EE 703: Digital Message Transmission

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End-semester Exam : **42 points** (180 min) November 26, 2012

Each question is worth 6 points.

1. Suppose the input X and output Y to a channel are related by $Y = \rho X + N$ where N is a zero-mean Gaussian random variable with variance σ^2 and ρ is a random variable independent of the noise. Assume that X is equally likely to be $\pm A$.
   (a) If ρ is the constant 1, what is the optimal decision rule and the resulting error probability?
   (b) If ρ takes values ± 1 with equal probability, what is the optimal decision rule and the resulting error probability?
   (c) If ρ takes values 0 and 1 with equal probability, what is the optimal decision rule and the resulting error probability?

2. Derive the following for a QPSK constellation used over a passband AWGN channel with power spectral density $\frac{N_0}{2}$ when the ML receiver is used.
   (a) The exact symbol error probability as a function of E_b and N_0
   (b) The union bound on the symbol error probability
   (c) The intelligent union bound on the symbol error probability
   (d) The nearest neighbor approximation of the symbol error probability

3. A 16-QAM constellation is used for transmission over a passband AWGN channel with power spectral density A shown in Figure 1.

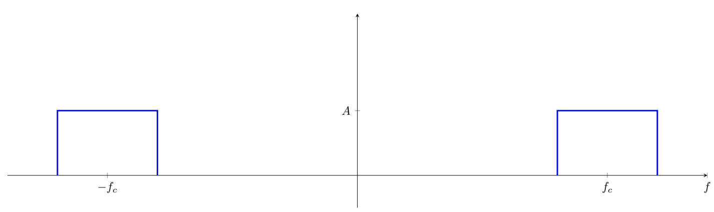

Figure 1: Passband AWGN channel power spectral density

   Assuming all the symbols are equally likely to be transmitted, calculate the following in terms of E_b and A where E_b is the average energy per bit.
   (a) The exact symbol error probability
   (b) The intelligent union bound on the symbol error probability
   (c) The nearest neighbor approximation of the symbol error probability

4. Let $s_i(t)$, $i = 1, \ldots, M$ be known **real** signals having finite energy and $n(t)$ be a **real** white Gaussian noise process. Prove that for the M-ary hypothesis testing problem given by

$$\begin{aligned} H_1 &: \quad y(t) = s_1(t) + n(t) \\ &\vdots \qquad\qquad \vdots \\ H_M &: \quad y(t) = s_M(t) + n(t) \end{aligned}$$

there is no loss in detection performance by using the optimal decision rule for the following M-ary hypothesis testing problem

$$\begin{aligned} H_1 &: \quad \mathbf{Y} = \mathbf{s}_1 + \mathbf{N} \\ &\vdots \qquad\quad \vdots \\ H_M &: \quad \mathbf{Y} = \mathbf{s}_M + \mathbf{N} \end{aligned}$$

where $\mathbf{Y}$, $\mathbf{s}_i$ and $\mathbf{N}$ are the projections of $y(t)$, $s_i(t)$ and $n(t)$ respectively onto the signal space spanned by $\{s_i(t)\}$. You should show how to obtain $\mathbf{Y}$ and $\mathbf{s_i}$ from $y(t)$ and $s_i(t)$'s. You should also specify the distribution of $\mathbf{N}$. You can use the results related to passing WGN through correlators without proof.

5. Derive the spectral efficiencies of BPSK, QPSK and 16-QAM. In each case, assume the complex envelope of the transmitted signal is given by

$$s(t) = \sum_{n=-\infty}^{\infty} b_n p(t - nT)$$

where $p(t) = I_{[0,T]}(t)$ and b_n's are the transmitted symbols which can be real or complex depending on the situation. Recall that the PSD of the complex envelope is given by

$$S(f) = S_b\left(e^{\,j2\pi fT}\right)\frac{|P(f)|^2}{T}$$

where $S_b(z) = \sum_{k=-\infty}^{\infty} R_b[k]z^{-k}$. Here $R_b[k]$ is the autocorrelation function of the symbol sequence.

6. Suppose we observe a sequence of real values $Y_1, Y_2, \ldots, Y_n$ given by

$$Y_k = \theta s_k + N_k, \quad k = 1, 2, \ldots, n$$

where $\mathbf{N} = \begin{bmatrix} N_1 & N_2 & \cdots & N_n \end{bmatrix}^T$ is a zero-mean Gaussian vector with known covariance matrix $\mathbf{\Sigma}$ which is a positive definite matrix. The sequence $s_1, \ldots, s_n$ is a known signal sequence and θ is an unknown parameter.

   (a) Find the ML estimate $\hat{\theta}_{ML}(\mathbf{Y})$ of the parameter θ.

   (b) Find the mean and variance of $\hat{\theta}_{ML}(\mathbf{Y})$.

Recall that the pdf of a real $n \times 1$ Gaussian vector $\mathbf{x}$ with mean vector $\mathbf{m}$ and covariance matrix $\mathbf{C}$ is given by

$$p(\mathbf{x}) = \frac{1}{\sqrt{(2\pi)^n \det(\mathbf{C})}} \exp\left(-\frac{1}{2}(\mathbf{x}-\mathbf{m})^T\mathbf{C}^{-1}(\mathbf{x}-\mathbf{m})\right)$$

7. Consider the complex baseband received signal

$$y(t) = As(t-\tau)e^{\,j\theta} + n(t)$$

where $n(t)$ is a complex WGN process with known PSD σ^2. The parameters A, τ and θ are unknown while $s(t)$ is known. Derive the ML estimates of A, τ and θ.