EE 325: Probability and Random Processes (Spring 2014)
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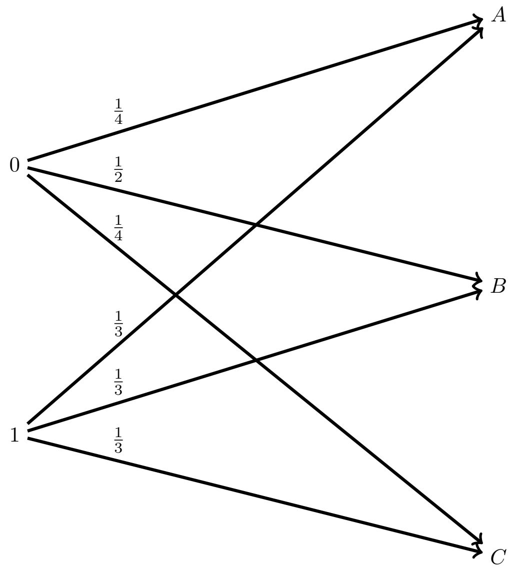
Quiz 2 : 16 points March 26, 2014

1. (2 points) Consider the binary input ternary output channel below. The inputs are either 0 or 1. The prior probability of the input being 0 is 0.4. The output is one of three symbols A, B, C.
   (a) Find the decision rule δ_{MPE} which minimizes the decision error probability.
   (b) Find the decision error probability when δ_{MPE} is used.



2. (4 points) Consider the following binary hypothesis testing problem where the hypotheses are equally likely.

$$
\begin{aligned}
H_0 &: Y \sim U\left[-\sqrt{\frac{e^2\pi}{2}}, \sqrt{\frac{e^2\pi}{2}}\right] \\
H_1 &: Y \sim \mathcal{N}(0,1)
\end{aligned}
$$

   U denotes the uniform distribution, $\mathcal{N}$ denotes the Gaussian distribution and e is the base of the natural logarithm.
   (a) Find the decision error probability of the rule which decides H_1 is true if $|Y| > \sqrt{\frac{e^2\pi}{2}}$ and decides H_0 is true if $|Y| \leq \sqrt{\frac{e^2\pi}{2}}$. Express your answer in terms of the Q function.
   (b) Find the decision error probability of the optimal decision rule. Express your answer in terms of the Q function.

3. (2 points) Suppose observations Y_i, $i = 1, 2, \ldots, N$ are Poisson distributed with parameter λ. Assume that the Y_i's are independent.
   (a) Derive the ML estimator for λ.
   (b) Find the mean and variance of the ML estimate.

4. (2 points) Suppose we observe Y_i, $i = 1, 2, \ldots, M$ such that

$$Y_i \sim U[-\theta, \theta]$$

   where Y_i's are independent and θ is unknown. Assume $\theta > 0$. Derive the ML estimator of θ.

5. (2 points) Suppose observations X_i and Y_i $(i = 1, \ldots, N)$ depend on an unknown parameter A as per the following distributions.

$$
\begin{aligned}
X_i &\sim \mathcal{N}(A, \sigma^2), \quad i = 1, 2, \ldots, N \\
Y_i &\sim \mathcal{N}(A, 2\sigma^2), \quad i = 1, 2, \ldots, N
\end{aligned}
$$

   Note that the variance of Y_i is twice the variance of X_i. Assume that X_i and X_j are independent for $i \neq j$. Assume that Y_i and Y_j are independent for $i \neq j$. Assume that X_i and Y_j are independent for all i, j. Assume σ^2 is known.

(a) Derive the ML estimator for A.

(b) Find the mean and variance of the ML estimate.

6. (2 points) Specify a method to generate a random variable with Rayleigh distribution which is a continuous random variable with probability distribution function given by

$$F(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 - e^{-\frac{x^2}{2\sigma^2}} & \text{otherwise} \end{cases}$$

where σ is a known parameter.

7. (2 points) Suppose a fair coin is repeatedly tossed. Specify a method to generate a random variable which corresponds to the number of tosses until a heads appears. Prove that the generated random variable has the correct distribution.

Useful Formulas

- Gaussian vector density $p(\mathbf{x}) = \frac{1}{\sqrt{(2\pi)^n \det(\mathbf{C})}} \exp\left(-\frac{1}{2}(\mathbf{x}-\mathbf{m})^T \mathbf{C}^{-1}(\mathbf{x}-\mathbf{m})\right)$ where $\mathbf{m} = E[\mathbf{X}], \mathbf{C} = E\left[(\mathbf{X}-\mathbf{m})(\mathbf{X}-\mathbf{m})^T\right]$
- Minimum probability of error rule

$$\delta_{\text{MPE}}(\mathbf{y}) = \arg \max_{1 \le i \le M} \pi_i p_i(\mathbf{y})$$

- The MAP decision rule is given by

$$\delta_{\text{MAP}}(\mathbf{y}) = \arg \max_{1 \le i \le M} P\left[H_i \text{ is true} \middle| \mathbf{y}\right]$$

- The ML decision rule is given by

$$\delta_{\text{ML}}(\mathbf{y}) = \arg \max_{1 \le i \le M} p_i(\mathbf{y})$$