Solutions of Boolean equations by orthonormal expansion

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1 Introduction

Developments in solving equations over Boolean algebras and their applications are as old as George Boole's monograph [1], while Shannon pioneered applications of Boolean logic to switching circuits. Boolean equations and their solutions are of central importance to many problems across Sciences such as Chemistry, Biology or Medicine while traditionally Boolean problems arise in domains such as verification and design of logic circuits, software verification and Artificial Intelligence in Computer Science as well as Decision Sciences such as Operational Research. Search for assignments of Boolean variables over the two element Boolean algebra $B_0 = \{0, 1\}$ for which given Boolean constraints hold true or function has specified value are known as satisfiability SAT problems and has lead to development of efficient algorithms such as DPLL and other approaches [4, 5] over past half century. Boolean equation solving methods of [3, 2] on the other hand are applicable for solving equations over general Boolean algebras. These two developments have largely evolved independently. The potential of Boolean equation solving methods is yet to be fully explored as is apparent from literature [4, 5]. Similarly development of efficient and scalable parallel computational algorithms for the problem of solving systems of Boolean equations in large number of unknowns over large number of processors is yet to mature fully. In this report we shall explore utilization of expansion in terms of orthonormal systems [2] and solutions of Boolean equations in orthonormal variables developed in [3]. Boolean equations in orthonormal variables are always linear in either the Boolean disjunction $+$ (denoting OR) or the ring operation $\oplus$ (denoting XOR). Hence these are rich systems for algorithmic study. However it is not clear which applications involve orthonormal variables naturally. In this report we explore one indirect application that of studying consistency of Boolean equations.

1.1 Motivation

One of the principal methods of solving Boolean equations over general Boolean algebras is the method of elimination of variables [3, 2]. An elimination of one variable x in a Boolean equation $f(x) = 0$ (respectively $f(x) = 1$) results from consistency condition for the equation in terms of (Shannon) expansion of $f(x)$ with respect to x and x'. Elimination of x also gives rise to inequalities on x in terms of Boolean functions in the remaining variables. Thus elimination of any number of variables is possible in one equation. Hence elimination theory in Boolean analysis is not comparable with well known elimination theory over fields known in algebra where elimination of k variables requires k independent equations. Moreover the x and x' are special cases of more general systems of functions (called orthonormal) in the Boolean algebra of functions. Our aim in this article is to explore generalization of elimination method with respect to expansion in terms of more general orthonormal functions (in many variables).

Elimination based methods of solving Boolean equations, although applicable for general Boolean algebras, are equivalent to resolution of clauses in CNF-SAT problems. Resolutions lead to requirement of high memory and hence have not been considered very suitable for sequential computation. Some of the deficiencies of elimination are pointed out in [5]. However it is important to realize that there is as yet no final word on scalability of parallel algorithms for solving Boolean equations. Algorithms which may not be known to be fast or memory efficient when executed sequentially might have superior properties suitable for parallel computation. Hence for solving large data problems using large number of parallel processors it is not the sequential speed or memory but the scalability of the algorithm under decomposition that is of prime importance and a well scalable algorithm can eventually give fast and efficient solution for such large sized problems when computations are distributed over large number of processors. Once the problem is decomposed the memory requirement for resolution can also be brought down considerably. Most of the Boolean analysis problems arising in applications also have inherent features of local connectivity due to which only local variables appear together in clauses. (There can be notable exceptions of SAT problems expressing equivalent number theory problems).

It is for this reason that search for new methods of solving Boolean equations is important. Technology of parallel computers has made rapid strides in recent time and computers with large memories and number of processor reaching up to million shall be available at affordable price. Hence in the near future methods of solving large sized challenge problems which can scale up with good efficiency even for large number of processors shall gain importance as much as (or perhaps more than) fast sequential methods workable for small data or scalable only over small number of processors. Scalability with respect to data size and number of processors should also give a practical measure to compare two algorithms both of which are likely to have exponential order of complexity (since the SAT problem is NP complete) asymptotically but are always used for solving finite size problems.

