Optimal Decoding of Error Correcting Codes

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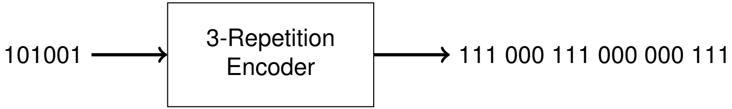
February 15, 2013

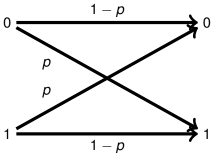
The 3-Repetition Code

- Given a block of message bits, each 0 is replaced with three 0's and each 1 is replaced with three 1's

$$0 \to 000, 1 \to 111$$

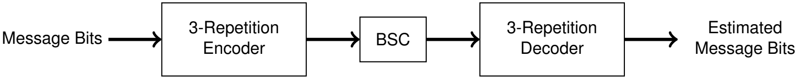


- Suppose the encoder output is sent through a binary symmetric channel



- What is the best way to build the decoder?

Optimal Decoder for the 3-Repetition Code



- We have to first decide on the optimality criterion
- Let us first associate a cost with decoder decisions
- Let X be the transmitted bit and $\hat{X}$ be its estimate by the decoder

$$C(X, \hat{X}) = \begin{cases} 0 & \text{if } \hat{X} = X \\ 1 & \text{if } \hat{X} \neq X \end{cases}$$

- Let $\mathbf{Y} = (Y_1, Y_2, Y_3)$ be the channel output corresponding to bit X
- The decoder's estimate $\hat{X}$ is a function of $\mathbf{Y}$
- What should the value of $\hat{X}$ be when $\mathbf{Y} = 000$? What is the cost of this decision?
- The costs are different depending on the value of X and both values of X can result in the same $\mathbf{Y}$

Expected Cost to the Rescue

- The solution is to calculate the expected cost as a function of the decoder
- The optimal decoder is the one which minimizes the expected cost
- What is a decoder? A partition of the space of binary 3-tuples
- Let Γ_0 and Γ_1 be a partition of $\Gamma = \{0,1\} \times \{0,1\} \times \{0,1\}$
- The decoder will decide according to the set $\mathbf{Y}$ belongs to

$$\hat{X} = \begin{cases} 0 & \text{if } \mathbf{Y} \in \Gamma_0 \\ 1 & \text{if } \mathbf{Y} \in \Gamma_1 \end{cases}$$

- Let $\pi_0 = P(X = 0)$ and $\pi_1 = P(X = 1)$
- The expected cost is given by

$$\begin{aligned} E\left[C(X,\hat{X})\right] &= \pi_0 E\left[C(X,\hat{X})\middle| X=0\right] + \pi_1 E\left[C(X,\hat{X})\middle| X=1\right] \\ &= \pi_0 E\left[C(0,\hat{X})\middle| X=0\right] + \pi_1 E\left[C(1,\hat{X})\middle| X=1\right] \\ &= \pi_0 P\left(\mathbf{Y} \in \Gamma_1 \middle| X=0\right) + \pi_1 P\left(\mathbf{Y} \in \Gamma_0 \middle| X=1\right) \end{aligned}$$

Minimizing the Expected Cost of a Decoder

$$
\begin{aligned}
E\left[C(X,\hat{X})\right] &= \pi_0 P\left(\mathbf{Y}\in\Gamma_1 \middle| X=0\right) + \pi_1 P\left(\mathbf{Y}\in\Gamma_0 \middle| X=1\right) \\
&= \pi_0 \sum_{\mathbf{y}\in\Gamma_1} P(\mathbf{Y}=\mathbf{y}|X=0) + \pi_1 \sum_{\mathbf{y}\in\Gamma_0} P(\mathbf{Y}=\mathbf{y}|X=1)
\end{aligned}
$$

- How can we minimize the expected cost as a function of the partition $\{\Gamma_0, \Gamma_1\}$?
- To minimize the cost, we choose the partition in the following manner

$$
\begin{aligned}
\Gamma_0 &= \left\{\mathbf{y}\in\Gamma \middle| \pi_1 P(\mathbf{Y}=\mathbf{y}|X=1) \le \pi_0 P(\mathbf{Y}=\mathbf{y}|X=0)\right\} \\
\Gamma_1 &= \left\{\mathbf{y}\in\Gamma \middle| \pi_1 P(\mathbf{Y}=\mathbf{y}|X=1) > \pi_0 P(\mathbf{Y}=\mathbf{y}|X=0)\right\}
\end{aligned}
$$

Optimal Decoder for Equally Likely Inputs

- Suppose $\pi_0 = \pi_1 = \frac{1}{2}$

$$
\begin{aligned}
\Gamma_0 &= \left\{ \mathbf{y} \in \Gamma \,\middle|\, P(\mathbf{Y} = \mathbf{y} | X = 1) \le P(\mathbf{Y} = \mathbf{y} | X = 0) \right\} \\
\Gamma_1 &= \left\{ \mathbf{y} \in \Gamma \,\middle|\, P(\mathbf{Y} = \mathbf{y} | X = 1) > P(\mathbf{Y} = \mathbf{y} | X = 0) \right\}
\end{aligned}
$$

- $P(\mathbf{Y} = 111 | X = 1) = (1-p)^3$, $P(\mathbf{Y} = 101 | X = 1) = p(1-p)^2$
- Let $d(\mathbf{y}, 111)$ be the Hamming distance between $\mathbf{y}$ and 111
  Let $d(\mathbf{y}, 000)$ be the Hamming distance between $\mathbf{y}$ and 000

$$
\begin{aligned}
P(\mathbf{Y} = \mathbf{y} | X = 1) &= p^{d(\mathbf{y},111)} (1-p)^{3-d(\mathbf{y},111)} \\
P(\mathbf{Y} = \mathbf{y} | X = 0) &= p^{d(\mathbf{y},000)} (1-p)^{3-d(\mathbf{y},000)}
\end{aligned}
$$

- If $p < \frac{1}{2}$, then

$$
\begin{aligned}
\Gamma_0 &= \left\{ \mathbf{y} \in \Gamma \,\middle|\, d(\mathbf{y}, 000) \le d(\mathbf{y}, 111) \right\} \\
\Gamma_1 &= \left\{ \mathbf{y} \in \Gamma \,\middle|\, d(\mathbf{y}, 000) > d(\mathbf{y}, 111) \right\}
\end{aligned}
$$

Optimal Decoder for Inputs Not Equally Likely

- Suppose $\pi_0 \neq \pi_1$

$$\begin{aligned}\Gamma_0 &= \left\{\mathbf{y} \in \Gamma \middle| \pi_1 P(\mathbf{Y} = \mathbf{y}|X = 1) \leq \pi_0 P(\mathbf{Y} = \mathbf{y}|X = 0)\right\} \\ \Gamma_1 &= \left\{\mathbf{y} \in \Gamma \middle| \pi_1 P(\mathbf{Y} = \mathbf{y}|X = 1) > \pi_0 P(\mathbf{Y} = \mathbf{y}|X = 0)\right\}\end{aligned}$$

- If $\pi_0 > \pi_1$, the optimal decoder may favor 0 even if **y** is closer to 111

Questions?