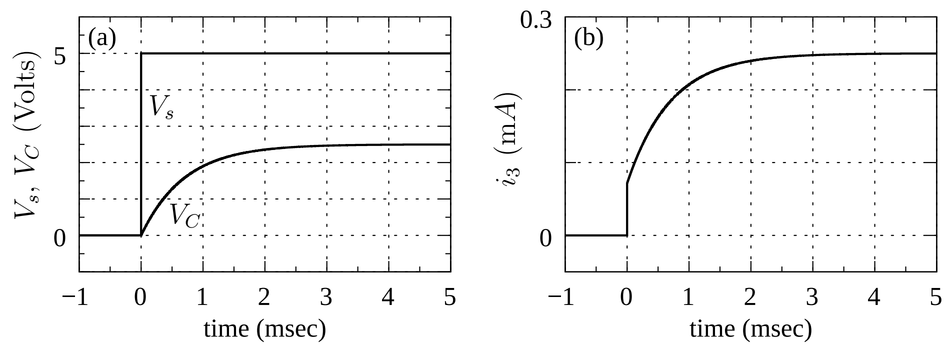

Figure 2.8: (a) $V_s(t)$, $V_C(t)$, (b) $i_3(t)$ for the circuit of Example 1.

The constants A_1 and B_1 are obtained using the values of $i_3(0^+) = 0.071\,\text{m}A$ and $i_3(\infty) = 0.25\,\text{m}A$, which are found from Figs. 2.6 (b) and 2.6 (c), respectively.

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2. For the circuit shown in Fig. 2.9, $R_1 = 10\,\Omega$, $R_2 = 40\,\Omega$, $L = 0.8\,H$, $t_0 = 0\,s$, and $t_1 = 0.1\,s$.

   (a) Find $i_L(t)$ and $i_s(t)$.

   (b) Simulate the circuit and verify that the total power absorbed is equal to the total power delivered at all times.

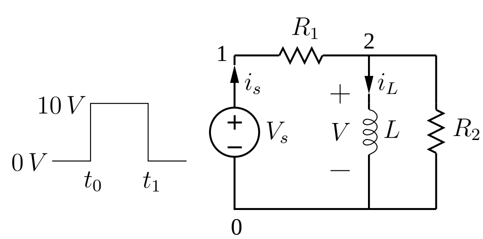

Figure 2.9: Circuit for Example 2.

The above problem is equivalent to three problems, each with a DC source, as shown in Fig. 2.10. The circuit time constant is the same in all cases, since the Thevenin

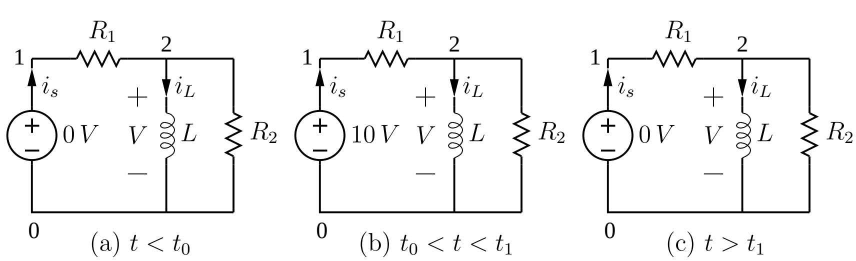

Figure 2.10: Circuit of Fig. 2.9 for three intervals: (a) $t < t_0$, (b) $t_0 < t < t_1$, and (c) $t > t_1$.

resistance seen by the inductor remains the same, viz., $R_{Th} = R_1 \parallel R_2 = 8\,\Omega$, giving

$\tau = L/R_{Th} = 0.1\,s$. The general form, $x(t) = A\,e^{-t/\tau} + B$, applies in all three cases for any current or voltage, with the constants A and B computed appropriately for the variable of interest.

