Systems theory approach to analog circuit design

Virendra Sule
Department of Electrical Engineering
Indian Institute of Technology Bombay
Powai, Mumbai 400076, India
(vrs@ee.iitb.ac.in)

January 16, 2013

1 Introduction

Analog circuits and and their interconnections, broadly called analog systems, are indispensable components of modern communication and electronic systems as well as electronic converters used in power systems. Almost all such circuits are built using active devices which require an external excitation for their operation. Active devices exhibit large sensitivity of performance to variations of sources, device parameters, operating conditions such as temperature and their operation is often unstable or required to be kept sufficiently stable. Active devices are also difficult to model accurately especially at high frequencies where they are often operated. To harness superior properties of active devices and circuits it is thus necessary to embed such circuits within systems which provide stable operation along with desired level of sensitivity reduction and desired response over a wide band. While interconnections of passive systems is always stable this is not so for active systems. This is in fact the reason why stability is never a requirement in passive circuit design. On the other hand an improperly designed interconnection of stable active circuits may even result in deterioration of stability and sensitivity. Hence for active circuit design it is necessary to build a mathematically rigorous theory of designing interconnections which will provide structure of interconnections for stablility and facilitate solution of performance optimization problems. For passive circuit synthesis a state space based systems theory approach is proposed in [4]. In case of active networks the problem of compensation design is also important apart from a purely synthesis problem, both of which require a rigorous stabilization theory. Purpose of this article is to propose such a stabilization theory in the formalism of factorization theory of feedback systems and show how this leads to an H_∞ formulation of the sensitivity reduction problem.

Application of fedback control for design of amplifiers has a long history even before the book by Bode [1]. This work can be considered as one of the early attempts at designing active circuits using feedback control concepts. Feedback theory has subsequently enormously advanced and has provided solutions of several problems of control theory (broadly known as H_∞ feedback control theory [8]). This theory has also found wide and successful applications in engineering. However an application of these advances to the problem of designing active circuits and systems is seldom found in the literature. Much of the analog system design is based on designer's experience. A systems theory approach shall facilitate formulation of designer's experience into mathematical steps and optimization problems. Feedback control approach of Bode has also been applied to design wideband active circuits with modern semiconductor devices, however the methods are effective only upto single loop designs [2]. Problems of multiport circuit compensation or typically multistage designs are complex enough for which a systematic approach (such as H_∞) is warranted. Due to lack of a systematic approach for multiport active circuit design it is difficult to answer several questions such as, how close is the performance of a designed system to the optimal under the given freedom or how it can be improved, whether the desired performances specifications are achievable under the given freedom of design, which requirements are mutually conflicting etc.

1.1 Representation of circuits and systems

It is often invariably presumed that design of feedback compensation can only be done with the help of representing an interconnection in terms of signal flow graph of an explicit feedback system. On the other hand a connection of circuits is always performed at ports of the same type (either voltage or current) and follows Kirchoff's laws at every such connection. In such connections independent sources can be treated as inputs and responses to these sources as outputs. Hence there is a natural pairing of sources and responses as physical quantities. In control systems there are either inputs and outputs and often latent variables which are neither. Outputs of a control system are not paired with external inputs. An interconnection of control systems follows rules of signal flow graph. Hence if feedback quantities are to be explicitly utilized to design an interconnected circuits then it is necessary to obtain the signal flow graph and appropriately define the loop transfer function quantities. This process need not always be convenient as networks increase in complexity. Defining loop transfer functions is extremely impractical when there are multiple loops. Hence it is necessary to utilize for design the most natural mathematical representation of interconnections of circuits in terms of multiport representations directly without invoking the signal flow graph[1]. To conclude, we make following key observations to highlight the differences between the representation of interconnected circuits via circuit connections and by signal flow graph.

1. Applying signal flow graph as required by feedback theory to circuit design imposes restrictions of signal flow graph which are not part of circuit's physical laws. Feedback structure is a mathematical representation and not natural to circuit interconnections.

