Repetition Code

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3-Repetition Code

- Each message bit is repeated 3 times

$$101001 \longrightarrow \boxed{\text{3-Repetition Encoder}} \longrightarrow 111\ 000\ 111\ 000\ 000\ 111$$

- How many errors can it correct?
- How many errors can the following code correct?

$$0 \rightarrow 101, 1 \rightarrow 010$$

- What about this code?

$$0 \rightarrow 101, 1 \rightarrow 110$$

- Error correcting capability depends on the distance between the codewords

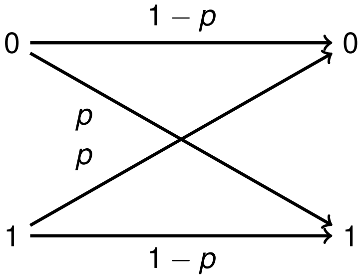
5-Repetition Code

- Each message bit is repeated 5 times
- How many errors can it correct?
- Is it better than the 3-repetition code?
- A code has rate $\frac{k}{n}$ if it maps k-bit messages to n-bit codewords
- There is a tradeoff between rate and error correcting capability

Decoder

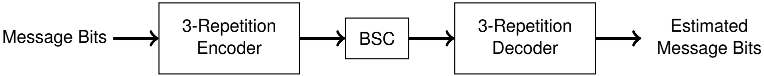
- Majority decoder was used for decoding repetition codes
- How do we know the majority decoder is the best?
- Consider a channel which flips all input bits. Does the majority decoder work?
- Consider a channel which causes burst errors. What is the best decoder?
- The optimal decoder depends on the channel

Binary Symmetric Channel



- p is called the crossover probability
- Abstraction of a modulator-channel-demodulator sequence
- Any error pattern is possible
- It is impossible to correct all errors

Optimal Decoder for 3-Repetition Code over BSC



- Let X be the transmitted bit and $\hat{X}$ be the decoded bit
- What is a decoder?
- Let Γ_0 and Γ_1 be a partition of $\Gamma = \{0, 1\}^3$
- If $\mathbf{Y}$ is the received 3-tuple then

$$\hat{X} = \begin{cases} 0 & \text{if } \mathbf{Y} \in \Gamma_0 \\ 1 & \text{if } \mathbf{Y} \in \Gamma_1 \end{cases}$$

- How can we compare decoders?
- Probability of correct decision = $\Pr\left(\hat{X} = X\right)$

Maximizing Probability of Correct Decision

Let $\pi_0 = \Pr(X = 0)$ and $\pi_1 = \Pr(X = 1)$

$$
\begin{aligned}
& \Pr\left(\hat{X} = X\right) \\
&= \pi_0 \Pr\left(\mathbf{Y} \in \Gamma_0 \middle| X = 0\right) + \pi_1 \Pr\left(\mathbf{Y} \in \Gamma_1 \middle| X = 1\right) \\
&= \pi_0 \left[1 - \Pr\left(\mathbf{Y} \in \Gamma_1 \middle| X = 0\right)\right] + \pi_1 \Pr\left(\mathbf{Y} \in \Gamma_1 \middle| X = 1\right) \\
&= \pi_0 + \sum_{\mathbf{y} \in \Gamma_1} \left[\pi_1 \Pr(\mathbf{Y} = \mathbf{y} | X = 1) - \pi_0 \Pr(\mathbf{Y} = \mathbf{y} | X = 0)\right]
\end{aligned}
$$

Maximizing as a function of Γ_1 gives us the following partitions

$$
\begin{aligned}
\Gamma_0 &= \left\{\mathbf{y} \in \Gamma \middle| \pi_1 \Pr(\mathbf{Y} = \mathbf{y} | X = 1) < \pi_0 \Pr(\mathbf{Y} = \mathbf{y} | X = 0)\right\} \\
\Gamma_1 &= \left\{\mathbf{y} \in \Gamma \middle| \pi_1 \Pr(\mathbf{Y} = \mathbf{y} | X = 1) \geq \pi_0 \Pr(\mathbf{Y} = \mathbf{y} | X = 0)\right\}
\end{aligned}
$$

Optimal Decoder for Equally Likely Inputs

- Suppose $\pi_0 = \pi_1 = \frac{1}{2}$
- Let $d(\mathbf{y}, \mathbf{x})$ be the Hamming distance between $\mathbf{y}$ and $\mathbf{x}$

$$
\begin{aligned}
\Pr(\mathbf{Y} = \mathbf{y} | X = 1) &= p^{d(\mathbf{y},111)}(1-p)^{3-d(\mathbf{y},111)} \\
\Pr(\mathbf{Y} = \mathbf{y} | X = 0) &= p^{d(\mathbf{y},000)}(1-p)^{3-d(\mathbf{y},000)}
\end{aligned}
$$

- If $p < \frac{1}{2}$, then

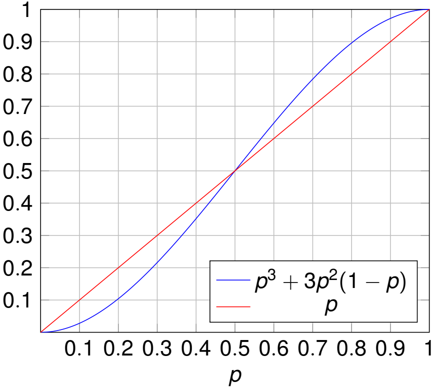
$$
\begin{aligned}
\Gamma_0 &= \left\{ \mathbf{y} \in \Gamma \middle| d(\mathbf{y}, 000) < d(\mathbf{y}, 111) \right\} = \{000, 100, 010, 001\} \\
\Gamma_1 &= \left\{ \mathbf{y} \in \Gamma \middle| d(\mathbf{y}, 000) \geq d(\mathbf{y}, 111) \right\} = \{111, 011, 101, 110\}
\end{aligned}
$$

- The majority decoder is optimal for a BSC if $p < \frac{1}{2}$ and inputs are equally likely

Error Analysis for 3-Repetition Code

$\Gamma_0 = \{000, 100, 010, 001\}, \Gamma_1 = \{111, 011, 101, 110\}$

$$\begin{aligned}\Pr\left(\hat{X} \neq X\right) &= \pi_0 \Pr\left(\mathbf{Y} \in \Gamma_1 \middle| X = 0\right) + \pi_1 \Pr\left(\mathbf{Y} \in \Gamma_0 \middle| X = 1\right) \\ &= p^3 + 3p^2(1-p)\end{aligned}$$



Simulation of 3-Repetition Code Performance

- Simulations are useful to verify analysis or when analysis is intractable
- Simulation procedure for 3-repetition code
    1. Generate a message bit X
    2. Encode bit to get codeword
    3. Generate errors in the codeword
    4. Decode corrupted codeword to get $\hat{X}$
    5. Increment number of decision errors E if $\hat{X} \neq X$
    6. Repeat steps 1 to 5 N times
    7. Simulated value of $\Pr\left(\hat{X} \neq X\right)$ is $\frac{E}{N}$
- How to do steps 1 and 3?
- How to choose N in step 6?

Error Analysis and Simulations for n-Repetition Code

Assignment 1

Questions?