1.2 Notations and background

We shall follow background on elimination theory of Boolean equations, general theory of Boolean functions and orthonormal functions from [3, 2]. However some changes in notations are made in view of recent literature [5] and SAT methodologies [4]. Denote the Boolean algebra of functions $f(x_1 \dots x_n)$ in n variables with co-efficients in a Boolean algebra $(B, +, ., ', 0, 1)$ by $B(n)$ which consists of the set $\{B, x_1, \dots, x_n\}$ and all functions defined by Boolean expressions developed using formal operations $(+, ., ')$ which satisfy rules of a Boolean algebra. At times for clarity, we also denote the set of variables by X and this algebra of functions as $B[X]$. Boolean functions are precisely those maps between $B^n \to B$ which have well known minterm canonical form, conjunctive normal form (CNF) and disjunctive normal form (DNF). A Boolean ring associated to B is denoted by $(R, \oplus, ., 0, 1)$ with well known relations between $+$ and $\oplus$ (also commonly known as OR and XOR in the two element Boolean algebra denoted $B_0 = (0, 1, +, ., ')$). We denote the Boolean ring associated to $B(n)$ by $R(n)$. The Boolean functions $R^n \to R$ are precisely those that have an algebraic normal form (ANF). We refer to [2] for background on Boolean functions.

1.3 Boole-Shannon expansion, consistency and elimination theory

We now briefly touch upon the background on elimination theory. Let f in $B(1)$ be a Boolean function in one variable over B. Then f has following representations called Shannon expansions

$$\begin{aligned} f(x) &= f(1)x + f(0)x' \\ f(x) &= [f(0) + x][f(1) + x'] \end{aligned} \tag{1}$$

see [2] for proof. These were proposed by both Boole and Shannon (hence we call these Boole-Shannon expansions). The Boolean equation

$$f(x) = 0$$

is said to be *consistent* if there is an a in B such that $f(a) = 0$. The necessary and sufficient condition for consistency [3] of this equation is that

$$f(0)f(1) = 0 \tag{2}$$

When the equation is consistent the set $S = \{f(1) + pf(0)'\}$ is the set of all solutions of this equation where p is an arbitrary element of B. The equation $f(0)f(1) = 0$ is independent of the variable x and is called the disjunctive *eliminant* of f. The solutions x of $f(x) = 0$ satisfy

$$f(0) \leq x \leq f(1)' \tag{3}$$

On similar lines, it follows that if $f(x) = 1$ is consistent then each of $(f(0) + x) = 1$, $(f(1) + x') = 1$ hold this implies $x' \leq f(0)$ and $x \leq f(1)$ hence

$$f(0) + f(1) = 1 \tag{4}$$

and we recover the above inequality on x. Conversely if this identity holds then $x = f(1)$ is a solution hence this condition is necessary and sufficient. These two expansions of a function $f(x)$ show that consistency conditions for $f(x) = 0$ (or $f(x) = 1$) are new Boolean identities to be be satisfied in which x is eliminated. We shall extend this process of elimination over what are called orthonormal functions.

More generally a system of m Boolean equations n variables

$$f_j(x_i, \dots, x_n) = g_j(x_1, \dots, x_n)$$

$j = 1, \dots, m$ where f_j, g_j are Boolean functions in $B(n)$ is said to be consistent if there exist elements $a_1, a_2, \dots, a_n$ in B such that all the equations are satisfied. Consistency of such a system of equations is equivalent to that of the single equation

$$\sum_j (f_j' g_j + f_j g_j') = 0$$

1.4 Orthonormal systems and expansions

In a Boolean algebra B a set of elements $\{a_1, \dots, a_m\}$ is called *orthogonal* (OG) of order m if $a_i a_j = 0$ for $i \neq j$. An OG set is called *orthonormal* (ON) if

$$\sum_{i=1}^{m} a_i = 1 \tag{5}$$

In $B(n)$ we consider OG and ON functions of n-variables. The system of *minterms* in n-variables in $B(n)$ is defined as

$$m_A \stackrel{\triangle}{=} X^A = x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n} \tag{6}$$

for $A = (\alpha_1 \dots \alpha_n)$ in $\{0,1\}^n$ where x^α is defined by $x^0 = x'$, $x^1 = x$. Set of all minterms is a 2^n order ON system in n variables. However following examples show ON systems of (polynomial) orders other than 2^n in three variables x, y, z in B_3,

order	system
4	$\{x, x'y, x'y'z, x'y'z'\}$
3	$\{xyz, x'yz + xy'z + xyz', x'y'z + xy'z' + x'yz' + x'y'z'\}$

In $B(n)$ functions which are products of a subset of n variables and their complements are called terms. For instance $x'y$, $xy'z$, $y'z'$ are terms in $B(3)$. Thus the first ON system above is a system of ON terms while the second one is not.