2. The connections of elements in a system described by a signal flow graph are cascade, sum or feedback type. These are not equivalent to circuit element interconnections such as parallel and series. The mathematical rule of cascade connection as product of functions cascaded is not valid for cascade connection of arbitrary network func-

[1]In behavioral system theory [9] notion of inputs are considered more generally from their mathematical property of freeness. Behavioral consideration of variables and interconnection utilizes mathematical models arising from physical laws hence can formulate problems of interconnection from the most general point of view. For circuit connections however the inputs and outputs are in terms of sources and responses and hence a natural choice

tions cascaded at ports. Such rules can be applied only for special transfer function parameters of multiport networks.

3. Identifying feedback loops and elements is not convenient (both theoretically and computationally) for multi port design problems.

4. Even if a design is completed by identifying elements of a signal flow graph, its realization is in circuit form. Hence a circuit compensation and interconnection may be designed directly without identifying feedback structure.

Giving up explicit feedback structure in circuit design appears to create the disadvantage that the traditional Nyquist criterion for stability is no more applicable for stability. However, note that Nyquist criterion, or for that matter Routh Hurwitz criterion are useful for stability analysis of feedback systems and can be used for determining number of roots of polynomials in specific regions of the complex plane. These criteria DO NOT resolve the problem of stabilization or that of determining a stabilizing controller in feedback systems. A rigorous stabilization theory is well developed in terms of factorization approach as well as in solutions to problems of state space theory such as observer and spectral assignment [5, 8, 7]. In this article we show precisely that the present problem of circuit design or compensation can be formulated in the mathematical formalism of these approaches and there is no explicit need for feedback structure to do stabilization of active circuits. We also show that the H_∞ optimization theory can be utilized for sensitivity optimization without the explicit feedback structure.

We shall follow notations and mathematical background of linear time invariant circuits from [3]. A network function is identified with the ratio of Laplace transform of two signals which will be a rational function of a complex variable with real co-efficients. The definitions of impedance and admittance functions are standard. Formulas for network functions of series and parallel connections are also standard for multi port circuits.

1.2 Stabilization problem in single port

The simplest of circuit connections are defined at single ports such as series and parallel single port connections of impedances and admittances. Consider a voltage source v_s connected to an active impedance Z at the port and denote by V_s its Laplace transform. In transformed quantities we have $I_r = Z^{-1}V_s$ where I_r is the Laplace transform of the response current. There is a standard definition [3] of stability for such a circuit. A voltage source fed circuit is called *short circuit stable* (SCS) if under zero initial conditions a bounded voltage source (as a function of time) has a bounded current response. Thus the single port voltage fed impedance network is SCS iff the network function Z^{-1} is holomorphic in the closed right half of the complex plane $\bar{\mathbb{C}}_+$. Such network functions are called *stable* functions.

1.2.1 Short circuit stabilization

If the voltage fed impedance is to be compensated, the only compensation which can affect the voltage through Z is a series compensation impedance Z_c. The (controlled) current in Z due to this compensation is $I_r = V_s/(Z + Z_c)$. hence following the definition of SC stability we can pose the stabilization problem in one port to be that of finding all impedances Z_c such that $(Z + Z_c)^{-1}$ is stable. However formally we choose a stronger requirement in defining stabilization as follows.

Problem 1 (Short circuit stabilization)**.** Given a one port circuit with impedance function Z, which is fed by a voltage source, find all impedance functions Z_c such that the impedance of the series connection $Z_T = (Z + Z_c)$ satisfies

1. Z_T^{-1} is holomorphic in the $\bar{\mathbb{C}}_+$ i.e. a stable function.
2. $\tilde{Z}_T^{-1}$ is a stable function where $\tilde{Z}_T = \tilde{Z} + Z_c$ for all $\tilde{Z}$ in a sufficiently small neighborhood of Z.

If Z_c satisfies the conditions above we call Z_c a *short circuit stabilizing compensation* of Z.

Remark 1. The second condition in the definition of the short circuit stabilization is important for practical reason. Since models of impedances are always approximate we expect a compensator to provide stable interconnection at least over a small neighborhood of a nominal model impedance Z of the circuit. More generally we expect a stable network to remain stable for sufficiently small perturbations of both, the network function Z and the function Z_c of the compensator. The second condition suffices to meet this objective.

