Gaussian Random Variables and Processes

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Gaussian Random Variables

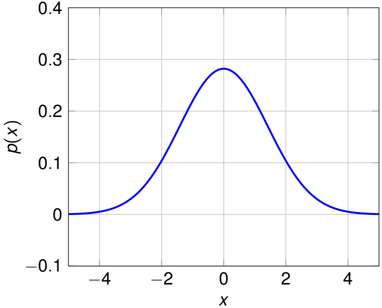
Gaussian Random Variable

Definition

A continuous random variable with pdf of the form

$$p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right), \quad -\infty < x < \infty,$$

where μ is the mean and σ^2 is the variance.



Notation

- $N(\mu, \sigma^2)$ denotes a Gaussian distribution with mean μ and variance σ^2
- $X \sim N(\mu, \sigma^2) \Rightarrow X$ is a Gaussian RV with mean μ and variance σ^2
- $X \sim N(0, 1)$ is termed a standard Gaussian RV

Affine Transformations Preserve Gaussianity

Theorem

If X is Gaussian, then $aX + b$ is Gaussian for $a, b \in \mathbb{R}$.

Remarks

- If $X \sim N(\mu, \sigma^2)$, then $aX + b \sim N(a\mu + b, a^2\sigma^2)$.
- If $X \sim N(\mu, \sigma^2)$, then $\frac{X-\mu}{\sigma} \sim N(0, 1)$.

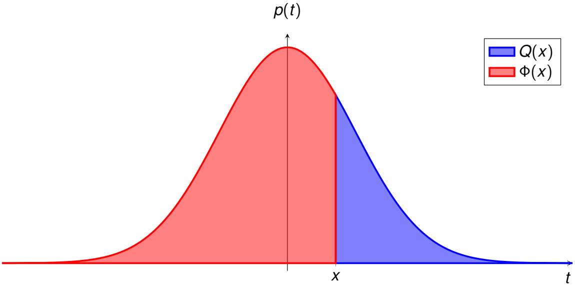
CDF and CCDF of Standard Gaussian

- Cumulative distribution function

$$\Phi(x) = P\left[N(0,1) \leq x\right] = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-t^2}{2}\right) dt$$

- Complementary cumulative distribution function

$$Q(x) = P\left[N(0,1) > x\right] = \int_{x}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-t^2}{2}\right) dt$$



Properties of $Q(x)$

- $\Phi(x) + Q(x) = 1$
- $Q(-x) = \Phi(x) = 1 - Q(x)$
- $Q(0) = \frac{1}{2}$
- $Q(\infty) = 0$
- $Q(-\infty) = 1$
- $X \sim N(\mu, \sigma^2)$

$$P[X > \alpha] = Q\left(\frac{\alpha - \mu}{\sigma}\right)$$

$$P[X < \alpha] = Q\left(\frac{\mu - \alpha}{\sigma}\right)$$

Jointly Gaussian Random Variables

Definition (Jointly Gaussian RVs)

Random variables $X_1, X_2, \ldots, X_n$ are jointly Gaussian if any non-trivial linear combination is a Gaussian random variable.

$$a_1 X_1 + \cdots + a_n X_n \text{ is Gaussian for all } (a_1, \ldots, a_n) \in \mathbb{R}^n \setminus \mathbf{0}$$

Example (Not Jointly Gaussian)

$X \sim N(0, 1)$

$$Y = \begin{cases} X, & \text{if } |X| > 1 \\ -X, & \text{if } |X| \leq 1 \end{cases}$$

$Y \sim N(0, 1)$ and $X + Y$ is not Gaussian.

Gaussian Random Vector

Definition (Gaussian Random Vector)

A random vector $\mathbf{X} = (X_1, \ldots, X_n)^T$ whose components are jointly Gaussian.

Notation

$\mathbf{X} \sim N(\mathbf{m}, \mathbf{C})$ where

$$\mathbf{m} = E[\mathbf{X}], \;\; \mathbf{C} = E\left[(\mathbf{X} - \mathbf{m})(\mathbf{X} - \mathbf{m})^T\right]$$

Definition (Joint Gaussian Density)

If $\mathbf{C}$ is invertible, the joint density is given by

$$p(\mathbf{x}) = \frac{1}{\sqrt{(2\pi)^m \det(\mathbf{C})}} \exp\left(-\frac{1}{2}(\mathbf{x} - \mathbf{m})^T \mathbf{C}^{-1}(\mathbf{x} - \mathbf{m})\right)$$