We can also define dual notions of OG and ON systems. Call a_i co-OG if $a_i + a_j = 1$ for $i \neq j$ and co-OG set as co-ON if

$$\prod_{i=1}^{m} a_i = 0$$

If $\{a_1, \dots, a_m\}$ are OG (respectively ON) then $\{a_1', \dots, a_m'\}$ are co-OG (respectively co-ON).

1.5 Orthonormal expansion

Consider the Boolean algebra $B(n)$ of functions $f(x_1, \dots, x_n) : B^n \to B$ of n variables over a Boolean algebra B. We denote these in short notation as $f(X)$, denoting variables $\{x_1, x_2, \dots, x_n\}$ by X. Let $\{\phi_1(X), \dots, \phi_m(X)\}$ be a system of ON functions. Then $f(X)$ has a representation

$$f(X) = \sum_i a_i(X) \phi_i(X) \tag{7}$$

iff $a_i(X)$ satisfy the relations

$$f(X)\phi_i(X) = a_i(X)\phi_i(X)$$

See [2, Proposition 3.14.i] for the proof. ON expansions are useful in decomposing computations due to following properties [2]. If $g(X)$ has expansion

$$g(X) = \sum_i b_i(X)\phi_i(X)$$

then $f+g$, fg, f', $f \oplus g$ have expansions

$$\begin{array}{rcl} f+g &=& \sum_i (a_i + b_i)\phi_i \\ fg &=& \sum_i a_i b_i \phi_i \\ f' &=& \sum_i a_i' \phi_i \\ f \oplus g &=& \sum_i (a_i \oplus b_i)\phi_i \end{array}$$

If g is a function of one variable over B and $f : B^n \to B$ is a function of n-variables with above expansion in an ON system then the composition $g(f(X))$ has expression

$$g(f(X)) = \sum_i g(a_i(X))\phi_i(X)$$

Similar compositions in multiple variables are also true as shown in [2, chapter 3]. These expressions show a far reaching role for ON systems in Boolean analysis and construction of computational algorithms for solving Boolean equations.

1.6 Expansion in ON terms

Consider now a special kind of expansion when ϕ_i is a system of ON terms in n variables X.

$$t_i(X) = x_{i1}^{a_{i1}} x_{i2}^{a_{i2}} \dots x_{ik}^{a_{ik}}$$

for $(a_{i1}, \dots, a_{ik})$ to be k components of A in $\{0,1\}^n$ then the expansion of $f(X)$ in $t_i(X)$ is

$$f(X) = \sum_i a_i t_i \tag{8}$$

where $a_i = f(t_i(X) = 1)$, see [2, Theorem 3.15.1]. The term $f(t_i(X) = 1)$ is also called the ratio denoted by f/t_i. Hence when the ON system is minterms in X then a_i are constants in B. This expansion also gives a way to express the variables in terms of any ON system of terms t_i as follows.

$$x_i = \sum_j \alpha_j t_j(X) \tag{9}$$

where $\alpha_j = 1$ when x_j belongs to t_j, $\alpha_j = x_i$ when x_i does not belong to t_j and 0 when x_i' belongs to t_j. Such expansion is useful mainly when t_i are minterms as α_j are constant 0 or 1 when t_j are minterms since either x_i' is present in t_j or x_i is present in t_j. (Similarly it follows that if ξ_i is a co-ON system of minterms such as

$$\xi_i(X) = x_1^{a_1} + x_2^{a_2} + \dots + x_n^{a_n}$$

for $A = (a_1, \ldots, a_n)$ in $\{0,1\}^n$ we can write

$$x_i = \prod_j (a_j + \xi_j(X)) \tag{10}$$

where $a_j = 0$ when x_i belongs to ξ_j and 1 otherwise). Expansion of a function (8) in ON terms can be seen as a generalization of the well known Shannon expansion.

2 Consistency with respect to ON systems of functions

We now come to our problem. Let B be a Boolean algebra and $f : B^n \to B$ be a Boolean function of n variables as described in the previous section. Our main problem is to determine the condition for consistency of the Boolean equation

$$f(X) = 0 \tag{11}$$

in terms of special ON system $t_1, \ldots, t_m$ of terms denoted by T in $B(n)$. We shall derive such conditions for a special class of functions f whose expansion in T involves constant co-efficients i.e. in B. We identify this class of functions by

Definition 1 (Class $B(T)$)**.** Let T be an ON system of functions in $B(n)$. A function f in $B(n)$ is said to be of class $B(T)$ if $f(X)$ has expansion

$$f(X) = \sum_i a_i t_i(X) \tag{12}$$

where a_i belong to B. Thus when T is a system of ON terms then f belongs to $B(T)$ when $f(X)/t_i$ are constants in B for all i.