1.2.2 Open circuit stabilization

Analogous open circuit stability and stabilization problem can be defined for a current fed circuit. For instance if an admittance Y is being fed by a bounded current source whose Laplace transform is I_s, then it is called *open circuit stable*(OCS) if the voltage response $V_r = Y^{-1}I_s$ across it is a Laplace transform of a bounded function under zero initial conditions. Thus the circuit is OCS iff Y^{-1} is a stable network function. Now the current through Y can be controlled only by a parallel compensator admittance Y_c and then the parallel connected circuit is stable if the combined admittance function $Y_T = Y + Y_c$ of the parallel connection has Y_T^{-1} a stable function. Hence we define the *open circuit stabilization* problem as follows.

Problem 2 (Open circuit stabilization)**.** Given a one port circuit with admittance function Y, which is fed by a current source, find all admittance functions Y_c such that the admittane of the parallel connection $Y_T = Y + Y_c$ satisfies

1. Y_T^{-1} is a stable function.
2. $\tilde{Y}_T^{-1}$ is a stable function where $\tilde{Y}_T = \tilde{Y} + Y_c$ for all $\tilde{Y}$ in a sufficiently small neighborhood of Y.

If Y_c satisfies the condition above we call Y_c an *open circuit stabilizing compensation* of Y.

These notions of stability and stabilization shall lead us to build a theory for stabilization of multi-port active linear time invariant (LTI) circuits to formulate and solve sensitivity and performance optimization problems. In the literature on circuits, open and short circuit stability of one port networks have been well known [3], however surprisingly the corresponding stabilization problems do not appear to have been defined in the above systems theoretic sense in the literature. Since the short circuit and open circuit stability concepts are mutually exclusive the two stabilization problems are also mutually exclusive. We show in the next section how such a problem can be formulated and solved with the help of factorization approach which originally arose for stabilization of feedback systems [5, 7] but can be applied to our problem due to mathematically similar structure of the problem but without any explicit conversion to feedback signal flow graph.

2 Structure of the stabilizing compensator for single port stabilization

We now briefly explain the basic mathematical premise of our approach by solving the single port case of short circuit stabilization problem. The setting is based on the factorization approach [5]. First consider the algebra $\mathcal{S}$ of stable network functions. For the present purpose we shall take $\mathcal{S}$ to be the algebra of rational functions of one complex variable with real co-efficients which are holomorphic in $\bar{\mathbb{C}}_+$. This algebra is the set of all stable network functions of interest. A general network function Z is considered to be an element of the ring of fractions $R = \{U/V, U, V \in \mathcal{S}, V \neq 0\}$. The algebra $\mathcal{S}$ is known to be a Euclidean domain with degree of an element U to be the number of zeros of U in $\bar{\mathbb{C}}_+$ and hence every pair of elements U, V have the greatest common divisor (gcd) d which is unique modulo invertible elements (called *units*) in $\mathcal{S}$ and can be expressed as

$$d = UX + VY$$

for some X, Y in $\mathcal{S}$. A fraction U/V can be expressed with suitable coprime fractions U, V whose gcd is a unit which can be taken as the constant function 1. For the single port case we have two stabilization problems and corresponding structures of stabilizing compensators as proved below.

2.1 Short circuit stabilization

Consider an impedance function Z being fed by a voltage source. A stabilizing compensation Z_c is a series impedance which satisfies the conditions of the short circuit stabilization problem obove. Such a set of all stabilizing Z_c is given by

Theorem 1. Let Z has a coprime fractional representation $Z = U/V$ and X, W in $\mathcal{S}$ satisfy the identitiy $UW + VX = 1$. Then the set of all functions Z_c which are short circuit stabilizing compensators of Z are given by fractional representation

$$Z_c = (X + QU)(W - QV)^{-1} \tag{1}$$

where Q is an arbitrary element of $\mathcal{S}$ such that $W - QV \neq 0$.