At $t = t_0^-$, since the source has been $0\,V$ for a long time, we have $i_L(0^-) = 0\,A$, and therefore $i_L(0^+) = i_L(0^-) = 0\,A$. Now, consider the circuit in Fig. 2.10 (b) for $t_0 < t < t_1$. If the voltage source were to remain at $10\,V$ beyond t_1, then, as $t \rightarrow \infty$, we would have[1] $V = L\,\dfrac{di_L}{dt} = 0\,V$, and by solving the circuit equations with $V_L = 0\,V$, we get $i_L(\infty) = 10\,V/R_1 = 1\,A$. Using these values of $i_L(0^+)$ and $i_L(\infty)$, we get

$$i_L(t) = 0.1\,\left(1 - e^{-t/\tau}\right)\,, \qquad t_0 < t < t_1\,. \tag{2.11}$$

With the above equation, we obtain, $i_L(t_1^-) = 0.632\,A$. In reality, however, the dc source changes back to $0\,V$ at $t = t_1$, leading to the circuit of Fig. 2.10 (c). We can now use the condition, $i_L(t_1^+) = i_L(t_1^-) = 0.632\,A$, to proceed further to the final interval, $t > t_1$. Since the source voltage is $0\,V$ in this interval, we have $i_L(\infty) = 0\,A$. Using these conditions, we get, $A = i_L(t_1^+)$, $B = 0$, leading to

$$i_L(t) = A\,e^{-t/\tau} = i_L(t_1^+)\,e^{-(t-t_1)/\tau}\,, \qquad t > t_1\,. \tag{2.12}$$

Combining Eqs. 2.11 and 2.12, we get the complete solution, as shown in Fig. 2.11.

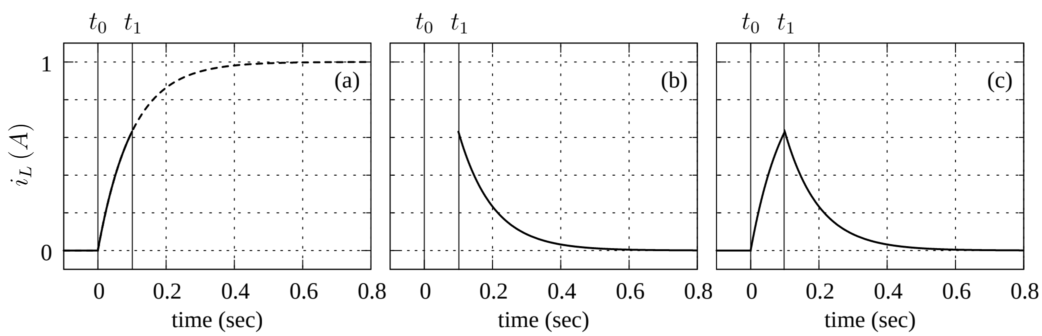

Figure 2.11: i_L versus time for the circuit of Example 2: (a) $t < t_1$, (b) $t > t_1$, (c) complete solution.

Having obtained $i_L(t)$, we can get the source current, $i_s(t)$:

$$i_s(t) = i_L(t) + \frac{1}{R_2}\,V = i_L(t) + \frac{1}{R_2}\,L\,\frac{di_L}{dt}\,. \tag{2.13}$$

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[1]In simple terms, in the interval $t_0 < t < t_1$, the circuit sees a constant voltage source and does not *know* that things are going to change at t_1. Therefore, up to $t = t_1$, its behaviour is *independent* of the change that actually occurs at $t = t_1$.

3. For the circuit shown in Fig. 2.12, $R = 1\,\mathrm{k}\Omega$ and $C = 0.1\,\mu F$.

   (a) Obtain $V_2(t)$, assuming that the diode has a negligible on resistance, infinite off resistance, and a voltage drop of $0.7\,V$ while conducting.

   (b) Repeat (a) with the diode reversed.

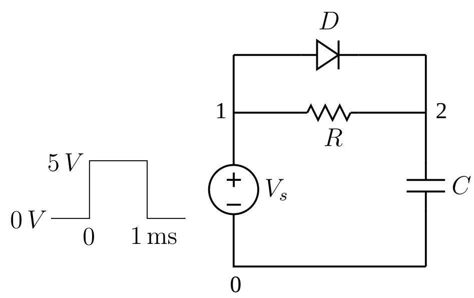

Figure 2.12: Circuit for Example 3.