Associated to the equation (11) for functions of class $B(T)$ we also need to consider the equation

$$\sum_i a_i \chi_i = 0 \tag{13}$$

over B in terms of ON unknown variables χ_i in view of expansion in ON terms.

2.1 Consistency of special ON systems

We now consider special ON systems in $B(n)$ and develop results on consistency of systems of equations defined by their values. These two special systems are

1. ON terms of order $n+1$ in n variables. For a fixed indexing of variables $x_1, \ldots, x_n$, define

$$\begin{aligned} t_1 &= x_1 \\ t_i &= x'_1 \ldots x'_{i-1} x_i \quad \text{for } i = 2, \ldots, n \\ t_{n+1} &= x'_1 \ldots x'_n \end{aligned} \tag{14}$$

2. ON minterms in n variables. These are ON terms of order 2^n denoted as $m_A(X) = X^A$ for A in $\{0,1\}^n$ defined in (6).

First we observe consistency conditions of systems of equations defined by special ON terms above.

Proposition 1. Let $\{\beta_1, \ldots, \beta_{n+1}\}$ be elements of B and T be an ON system (14) of terms of order $n+1$ in $B(n)$. Then the system of equations

$$t_i(X) = \beta_i,\ i = 1, \ldots, n+1 \tag{15}$$

is consistent iff $\{\beta_1, \ldots, \beta_{n+1}\}$ is an ON system of order $n+1$ in B.

Proof: Necessity is obvious. Let the constants $\{\beta_i\}$ be an ON system of order $n+1$ in B. Then β_i satisfy for $i < j$, $\beta_i\beta_j = 0$ which is $\beta_j \leq \beta_i'$. Hence $\beta_j \leq \beta_1' \ldots \beta_{j-1}'$ which implies

$$\begin{array}{rcl} \beta_1'\beta_2 & = & \beta_2 \\ \beta_1'\beta_2'\beta_3 & = & \beta_3 \\ \vdots & & \vdots \\ \beta_1' \ldots \beta_{n+1}' & = & 0 \end{array}$$

The $n+1$ order system of ON terms t_i defined in (14) the equations $t_i = \beta_i$ are equivalent to

$$\begin{array}{rcl} x_1 & = & \beta_1 \\ x_1'x_2 & = & \beta_2 \\ x_1'x_2'x_3 & = & \beta_3 \\ \vdots & & \vdots \\ x_1' \ldots x_n & = & \beta_n \end{array}$$

From the relations satisfied by β_i above it follows that the above equations are satisfied by $x_i = \beta_i$ for $i = 1, \ldots, n$ while

$$\begin{array}{rcl} t_{n+1} & = & x_1'x_2' \ldots x_n' \\ & = & \beta_1'\beta_2' \ldots \beta_n' \\ & = & \beta_{n+1} \end{array}$$

holds as β_i are ON. Thus the system (15) is consistent. □

Next we show similar consistency condition for equations in minterms.

Proposition 2. Let $\{\beta_1, \ldots, \beta_N\}$ be $N = 2^n$ elements of B and T be an ON system of minterms in n variables in $B(n)$. Then the system of equations

$$t_i(X) = \beta_i,\ i = 1, \ldots, N \tag{16}$$

is consistent iff $\{\beta_1, \ldots, \beta_N\}$ is an ON system in B.

Proof: Necessity is obvious. If the equations above hold, the expressions for variables x_i in terms of t_i given by

$$x_i = \sum_{t_j \leq x_i} t_j$$

give solutions

$$x_i = \sum_{j, t_j \leq x_i} \beta_j$$

where the sum is taken over all indices j such that the minterms $t_j \leq x_i$ i.e. x_i appear in terms t_j. This proves that the equations are consistent. □

2.2 Consistency of an equation in terms of expansion in special ON systems

Now consider a Boolean function $f(X)$ in $B(n)$ of class $B(T)$ where T is an ON system of order N of any of the two special types above (hence N is either $n+1$ or 2^n). Consider the representation of $f(X)$ in T

$$f(X) = \sum_i a_i t_i \tag{17}$$

Theorem 1. Following statements are equivalent for a function defined in (17)

