Projective Geometry

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The Projective Plane

First Definition

- Let a, b, c, a', b', c' be real numbers
- Consider the equivalence relation $\sim$ given by $[a, b, c] \sim [a', b', c']$ if there exists a $t \in \mathbb{R} \setminus \{0\}$ such that

$$a = ta', b = tb', c = tc'.$$

- The **projective plane** $\mathbb{P}^2$ is defined as the set of equivalence classes $[a, b, c]$, excluding the triple $[0, 0, 0]$
- The points in $\mathbb{P}^2$ can be interpreted as vectors emanating from the origin in $\mathbb{R}^3$
- The numbers a, b, c are called the **homogeneous coordinates** of the point $[a, b, c]$
- A **line** in $\mathbb{P}^2$ is defined as the set of points $[a, b, c] \in \mathbb{P}^2$ whose coordinates satisfy an equation of the form

$$\alpha X + \beta Y + \gamma Z = 0$$

  for some real constants α, β, γ not all zero
  - Any representative of $[a, b, c]$ can be used to check if it lies on a line

Second Definition

- Consider the following geometric facts in the Euclidean plane
  - Two distinct points determine a unique line
  - Two distinct lines determine a unique point, unless they are parallel
- Can we add points to the Euclidean plane that can become the intersection points of parallel lines?
  - Like $\sqrt{-1}$ was added to $\mathbb{R}$ to obtain all the roots of polynomials
- How many points do we need to add?
  - Can we add only one point and designate it as the intersection point of all parallel lines?
  - No, we need one extra point for each direction
  - A direction is a collection of all lines parallel to a given line
- Let the **affine plane** $\mathbb{A}^2$ denote the usual Euclidean plane $\mathbb{R}^2$
- The second definition of the **projective plane** is given by

$$\mathbb{P}^2 = \mathbb{A}^2 \cup \left\{\text{the set of directions in } \mathbb{A}^2\right\}$$

- The points in $\mathbb{P}^2$ not in $\mathbb{A}^2$ are called the **points at infinity**
- The set of all points at infinity is itself considered to be a line, denoted by L_∞
  - The two geometric facts above hold in $\mathbb{P}^2$ without qualification

Refining the Second Definition

- We want to show that the two definitions of the projective plane are equivalent
- We need a more precise definition of the set of directions
- We can use the lines passing through the origin in $\mathbb{A}^2$ to specify the directions, i.e. the lines

$$Ay = Bx$$

  where both A and B are **not both zero**
- Every line in $\mathbb{A}^2$ is parallel to a unique line through the origin
- Two pairs (A, B) and (A', B') give the same line if and only if there is a $t \in \mathbb{R} \setminus \{0\}$ such that $A = tA'$ and $B = tB'$
- The set of directions in $\mathbb{A}^2$ is thus specified by the points of the projective line $\mathbb{P}^1$
- The second definition of the **projective plane** is then given by

$$\mathbb{P}^2 = \mathbb{A}^2 \cup \mathbb{P}^1$$

Equivalence of the Two Definitions of $\mathbb{P}^2$

- **Algebraic Definition**: Set of equivalence classes $[a, b, c]$, excluding the triple $[0, 0, 0]$
- **Geometric Definition**: $\mathbb{A}^2 \cup \mathbb{P}^1$
- Mapping equivalence classes to points in $\mathbb{A}^2 \cup \mathbb{P}^1$

$$[a, b, c] \to \begin{cases} \left(\frac{a}{c}, \frac{b}{c}\right) \in \mathbb{A}^2 & \text{if } c \neq 0, \\ [a, b] \in \mathbb{P}^1 & \text{if } c = 0. \end{cases}$$

- Mapping points in $\mathbb{A}^2 \cup \mathbb{P}^1$ to equivalence classes

$$(x, y) \in \mathbb{A}^2 \to [x, y, 1]$$
$$[A, B] \in \mathbb{P}^1 \to [A, B, 0]$$

