EE 325: Probability and Random Processes

Instructor: Saravanan Vijayakumaran

Indian Institute of Technology Bombay

Spring 2013

Midsem : **30 points** (120 min) February 22, 2013

Each question is worth 2 points.

1. Suppose independent trials of an experiment are performed until the first success. In each trial, the probability that the experiment succeeds is p. Let X denote the number of trials required to get the first success. Find the probability mass function, expected value and variance of X. Such a random variable is called a geometric random variable.

2. In answering a multiple-choice question, a student either knows the answer or guesses. Let p be the probability that he knows the answer and $1-p$ be the probability that he guesses. Assume that a student who guesses at the answer will be correct with probability $\frac{1}{m}$ where m is the number of options in the multiple-choice question. What is the probability that a student knew the answer to the question given that he answered correctly?

3. Let X and Y be independent Bernoulli random variables which are equally likely to take the value 0 or 1. Show that $X+Y$ and $|X-Y|$ are not independent random variables but are uncorrelated random variables.

4. Suppose X and Y are random variables which take values in the set $\{0,1\}$ with joint probability mass function $P(X=x, Y=y) = f(x,y)$. Let $f(0,0)=a$, $f(0,1)=b$, $f(1,0)=c$ and $f(1,1)=d$. Find necessary and sufficient conditions for X and Y to be

   (a) uncorrelated

   (b) independent

5. If X is a discrete random variable which takes non-negative integer values show that

$$E(X) = \sum_{n=0}^{\infty} P(X > n)$$

6. (a) A biased coin has a probability p of showing heads when tossed. In n independent tosses of the coin, what is the probability that the kth heads appears on the nth toss?

   (b) One hundred fish are caught from a pond and returned to the pond after they are tagged. Later another 100 fish are caught from the same pond and are found to contain 10 tagged ones. What is the probability of this event if the pond has n fish?

7. A fair coin is tossed n times. Let Y_n denote the number of heads minus the number of tails. Find the probability mass function of Y_n and its expected value.

8. An fair die is thrown n times. Let M denote the maximum value obtained in the n throws and let m denote the minimum value obtained in the n throws. Assuming that the throws are independent of each other, find $P(m=2, M=5)$. *Hint: Write* $\{m \geq 2, M \leq 5\}$ *as a disjoint union.*

9. A fair die is thrown and as many fair coins as the number shown on the die are tossed.

   (a) What is the probability of obtaining k heads?

   (b) If 3 heads are obtained, what is the probability that the die showed the value n? You only know that number of heads obtained here.

10. Let $X_1, \ldots, X_m$ be m independent discrete random variables taking only non-negative integer values. Let all of them have the same probability mass function $P(X=n) = p_n$ for $n \geq 0$. What is the expected value of the minimum of $X_1, \ldots, X_m$?

11. Suppose the number of people who visit a bank in a day is a Poisson[1] random variable with parameter λ. Suppose that each person who visits the bank is female with probability p and male with probability $1-p$ independently of the others.

    (a) Find the joint probability that exactly n women and m men will visit the bank on a particular day.

    (b) Find the probability mass function of the number of women who will visit the bank on a particular day.

12. (a) At a party N men throw their hats into the center of the room. The hats are mixed up and each man randomly selects one. Find the expected number of men who select their own hats.

    (b) A box of cereal comes with a free toy inside. There are 25 different types of toys. In each box of cereal, the toy is equally likely to be any of the 25 types. The toys in different boxes are independent of each other. Find the expected number of different toys that are contained in a set of 10 cereal boxes.

13. The covariance of two random variables X and Y is defined as $\text{cov}(X, Y) = E(XY) - E(X)E(Y)$. Let $X_1, \ldots, X_n$ be independent random variables with the same probability mass function. Let the expected value of each X_i be μ and variance be σ^2. Prove the following where $\bar{X} = \frac{\sum_{i=1}^n X_i}{n}$.

    (a) $E(\bar{X}) = \mu$, $\text{var}(\bar{X}) = \frac{\sigma^2}{n}$

    (b) $\text{cov}(\bar{X}, X_i - \bar{X}) = 0$ for all $i = 1, \ldots, n$

14. Let $X_1, \ldots, X_n$ be independent continuous random variables with the same probability distribution function F. Let $X_{(i)}$ denote the ith smallest of these random variables. Then $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ are called the order statistics. Find the probability distribution function of $X_{(i)}$ in terms of F.

15. Let X be a discrete random variable which takes only nonnegative values. Prove that for any value $a > 0$

$$P(X \geq a) \leq \frac{E(X)}{a}.$$

[1] $P[X = k] = \frac{\lambda^k}{k!} e^{-\lambda}, k = 0, 1, 2, \ldots$