Cryptographic Hash Functions

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Cryptographic Hash Functions

- Important building block in cryptography
- Provide data integrity by construction of a short fingerprint or *message digest*
- Map arbitrary length inputs to fixed length outputs
  - For example, output length can be 256 bits
- Applications
  - Password hashing
  - Digital signatures on arbitrary length data
  - Commitment schemes

Properties

- Let $H : \mathcal{X} \mapsto \mathcal{Y}$ denote a cryptographic hash function
- $\mathcal{X}$ and $\mathcal{Y}$ are subsets of $\{0, 1\}^*$
- $H(x)$ can be computed efficiently for all $x \in \mathcal{X}$
- If H is considered secure, three problems are difficult to solve
  - Preimage
    - Given $y \in \mathcal{Y}$, find $x \in \mathcal{X}$ such that $H(x) = y$
  - Second Preimage
    - Given $x \in \mathcal{X}$, find $x' \in \mathcal{X}$ such that $x' \neq x$ and $H(x) = H(x')$
  - Collision
    - Find $x, x' \in \mathcal{X}$ such that $x' \neq x$ and $H(x) = H(x')$
- If $|\mathcal{X}| \geq 2|\mathcal{Y}|$, then we have

  Collision resistance $\implies$ Second preimage resistance $\implies$ Preimage resistance

  (Proof in Section 4.2, Stinson, 3rd edition)

SHA-256

- SHA = Secure Hash Algorithm, 256-bit output length
- Accepts bit strings of length upto $2^{64} - 1$
- Announced in 2001 by NIST, US Department of Commerce
- Output calculation has two stages
  - Preprocessing
  - Hash Computation
- Preprocessing
  1. The input M is padded to a length which is a multiple of 512
  2. A 256-bit state variable $H^{(0)}$ is set to

$$
\begin{aligned}
H_0^{(0)} &= \texttt{0x6A09E667}, & H_1^{(0)} &= \texttt{0xBB67AE85}, \\
H_2^{(0)} &= \texttt{0x3C6EF372}, & H_3^{(0)} &= \texttt{0xA54FF53A}, \\
H_4^{(0)} &= \texttt{0x510E527F}, & H_5^{(0)} &= \texttt{0x9B05688C}, \\
H_6^{(0)} &= \texttt{0x1F83D9AB}, & H_7^{(0)} &= \texttt{0x5BE0CD19}.
\end{aligned}
$$

SHA-256 Input Padding

- Let input M be l bits long
  - Find smallest non-negative k such that

$$k + l + 65 = 0 \bmod 512$$

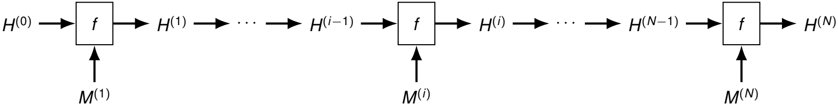
  - Append $k + 1$ bits consisting of single 1 and k zeros
  - Append 64-bit representation of l
- Example: $M = 101010$ with $l = 6$
  - $k = 441$
  - 64-bit representation of 6 is $000 \cdots 00110$
  - 512-bit padded message

$$\underbrace{101010}_{M}\ 1\ \underbrace{00000 \cdots 00000}_{441 \text{ zeros}}\ \underbrace{00 \cdots 00110}_{l}.$$

SHA-256 Hash Computation

1. Padded input is split into N 512-bit blocks $M^{(1)}, M^{(2)}, \ldots, M^{(N)}$
2. Given $H^{(i-1)}$, the next $H^{(i)}$ is calculated using a function f

$$H^{(i)} = f(M^{(i)}, H^{(i-1)}), \quad 1 \leq i \leq N.$$



3. f is called a *compression function*
4. $H^{(N)}$ is the output of SHA-256 for input M