Since the diode is a non-linear circuit element, it would appear that our analysis would not be valid for this circuit. However, when the diode is replaced with the approximate equivalent circuits shown in Figs. 2.13 (a) and (b), the capacitor sees a linear circuit in both the on and off cases. Our expression $x(t) = A\,e^{-t/\tau} + B$ is therefore valid in both

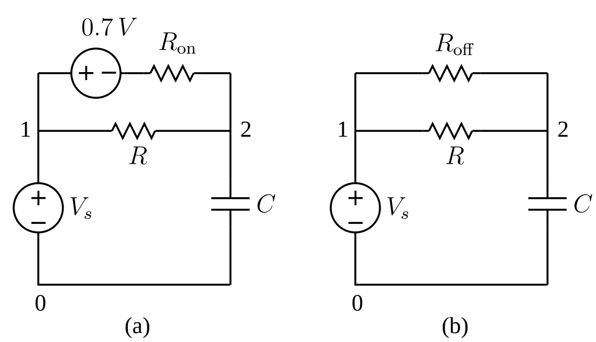

Figure 2.13: Approximate description of the circuit of Fig. 2.12: (a) diode conducting, (b) diode not conducting. ($R_{\mathrm{on}} \rightarrow 0\,\Omega$, $R_{\mathrm{off}} \rightarrow \infty\,\Omega$).

cases, the major difference between the two cases being the magnitude of the time constants. When D conducts (Fig. 2.13 (a)), the Thevenin resistance seen by the capacitor is $R_{Th} = R \parallel R_{\mathrm{on}} \approx R_{\mathrm{on}}$, a very small resistance. When D does not conduct (Fig. 2.13 (b)), $R_{Th} = R \parallel R_{\mathrm{off}} \approx R$. Thus, the time constants $\tau_1 = R_{\mathrm{on}}\,C$ and $\tau_2 = R\,C$ in the two cases are vastly different, with $\tau_1 \ll \tau_2$, and on the time scale of the given circuit, we can say that $\tau_1 \approx 0\,\mathrm{sec}$.

When the source voltage rises to $5\,V$, the capacitor charges instantaneously (since $\tau_1 \approx 0\,\mathrm{s}$) up to about $4.3\,V$, i.e., $0.7\,V$ less than $5\,V$ (see Fig. 2.14). Beyond this point, the diode turns off, and the capacitor charges slowly, with a time constant of $\tau_2 = R\,C = 0.1\,\mathrm{ms}$. Since the pulse duration ($1\,\mathrm{ms}$) is larger than $5\,\tau = 0.5\,\mathrm{ms}$, the charging process is completed, and V_2 reaches $5\,V$.

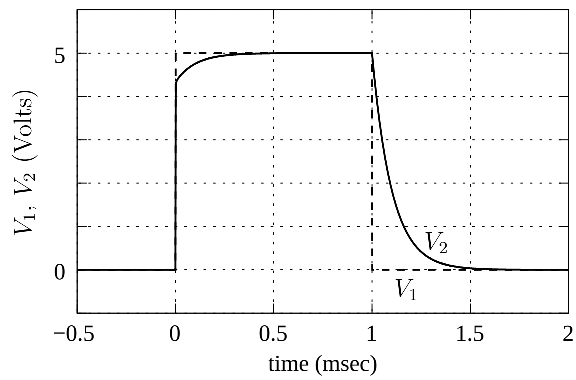

Figure 2.14: V_1 and V_2 versus time for the circuit of Example 3.

At $t = 1$ ms, the capacitor discharges with a time constant $\tau_2 = 0.1$ ms, and after about five time constants, V_2 reaches its steady-state value of $0\,V$.

When the diode is reversed, the capacitor gets charged with $\tau = RC$ and discharged with $\tau \approx 0$ s. The reader is encouraged to work out the waveform for V_2 in this case and compare it with simulation results.

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4. For the circuit shown in Fig. 2.15 (a), the switch opens at $t = 0$ s. The component values are $R_1 = 5\,\text{k}\Omega$, $R_2 = 1\,\text{k}\Omega$, $R_3 = 5\,\text{k}\Omega$, and $C = 5\,\mu F$. Find $V_C(t)$ and $i(t)$.