Uncorrelated Random Variables

Definition

X_1 and X_2 are uncorrelated if $\mathrm{cov}(X_1, X_2) = 0$

Remarks

For uncorrelated random variables $X_1, \ldots, X_n$,

$$\mathrm{var}(X_1 + \cdots + X_n) = \mathrm{var}(X_1) + \cdots + \mathrm{var}(X_n).$$

If X_1 and X_2 are independent,

$$\mathrm{cov}(X_1, X_2) = 0.$$

Correlation coefficient is defined as

$$\rho(X_1, X_2) = \frac{\mathrm{cov}(X_1, X_2)}{\sqrt{\mathrm{var}(X_1)\,\mathrm{var}(X_2)}}.$$

Uncorrelated Jointly Gaussian RVs are Independent

If $X_1, \ldots, X_n$ are jointly Gaussian and pairwise uncorrelated, then they are independent.

$$\begin{aligned} p(\mathbf{x}) &= \frac{1}{\sqrt{(2\pi)^m \det(\mathbf{C})}} \exp\left(-\frac{1}{2}(\mathbf{x}-\mathbf{m})^T \mathbf{C}^{-1} (\mathbf{x}-\mathbf{m})\right) \\ &= \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma_i^2}} \exp\left(-\frac{(x_i - m_i)^2}{2\sigma_i^2}\right) \end{aligned}$$

where $m_i = E[X_i]$ and $\sigma_i^2 = \text{var}(X_i)$.

Uncorrelated Gaussian RVs may not be Independent

Example

- $X \sim N(0, 1)$
- W is equally likely to be +1 or -1
- W is independent of X
- $Y = WX$
- $Y \sim N(0, 1)$
- X and Y are uncorrelated
- X and Y are not independent

Complex Gaussian Random Vectors

Complex Gaussian Random Variable

Definition (Complex Random Variable)

A complex random variable $Z = X + jY$ is a pair of real random variables X and Y.

Remarks

- The pdf of a complex RV is the joint pdf of its real and imaginary parts.
- $E[Z] = E[X] + jE[Y]$
- $\text{var}[Z] = E[|Z|^2] - |E[Z]|^2 = \text{var}[X] + \text{var}[Y]$

Definition (Complex Gaussian RV)

If X and Y are jointly Gaussian, $Z = X + jY$ is a complex Gaussian RV.

Complex Random Vectors

Definition (Complex Random Vector)

A complex random vector is defined as $\mathbf{Z} = \mathbf{X} + j\mathbf{Y}$ where $\mathbf{X}$ and $\mathbf{Y}$ are real random vectors having dimension $n \times 1$.

- There are four matrices associated with $\mathbf{X}$ and $\mathbf{Y}$

$$\begin{aligned}
\mathbf{C}_{\mathbf{X}} &= E\left[(\mathbf{X} - E[\mathbf{X}])(\mathbf{X} - E[\mathbf{X}])^T\right] \\
\mathbf{C}_{\mathbf{Y}} &= E\left[(\mathbf{Y} - E[\mathbf{Y}])(\mathbf{Y} - E[\mathbf{Y}])^T\right] \\
\mathbf{C}_{\mathbf{XY}} &= E\left[(\mathbf{X} - E[\mathbf{X}])(\mathbf{Y} - E[\mathbf{Y}])^T\right] \\
\mathbf{C}_{\mathbf{YX}} &= E\left[(\mathbf{Y} - E[\mathbf{Y}])(\mathbf{X} - E[\mathbf{X}])^T\right]
\end{aligned}$$

- The pdf of $\mathbf{Z}$ is the joint pdf of its real and imaginary parts i.e. the pdf of

$$\tilde{\mathbf{Z}} = \begin{bmatrix} \mathbf{X} \\ \mathbf{Y} \end{bmatrix}$$

Covariance and Pseudocovariance of Complex Random Vectors

- Covariance of $\mathbf{Z} = \mathbf{X} + j\mathbf{Y}$

$$\begin{aligned}\mathbf{C_Z} &= E\left[(\mathbf{Z} - E[\mathbf{Z}])(\mathbf{Z} - E[\mathbf{Z}])^H\right] \\ &= \mathbf{C_X} + \mathbf{C_Y} + j\left(\mathbf{C_{YX}} - \mathbf{C_{XY}}\right)\end{aligned}$$

- Pseudocovariance of $\mathbf{Z} = \mathbf{X} + j\mathbf{Y}$

$$\begin{aligned}\tilde{\mathbf{C}}_\mathbf{Z} &= E\left[(\mathbf{Z} - E[\mathbf{Z}])(\mathbf{Z} - E[\mathbf{Z}])^T\right] \\ &= \mathbf{C_X} - \mathbf{C_Y} + j\left(\mathbf{C_{XY}} + \mathbf{C_{YX}}\right)\end{aligned}$$

