Convergence of Random Variables

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Motivation

Theorem (Weak Law of Large Numbers)

Let $X_1, X_2, \ldots$ be a sequence of independent identically distributed random variables with finite means μ. Their partial sums $S_n = X_1 + X_2 + \cdots + X_n$ satisfy

$$\frac{S_n}{n} \xrightarrow{P} \mu \qquad \text{as } n \to \infty.$$

Theorem (Central Limit Theorem)

Let $X_1, X_2, \ldots$ be a sequence of independent identically distributed random variables with finite means μ and finite non-zero variance σ^2. Their partial sums $S_n = X_1 + X_2 + \cdots + X_n$ satisfy

$$\frac{S_n - n\mu}{\sqrt{n\sigma^2}} \xrightarrow{D} N(0,1) \qquad \text{as } n \to \infty.$$

Modes of Convergence

- Let $X, X_1, X_2, \ldots$ be random variables on some probability space $(\Omega, \mathcal{F}, P)$
- There are four ways of defining $X_n \to X$ as $n \to \infty$
  - Convergence almost surely
  - Convergence in rth mean
  - Convergence in probability
  - Convergence in distribution

Convergence Almost Surely

- $X_n \to X$ almost surely if $\{\omega \in \Omega : X_n(\omega) \to X(\omega) \text{ as } n \to \infty\}$ is an event whose probability is 1
- "$X_n \to X$ almost surely" is abbreviated as $X_n \xrightarrow{\text{a.s.}} X$
- Other notations are
  - $X_n \to X$ almost everywhere or $X_n \xrightarrow{\text{a.e.}} X$
  - $X_n \to X$ with probability 1 or $X_n \to X$ w.p. 1

Convergence in rth Mean

- $X_n \to X$ in rth mean if $E\left[|X|^r\right] < \infty$ for all n and

  $$E\left(|X_n - X|^r\right) \to 0 \text{ as } n \to \infty$$

  where $r \geq 1$
- "$X_n \to X$ in rth mean" is abbreviated as $X_n \xrightarrow{r} X$
- For $r = 1$, $X_n \xrightarrow{1} X$ is written as "$X_n \to X$ in mean"
- For $r = 2$, $X_n \xrightarrow{2} X$ is written as "$X_n \to X$ in mean square" or $X_n \xrightarrow{\text{m.s.}} X$

Convergence in Probability

- $X_n \to X$ in probability if

$$P(|X_n - X| > \epsilon) \to 0 \text{ as } n \to \infty \text{ for all } \epsilon > 0$$

- "$X_n \to X$ in probability" is abbreviated as $X_n \xrightarrow{P} X$

Convergence in Distribution

- $X_n \to X$ in distribution if

$$P(X_n \leq x) \to P(X \leq x) \text{ as } n \to \infty$$

  for all points x where $F_X(x) = P(X \leq x)$ is continuous
- "$X_n \to X$ in distribution" is abbreviated as $X_n \xrightarrow{D} X$
- Convergence in distribution is also termed weak convergence

Example

Let X be a Bernoulli RV taking values 0 and 1 with equal probability $\frac{1}{2}$.
Let $X_1, X_2, X_3, \dots$ be identical random variables given by $X_n = X$ for all n.
The X_n's are not independent but $X_n \xrightarrow{D} X$.
Let $Y = 1 - X$. Then $X_n \xrightarrow{D} Y$.
But $|X_n - Y| = 1$ and the X_n's do not converge to Y in any other mode.

Relations between Modes of Convergence

Theorem

$$
\begin{array}{ccccc}
(X_n \xrightarrow{a.s.} X) & & & & \\
& \searrow & & & \\
& & (X_n \xrightarrow{P} X) & \Rightarrow & (X_n \xrightarrow{D} X) \\
& \nearrow & & & \\
(X_n \xrightarrow{r} X) & & & &
\end{array}
$$

for any $r \geq 1$.

Theorem

If $X_n \xrightarrow{D} c$, *where* c *is a constant, then* $X_n \xrightarrow{P} c$.

Weak Law of Large Numbers

Let $X_1, X_2, \ldots$ be a sequence of independent identically distributed random variables with finite means μ. Their partial sums $S_n = X_1 + X_2 + \cdots + X_n$ satisfy

$$\frac{S_n}{n} \xrightarrow{P} \mu \qquad \text{as } n \to \infty.$$

Proof.

- Since μ is a constant, it is enough to show convergence in distribution
- It is enough to show that the characteristic functions of $\frac{S_n}{n}$ converge to the characteristic function of μ
- By Taylor's theorem, the characteristic function of the X_n's is

$$\phi_X(t) = E\left[e^{itX}\right] = 1 + i\mu t + o(t)$$

- The characteristic function of $\frac{S_n}{n}$ is

$$\phi_n(t) = \left[\phi_X\left(\frac{t}{n}\right)\right]^n = \left[1 + i\mu\frac{t}{n} + o\left(\frac{t}{n}\right)\right]^n \to \exp(it\mu)$$

Strong Law of Large Numbers

Let $X_1, X_2, \ldots$ be a sequence of independent identically distributed random variables. Then

$$\frac{1}{n}\sum_{i=1}^{n} X_i \to \mu \quad \text{almost surely, as } n \to \infty.$$

for some constant μ, if and only if $E|X_1| < \infty$. In this case, $\mu = E[X_1]$.

Central Limit Theorem

Let $X_1, X_2, \ldots$ be a sequence of independent identically distributed random variables with finite means μ and finite non-zero variance σ^2. Their partial sums $S_n = X_1 + X_2 + \cdots + X_n$ satisfy

$$\frac{S_n - n\mu}{\sqrt{n\sigma^2}} \xrightarrow{D} N(0,1) \qquad \text{as } n \to \infty.$$

Proof.

- It is enough to show that the characteristic functions of $\frac{S_n - n\mu}{\sqrt{n\sigma^2}}$ converge to the characteristic function of $Z \sim N(0,1)$ which is $e^{-\frac{t^2}{2}}$
- Let $\phi_Y(t)$ be the characteristic function of $Y_n = \frac{X_n - \mu}{\sigma}$
- By Taylor's theorem, the characteristic function of the Y_n's is

$$\phi_Y(t) = E\left[e^{itY}\right] = 1 - \frac{t^2}{2} + o(t^2)$$

- The characteristic function of $\frac{S_n - n\mu}{\sqrt{n\sigma^2}} = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} Y_j$ is

$$\psi_n(t) = \left[\phi_Y\left(\frac{t}{\sqrt{n}}\right)\right]^n = \left[1 - \frac{t^2}{2n} + o\left(\frac{t^2}{n}\right)\right]^n \to \exp\left(-\frac{t^2}{2}\right)$$

Questions?