Lines in the Projective Plane

- A line L in $\mathbb{P}^2$ is the set of points $[a, b, c] \in \mathbb{P}^2$ that satisfy

$$\alpha X + \beta Y + \gamma Z = 0$$

  for some real constants α, β, γ not all zero
- Suppose that $\alpha \neq 0$ and $\beta \neq 0$
  - Any point $[a, b, c] \in L$ with $c \neq 0$ is sent to the point $\left(\frac{a}{c}, \frac{b}{c}\right)$ on the line

$$\alpha X + \beta Y + \gamma = 0 \text{ in } \mathbb{A}^2$$

  - The point $[-\beta, \alpha, 0] \in L$ is sent to the point $[-\beta, \alpha] \in \mathbb{P}^1$, which corresponds to the direction of the line $-\beta y = \alpha x$
- Suppose both $\alpha = 0$ and $\beta = 0$
  - The line L has the equation $Z = 0$
  - Every point on L is sent to a direction in $\mathbb{P}^1$
  - Thus, the line $Z = 0$ corresponds to the line at infinity L_∞

Curves in the Projective Plane

- An **algebraic curve** in the affine plane $\mathbb{A}^2$ is defined to be the set of solutions to a polynomial equation in two variables

$$f(x, y) = 0$$

- A polynomial $F(X, Y, Z)$ is called a **homogeneous polynomial of degree** d if it satisfies the identity

$$F(tX, tY, tZ) = t^d F(X, Y, Z)$$

- A **projective curve** in $\mathbb{P}^2$ is the set of solutions of

$$F(X, Y, Z) = 0$$

  where F is a non-constant homogeneous polynomial
  - Examples: $X^2 + Y^2 - Z^2 = 0$, $Y^2Z - X^3 - XZ^2 = 0$
- The **affine part** of a projective curve is given by the solutions of

$$f(x, y) = F(x, y, 1)$$

Example of a Projective Curve

- Consider the projective curve C given by

$$X^2 - Y^2 - Z^2 = 0$$

- There are two points on C with $Z = 0$ namely $[1, 1, 0]$ and $[1, -1, 0]$
  - These correspond to the points at infinity at $[1, 1], [1, -1] \in \mathbb{P}^1$
  - These points correspond to directions given by $y = x$ and $y = -x$
- The affine part of C is given by the hyperbola

$$x^2 - y^2 - 1 = 0$$

- The lines $y = \pm x$ correspond to the asymptotes of this hyperbola

Homogenization of Affine Polynomials

- Given a degree d polynomial $f(x, y) = \sum_{i,j} a_{ij} x^i y^j$, its homogenization is given by

$$F(X, Y, Z) = Z^d f\left(\frac{X}{Z}, \frac{Y}{Z}\right) = \sum_{i,j} a_{i,j} X^i Y^j Z^{d-i-j}$$

- Examples
  - $y - ax - b$ is homogenized to $Y - aX - Z$
  - $x^2 + y^2 - 1$ is homogenized to $X^2 + Y^2 - Z^2$
  - $y^2 + x^3 - 1$ is homogenized to $Y^2 Z + X^3 - Z^3$
- Application
  - Suppose we want to find the intersection of $y = ax + b$ and $y = ax + b'$ where $b \neq b'$
  - No intersection points in the affine plane
  - Homogenization gives us lines $Y - aX + bZ = 0$ and $Y - aX + b'Z = 0$
  - These lines intersect at the point $[1, a, 0]$

Intersections of Projective Curves

- **Claim:** A line and a degree-2 curve (a conic) intersect in two points in the projective plane
  - The line and conic should not have common components
  - We have to allow complex coordinates and count multiplicities of intersection
- Examples
  - $x + y + 1 = 0$ and $x^2 + y^2 = 1$
  - $x + y + 2 = 0$ and $x^2 + y^2 = 1$
  - $x + 1 = 0$ and $x^2 - y = 0$
  - $x + y = 2$ and $x^2 + y^2 = 2$
- **Bezout's Theorem:** Let C_1 and C_2 be projective curves with no common components. Then they intersect in $(\deg C_1)(\deg C_2)$ points.

References

- Appendix A of *Rational Points on Elliptic Curves*, Joseph H. Silverman, John T. Tate, 2nd Edition, 2015