- A complex random vector $\mathbf{Z}$ is called proper if its pseudocovariance is zero

$$\begin{aligned}\mathbf{C_X} &= \mathbf{C_Y} \\ \mathbf{C_{XY}} &= -\mathbf{C_{YX}}\end{aligned}$$

Motivating the Definition of Proper Random Vectors

- For $n = 1$, a proper complex RV $Z = X + jY$ satisfies

$$\begin{aligned} \text{var}(X) &= \text{var}(Y) \\ \text{cov}(X, Y) &= -\text{cov}(Y, X) \end{aligned}$$

- Thus $\text{cov}(X, Y) = 0$
- If Z is a proper complex Gaussian random variable, its real and imaginary parts are independent

Proper Complex Gaussian Random Vectors

For random vector $\mathbf{Z} = \mathbf{X} + j\mathbf{Y}$ and $\tilde{\mathbf{Z}} = \begin{bmatrix} \mathbf{X} & \mathbf{Y} \end{bmatrix}^T$, the pdf is given by

$$p(\mathbf{z}) = p(\tilde{\mathbf{z}}) = \frac{1}{(2\pi)^n (\det(\mathbf{C}_{\tilde{\mathbf{Z}}}))^{\frac{1}{2}}} \exp\left(-\frac{1}{2}(\tilde{\mathbf{z}} - \tilde{\mathbf{m}})^T \mathbf{C}_{\tilde{\mathbf{Z}}}^{-1} (\tilde{\mathbf{z}} - \tilde{\mathbf{m}})\right)$$

If $\mathbf{Z}$ is proper, the pdf is given by

$$p(\mathbf{z}) = \frac{1}{\pi^n \det(\mathbf{C}_{\mathbf{Z}})} \exp\left(-(\mathbf{z} - \mathbf{m})^H \mathbf{C}_{\mathbf{Z}}^{-1} (\mathbf{z} - \mathbf{m})\right)$$

Random Processes

Random Process

Definition

An indexed collection of random variables $\{X(t) : t \in \mathcal{T}\}$.

Discrete-time Random Process $\mathcal{T} = \mathbb{Z}$ or $\mathbb{N}$

Continuous-time Random Process $\mathcal{T} = \mathbb{R}$

Statistics

Mean function

$$m_X(t) = E[X(t)]$$

Autocorrelation function

$$R_X(t_1, t_2) = E[X(t_1)X^*(t_2)]$$

Autocovariance function

$$C_X(t_1, t_2) = E\left[(X(t_1) - m_X(t_1))\,(X(t_2) - m_X(t_2))^*\right]$$

Crosscorrelation and Crosscovariance

Crosscorrelation

$$R_{X_1,X_2}(t_1,t_2) = E[X_1(t_1)X_2^*(t_2)]$$

Crosscovariance

$$\begin{aligned} C_{X_1,X_2}(t_1,t_2) &= E\left[\left(X_1(t_1) - m_{X_1}(t_1)\right)\left(X_2(t_2) - m_{X_2}(t_2)\right)^*\right] \\ &= R_{X_1,X_2}(t_1,t_2) - m_{X_1}(t_1)m_{X_2}^*(t_2) \end{aligned}$$

Stationary Random Process

Definition

A random process which is statistically indistinguishable from a delayed version of itself.

Properties

- For any $n \in \mathbb{N}$, $(t_1, \ldots, t_n) \in \mathbb{R}^n$ and $\tau \in \mathbb{R}$, $(X(t_1), \ldots, X(t_n))$ has the same joint distribution as $(X(t_1 - \tau), \ldots, X(t_n - \tau))$.
- $m_X(t) = m_X(0)$
- $R_X(t_1, t_2) = R_X(t_1 - \tau, t_2 - \tau) = R_X(t_1 - t_2, 0)$

Wide Sense Stationary Random Process

Definition

A random process is WSS if

$$\begin{aligned} m_X(t) &= m_X(0) \quad \text{for all } t \text{ and} \\ R_X(t_1, t_2) &= R_X(t_1 - t_2, 0) \quad \text{for all } t_1, t_2. \end{aligned}$$

Autocorrelation function is expressed as a function of $\tau = t_1 - t_2$ as $R_X(\tau)$.

