Generating Functions

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March 22, 2013

Generating Functions

Definition

The generating function of a sequence of real numbers $\{a_i : i = 0, 1, 2, \ldots\}$ is defined by

$$G(s) = \sum_{i=0}^{\infty} a_i s^i$$

for $s \in \mathbb{R}$ for which the sum converges.

Example

Consider the sequence $a_i = 2^{-i}, i = 0, 1, 2, \ldots$.

$$G(s) = \sum_{i=0}^{\infty} \left(\frac{s}{2}\right)^i = \frac{1}{1 - \frac{s}{2}} \qquad \text{for } |s| < 2.$$

Definition

Suppose X is a discrete random variable taking non-negative integer values $\{0, 1, 2, \ldots\}$. The generating function of X is the generating function of its probability mass function.

$$G(s) = \sum_{i=0}^{\infty} P[X = i] s^i = E[s^X]$$

Examples of Generating Functions

- **Constant RV:** Suppose $P(X = c) = 1$ for some fixed $c \in \mathbb{Z}^+$

$$G(s) = E(s^X) = s^c$$

- **Bernoulli RV:** $P(X = 1) = p$ and $P(X = 0) = 1 - p$

$$G(s) = 1 - p + ps$$

- **Geometric RV:** $P(X = k) = p(1 - p)^{k-1}$ for $k \geq 1$

$$G(s) = \sum_{k=1}^{\infty} s^k p(1 - p)^{k-1} = \frac{ps}{1 - s(1 - p)}$$

- **Poisson RV:** $P[X = k] = \frac{e^{-\lambda}\lambda^k}{k!}$ for $k \geq 0$

$$G(s) = \sum_{k=0}^{\infty} s^k \frac{e^{-\lambda}\lambda^k}{k!} = e^{\lambda(s-1)}$$

Moments from the Generating Function

Theorem

If X has generating function $G(s)$ then

- $E[X] = G^{(1)}(1)$
- $E[X(X-1)\cdots(X-k+1)] = G^{(k)}(1)$

where $G^{(k)}$ is the kth derivative of $G(s)$.

Result

$\mathrm{var}(X) = G^{(2)}(1) + G^{(1)}(1) - G^{(1)}(1)^2$

Example (Geometric RV)

A geometric RV X has generating function $G(s) = \frac{ps}{1-s(1-p)}$. $\mathrm{var}(X) = ?$

$$\begin{aligned}
G^{(1)}(1) &= \frac{\partial}{\partial s}\frac{ps}{1-s(1-p)}\bigg|_{s=1} = \frac{1}{p} \\
G^{(2)}(1) &= \frac{\partial^2}{\partial s^2}\frac{ps}{1-s(1-p)}\bigg|_{s=1} = \frac{2(1-p)}{p} + \frac{2(1-p)^2}{p^2} \\
\mathrm{var}(X) &= \frac{1-p}{p^2}
\end{aligned}$$

Generating Function of a Sum of Independent RVs

Theorem

If X and Y are independent, $G_{X+Y}(s) = G_X(s)G_Y(s)$

Example (Binomial RV)

Using above theorem, how can we find the generating function of a binomial random variable?

A binomial random variable with parameters n and p is a sum of n independent Bernoulli random variables.

$$S = X_1 + X_2 + \cdots + X_{n-1} + X_n$$

where each X_i has generating function $G(s) = 1 - p + ps = q + ps$.

$$G_S(s) = [G(s)]^n = [q + ps]^n$$

Example (Sum of independent Poisson RVs)

Let X and Y be independent Poisson random variables with parameters λ and μ respectively. What is the distribution of $X + Y$?

Poisson with parameter $\lambda + \mu$

Sum of a Random Number of Independent RVs

Theorem

Let $X_1, X_2, \ldots$ is a sequence of independent identically distributed (iid) random variables with common generating function $G_X(s)$. Let N be a positive integer-valued random variable which is independent of the X_i's and has generating function $G_N(s)$. Then

$$S = X_1 + X_2 + \cdots + X_N$$

has generating function given by

$$G_S(s) = G_N\left(G_X(s)\right)$$

Example

A group of hens lay N eggs where N has a Poisson distribution with parameter λ. Each egg results in a healthy chick with probability p independently of the other eggs. Let K be the number of healthy chicks. Find the distribution of K.
Solution Poisson with parameter λp

Questions?