Proof: Consider a compensator with coprime fractional representation over $\mathcal{S}$ as $Z_c = U_c V_c^{-1}$. Then Z_c is a stabilizing compensator (of the short circuit stabilization problem) iff

$$Z_T^{-1} = (Z + Z_c)^{-1} = V(U_c V + V_c U)^{-1} V_c = V \Delta^{-1} V_c$$

where

$$\Delta = U_c V + V_c U$$

is stable as well as $\tilde{Z}_T^{-1}$ is stable for $\tilde{Z}$ perturbed sufficiently small way from Z. By the first condition of the short circuit stabilization problem, the function Δ can have zeros in $\bar{\mathbb{C}}_+$ exactly at the zeros of V or V_c or both of the same multiplicity. However when Z perturbs to $\tilde{Z}$ these zeros may not get canceled to make $\tilde{Z}_T^{-1}$ stable. Hence Z_c is stabilizing iff Δ has no zeros in $\bar{\mathbb{C}}_+$. In other words Δ is a unit of $\mathcal{S}$. Hence we can write modified fractional representation $Z_c = (U_c\Delta^{-1})(V_c\Delta^{-1})^{-1}$ for the stabilizing controller. For such a representation of Z_c the function $\Delta = 1$. Let W, X be solutions of the identity $UW + VX = 1$ then all other solutions are $W_1 = W - QV$, $X_1 = X + QU$. This proves the formula claimed for all Z_c which provide a stable compensated voltage fed impedance Z_T $\square$.

In practice we also need to impose the restriction on Q that $X + QU \neq 0$. One consequence of the the above theorem is that the impedance Z of the circuit and Z_c that of the stabilizing compensator, cannot share a common zero in $\bar{\mathbb{C}}_+$ (called a non-minimum phase (NMP) zero). Even their closer proximity in $\bar{\mathbb{C}}_+$ would mean that the connected circuit has poor stability margin.

2.2 Open circuit stabilization

We can now briefly state the analogous formula for stabilizing Y_c. Due to the algebraic duality with the formulas in short circuit stabilization we can immediately state the following theorem for the structure of admittance of a stabilizing compensator Y_c which is connected in parallel to Y for the purpose of controlling the current in Y.

Theorem 2. Let Y has a coprime fractional representation $Y = U/V$ and W, X in S satisfy the identity $UW + VX = 1$. Then the set of all functions Y_c which are open circuit stabilizing compensators of Y are given by fractional representation

$$Y_c = (X + QU)(W - QV)^{-1}$$

where Q is an arbitrary element of S such that $W - QV \neq 0$.

The proof is identical to that of the earlier theorem. Analogous to the short circuit stabilization, open circuit stabilization implies that the circuit admittance Y and that of the stabilizing compensator Y_c cannot share a NMP zero in $\bar{\mathbb{C}}_+$, moreover even a close proximity of such zeros would imply poor stability margin.

For completeness we can try to see the impedance formula for open circuit stabilization. Consider the current fed impedance Z_c forming an open circuit stable circuit with the impedance Z where Z and Z_c are connected in parallel. The required network function for OCS is

$$Y_T^{-1} = Z_T = ZZ_c/(Z + Z_c)$$

Then in terms of fractional representations above

$$Z_T = U(U_c V + V_c U)^{-1} U_c$$

Hence the formula for stabilizing Z_c remains the same as above in the case of SCS.

Note that the structure of the stabilizing compensator in either of the two problems above is closely related to the coprime fractional representation of the impedance Z and admittance Y functions. Hence although an active circuit can have stable impedance but unstable admittance (or vice versa) thereby making it OCS but not SCS, it is both open and short circuit stabilizable.

2.3 Sensitivity reduction problem for single ports

A primary reason for use of feedback in control is reduction of sensitivity of the system to variations in parameters (at least over a range of frequencies of operation). Bode had recognized this fact and explicitly related extent of feedback to uncertainty [1]. The modern tradition of robust control [5, 8, 7] is founded on Bode's ideas, a systematic development of a theory of feedback stabilization, H_∞ optimization and state space theory. In the present case of active compensation at one port we can define analogous sensitivity function and formulate the sensitivity reduction problem as follows.

