EE 325: Probability and Random Processes (Spring 2015)
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Assignment 2: 25 points — Due Date: February 13, 2015

1 Problems

Let $F : \mathbb{R} \mapsto [0,1]$ be the distribution function a random variable X. Prove the following using the definitions given in the next section.

1. [5 points] $\lim_{x \to -\infty} F(x) = 0$.
2. [5 points] $\lim_{x \to \infty} F(x) = 1$.
3. [5 points] F is right continuous, $F(x+h) \to F(x)$ as $h \downarrow 0$.
4. [5 points] F is not **always** left continuous i.e. it is not always true that $F(x+h) \not\to F(x)$ as $h \uparrow 0$.
5. [5 points] $P(X = x) = F(x) - \lim_{y \uparrow x} F(y)$.

2 Definitions

Definition 1. Given a function $f : \mathbb{R} \mapsto \mathbb{R}$ and $c \in \mathbb{R}$, we say that $\lim_{x \to c} f(x) = L$ if for all $\varepsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - L| < \varepsilon$ whenever $0 < |x - c| < \delta$.

Remark. Note that the restriction $|x - c| > 0$ means that the value of the function at c i.e. $f(c)$ has no role in detemining the convergence of a function at c. See `http://en.wikipedia.org/wiki/Limit_of_a_function` and `http://en.wikibooks.org/wiki/Real_Analysis/Limits` for a more detailed explanation and examples.

Definition 2. Given a function $f : \mathbb{R} \mapsto \mathbb{R}$ and $c \in \mathbb{R}$, we say that $\lim_{x \to c} f(x) = L$ if for all sequences $\{x_n : n \in \mathbb{N}\}$ with $x_n \neq c$ and $\lim_{n \to \infty} x_n = c$ we have $\lim_{n \to \infty} f(x_n) = L$.

Remarks. The following remarks apply to Definition 2.

- Definition 2 says that convergence of $\{f(x_n)\}$ for all sequences $\{x_n\}$ with $x_n \neq c$ is equivalent to convergence of $f(x)$ as defined in Definition 1.
- The $x_n \neq c$ restriction is to enforce the condition $|x_n - c| > 0$.
- When we write $\lim_{n \to \infty} x_n = c$, we mean that for all $\varepsilon > 0$ there exists an $n_0 \in \mathbb{N}$ such that $|x_n - c| < \varepsilon$ for all $n \geq n_0$.
- When we write $\lim_{n \to \infty} f(x_n) = L$, we mean that for all $\varepsilon > 0$ there exists an $n_0 \in \mathbb{N}$ such that $|f(x_n) - L| < \varepsilon$ for all $n \geq n_0$.

Definition 3. Given a function $f : \mathbb{R} \mapsto \mathbb{R}$, we say that $\lim_{x \to \infty} f(x) = L$, if for all $\varepsilon > 0$ there exists an $M \in \mathbb{R}$ such that $|f(x) - L| < \varepsilon$ whenever $x > M$.

Definition 4. Given a function $f : \mathbb{R} \mapsto \mathbb{R}$, we say that $\lim_{x \to -\infty} f(x) = L$, if for all $\varepsilon > 0$ there exists an $M \in \mathbb{R}$ such that $|f(x) - L| < \varepsilon$ whenever $x < M$.

Remarks. The following remarks apply to Definitions 3 and 4.

- We cannot use Definition 1 with $c = \pm\infty$ to get Definitions 3 and 4 because c is required to belong to $\mathbb{R}$.

- The following points are analogous to Definition 2.
  - We say $\lim_{x \to \infty} f(x) = L$ if for all real sequences $\{x_n : n \in \mathbb{N}\}$ with $\lim_{n \to \infty} x_n = \infty$ we have $\lim_{n \to \infty} f(x_n) = L$.
  - We say $\lim_{x \to -\infty} f(x) = L$ if for all real sequences $\{x_n : n \in \mathbb{N}\}$ with $\lim_{n \to \infty} x_n = -\infty$ we have $\lim_{n \to \infty} f(x_n) = L$.
- We say $\lim_{n \to \infty} x_n = \infty$ if for every $M \in \mathbb{R}$ there exists an $n_0 \in \mathbb{N}$ such that $x_n \geq M$ for all $n \geq n_0$.
- We say $\lim_{n \to \infty} x_n = -\infty$ if for every $M \in \mathbb{R}$ there exists an $n_0 \in \mathbb{N}$ such that $x_n \leq M$ for all $n \geq n_0$.

Definition 5. Given a function $f : \mathbb{R} \mapsto \mathbb{R}$ and $c \in \mathbb{R}$, we say that $\lim_{x \downarrow c} f(x) = L$ if for all $\varepsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - L| < \varepsilon$ whenever $c < x < c + \delta$.

Definition 6. Given a function $f : \mathbb{R} \mapsto \mathbb{R}$ and $c \in \mathbb{R}$, we say that $\lim_{x \uparrow c} f(x) = L$ if for all $\varepsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - L| < \varepsilon$ whenever $c - \delta < x < c$.

Remarks. The following remarks apply to Definitions 5 and 6.

- See `http://en.wikipedia.org/wiki/One-sided_limit` for more details and an example.
- The "for all sequences" interpretation as in Definition 2 holds for Definitions 5 and 6 as well.