1. The equation $f(X) = 0$ is consistent
2. The associated equation (13) which is
$$\sum_i a_i \chi_i = 0$$
in ON variables χ_i is consistent over B.
3. $\prod_{i=1}^{N} a_i = 0$
4. A solution for ON variables is
$$\begin{aligned} \chi_1 &= \beta_1 \stackrel{\triangle}{=} a_1' \\ \chi_i &= \beta_i \stackrel{\triangle}{=} a_1 a_2 \ldots a_{i-1} a_i' \; i = 2, \ldots, N-1 \\ \chi_N &= \beta_N \stackrel{\triangle}{=} a_1 a_2 \ldots a_{N-1} \end{aligned}$$
5. A solution for variables X in the two cases is
   (a) Case $N = n+1$
   $$x_i = a_i' \; i = 1, \ldots, n$$
   (b) Case $N = 2^n$
   $$x_i = \sum_{j, t_j \le x_i} \beta_j$$

Proof: 1⇒2. If $f(X) = 0$ is consistent, since f is in $B(T)$ there exists an n-tuple $X = \Gamma$ in B^n such that

$$f(\Gamma) = \sum_i a_i t_i(\Gamma) = 0$$

Thus the associated system in ON variables has a solution $\chi_i = t_i(\Gamma)$.

2⇒3. If $\chi_i = \beta_i$ is a solution for the ON variables χ_i then β_i is an ON system in B and $a_i \beta_i = 0$ for all i. Thus

$$\begin{aligned} \beta_i &\le a_i' \forall i \\ \Rightarrow \; \sum_i \beta_i &\le \sum_i a_i' \\ \Rightarrow \; \sum_i a_i' &= 1 \end{aligned}$$

which is 3.

3⇒4. If a_i satisfy above condition, the constants β_i defined are ON and satisfy the equation for variables χ_i.

4⇒5. Since β_i are ON the system $t_i(X) = \beta_i$ for $i = 1, \dots, N$ is consistent from propositions (1) or (2) as appropriate for N. Solutions of x_i as given are derived in the proofs of propositions (1) and (2).

5⇒1. The solutions of x_i obtained satisfy

$$\sum_i a_i t_i(X) = 0$$

which shows x_i given is a solution of $f(X) = 0$. □

2.3 Elimination interpretation

The consistency theorem above for the equation $f(X) = 0$ is a generalization of the consistency condition of an equation of one variable $f(x) = 0$ where $f(x)$ is a Boolean function of one variable in the sense that the ON system in one variable x, x' in $B(1)$ is generalized in terms of expansion of f in $B(n)$ in terms of special ON systems. Consider now a function $f(X, Y)$ in $B(n+m)$ where $X = \{x_1, \dots, x_n\}$, $Y = \{y_1, \dots, y_m\}$ are indexed sets of variables. Then f can be considered as an element of $\tilde{B}(n)$ where $\tilde{B}$ is the Boolean algebra $B(m)$ of functions in m-variables. Let T denote one of the ON system of terms $t_i(X)$ in X and assume that f belong to the class $\tilde{B}(T)$. For such functions we have ON expansion

$$f(X, Y) = \sum_i a_i(Y) t_i(X)$$

From the above consistency condition there follows

Corollary 1. The equation

$$f(X, Y) = 0$$

is consistent iff

$$\prod_i a_i(Y) = 0$$

is consistent.

Proof: Let the equation $f = 0$ be consistent, then there exist a solution $X = \Gamma$ in B^n and $Y = \Delta$ in B^m which satisfies the equation. Hence the equation

$$\sum_i a_i(\Delta) \chi_i = 0$$

is consistent over B with a solution $\chi_i = t_i(\Gamma)$. Hence from the above theorem we have

$$\prod_i a_i(\Delta) = 0$$

which implies that

$$\prod_i a_i(Y) = 0$$

is consistent. Conversely if this equation is consistent with a solution Δ the the above equation in ON variables χ_i is consistent. For an ON solution β_i of these χ_i variables we get consistency of the ON system $t_i(X) = \beta_i$ proved in the propositions 1, 2. This implies that the equation

$$\sum_i a_i(\Delta)t_i(X) = 0$$

is consistent in X variables or what is the same, $f(X, \Delta) = 0$ is consistent which implies $f = 0$ is consistent. $\square$

From this corollary we get the interpretation of elimination of X variables since the consistency of f is equivalent to that of

$$\text{ONelimD}(f, X) \triangleq \prod_i a_i(Y) = 0$$

which is a function of only Y variables. We can call ONelimD(f, X) the *ON disjunctive eliminant* of f following the notations of [2].

