EE 325: Probability and Random Processes (Spring 2015)
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Midsemester Exam : 30 points — February 20, 2015

1. (3 points) Let $F : \mathbb{R} \mapsto [0,1]$ be the distribution function of a random variable X. Prove that $\lim_{x\to\infty} F(x) = 1$.

2. (3 points) Let $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \dots$ be a sequence of collections of subsets of Ω such that $\mathcal{F}_n \subseteq \mathcal{F}_{n+1}$ for each n.
   (a) Suppose each $\mathcal{F}_i$ is a field. Prove that $\bigcup_{i=1}^{\infty} \mathcal{F}_i$ is also a field.
   (b) Suppose each $\mathcal{F}_i$ is a σ-field. Using a counterexample, show that $\bigcup_{i=1}^{\infty} \mathcal{F}_i$ might not be a σ-field. *Hint: Let $\Omega = \mathbb{N}$ and try to construct an increasing sequence of σ-fields.*

3. (3 points) Let X and Y have joint probability density function $f(x,y) = 2e^{-x-y}$, $0 < x < y < \infty$. Find the expected values of X and Y.

4. (3 points) An urn contains n tickets numbered 1 to n. Two tickets are drawn without replacement. Let X denote the smaller and Y the larger of the two numbers so obtained.
   (a) Find the joint probability mass function of X and Y.
   (b) Find the marginal probability mass functions of X and Y.

5. (3 points) A says B told him that C had lied. If each of these persons tells the truth with probability p independently of the others, what is the probability that C did lie? Assume that B knows if C is lying or not but decides to tell a lie himself with probability $1-p$. On the other hand, A will repeat what B said with probability p or lie about what B said with probability $1-p$. *Hint: What is the sample space of outcomes? Try writing the evidence and the event that C lied using these outcomes.*

6. (3 points) A coin having probability p of showing heads is repeatedly tossed. Let $P_j(n)$ denote the probability that a run of j consecutive heads occurs within the first n tosses. Find the function $g(n,j,p)$ such that $P_j(n) = P_j(n-1) + g(n,j,p)$. *Hint: Express the event of j consecutive heads within the first n tosses as a disjoint union of two events. The probability of one of these events should be $P_j(n-1)$.*

7. (3 points) Let X, Y, Z be independent random variables which are uniformly distributed on $[0,1]$. Find the probability density function of $X+Y+Z$.

8. (3 points) A gambler plays the following game at a casino. The dealer tosses a biased coin. If the coin shows heads, the dealer gives the gambler one rupee. If the coin shows tails, the dealer takes one rupee from the gambler. The coin shows heads with probability p and tails with probability $1-p$. Let S_0 be the amount of money which the gambler has before he starts playing and let S_n be the amount of money the gambler has after n tosses. You can assume that the tosses are independent. Prove the following.
   (a) $P(S_n = j | S_0 = a) = P(S_n = j + b | S_0 = a + b)$ for all $a, b, j \in \mathbb{Z}$ and $n \in \mathbb{N}$.
   (b) $P(S_n = j | S_0 = a) = P(S_{m+n} = j | S_m = a)$ for all $a, j \in \mathbb{Z}$ and $m, n \in \mathbb{N}$.

9. (3 points) For discrete random variables X and Y, prove that $E\left[E[X|Y]\right] = E[X]$.

10. (3 points) Let X be a continuous random variable with probability density function given by
$$f_X(x) = \lambda e^{-\lambda x}, \qquad 0 < x < \infty$$
where $\lambda > 0$. Find $E[X|X>1]$. *Hint: You need the conditional probability density function of X given $X > 1$. What is the conditional distribution function of X given $X > 1$?*