## **Limit Theorems**

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### **Limit Theorems**

## Theorem (Weak Law of Large Numbers)

Let  $X_1, X_2, \ldots$  be a sequence of independent identically distributed random variables with finite means  $\mu$ . Their partial sums  $S_n = X_1 + X_2 + \cdots + X_n$  satisfy

$$\frac{S_n}{n} \xrightarrow{P} \mu$$
 as  $n \to \infty$ .

## Theorem (Central Limit Theorem)

Let  $X_1, X_2, \ldots$  be a sequence of independent identically distributed random variables with finite means  $\mu$  and finite non-zero variance  $\sigma^2$ . Their partial sums  $S_n = X_1 + X_2 + \cdots + X_n$  satisfy

$$\frac{S_n - n\mu}{\sqrt{n\sigma^2}} \xrightarrow{D} N(0,1) \quad \text{as } n \to \infty.$$

# Characteristic Functions

## Characteristic Functions

#### Definition

For a random variable X, the characteristic function is given by

$$\phi(t) = E(e^{itX})$$

## Examples

• Bernoulli RV: P(X = 1) = p and P(X = 0) = 1 - p $\phi(t) = 1 - p + pe^{it} = q + pe^{it}$ 

• Gaussian RV: Let  $X \sim N(\mu, \sigma^2)$ 

$$\phi(t) = \exp\left(i\mu t - \frac{1}{2}\sigma^2 t^2\right)$$

# Properties of Characteristic Functions

#### **Theorem**

If X and Y are independent, then

$$\phi_{X+Y}(t) = \phi_X(t)\phi_Y(s).$$

## Example (Binomial RV)

$$\phi(t) = \left(q + pe^{it}\right)^n$$

## Example (Sum of Independent Gaussian RVs)

Let  $X \sim N(\mu_1, \sigma_1^2)$  and  $Y \sim N(\mu_2, \sigma_2^2)$  be independent. What is the distribution of X + Y?

#### **Theorem**

If  $a, b \in \mathbb{R}$  and Y = aX + b, then

$$\phi_Y(t)=e^{itb}\phi_X(at).$$

# **Inversion and Continuity Theorems**

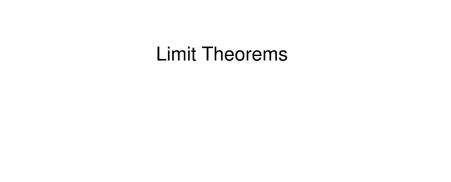
#### **Theorem**

Random variables X and Y have the same characteristic function if and only if they have the same distribution function.

#### **Theorem**

Suppose  $F_1, F_2, ...$  is a sequence of distribution functions with corresponding characteristic functions  $\phi_1, \phi_2, ...$ 

- If F<sub>n</sub> → F for some distribution function F with characteristic function φ, then φ<sub>n</sub>(t) → φ(t) for all t.
- Conversely, if  $\phi(t) = \lim_{n \to \infty} \phi_n(t)$  exists and is continuous at t = 0, then  $\phi$  is the characteristic function of some distribution function F, and  $F_n \to F$ .



# Weak Law of Large Numbers

Let  $X_1, X_2, \ldots$  be a sequence of independent identically distributed random variables with finite means  $\mu$ . Their partial sums  $S_n = X_1 + X_2 + \cdots + X_n$  satisfy

$$\frac{S_n}{n} \xrightarrow{P} \mu$$
 as  $n \to \infty$ .

#### Proof.

- Since  $\mu$  is a constant, it is enough to show convergence in distribution
- It is enough to show that the characteristic functions of  $\frac{S_n}{n}$  converge to the characteristic function of  $\mu$
- By Taylor's theorem, the characteristic function of the  $X_n$ 's is

$$\phi_X(t) = E\left[e^{itX}\right] = 1 + i\mu t + o(t)$$

• The characteristic function of  $\frac{S_n}{n}$  is

$$\phi_n(t) = \left[\phi_X\left(\frac{t}{n}\right)\right]^n = \left[1 + i\mu\frac{t}{n} + o\left(\frac{t}{n}\right)\right]^n \to \exp(it\mu)$$

# Strong Law of Large Numbers

Let  $X_1, X_2, \ldots$  be a sequence of independent identically distributed random variables. Then

$$\frac{1}{n}\sum_{i=1}^{n}X_{i}\to\mu$$
 almost surely, as  $n\to\infty$ .

for some constant  $\mu$ , if and only if  $E|X_1| < \infty$ . In this case,  $\mu = E[X_1]$ .

## Central Limit Theorem

Let  $X_1, X_2, \ldots$  be a sequence of independent identically distributed random variables with finite means  $\mu$  and finite non-zero variance  $\sigma^2$ . Their partial sums  $S_n = X_1 + X_2 + \cdots + X_n$  satisfy

$$\frac{S_n - n\mu}{\sqrt{n\sigma^2}} \xrightarrow{D} N(0,1) \quad \text{as } n \to \infty.$$

#### Proof.

- It is enough to show that the characteristic functions of  $\frac{S_n n\mu}{\sqrt{n\sigma^2}}$  converge to the characteristic function of  $Z \sim N(0,1)$  which is  $e^{-\frac{j^2}{2}}$
- Let  $\phi_Y(t)$  be the characteristic function of  $Y_n = \frac{X_n \mu}{\sigma}$
- By Taylor's theorem, the characteristic function of the  $Y_n$ 's is

$$\phi_Y(t) = E\left[e^{itY}\right] = 1 - \frac{t^2}{2} + o(t^2)$$

• The characteristic function of  $\frac{S_n - n\mu}{\sqrt{n}\sigma^2} = \frac{1}{\sqrt{n}} \sum_{i=1}^n Y_i$  is

$$\psi_n(t) = \left[\phi_Y\left(\frac{t}{\sqrt{n}}\right)\right]^n = \left[1 - \frac{t^2}{2n} + o\left(\frac{t^2}{n}\right)\right]^n \to \exp\left(-\frac{t^2}{2}\right)$$

## Reference

• Chapter 5, *Probability and Random Processes*, Grimmett and Stirzaker, Third Edition, 2001.