2.4 Sensitivity of a voltage fed one port

Consider an impedance Z being fed by a voltage source V which has response current $I = V/Z$. If the impedance perturbs to $\tilde{Z}$ the current perturbs to $\tilde{I} = V/\tilde{Z}$. If the circuit is compensated as above by a series impedance Z_c, the current in the network is $I_c = V/(Z + Z_c)$ which perturbs to $\tilde{I}_c = V/(\tilde{Z} + Z_c)$. Following the sensitivity formula of Bode [1, 7] we define the sensitivity of the current to be

$$\begin{aligned} S &= \lim_{\tilde{Z}\to Z} \frac{\text{Per unit change in current in compensated network}}{\text{Per unit change in current without compensation}} \\ &= \lim_{\tilde{Z}\to Z} \frac{(\tilde{I}_c - I_c)/I_c}{(\tilde{I} - I)/I} \\ &= \lim_{\tilde{Z}\to Z} \frac{(\tilde{Z}+Z_c)^{-1} - (Z+Z_c)^{-1}}{(\tilde{Z}^{-1} - Z^{-1})} \\ &= \frac{Z}{Z+Z_c} \end{aligned}$$

Similar sensitivity formula can be derived for the current fed one port.

2.4.1 Sensitivity minimization problem

The stabilizing compensator formula (1) can now be used to get a formula for the sensitivity function in terms of the free parameter function Q. Substituting for Z_c in (1) gives

$$S = U(W - QV)$$

The problem of designing a compensator Z_c to achieve the desired frequency domain profile defined by a magnitude function $\phi(\omega)$ as

$$|S(j\omega)| \leq \phi(\omega) \tag{2}$$

is then equivalent to finding a Q in S such that

$$||\Gamma U(W - QV)||_\infty \leq 1$$

where Γ is a stable function such that $|\Gamma(j\omega)| = 1/\phi(\omega)$. Thus the sensitivity reduction objective of compensation leads to solving the H_∞ minimiation problem. Find

$$\gamma_0 = \min_Q ||\Gamma U(W - QV)||_\infty$$

The sensitivity objective (2) is satisfied iff $\gamma_0 \leq 1$. Several other performance objectives can be considered but sensitivity is of prime importance which warrants feedback compensation. Other performance objectives in design can be achieved by compensation whose port connections need not amount to closed loop control. Hence a two degree of freedom compensation can be considered in which the two stages of compensation one for sensitivity reduction and the other for response improvement can be separately designed as is well known in feedback system design [7].

2.4.2 Sensitivity tradeoff for the voltage fed impedance

In feedback theory there are well known results known as sensitivity tradeoffs [7]. These are all applicable in the present design case as well. We shall briefly mention these here leaving details to be developed elsewhere. For instance if the circuit is fed by a voltage source and has impedance Z as considered above, even if the circuit is SCS i.e. Z^{-1} is stable but not OCS i.e. Z is not a stable function (has a pole in $\bar{\mathbb{C}}_+$) then in coprime fractions $Z = UV^{-1}$ the function V has a NMP zero say $\lambda \in \bar{\mathbb{C}}_+$. For such an impedance the weighted sensitivity function ΓS has the lower bound

$$||\Gamma S||_\infty \geq \Gamma(\lambda)U(\lambda) = \delta$$

Hence when $\delta > 1$ the sensitivity cannot be made small over the bandwidth as modeled by Γ. In such cases both the magnitude of sensitivity may have to be compromised by an increase and bandwidth may have to be compromised by decrease. Other tradeoffs such as waterbed effect known in feedback theory shall also be applicable in this design problem. For feedback systems details of such tradeoffs are well known and discussed in [7, 6].

This completes a brief account of systems theory approach for a single port circuit design case. The multi port case can be built on similar lines however involves much algebraic complication due to the matrix valued nature of hybrid parameters of multi ports and matrix version of sensitivity minimization. This theory is described in the next section.

3 Multi port stabilization for Bounded Source Bounded Response (BSBR) stability

For multi port circuits, the open and short circuit stability both need to be considered simultaneously since voltage and current independent sources can exist at the ports simultaneously. Hence an analogous stability amounting to boundedness of responses for bounded sources and the corresponding stabilization problem are the appropriate generalizations of the single port case. We thus begin from such a definition. Consider a linear time invariant circuit with a hybrid representation

$$Y_r = HU_s \tag{3}$$

where U_s denotes the vector of Laplace transforms of independent sources of the circuit while Y_r denotes the vector of Laplace transforms of the responses at the these ports (respecting the indices). We call this circuit *bounded source bounded response* (BSBR) stable if for zero initial conditions of the network's capacitances and inductors, uniformly bounded sources have uniformly bounded responses. This is the case iff the network function (matrix) H is stable i.e. H has every entry belonging to $\mathcal{S}$. (We shall denote all matrices (of size clear from context) whose entries are stable by $M(\mathcal{S})$).