SHA-256 Compression Function Building Blocks

- U, V, W are 32-bit words
- $U \wedge V$, $U \vee V$, $U \oplus V$ denote bitwise AND, OR, XOR
- $U + V$ denotes integer sum modulo 2^{32}
- $\neg U$ denotes bitwise complement
- For $1 \leq n \leq 32$, the shift right and rotate right operations

$$\mathrm{SHR}^n(U) = \underbrace{000\cdots000}_{n\text{ zeros}}\, u_0 u_1 \cdots u_{30-n} u_{31-n},$$

$$\mathrm{ROTR}^n(U) = u_{31-n+1} u_{31-n+2} \cdots u_{30} u_{31} u_0 u_1 \cdots u_{30-n} u_{31-n},$$

- Bitwise choice and majority functions

$$\mathrm{Ch}(U, V, W) = (U \wedge V) \oplus (\neg U \wedge W),$$
$$\mathrm{Maj}(U, V, W) = (U \wedge V) \oplus (U \wedge W) \oplus (V \wedge W),$$

- Let

$$\Sigma_0(U) = \mathrm{ROTR}^2(U) \oplus \mathrm{ROTR}^{13}(U) \oplus \mathrm{ROTR}^{22}(U)$$
$$\Sigma_1(U) = \mathrm{ROTR}^6(U) \oplus \mathrm{ROTR}^{11}(U) \oplus \mathrm{ROTR}^{25}(U)$$
$$\sigma_0(U) = \mathrm{ROTR}^7(U) \oplus \mathrm{ROTR}^{18}(U) \oplus \mathrm{SHR}^3(U)$$
$$\sigma_1(U) = \mathrm{ROTR}^{17}(U) \oplus \mathrm{ROTR}^{19}(U) \oplus \mathrm{SHR}^{10}(U)$$

SHA-256 Compression Function Calculation

- Maintains internal state of 64 32-bit words $\{W_j \mid j = 0, 1, \ldots, 63\}$
- Also uses 64 constant 32-bit words $K_0, K_1, \ldots, K_{63}$ derived from the first 64 prime numbers $2, 3, 5, \ldots, 307, 311$
- $f(M^{(i)}, H^{(i-1)})$ proceeds as follows

  1. Internal state initialization

  $$W_j = \begin{cases} M_j^{(i)} & 0 \leq j \leq 15, \\ \sigma_1(W_{j-2}) + W_{j-7} + \sigma_0(W_{j-15}) + W_{j-16} & 16 \leq j \leq 63. \end{cases}$$

  2. Initialize eight 32-bit words

  $$(A, B, C, D, E, F, G, H) = \left(H_0^{(i-1)}, H_1^{(i-1)}, \ldots, H_6^{(i-1)}, H_7^{(i-1)}\right).$$

  3. For $j = 0, 1, \ldots, 63$, iteratively update $A, B, \ldots, H$

  $$\begin{aligned} T_1 &= H + \Sigma_1(E) + \mathrm{Ch}(E, F, G) + K_j + W_j \\ T_2 &= \Sigma_0(A) + \mathrm{Maj}(A, B, C) \\ (A, B, C, D, E, F, G, H) &= (T_1 + T_2, A, B, C, D + T_1, E, F, G) \end{aligned}$$

  4. Calculate $H^{(i)}$ from $H^{(i-1)}$

  $$(H_0^{(i)}, H_1^{(i)}, \ldots, H_7^{(i)}) = \left(A + H_0^{(i-1)}, B + H_1^{(i-1)}, \ldots, H + H_7^{(i-1)}\right).$$

The Merkle-Damgård Transform

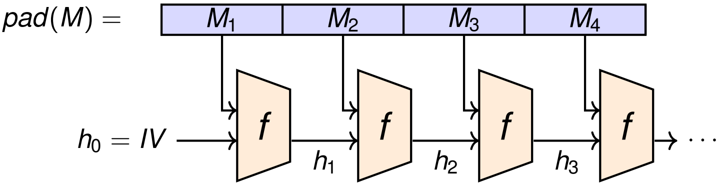


Figure source: `https://www.iacr.org/authors/tikz/`

- The SHA-256 construction is an example of the MD transform
- Typical hash function design
  - Construct collision-resistant compression function
  - Extend the domain using MDT to get collision-resistant hash function