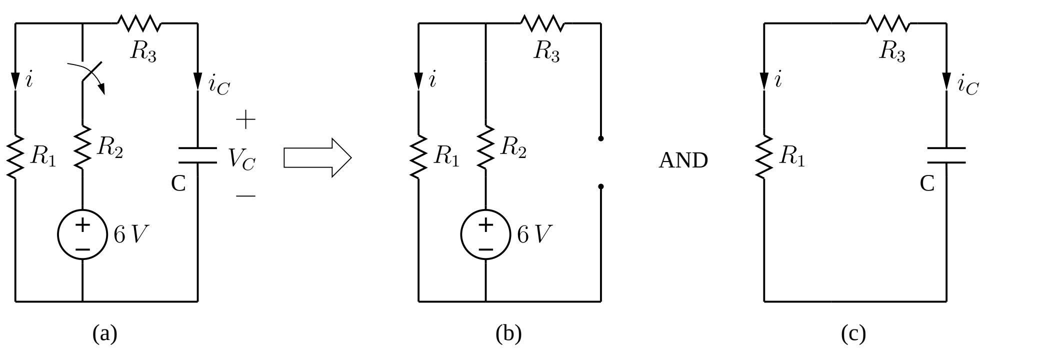

Figure 2.15: (a) Circuit for Example 4, (b) Circuit for $t < 0$, (c) Circuit for $t > 0$.

Assuming that the switch has been closed for a long time before $t = 0$ s, we have $i_C = C\,\dfrac{dV_C}{dt} = 0\,A$ (i.e., the capacitor is an open circuit), leading to $i(0^-) = \dfrac{6\,V}{R_1 + R_2} = 1\,\text{m}A$, and $V_C(0^-) = R_1\,i = 5\,V$ (see Fig. 2.15 (b)). By continuity of V_C, we get $V_C(0^+) = V_C(0^-) = 5\,V$.

At $t = 0$, the switch opens, and the Thevenin resistance seen by the capacitor for $t > 0$ is $R_{Th} = R_1 + R_3 = 10\,\text{k}\Omega$, resulting in a time constant $\tau = R_{Th}\,C = 50$ ms (see Fig. 2.15 (c)). As $t \to \infty$, $V_C \to 0\,V$, and $i \to 0\,A$.

Let $V_C(t) == A_1\,e^{-t/\tau} + B_1$ for $t > 0$. Substituting $V_C(0^+) = 5\,V$ and $V_C(\infty) = 0\,V$, we can get A_1 and B_1.

Let $i(t) = A_2\, e^{-t/\tau} + B_2$ for $t > 0$. Since $V_C(0^+) = 0\,V$, we get (see Fig. 2.15 (c)), $i(0^+) = \dfrac{5\,V}{R_1 + R_3} = 0.5\,\mathrm{m}A$. Using $i(0^+)$ and $i(\infty)$, we can get A_2 and B_2.

Fig. 2.16 shows V_C and i as a function of time.

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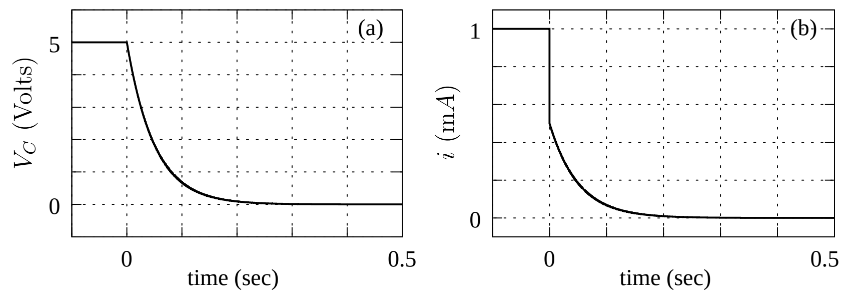

Figure 2.16: (a) $V_C(t)$, (b) $i(t)$ for the circuit of Example 4.

5. For the circuit shown in Fig. 2.17, the switch opens at $t = 0\,$s. The component values are $R_1 = 2\,\Omega$, $R_2 = 5\,\Omega$, $R_3 = 3\,\Omega$, and $C = 1\,\mathrm{m}F$. Find $i_C(t)$.

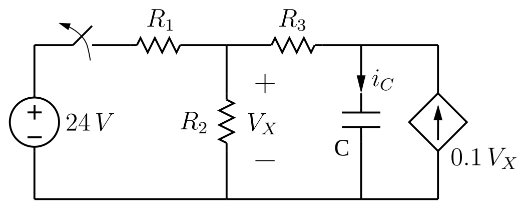

Figure 2.17: Circuit for Example 5.