3.1 Stabilization problem

Consider a compensation network of same number and type of independent sources as the circuit in (3) to be compensated.

$$Y_{cr} = H_c U_{cs} \tag{4}$$

Note that if the compensation is to have any controlling effect on currents (respectively voltages) of (3) in response to voltage (respectively current) sources, then the compensator must be connected to the circuit in such a way that the voltage ports of individual networks are connected in series and current ports of individual networks are connected in parallel. Hence the vector of independent voltage sources to the interconnected network, denoted V, must be the sum of voltages at the voltage ports while the vector of independent current sources to the interconnected network, denoted I must be sum of the currents in the current ports of the two networks. This shows that the independent source vector of the interconnected network is given by (in Laplace transform)

$$U = U_s + U_{cs} \tag{5}$$

and the response vector Y of the interconnected network satisfies

$$Y = Y_r = Y_{cr} \tag{6}$$

We now consider a condition of non-singularity on all multiport circuits to be considered in our theory. This condition is the assumption that the hybrid representations (3) satisfy $\det H \neq 0$ as network functions. Due to this for every network function matrix H there is a unique network function matrix $G = H^{-1}$. In our circuit, denote $G = H^{-1}$ and $G_c = H_c^{-1}$. Then from (5), (6) it follows that

$$Y = (G + G_c)^{-1} U \tag{7}$$

is the hybrid representation of the interconnected circuit. Clearly the interconnected circuit is BSBR stable iff $(G+G_c)^{-1}$ is a matrix of stable network functions. Using this form of the representation of the interconnected circuit we define the stabilization problem as follows.

Problem 3 (Stabilization problem for circuit interconnection)**.** Given a nonsingular representation (3) of a linear time invariant circuit with $G = H^{-1}$, find all nonsingular representations H_c with $G_c = H_c^{-1}$ such that

1. $(G + G_c)^{-1}$ belongs to $M(\mathcal{S})$
2. $(\tilde{G} + G_c)^{-1}$ belongs to $M(\mathcal{S})$ for $\tilde{G}$ sufficiently close to G

Any G_c satisfying the above is called the *stabilizing compensator* of (3).

A solution of the stabilization problem in terms of the formula for all stabilizing compensators which was derived for one port case above shall require extension of the coprime factorization to matrix case. Such a theory is well known for matrix functions in [5].

3.2 Structure of the stabilizing compensator

Extending the above approach for stabilization in the multiport case is much complex as compared to the single port case as this requires coprime factorization of hybrid matrix functions. The structure of the stabilizing compensator involves both left as well as right coprime matrix fractions. As above we start with $\mathcal{S}$ the ring of stable matrix functions (which are rational functions of a complex variable holomorphic in $\bar{\mathbb{C}}_+$) and denote by $M(\mathcal{S})$ matrices over $\mathcal{S}$ when their sizes are clear from the context. We shall follow developments in [5] for matrix coprime factorization over $\mathcal{S}$. The concept of doubly coprime fractions of rational matrices shall be most suitable for our purpose.

3.2.1 Doubly coprime factorization and stabilizing compensator

Any rational function matrix G (a network function in our case), has the doubly coprime factorization (DCF) which we now describe. For proof of existence of such a factorization see [5]. Given a rational matrix H of a complex variable with real co-efficients, there exist pairs (N, D), $\tilde{N}, \tilde{D})$ in $M(\mathcal{S})$ such that $\tilde{D}$, D are square and nonsingular, $G = ND^{-1} = \tilde{D}^{-1}\tilde{N}$ and there exist pairs of matrices (W, X) and $(\tilde{W}, \tilde{X})$ in $M(\mathcal{S})$ such that

$$\begin{bmatrix} X & W \\ -\tilde{N} & \tilde{D} \end{bmatrix} \begin{bmatrix} D & -\tilde{W} \\ N & \tilde{X} \end{bmatrix} = I \tag{8}$$

The formula for stabilizing compensation obtained in the single port case can be extended to the multiport case using the DCF of the network function G (or the hybrid function H of the multiport circuit). We state this theorem below and outline the proof in Appendix.