Definition (Power Spectral Density of a WSS Process)

The Fourier transform of the autocorrelation function.

$$S_X(f) = \mathcal{F}\left(R_X(\tau)\right)$$

Energy Spectral Density

Definition

For a signal $s(t)$, the energy spectral density is defined as

$$E_s(f) = |S(f)|^2.$$

Motivation

Pass $s(t)$ through an ideal narrowband filter with response

$$H_{f_0}(f) = \begin{cases} 1, & \text{if } f_0 - \frac{\Delta f}{2} < f < f_0 + \frac{\Delta f}{2} \\ 0, & \text{otherwise} \end{cases}$$

Output is $Y(f) = S(f)H_{f_0}(f)$. Energy in output is given by

$$\int_{-\infty}^{\infty} |Y(f)|^2 \, df = \int_{f_0 - \frac{\Delta f}{2}}^{f_0 + \frac{\Delta f}{2}} |S(f)|^2 \, df \approx |S(f_0)|^2 \Delta f$$

Power Spectral Density

Motivation

PSD characterizes spectral content of random signals which have infinite energy but finite power

Example (Finite-power infinite-energy signal)

Binary PAM signal

$$x(t) = \sum_{n=-\infty}^{\infty} b_n p(t - nT)$$

Power Spectral Density of a Realization

Time windowed realizations have finite energy

$$\begin{aligned} x_{T_o}(t) &= x(t) I_{[-\frac{T_o}{2}, \frac{T_o}{2}]}(t) \\ S_{T_o}(f) &= \mathcal{F}(x_{T_o}(t)) \\ \hat{S}_x(f) &= \frac{|S_{T_o}(f)|^2}{T_o} \quad \text{(PSD Estimate)} \end{aligned}$$

Definition (PSD of a realization)

$$\bar{S}_x(f) = \lim_{T_o \to \infty} \frac{|S_{T_o}(f)|^2}{T_o}$$

Autocorrelation Function of a Realization

Motivation

$$\hat{S}_x(f) = \frac{|S_{T_o}(f)|^2}{T_o} \quad \rightleftharpoons \quad \frac{1}{T_o}\int_{-\infty}^{\infty} x_{T_o}(u)x^*_{T_o}(u-\tau)\,du$$

$$= \frac{1}{T_o}\int_{-\frac{T_o}{2}}^{\frac{T_o}{2}} x_{T_o}(u)x^*_{T_o}(u-\tau)\,du$$

$$= \hat{R}_x(\tau) \qquad \text{(Autocorrelation Estimate)}$$

Definition (Autocorrelation function of a realization)

$$\bar{R}_x(\tau) = \lim_{T_o\to\infty} \frac{1}{T_o}\int_{-\frac{T_o}{2}}^{\frac{T_o}{2}} x_{T_o}(u)x^*_{T_o}(u-\tau)\,du$$

The Two Definitions of Power Spectral Density

Definition (PSD of a WSS Process)

$$S_X(f) = \mathcal{F}\left(R_X(\tau)\right)$$

where $R_X(\tau) = E\left[X(t)X^*(t-\tau)\right]$.

Definition (PSD of a realization)

$$\bar{S}_x(f) = \mathcal{F}\left(\bar{R}_x(\tau)\right)$$

where

$$\bar{R}_x(\tau) = \lim_{T_o \to \infty} \frac{1}{T_o} \int_{-\frac{T_o}{2}}^{\frac{T_o}{2}} x_{T_o}(u) x_{T_o}^*(u-\tau)\, du$$

Both are equal for ergodic processes

Ergodic Process

Definition

A stationary random process is ergodic if time averages equal ensemble averages.

- Ergodic in mean

$$\lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)\, dt = E[X(t)]$$

- Ergodic in autocorrelation

$$\lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)x^*(t-\tau)\, dt = R_X(\tau)$$

Gaussian Random Processes

Gaussian Random Process

Definition

A random process $\{X(t) : t \in \mathcal{T}\}$ is Gaussian if its samples $X(t_1), \ldots, X(t_n)$ are jointly Gaussian for any $n \in \mathbb{N}$.

Properties

- The mean and autocorrelation functions completely characterize a Gaussian random process.
- Gaussian WSS processes are stationary.
- If the input to an LTI system is a Gaussian RP, the output is also a Gaussian RP.

White Gaussian Noise

Definition

A zero mean WSS Gaussian random process with power spectral density

$$S_n(f) = \frac{N_0}{2}.$$

Remarks

- $R_n(\tau) = \frac{N_0}{2}\delta(\tau)$
- $\frac{N_0}{2}$ is termed the two-sided PSD and has units Watts per Hertz.

Thanks for your attention