We begin by finding V_C at $t = 0^-$ for which the circuit is shown in Fig. 2.18 (a). Assuming that the switch has been in the closed state for a long time prior to $t = 0\,$s, We

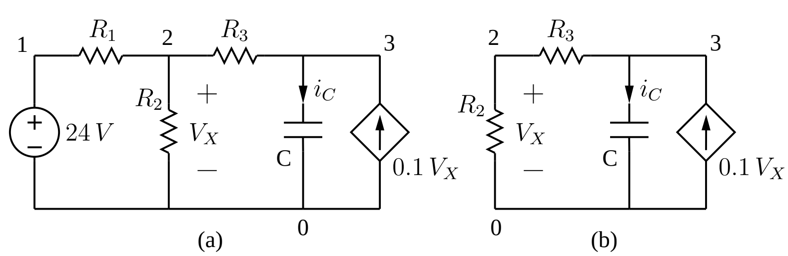

Figure 2.18: Circuit of Example 5 for (a) $t < 0$, (b) $t > 0$.

have $i_C = 0\,A$, i.e., the capacitor is an open circuit. KCL at node 2 gives,

$$\frac{1}{R_1}\,(V_2 - 24) + \frac{1}{R_2}\,V_2 - 0.1\,V_2 = 0\,, \tag{2.14}$$

yielding $V_2 = 20\,V$, and $V_C(0^-) = V_2 + 0.1\,V_2\,R_3 = 26\,V$. Using continuity of V_C, we get $V_C(0^+) = V_C(0^-) = 26\,V$.

To find the time constant of the circuit for $t > 0\,\mathrm{s}$ (see Fig. 2.18 (b)), we redraw the circuit as shown Fig. 2.19 (a). The Thevenin resistance as seen by the capacitor can be

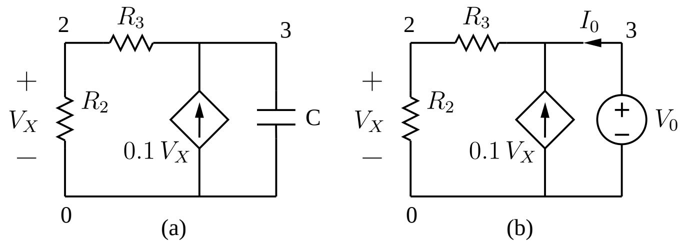

Figure 2.19: (a) Circuit of Fig. 2.18 (b) redrawn, (b) Circuit in (a) with capacitor replaced with a test voltage source.

obtained by applying a test source as shown in Fig. 2.19 (b) and computing $R_{Th} = V_0/I_0$. Writing KCL at node 3,

$$-I_0 - 0.1\,V_0\,\frac{R_2}{R_2+R_3} + \frac{V_0}{R_2+R_3} = 0\,, \tag{2.15}$$

$$R_{Th} = \frac{V_0}{I_0} = \frac{R_2+R_3}{1-0.1\,R_2} = 16\,\Omega\,. \tag{2.16}$$

The circuit time constant for $t > 0$ is then $\tau = R_{Th}\,C = 16\,\mathrm{ms}$. Using the conditions, $V_C(0^+) = 26\,V$ and $v(\infty) = 0\,V$, we get

$$V_C(t) = 26\,\left(1 - e^{-t/\tau}\right)\,V \qquad t > 0\,. \tag{2.17}$$

The current $i_C(t)$ is therefore

$$i_C(t) = C\,\frac{dV_C}{dt} = -1.625\,e^{-t/\tau}\,A \qquad t > 0\,. \tag{2.18}$$

Fig. 2.20 shows V_C and i_C versus time.