Theorem 3. Let the hybrid representation of a given circuit be as in (3) and $G = H^{-1}$. Let G have a DCF (8). Then the set of all stabilizing compensators of the circuit represented by (4) have G_c in given by any one of the following fractional representations.

$$G_c = (W - Q\tilde{D})^{-1}(X + Q\tilde{N})$$

where Q is a freely chosen matrix in $M(\mathcal{S})$ such that $\det(W - Q\tilde{D}) \neq 0$. Alternatively G_c is also given by fractional representation

$$G_c = (\tilde{X} + NQ)(\tilde{W} - DQ)^{-1}$$

where Q is a freely chosen matrix in $M(\mathcal{S})$ such that $\det(\tilde{W} - DQ) \neq 0$.

Since we require that G_c is also non-singular so that the hybrid representation (4) exists, Q in above formulas must also satisfy non-singularity condition for matrices $(X + Q\tilde{N})$ and $(\tilde{X} + NQ)$. Conditions for existence of such Q in $M(\mathcal{S})$ may not pose much difficulty hence their discussion is omitted. The formula for the composite compensated circuit can also be considered in any one of the two forms as follows. The proof follows easily by substitution of the above parametrized formulas of G_c in $(G+G_c)^{-1}$.

$$(G+G_c)^{-1} = D(W - Q\tilde{D}) = (\tilde{W} - DQ)\tilde{D} \tag{9}$$

3.3 Sensitivity function

Definition of sensitivity in the multiport case needs proper interpretation since the response Y_r in this case is a vector quantity hence the notion of per unit change cannot be defined by taking ratios. Considering the hybrid representation (3) of the circuit if H perturbs to $\tilde{H}$ then the response Y_r perturbs to $\tilde{Y}_r = \tilde{H}U_s$. Hence we can define the per unit change in the response of the circuit as the matrix function R such that

$$\tilde{Y}_r - Y_r = RY_r$$

It follows that

$$\begin{aligned} \tilde{Y}_r - Y_r &= (\tilde{H} - H)U_s \\ &= (\tilde{H} - H)H^{-1}Y_r \end{aligned}$$

Hence we get

$$R = (\tilde{H} - H)H^{-1}$$

Now the response Y in the compensated circuit is $Y = (G+G_c)^{-1}U$ which perturbs to $\tilde{Y} = (\tilde{G}+G_c)^{-1}U$ where $\tilde{G} = \tilde{H}^{-1}$. Hence per unit change in the compensated network is taken as the matrix function $\hat{R}$ such that

$$\tilde{Y} - Y = \hat{R}Y$$

We can now define the *Bode sensitivity* of the composite network to be the matrix function S which satisfies

$$\lim_{\tilde{H}\to H} \hat{R} = S \lim_{\tilde{H}\to H} R \tag{10}$$

Consider computation of the formula for S. We have

$$\begin{aligned} \tilde{Y}_r - Y_r &= (\tilde{H} - H)U_s \\ &= (\tilde{H} - H)H^{-1}Y \end{aligned}$$

hence

$$R = (\tilde{H} - H)H^{-1}$$

Similarly, for the composite circuit we have

$$\begin{aligned} \tilde{Y} - Y &= [(\tilde{G}+G_c)^{-1} - (G+G_c)^{-1}]U \\ &= [(\tilde{G}+G_c)^{-1} - (G+G_c)^{-1}](G+G_c)Y \end{aligned}$$

hence

$$\begin{aligned} \hat{R} &= (\tilde{G}+G_c)^{-1}[G - \tilde{G}] \\ &= (\tilde{G}+G_c)^{-1}(H^{-1} - \tilde{H}^{-1}) \\ &= (\tilde{G}+G_c)^{-1}\tilde{H}^{-1}(\tilde{H} - H)H^{-1} \end{aligned}$$