2.4 Choice of special ON system

Consider a Boolean function $f(X)$ of n variables in $B(n)$. It would be useful to find out an indexing of variables X as $\{x_1, \ldots, x_m, x_{m+1}, \ldots x_n\}$ such that for the special ON system T of order $m+1$ given in (14) in $\{x_1, \ldots, x_m\}$ is such that f belongs to $\tilde{B}(T)$ where $\tilde{B} = B[x_{m+1}, \ldots, x_n]$. Clearly there would be many such systems while an optimal one can be considered as that system with largest m. An algorithm for choosing such indexing over B_0 is as follows.

Algorithm 1. Input : $f(X)$ in $B_0(n)$ given in DNF with the set of disjunctive terms $\{D_k\}$.

1. Assign x_1 arbitrary set $t_1 = x_1$.

2. Let X_1 denotes set of variables in D_i/t_1 union $\{x_1\}$. Choose x_2 arbitrary in $X - X_1$. set $t_2 = x_1'x_2$

3. Having chosen variables upto x_{k-1} and terms t_{k-1}. Let X_{k-1} as the variables in D_i/t_{k-1} union $\{x_{k-1}\}$. Choose x_k in $X - X_1 \cup \ldots \cup X_{k-1}$. Set $t_k = x_1'x_2' \ldots x_{k-1}'x_k$.

4. Repeat untill no variable is available to choose after the last step for choice of x_k or $X = X_1 \cup \ldots X_{k-1}$.

5. Set $t_{k+1} = x_1' \ldots x_k'$.

While the m variables are being chosen to form an ON system of $m+1$ order, the expansion of the function f can also be computed in ON terms t_i by computing the coefficients in the expansion

$$a_i = \sum_j (D_j/t_i)$$

In the notation of the last section on elimination interpretation, let $X = \{x_1, \ldots, x_m\}$ and Y denote the remaining variables. The eliminant

$$\mathrm{ONelimD}(f, X) = \prod_i a_i$$

the ON eliminant of f is a new Boolean function in $B_0[Y]$. The algorithm can be reapplied to find expansion of the eliminant above in Y variables to simplify further computations for consistency. A full algorithm for consistency based on these ideas shall be written in a separate report.

2.5 Application of ON expansion to systems of equations

Up till now we considered consistency condition for a single equation $f = 0$ in terms of ON expansion of f in ON terms T by computing the eliminant of f with respect to T. In principle this theory is sufficient for deciding consistency of a system of equations

$$f_j = g_j$$

where $j = 1, \ldots, m$ where f_j, g_j are Boolean functions in n variables in $B(n)$ by converting the consistency of the system in terms of a single Boolean equation

$$F = \sum_j (f_j \oplus g_j) = 0$$

However, from computational point of view it is important to realize as much parallelism in computation as possible. Expansion in ON terms suggests one way of achieving this. For this purpose consider expansions in terms of the ON system T

$$\begin{array}{rcl} f_j & = & \sum_k a_{jk} t_k \\ g_j & = & \sum_k b_{jk} t_k \end{array}$$

Then the eliminant of the system is given by

$$\prod_k \sum_j (a_{jk} \oplus b_{jk})$$

such a product of sum expression can be fruitfully utilized for decomposition of the computations involved by utilizing the functions f_j, g_j in a DNF and carrying out operations in terms of the disjunctive terms.

3 Conclusions

A condition for consistency of a Boolean equation $f(X) = 0$ is shown to be possible in terms of expansion of the Boolean function f in special ON systems in $B(n)$. The condition is expressed in terms of consistency of an equation in the eliminant of f with respect to the ON system. The formula for a special solution of the variables is also constructed. An algorithm for choice of variables for construction of the special ON system of order $m + 1$

in m variables is developed. The expansion in ON terms can be seen as a generalization of the well known Shannon expansion. Similarly the consistency condition can be seen as an extension of the single variable elimination method. ON expansion is a technique which can be useful for parallel computation of consistency checking and Boolean operations of functions in large number of variables. Such expansions can be valuable for computations with systems of Boolean equations without converting them to a single equation.

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