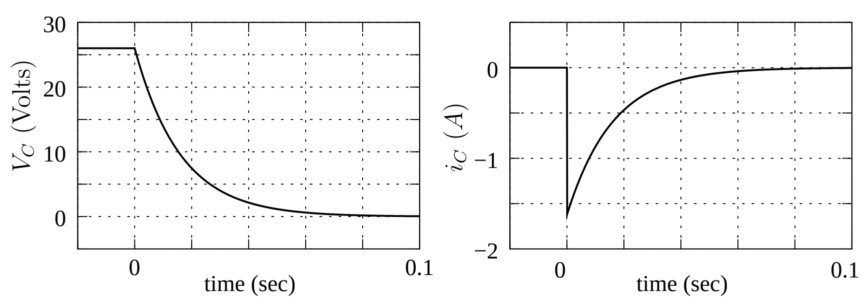

Figure 2.20: (a) $V_C(t)$, (b) $i_C(t)$ for the circuit of Example 5.

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2.3 Exercise Set:

1. For the circuit shown in Fig. 2.21,

   (a) Find $i(0^-)$, $i(0^+)$, and $i(\infty)$.

   (b) Find R_{Th} as seen by the inductor for $t > 0$.

   (c) Obtain an expression for $i(t)$, using the results of (a) and (b). Verify your result with simulation.

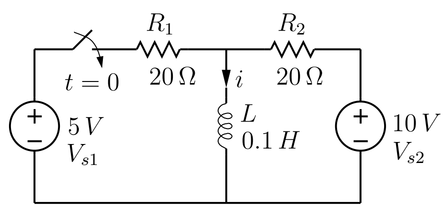

Figure 2.21: Circuit for Exercise 1.

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2. For the circuit shown in Fig. 2.22, show that the following results hold in the steady state:

   (a) $V_{\max} = V_0 \dfrac{1-k_1}{1-k_0}$, $V_{\min} = k_2 V_{\max}$, where $k_1 = e^{-T_1/\tau}$, $k_2 = e^{-T_2/\tau}$, $k_0 = k_1 k_2$, $\tau = RC$.
   Hint: Obtain $V_C(t)$ in the T_1 and T_2 intervals, use the condition of periodicity of V_C in the steady state.

   (b) The average value of V_C is the same as the average value of V_s. i.e.,
   $\dfrac{1}{T}\displaystyle\int_0^T V_s\,dt = \dfrac{1}{T}\int_0^T V_C\,dt.$
   Hint: write KVL for the circuit and integrate.

With simulation, verify the above results for various values of T_1 and T_2.

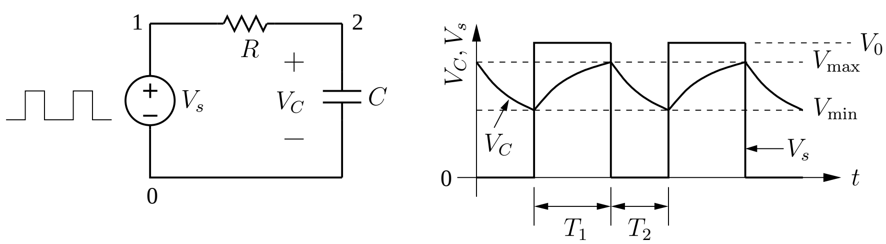

Figure 2.22: Circuit for Exercise 2.

SEQUEL file: `ee101_rc1b.sqproj`

3. For the circuit shown in Fig. 2.23,

   (a) Obtain $V_C(t)$ and hence $V_1(t)$.

   (b) Find $i_s(0^-)$, $i_s(0^+)$, $i_s(\infty)$ from $V_C(0^-)$, $V_C(0^+)$, $V_C(\infty)$, and use them to obtain $i_s(t)$ directly.

   (c) Verify your results with simulation.

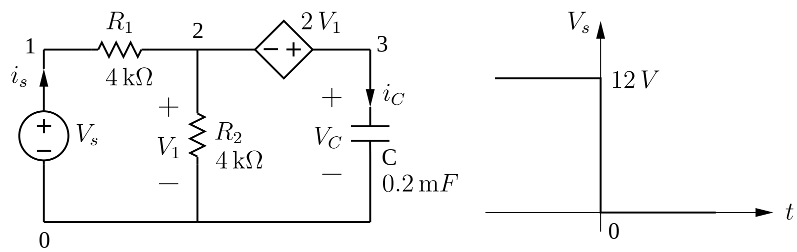

Figure 2.23: Circuit for Exercise 3.

SEQUEL file: `ee101_rc7.sqproj`