From the definition of the sensitivity function (10) we observe that since S must satisfy

$$\lim_{\tilde{H}\to H}(\tilde{G}+G_c)^{-1}\tilde{H}^{-1}(\tilde{H}-H)H^{-1} \quad = \quad S\lim_{\tilde{H}\to H}(\tilde{H}-H)H^{-1}$$

it follows on taking limits that

$$S = (G+G_c)^{-1}G \tag{11}$$

This way we obtain a generalization of the single port sensitivity formula derived in the previous section. Note that the formula can also be expressed as

$$S = H_c(H+H_c)^{-1}$$

Substituting now the formula for the stabilizing compensator we obtain

$$\begin{aligned} S &= (\tilde{D}^{-1}\tilde{N} + (\tilde{X}+NQ)(\tilde{W}-DQ)^{-1})^{-1}\tilde{D}^{-1}\tilde{N} \\ &= (\tilde{W}-DQ)\tilde{N} \end{aligned} \tag{12}$$

which is the parametrized form of the sensitivity function of the multiport composite circuit.

3.4 Sensitivity optimization problem

The parametrized formula for sensitivity now allows formulation of an H_∞ optimization problem. The sensitivity function is a matrix valued function and the generalization of the notion of gain in this case is the spectral norm $||S(j\omega)||$ at frequency ω. If $\phi(w)$ is a positive function defining the desired bound on sensitivity gain

$$||S(j\omega)|| \leq \phi(\omega)$$

and $\Gamma(s)$ is the minimum phase function for the boundary value $\phi(\omega)$

$$|\Gamma(j\omega)| = 1/\phi(\omega)$$

Then the specification

$$||S(j\omega)|| \leq \phi(\omega)$$

is satisfied by the H_∞ bound

$$||S\Gamma^{-1}||_\infty \leq 1$$

Hence achieving the sensitivity specification above is equivalent to computing the optimal value

$$\gamma_0 = \min_{Q\in M(\mathcal{S})} ||(\tilde{W}-DQ)\tilde{N}\Gamma^{-1}||_\infty$$

Then the sensitivity specification is satisfied iff $\gamma_0 \leq 1$. In practice more general matrix weighing Γ can be considered to provide different penalties at different ports. Typical design problems of multiports exercising such strategies shall be interesting case studies. These shall be pursued in further work.

4 Conclusions

This report proposes modern linear systems theory approach for design of compensation for active circuits. It is first observed that the compensation design for sensitivity reduction need not be explicitly formulated in terms of signal flow graph of a feedback system. A parametrization of all compensators which form a stable circuit is determined using the well known coprime factorization theory. The resulting parametrization of the sensitivity function leads to formulation of sensitivity reduction as an H_∞ optimization problem. This is thus a mathematically systematic approach to design of broadband analog systems which should facilitate precise statement of design objectives and analysis of design tradeoffs and work for multiport circuits. Compensation and synthesis of analog active circuits cannot be achieved without the constraint of stability which imposes structural restrictions on the circuit building blocks. Goals of sensitivity reduction by compensation can only be achieved under such constraints. These structures have been well understood in feedback control theory. This report shows that the underlying mathematical theory theory can be extended to multiport circuit connections without explicit formulation in terms of feedback structure.

References

[1] Bode H. W. Network analysis and feedback amplifier design. D. Van Nostrand, Princeton, NJ, 1945.

[2] Maclean D. J. H. Broadband feedback amplifiers. Research Studies Press, NY, 1982.

[3] Chua L. O., C. A. Desoer, E. S. Kuh. Linear and nonlinear circuits. McGraw-Hill Book Company, NY, 1987.

[4] Anderson B. D. O., S. Vongpanitlerd. Network analysis and synthesis. A modern systems theory approach. Dover Publications Inc., NY, 2006.

[5] Vidyasagar M. Control system synthesis. A factorization approach. MIT Press, Cambridge, MS, 1985.

[6] Francis B. A. A course in H_∞ control theory, vol. 88, Lecture Notes in Control and Information Sciences, Springer Verlag, 1987.

[7] Doyle J. C. B. A. Francis, A. R. Tannenbaum. Feedback control theory. Macmillan Publishing Company, NY, 1992.

[8] Zhou K. J. C. Doyle, K. Glover. Robust and optimal control. Printice Hall, 1996.

[9] J. W. Polderman, J. C. Willems. Introduction to mathematical systems theory. A behavioral approach. Text in Applied Mathematics no.26, Springer, 1998.