Birthday Attacks for Finding Collisions

- Birthday Problem: Given Q people, what is the probability of two of them having the same birthday?
- Suppose the size of $\mathcal{Y}$ is M. For SHA-256, $M = 2^{256}$.
- If we calculate H for Q inputs, the probability of a collision is

$$1 - \left(1 - \frac{1}{M}\right)\left(1 - \frac{2}{M}\right)\cdots\left(1 - \frac{Q-1}{M}\right) \approx 1 - \exp\frac{-Q(Q-1)}{2M}$$

- For success probability ε, the number of "queries" is

$$Q \approx \sqrt{2M \ln \frac{1}{1-\varepsilon}}$$

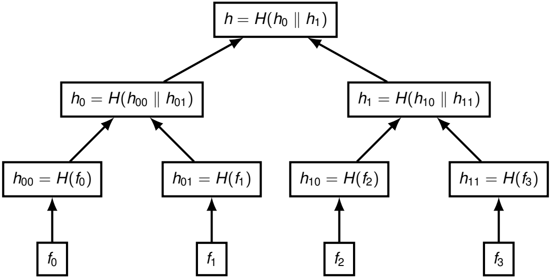
- For $\varepsilon = 0.5$, $Q \approx 1.17\sqrt{M}$
- For SHA-256, $Q \approx 2^{128}$

Applications

- Virus fingerprinting
- Data deduplication
- Digital signatures on arbitrary length data
- Password hashing
- Commitment schemes
  - A kind of digital envelope
  - Allows one party to “commit” to a message m by sending a commitment c to the counterparty
  - Set $c = H(m \| r)$ where r is a random n-bit string
  - **Hiding**: c reveals nothing about m
  - **Binding**: Infeasible for c to be opened to a different message m'

Merkle Trees

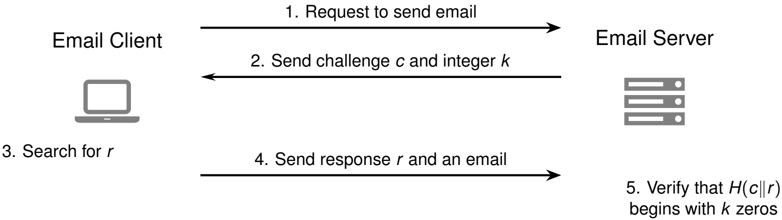
- Alternative to Merkle-Damgård transform for domain extension
- Suppose a client uploads multiple files to server
- Client wants to ensure file integrity at a later retrieval



- For N files, $\mathcal{O}(\log N)$ communication from server ensures integrity
- The communication is called a *Merkle proof*

Hashcash

- Hashcash was proposed in 1997 to prevent spam
- Protocol
  - Suppose an email client wants to send email to an email server
  - Client and server agree upon a cryptographic hash function H
  - Email server sends the client a challenge string c
  - Client needs to find a string r such that $H(c\|r)$ begins with k zeros



- The r is considered **proof-of-work (PoW)**; difficult to generate but easy to verify
- Demo

Difficulty Increases with k

- Let hash function output length n be 4 bits

Decimal	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Binary	0000	0001	0010	0011	0100	0101	0110	0111	1000	1001	1010	1011	1100	1101	1110	1111

$\underbrace{0000\ \ 0001}_{k=3}$

$\underbrace{0000\ \ 0001\ \ 0010\ \ 0011}_{k=2}$

$\underbrace{0000\ \ 0001\ \ 0010\ \ 0011\ \ 0100\ \ 0101\ \ 0110\ \ 0111}_{k=1}$

- Since H has pseudorandom outputs, probability of success in a single trial is

$$\frac{2^{n-k}}{2^n} = \frac{1}{2^k}$$

References

- Chapter 5 of *Introduction to Modern Cryptography*, J. Katz, Y. Lindell, 2nd edition
- Chapter 4 of *Cryptography: Theory and Practice*, Douglas R. Stinson, 3rd edition
- Chapter 8 of *A Graduate Course in Applied Cryptography*, D. Boneh, V. Shoup, `www.cryptobook.us`
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