Some Results on Jointly Distributed Discrete Random Variables

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Jointly Distributed Discrete Random Variables

Definition

The joint probability distribution function of discrete RVS X and Y is given by

$$F(x, y) = P(X \leq x \text{ and } Y \leq y).$$

The joint probability mass function is given by

$$f(x, y) = P(X = x \text{ and } Y = y).$$

Definition

Given the joint pmf, the marginal pmfs are given by

$$\begin{aligned} P(X = x) &= \sum_{y} f(x, y) \\ P(Y = y) &= \sum_{x} f(x, y) \end{aligned}$$

Result

If X and Y have a joint pmf f, then $E[g(X, Y)] = \sum_{x,y} g(x, y) f(x, y)$

Cauchy-Schwarz Inequality

Theorem

For random variables X and Y,

$$[E(XY)]^2 \leq E(X^2)E(Y^2)$$

Proof.

For $a \in \mathbb{R}$, let $Z = aX - Y$. Then

$$0 \leq E[Z^2] = a^2 E(X^2) - 2aE(XY) + E(Y^2)$$

The quadratic in a can have at most one real root. So the discriminant is non-positive.
If $E(X^2) > 0$, this gives

$$\frac{4\,[E(XY)]^2 - 4E(X^2)E(Y^2)}{2E(X^2)} \leq 0$$

which proves the result.
If $E(X^2) = 0$, then $P(X = 0) = 1$ and the result follows. □

Conditional Distribution

Definition

The conditional probability distribution function of Y given $X = x$ is defined as

$$F_{Y|X}(y|x) = P(Y \leq y|X = x)$$

for any x such that $P(X = x) > 0$.
The conditional probability mass function of Y given $X = x$ is defined as

$$f_{Y|X}(y|x) = P(Y = y|X = x)$$

Definition

The conditional expectation of Y given $X = x$ is defined as

$$E(Y|X = x) = \sum_{y} y f_{Y|X}(y|x)$$

The conditional expectation is a function of the conditioning random variable i.e. $\psi(X) = E(Y|X)$

Law of Iterated Expectation

Theorem

The conditional expectation $\psi(X) = E(Y|X)$ *satisfies*

$$E\left[E(Y|X)\right] = E(Y)$$

Example

A group of hens lay N eggs where N has a Poisson distribution with parameter λ. Each egg results in a healthy chick with probability p independently of the other eggs. Let K be the number of chicks. Find $E(K)$.

Theorem

The conditional expectation $\psi(X) = E(Y|X)$ *satisfies*

$$E\left[\psi(X)g(X)\right] = E\left[Yg(X)\right]$$

for any function g *for which both expectations exist.*

Sum of Random Variables

Theorem

For discrete random variables X and Y with joint pmf $f(x, y)$, the pmf of $X + Y$ is given by

$$P(X + Y = z) = \sum_x f(x, z - x) = \sum_y f(z - y, y)$$

If X and Y are independent, the pmf of $X + Y$ is the convolution of the pmfs of X and Y.

$$P(X + Y = z) = \sum_x f_X(x) f_Y(z - x) = \sum_y f_X(z - y) f_Y(y)$$

Questions?