EE 325: Probability and Random Processes (Spring 2014)
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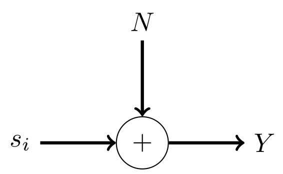
Endsemester Exam : 40 point April 26, 2014

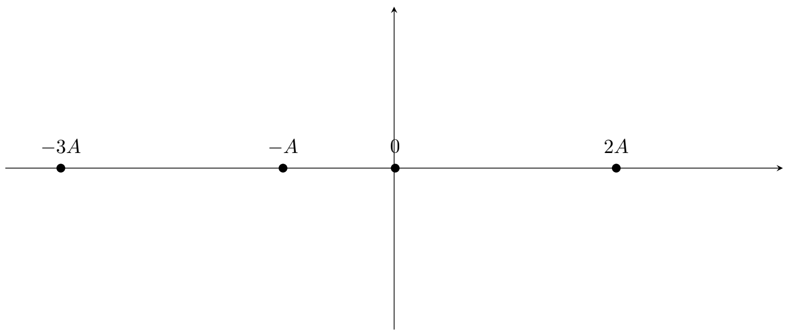
1. Suppose the number of people who visit a bank in a day is a Poisson random variable with parameter λ. Suppose that each person who visits the bank is female with probability p and male with probability $1-p$ independently of the others.
   (a) (2 points) Find the joint probability that exactly n women and m men will visit the bank on a particular day.
   (b) (2 points) Find the probability mass function of the number of women who will visit the bank on a particular day.

2. The covariance of a pair of random variables is defined as $\text{cov}(X,Y) = E\left[(X - E[X])\,(Y - E[Y])\right]$.
   (a) (1 point) Show that $\text{cov}(X, Y+Z) = \text{cov}(X,Y) + \text{cov}(X,Z)$.
   (b) (2 points) For n random variables $X_1, X_2, \dots, X_n$, calculate the variance of $\sum_{i=1}^{n} X_i$ as a function of the variances of X_i and the covariance between different X_i's.
   (c) (1 point) Using the above result, show that the variance of $\sum_{i=1}^{n} X_i$ is equal to the sum of the variances of X_i's if the X_i's are independent.

3. A point's location in the two-dimensional plane is given by the ordered pair (X,Y) where X and Y are independent Gaussian random variables with mean A and variance σ^2.
   (a) (1 point) What is the probability that the point lies on the left of the y-axis?
   (b) (1 point) What is the probability that the point lies below the x-axis?
   (c) (2 points) What is the probability that the point does not lie in the first quadrant?

4. (4 points) A transmitter sends of one of four possible signals $s_0 = -3A, s_1 = -A, s_2 = 0, s_3 = 2A$. The transmitted signal is corrupted by noise N which is a zero mean Gaussian random variable having variance σ^2. Assume all four signals are equally likely to be transmitted.
   (a) Find the optimal decision rule based on the observation Y.





   (b) Find the average probability of decision error for the optimal decision rule. Express your final answer in terms of the Q function.

5. (4 points) Suppose we observe $Y_i,\ i = 1, 2, \dots, M$ such that
$$Y_i \sim N(\mu, \sigma^2)$$
where the Y_i's are independent.
   (a) If μ is **unknown** and σ is **known**, derive the maximum likelihood estimator of μ.
   (b) If μ is **known** and σ is **unknown**, derive the maximum likelihood estimator of σ^2.

6. (4 points) Suppose $Y_1, Y_2, \dots, Y_n$ is a sequence of random variables, each taking values 0 and 1 with probabilities $\frac{1}{2}$. Consider the following two hypotheses concerning the Y_i's.
$$\begin{aligned} H_0 &: Y_1, Y_2, \dots, Y_n \text{ are independent} \\ H_1 &: p_1(y_k | y_{k-1}, y_{k-2}, \dots, y_1) = \begin{cases} \frac{3}{4} & \text{if } y_k = y_{k-1} \\ \frac{1}{4} & \text{if } y_k \neq y_{k-1} \end{cases} \\ & \quad k = 2, 3, \dots, n \end{aligned}$$
where $p_1(y_k|y_{k-1}, y_{k-2}, \dots, y_1) = \Pr\left[Y_k = y_k | Y_{k-1} = y_{k-1}, \dots, Y_1 = y_1\right]$. Find the optimal decision rule which minimizes the decision error probability under the assumption that the hypotheses are equally likely.

7. (a) (1 point) Let X be a non-negative random variable and $a > 0$. Prove Markov's inequality

$$\Pr(X \geq a) \leq \frac{E(X)}{a}.$$

   (b) (1 point) For $a > 0$ and any random variable X with finite mean and variance, prove Chebyshev's inequality

$$\Pr\left(|X - E[X]| \geq a\right) \leq \frac{\mathrm{var}(X)}{a^2}.$$

   (c) (2 points) Suppose $X_1, X_2, \ldots$ are independent and identically distributed random variables with finite mean μ and finite variance σ^2. Use Chebyshev's inequality to prove the weak law of large numbers. *Hint: Find an upper bound for* $\Pr\left(|\frac{1}{n}\sum_{i=1}^{n} X_i - \mu| \geq \varepsilon\right)$ *for* $\varepsilon > 0$.

8. (4 points) A fair coin is tossed repeatedly and a random process X_n for $n = 1, 2, 3, \ldots$ is generated according to the following rule.

$$X_n = \begin{cases} 2^n & \text{if the first } n \text{ tosses all result in heads} \\ 0 & \text{if any one of the first } n \text{ tosses results in tails} \end{cases}$$

   (a) Prove or disprove almost sure convergence of X_n.
   (b) Prove or disprove convergence in probability of X_n.
   (c) Prove or disprove convergence in distribution of X_n.
   (d) Prove or disprove convergence in mean of X_n.

9. (4 points) Let a random process be defined as $X(t) = A\cos(2\pi f_c t) + B\sin(2\pi f_c t)$ where f_c is a constant and A and B are independent random variables with mean zero and variance σ^2. Assume that all of the moments of A and B except the mean are non-zero.
   (a) Find the mean function of $X(t)$.
   (b) Find the autocorrelation function of $X(t)$.
   (c) Prove or disprove the wide-sense stationarity of $X(t)$.
   (d) Prove or disprove the strict-sense stationarity of $X(t)$. *Hint: Try calculating* $E\left[X^3(t)\right]$.

10. Consider the random process $X(t)$ resulting from an amplitude modulated pulse train given by

$$X(t) = \sum_{i=-\infty}^{\infty} A_i p(t - iT)$$

where the A_i's are independent and identically distributed discrete random variables which are equally likely to be ± 1 and $p(t)$ is a unit pulse of duration T i.e. $p(t) = 1$ for $t \in [0, T)$ and 0 otherwise.
   (a) (1 point) Prove that $X(t)$ is not wide-sense stationary.
   (b) (3 points) A random process $Y(t)$ is said to be wide-sense cyclostationary with respect to time interval T_0 if the mean and autocorrelation functions satisfy

$$\begin{aligned} m_Y(t) &= m_Y(t - T_0) \quad \text{for all } t, \\ R_Y(t_1, t_2) &= R_Y(t_1 - T_0, t_2 - T_0) \quad \text{for all } t_1, t_2. \end{aligned}$$

   Prove that $X(t)$ is wide-sense cyclostationary with respect to time interval T.